REGULARIZATION FOR $n$TH-ORDER LINEAR BOUNDARY VALUE PROBLEMS USING $m$TH-ORDER DIFFERENTIAL OPERATORS

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Abstract. Let $X$ and $Y$ denote real Hilbert spaces, and let $L: X \to Y$ be a closed densely-defined linear operator having closed range. Given an element $y \in Y$, we determine least squares solutions of the linear equation $Lx = y$ by using the method of regularization.

Let $Z$ be a third Hilbert space, and let $T: X \to Z$ be a linear operator with $\mathcal{D}(L) \subseteq \mathcal{D}(T)$. Under suitable conditions on $L$ and $T$ and for each $\alpha \neq 0$, we show that there exists a unique element $x_\alpha \in \mathcal{D}(L)$ which minimizes the functional $G_\alpha(x) = \|Lx - y\|^2 + \alpha^2\|Tx\|^2$, and the $x_\alpha$ converge to a least squares solution $x_0$ of $Lx = y$ as $\alpha \to 0$.

We apply our results to the special case where $L$ is an $n$th-order differential operator in $X = L^2[a, b]$, and we regularize using for $T$ an $m$th-order differential operator in $L^2[a, b]$ with $m \leq n$. Using an approximating space of Hermite splines, we construct numerical solutions to $Lx = y$ by the method of continuous least squares and the method of discrete least squares.

I. Introduction and general theory.


Throughout this paper we let $X$ and $Y$ denote real Hilbert spaces with inner products $(\cdot, \cdot)$ and norms $\|\cdot\|$, and we let $L: X \to Y$ be a closed densely-defined linear operator having closed range. Given an element $y \in Y$, we want to determine least squares solutions of the linear equation

$$(1.1) \quad Lx = y$$

by using the method of regularization.

To introduce regularization, let $Z$ be a third Hilbert space and let $T: X \to Z$ be a linear operator with $\mathcal{D}(L) \subseteq \mathcal{D}(T)$. For each real number $\alpha \neq 0$ let $G_\alpha$ be the functional defined on $\mathcal{D}(L)$ by

$$G_\alpha(x) = \|Lx - y\|^2 + \alpha^2\|Tx\|^2.$$
Under suitable conditions on $L$ and $T$ there exists a unique element $x_a \in \mathcal{D}(L)$ satisfying

\begin{equation}
G_a(x_a) = \inf_{x \in \mathcal{D}(L)} G_a(x),
\end{equation}

and the $x_a$ converge to a least squares solution of (1.1) as $\alpha \to 0$.

In the following exposition many well-known results regarding regularization are given. Most are included in §I, where we establish the general theory of regularization. In §II we study regularizing linear boundary value problems using similar differential operators, and in §III we examine regularization in a discrete setting. These two sections represent the principal contributions of this paper.

§I follows the work of Locker and Prenter [19, 21] by making appropriate assumptions on $L$ and $T$, the regularizing operator, and by viewing regularization as a least squares process, an idea introduced and outlined by Nashed [23]. For each real number $\alpha \neq 0$ we establish the existence and uniqueness of the regularized approximate solution $x_a$, and show that the $x_a$ converge to a least squares solution of (1.1) as $\alpha \to 0$. The technique employed to establish error estimates is the *alternative method*, surveys of which have been given by Cesari [4, 5] and Hale [10]. The special form of the alternative method used in this paper was developed by Kannan and Locker [13]. The rate of convergence will be shown to be of order $\alpha^2$, which is in contrast to the rate of convergence of order $|\alpha|$ obtained by Ivanov [12] with $T = I$, the identity operator.

In §§II and III we let $L$ be an $n$th-order differential operator in $X = L^2[a, b]$, and we regularize using for $T$ an $m$th-order differential operator in $L^2[a, b]$ with $m \leq n$. Using an approximating space of Hermite splines, we construct numerical solutions to (1.1) by the method of continuous least squares in §II, and by the method of discrete least squares in §III.

In both sections we establish error estimates by first showing superconvergence at the knots of the partition associated with the spline approximating space. This is known to occur with Galerkin approximates to particular two-point boundary value problems [9]. Developments in both sections extend work of Sammon [26] and Locker and Prenter [18] in a well-posed setting. The discrete least squares analysis in [18] is an alternate analysis of the method of collocation applied to well-posed ordinary differential equations, which was done by deBoor and Swartz [8]. Ascher [1] has studied discrete least squares together with Gaussian quadrature in a well-posed setting. The analysis is relatively complicated, but it allows for a more general spline approximating space than the Hermite splines used here.

An outline of §II follows: Part (A) describes the mathematical setting and introduces $\hat{x}_a$, the (continuous) least squares approximate to $x_a$ which is selected from a spline subspace. In (F) we establish the convergence of $\hat{x}_a$ to $x_0 = L^n y$, a particular least squares solution of (1.1).

To develop the error estimates, we introduce in (B) the generalized inverse $L^+$ and list properties of the associated generalized Green’s function $G(t, s)$, which has been characterized by Locker in [16, 17]. This information along with appropriate adjoints of $L$ and $A = T|\mathcal{D}(L)$, which are examined in (C), is used to establish smoothness
and uniform boundedness of the $x_\alpha$ in (D). In (E) we examine the linear operator $T_\alpha x = (Lx, \alpha Ax)$, explicitly characterizing the local and global smoothness of the associated representors for its generalized inverse. This parallels the representor theory of Locker and Prenter [19, 22].

Because of the exact integration involved in the computation of $\tilde{x}_\alpha$, the least squares approximate to $x_\alpha$, the method in §II is in general impractical. This is readily overcome by the implementation of discrete least squares and numerical quadrature in §III.

We begin in (A) of the third section by introducing $\hat{x}_\alpha$, the discrete least squares approximate to $x_\alpha$, proving its existence and uniqueness in (B) and (C). In (D) we introduce $\hat{x}_\alpha$, the quasi-discrete least squares approximate to $x_\alpha$, using it to establish error estimates for $\hat{x}_\alpha - x_0$ in (E).

(B) The method of regularization. To analyze the method of regularization, it is necessary to work with $D(L)$ under two different structures. The first is the graph norm structure of $L$:

$$(x, y)_L = (x, y) + (Lx, Ly), \quad \|x\|_L = (x, x)^{1/2}. $$

Because $L$ is a closed operator, it follows that $D(L)$ is a Hilbert space under this structure. We now make the following assumptions on $L$ and $T$:

(I) $\mathcal{N}(L) \cap \mathcal{N}(T) = \{0\}$.

(II) The restriction $A = T|D(L)$ is continuous from $D(L)$ under the graph norm structure of $L$ into $Z$ under its standard structure.

(III) There exists a constant $\beta > 0$ such that $\|Tx\| \geq \beta \|x\|$ for all $x \in \mathcal{N}(L)$.

The second structure on $D(L)$ is the $\ast$-structure:

$$(x, y)_\ast = (Lx, Ly) + (Ax, Ay), \quad \|x\|_\ast = (x, x)^{1/2}. $$

From [19, Lemma 3.3, p. 509] it can be shown that $(\ , \ )_\ast$ is an inner product on $D(L)$, $D(L)$ is a Hilbert space under the inner product $(\ , \ )_\ast$ and $\|\cdot\|_L$ and $\|\cdot\|_\ast$ are equivalent norms on $D(L)$. It follows that both $L$ and $A$ are bounded linear operators from $D(L)$ under the $\ast$-structure into $Y$ and $Z$ under their standard structures, resp. Thus, there exist bounded everywhere-defined adjoint operators

$L^\sharp: Y \to D(L)$ and $A^\sharp: Z \to D(L)$

given by the equations

$$(Lx, y) = (x, L^\sharp y)_\ast \quad \text{and} \quad (Ax, z) = (x, A^\sharp z)_\ast$$

for all $x \in D(L)$, $y \in Y$, and $z \in Z$.

We get the main existence and uniqueness theorem for the method of regularization from [19, Theorem 3.1, p. 508].

**Theorem 1.1.** For each $y \in Y$ and each $\alpha \neq 0$, the equation

$$L^\sharp Lx + \alpha^2 A^\sharp Ax = L^\sharp y$$

has a unique solution $x_\alpha \in D(L)$. Moreover, the element $x_\alpha$ is also characterized as the unique element in $D(L)$ satisfying $G_{\alpha}(x_\alpha) = \inf_{x \in D(L)} G_{\alpha}(x)$. 
It is possible to view regularization as a least squares process, an approach introduced by Nashed [23]. Let us use the standard product structure on $Y \oplus Z$:

$$\langle (u, v), (\xi, \eta) \rangle = (u, \xi) + (v, \eta), \quad \|(u, v)\| = \left(\|u\|^2 + \|v\|^2\right)^{1/2}$$

for $(u, v), (\xi, \eta) \in Y \oplus Z$. For each real $\alpha \neq 0$ let $T_\alpha$ be the linear operator from $\mathcal{D}(L)$ into $Y \oplus Z$ defined by $T_\alpha x = (Lx, \alpha Ax)$. From assumption (I) we see that $T_\alpha$ is a 1-1 mapping, and clearly $T_\alpha$ is bounded from $\mathcal{D}(L)$ under the $*$-structure into $Y \oplus Z$ under the product structure. Thus, $T_\alpha$ has a bounded, everywhere-defined adjoint $T_\alpha^*: Y \oplus Z \rightarrow \mathcal{D}(L)$ determined by the equation

$$\langle T_\alpha^* x, z \rangle = \langle x, T_\alpha^* z \rangle_\ast \quad \text{for all } x \in \mathcal{D}(L), z \in Y \oplus Z.$$

It follows from [19, Lemma 3.3, p. 509] that the range $\mathcal{R}(T_\alpha)$ is closed in $Y \oplus Z$ under the product structure. Let $Q_\alpha$ be the orthogonal projection from $Y \oplus Z$ onto $\mathcal{N}(T_\alpha^*)$ in terms of the product structure, so that $I - Q_\alpha$ is the orthogonal projection onto $\mathcal{R}(T_\alpha)$. Let $T_\alpha^+$ be the generalized inverse of $T_\alpha$ given by $T_\alpha^+ = T_\alpha^{-1}(I - Q_\alpha)$. From the Open Mapping Theorem the operator $T_\alpha^+: Y \oplus Z \rightarrow \mathcal{D}(L)$ is bounded from the product structure into the $*$-structure. More precisely, we have

$$\|T_\alpha^+\|_* = \|T_\alpha^{-1}\|_* \leq 1/|\alpha| \quad \text{for } 0 < |\alpha| \leq 1,$$

where the operator norm is taken from the product structure into the $*$-structure on $\mathcal{D}(T_\alpha) = \mathcal{D}(L)$.

If $y \in Y$ and if we set $\hat{y} = (y, 0) \in Y \oplus Z$, then $\|T_\alpha x - \hat{y}\|^2 = \|Lx - y\|^2 + \alpha^2\|Ax\|^2 = G_\alpha(x)$. We immediately get

**Lemma 1.2.** The element $x_\alpha$ in Theorem 1.1 is characterized as the unique least squares solution of the equation $T_\alpha x = \hat{y}$, i.e. $x_\alpha = T_\alpha^+ \hat{y}$.

(C) Error estimates in the $*$-norm for $x_\alpha - L^\mu y$. Next, we establish the convergence of the $x_\alpha$ to a least squares solution of (1.1) as $\alpha \rightarrow 0$ and develop the associated error estimates. Let $N_\perp^\ast$ denote the orthogonal complement of a subspace $N$ in $\mathcal{D}(L)$ with respect to the inner product $(,)_\ast$, and let $P_\ast$ and $I - P_\ast$ be the orthogonal projections from $\mathcal{D}(L)$ onto $\mathcal{N}(L^\perp)$ and $\mathcal{N}(L^\perp)^\perp = \mathcal{R}(L^\perp)$, resp. In terms of the standard inner product on $Y$, let $Q$ and $I - Q$ be the orthogonal projections from $Y$ onto $\mathcal{N}(L^\perp) = \mathcal{R}(L^\perp)$ and $\mathcal{N}(L^\perp)^\perp = \mathcal{R}(L)$, resp. Let $L^\mu: Y \rightarrow \mathcal{D}(L)$ be the generalized inverse of $L$, where we work with $\mathcal{D}(L)$ under the $*$-structure and with $Y$ under the standard structure:

$$L^\mu = \left[ L|\mathcal{D}(L) \cap \mathcal{N}(L)^\perp \right]^{-1}(I - Q).$$

Since we are assuming $\mathcal{R}(L)$ is closed in $Y$, the linear operator $L^\mu$ is continuous from $Y$ under its standard structure into $\mathcal{D}(L)$ under the $*$-structure. The range $\mathcal{R}(L)$ being closed in $Y$ also implies that the ranges $\mathcal{R}(L^\perp)$ and $\mathcal{R}(L^\perp L)$ are closed in $\mathcal{D}(L)$ under the $*$-structure, and $\mathcal{R}(L^\perp L) = \mathcal{R}(L^\perp)$ and $\mathcal{N}(L^\perp L) = \mathcal{N}(L)$ [3, pp. 481–482]. Form the generalized inverse $(L^\perp)^\mu: \mathcal{D}(L) \rightarrow Y$, which is continuous from $\mathcal{D}(L)$ under the $*$-structure into $Y$ under the standard structure

$$(L^\perp)^\mu = \left[ L^\perp|\mathcal{N}(L^\perp)^\perp \right]^{-1}(I - P_\ast).$$
Also, the operator $L^* L : \mathcal{D}(L) \to \mathcal{D}(L)$ is continuous under the $\ast$-structure, and it is easy to check that $L^* L$ is symmetric, and hence, is selfadjoint. Form its generalized inverse $K = (L^* L)^m : \mathcal{D}(L) \to \mathcal{D}(L)$, which is given by

$$K = (L^* L)^m = \left[ L^* L[\mathcal{D}(L) \cap \mathcal{N}(L)]^{-1} \right]^{-1}(I - P_{\ast}),$$

and which is continuous under the $\ast$-structure. The operator $K = (L^* L)^m = L^m(L^*)^m$ plays a key role in establishing the following uniform boundedness and convergence of the $x_\alpha$ [21].

**Theorem 1.3.** The element $x_\alpha \in \mathcal{D}(L) \cap \mathcal{N}'(L)^{-\ast}$, for each $\alpha \neq 0$, and the $x_\alpha$ are uniformly bounded and converge to $L^\alpha y$ with rate of convergence of order $\alpha^2$ as $\alpha \to 0$, with estimates

$$\|x_\alpha\|_* \leq \frac{s}{s} \|L^\alpha y\|_*$$

and

$$\|x_\alpha - L^\alpha y\|_* \leq \left[ \frac{32}{\bar{s}} \|K\| + \frac{4}{s} \right] \|L^\alpha y\|_* \alpha^2$$

for $0 < |\alpha| \leq \min\{\frac{1}{s}, \sqrt{3}/2\sqrt{2}\|K\|\}$.

**II. Regularization and nth-order linear boundary value problems with $T$ an mth-order differential operator.**

(A) *Mathematical preliminaries and the least squares approximate $x_\alpha$.*** In this section the regularization method is applied to the numerical solution of ill-posed two-point linear boundary value problems. This extends earlier work [18] for well-posed problems. We will work in the real Hilbert space $X = Y = Z = L^2[a, b]$ under the standard inner product and norm, and in the subspace

$$H^n[a, b] = \{ x \in C^{n-1}[a, b] | x^{n-1} \text{ is absolutely continuous on } [a, b], \text{ and } x^{(n)} \in L^2[a, b] \}.$$

Let $L$ be an $n$th-order differential operator in $L^2[a, b]$ determined by a formal differential operator $\tau = \sum_{i=0}^{n} a_i(t) (d/dt)^i$ with coefficients $a_i(t) \in C^\infty[a, b]$ and $a_i(t) \neq 0$ on $[a, b]$, and by a set $k_0 (0 \leq k_0 \leq 2n)$ linearly independent boundary values $B_1, \ldots, B_{k_0}$:

$$\mathcal{D}(L) = \{ x \in H^n[a, b] | B_i(x) = 0, i = 1, \ldots, k_0 \}, \quad Lx = \tau x.$$ 

Let $T$ be an $m$th-order differential operator in $L^2[a, b]$ with $m \leq n$ which is determined by a formal differential operator $\sigma = \sum_{i=0}^{m} b_i(t) (d/dt)^i$ with coefficients $b_i(t) \in C^\infty[a, b]$ and $b_m(t) \neq 0$ on $[a, b]$ and by a set of $k_1 (0 \leq k_1 \leq 2m)$ linearly independent boundary values $C_1, \ldots, C_{k_1}$:

$$\mathcal{D}(T) = \{ x \in H^m[a, b] | C_i(x) = 0, i = 1, \ldots, k_1 \}, \quad Tx = \sigma x.$$ 

It is well known that $L$ and $T$ are closed densely-defined linear operators in $L^2[a, b]$ with $\mathcal{R}(L)$ and $\mathcal{R}(T)$ closed subspaces in $L^2[a, b]$.

For a given function $y \in L^2[a, b]$ we want to numerically construct approximate least squares solutions of the linear boundary value problem $Lx = y$, using $T$ as the regularizing operator. In the sequel we assume that $\mathcal{D}(L) \subseteq \mathcal{D}(T)$ with $L$ and $T$ satisfying conditions (I)–(III) of §I.
In our analysis we will frequently use the norm
\[ \|x\|_{H^n} = \sum_{i=0}^{n-1} \|x^{(i)}\|_\infty + \|x^{(n)}\| \]
on \mathcal{D}(L), where \( \| \cdot \|_\infty \) is the standard \( L^\infty \)-norm. It is well known that \( \mathcal{D}(L) \) is a Banach space under the norm \( \| \cdot \|_{H^n} \), and \( \| \cdot \|_{H^n} \) is equivalent to the graph norm \( \| \cdot \|_L \) of \( L \). We will refer to \( \| \cdot \|_{H^n} \) as the \( H^n \)-norm and to the associated Banach space structure as the \( H^n \)-structure for \( \mathcal{D}(L) \).

Note that \( A = T|\mathcal{D}(L) \) is continuous from \( \mathcal{D}(L) \) under the \( H^n \)-structure into \( L^2[a, b] \) under its standard structure, and hence, condition (II) is automatically satisfied. Also, if condition (I) is satisfied, then \( T \) is 1-1 on the finite-dimensional subspace \( \mathcal{N}(L) \), implying that condition (III) is satisfied. Thus, conditions (I)–(III) hold whenever condition (I) is satisfied.

Treating \( L \) and \( T \) in the setting of unbounded linear operators, the adjoints \( L^* \) and \( T^* \) exist as closely densely-defined operators given by \( (Lx, z) = (x, L^*z) \) for all \( x \in \mathcal{D}(L) \) and \( z \in \mathcal{D}(L^*) \), and \( (Tx, z) = (x, T^*z) \) for all \( x \in \mathcal{D}(T) \) and \( z \in \mathcal{D}(T^*) \). We know that \( L^* \) is an \( n \)-th-order differential operator in \( L^2[a, b] \) determined by the formal adjoint \( \tau^* \) and by a set of \( 2n - k_0 \) adjoint boundary values \( B^*_1, \ldots, B^*_{2n-k_0} \), and \( T^* \) is an \( m \)-th-order differential operator in \( L^2[a, b] \) determined by \( \sigma^* \) and a set of \( 2m - k_1 \) adjoint boundary values \( C^*_1, \ldots, C^*_{2m-k_1} \). Also, the product operator \( L^*L \) is a \( 2n \)-th-order differential operator in \( L^2[a, b] \) determined by the formal differential operator \( \tau^*\tau \) and by the set of \( 2n \) boundary values \( B_1, \ldots, B_{k_0}, B_1^*\tau, \ldots, B_{2n-k_0}^*\tau \), and \( L^*L \) is selfadjoint [17, Chapter IV].

Let \( P \) denote the family of all partitions \( \Delta \) of \( [a, b] \). For \( \Delta \in P \) given by \( \Delta: a = t_0 < t_1 < \cdots < t_N = b \), let
\[ h = \max_{1 \leq i \leq N} (t_i - t_{i-1}) \quad \text{and} \quad h = \min_{1 \leq i \leq N} (t_i - t_{i-1}). \]
In terms of a partition \( \Delta \) we introduce the subspace
\[ C^{n,k}_\Delta[a,b] = \{ x \in H^n[a,b] | x \in C^{n+k}[t_{i-1}, t_i] \text{ for } i = 1, \ldots, N \} \]
for \( 0 \leq k \leq \infty \), which is a Banach space under the norm
\[ \|x\|_{C^{n,k}_\Delta} = \sum_{j=0}^{n+k} \|x^{(j)}\|_\infty \quad \text{for } 0 \leq k < \infty. \]

Let \( \text{Sp}(2n-1, \Delta, n-1) \) denote the space of Hermite splines of degree \( 2n - 1 \) having \( C^{n-1} \) global continuity. The cardinal basis
\[ \Phi_N = \{ \varphi_{ij} | 0 \leq i \leq N, 0 \leq j \leq n - 1 \} \]
for \( \text{Sp}(2n-1, \Delta, n-1) \) consists of the unique splines from \( \text{Sp}(2n-1, \Delta, n-1) \) solving the interpolation problems
\[ \varphi^{(m)}_{ij}(t_l) = \delta_{il}\delta_{jm}, \quad 0 \leq i, l \leq N; 0 \leq j, m \leq n - 1. \]
Each \( x \in \text{Sp}(2n-1, \Delta, n-1) \) has the unique representation
\[ x(t) = \sum_{i=0}^{N} \sum_{j=0}^{n-1} x^{(j)}(t_i) \varphi_{ij}(t), \]
which will be used later to establish certain error estimates.
Assume that the regularization parameter $\alpha \neq 0$ has been fixed so that the corresponding $x_\alpha$ is a good approximation to the least squares solution $x_0 = L^n y$ (see Theorem 1.3). We are going to use the method of least squares to approximate $x_\alpha$ by means of Hermite splines. Let $S_N = \text{Sp}(2n - 1, \Delta, n - 1) \cap \mathcal{D}(L)$ and let $\varphi_1, \ldots, \varphi_M$, where $M = n(N + 1) - k_0$, be a basis for $S_N$.

By Lemma 1.2 we know that
\[
(T_\alpha x_\alpha - \hat{y}) = \inf_{x \in \mathcal{D}(L)} |T_\alpha x - \hat{y}|, 
\]
or equivalently,
\[
\langle T_\alpha x_\alpha - \hat{y}, T_\alpha x \rangle \quad \text{for all } x \in \mathcal{D}(T_\alpha) = \mathcal{D}(L).
\]
Define the (continuous) least squares approximate $\tilde{x}_\alpha \in S_N$ to $x_\alpha$ by the condition
\[
|T_\alpha \tilde{x}_\alpha - \hat{y}| = \inf_{\varphi \in S_N} |T_\alpha \varphi - \hat{y}|, 
\]
or equivalently,
\[
\langle T_\alpha \tilde{x}_\alpha - \hat{y}, T_\alpha \varphi \rangle = 0 \quad \text{for all } \varphi \in S_N.
\]
Combining (2.3) and (2.5) we have
\[
|T_\alpha x_\alpha - T_\alpha \tilde{x}_\alpha| \leq |T_\alpha x_\alpha - T_\alpha \varphi| \quad \text{for all } \varphi \in S_N.
\]
If we write $\tilde{x}_\alpha = \sum_{j=1}^{M} c_j \varphi_j$, then from (2.6) the coefficients $c_1, \ldots, c_M$ satisfy the linear system
\[
\sum_{j=1}^{M} \left[ (L \varphi_i, L \varphi_j) + \alpha^2 (A \varphi_i, A \varphi_j) \right] c_j = (y, L \varphi_i), \quad 1 \leq i \leq M.
\]

(B) The generalized inverse $L^+$ and the generalized Green's function $G(t, s)$. The remainder of this section is devoted to establishing error estimates for $\tilde{x}_\alpha(t) - x_0(t)$, where $x_0 = L^n Y$ and $j = 0, 1, \ldots, n - 1$. This will require establishing smoothness and uniform boundedness of the $x_\alpha$, and investigating the operators $T_\alpha$ and their generalized inverses and associated representors. Toward this end we introduce the generalized inverse $L^+$ and the generalized Green's function $G(t, s)$ for $L$, and consider various adjoints of the operators $L$ and $A = T|\mathcal{D}(L)$.

Let $L^+$ be the generalized inverse of $L$ given by
\[
L^+ = \left[ L|\mathcal{D}(L) \cap \mathcal{N}(L) \right]^{-1} (I - Q) : L^2[a, b] \to \mathcal{D}(L),
\]
where $Q$ and $I - Q$ are the $L^2$-orthogonal projections from $L^2[a, b]$ onto $\mathcal{N}(L^*)$ and $\mathcal{R}(L)$, resp., and let $G$ be the generalized Green's function associated with $L$, which satisfies
\[
L^+ z(t) = \int_a^b G(t, s) z(s) \, ds \quad \text{for all } z \in L^2[a, b], a \leq t \leq b
\]
(see [16 or 17]). For each $t \in [a, b]$ we have $G_t(\cdot) = G(t, \cdot) \in \mathcal{R}(L)$ and
\[
\|G_t\| \leq \|L^+\|,
\]
where we work with $L^+$ from $L^2[a, b]$ under the standard structure into $\mathcal{D}(L)$ under the $H^n$-structure.
We have the following result concerning the boundedness of the operators \( L \) and \( L^+ \). See [17, Chapter IV].

**Lemma 2.1.** For \( \Delta \in P, 0 \leq j < \infty, 0 \leq k < \infty \), there exist positive constants \( \alpha_{jk} \) and \( \beta_{jk} \) independent of \( \Delta \) such that

(i) \[ \|Lx\|_{C^{j,k}} \leq \alpha_{jk} \|x\|_{C^{j,k}} \quad \text{for all } x \in \mathcal{D}(L) \cap C^{n+j,k}[a, b], \]

and

(ii) \[ \|L^+z\|_{C^{j,k}} \leq \beta_{jk} \|z\|_{C^{j,k}} \quad \text{for all } z \in C^{j,k}[a, b]. \]

(C) The adjoints \( L_\bullet \) and \( A_\bullet \) and their relationships with the adjoints \( L^\# \) and \( A^\# \). We will now examine the operator \( L \) and \( A \) on \( \mathcal{D}(L) \) with an emphasis on the graph norm structure of \( L \). We know that \( L \) and \( A \) are both bounded linear operators from the graph norm structure of \( L \) into the \( L^2 \)-structure, and hence, there exist adjoint operators \( L_*: L^2[a, b] \to \mathcal{D}(L) \) and \( A_*: L^2[a, b] \to \mathcal{D}(L) \) determined by the equations

\[
(Lx, z) = (x, L_* z)_L \\
(Ax, z) = (x, A_* z)_L
\]

for all \( x \in \mathcal{D}(L) \) and \( z \in L^2[a, b] \). Both \( L_* \) and \( A_* \) are bounded from the \( L^2 \)-structure into the graph norm structure of \( L \). We will characterize the adjoint operators \( L_* \) and \( A_* \) and exhibit their relationships with the adjoint operators \( L^\# \) and \( A^\# \). The following two lemmas are proved in a straightforward manner using the 1-1, selfadjoint operators \( L^*L + I \) and \( LL^* + I \) [17, Chapter I].

**Lemma 2.2.**

(i) \[ L_* z = (L^*L + I)^{-1}L^* z \quad \text{for all } z \in \mathcal{D}(L^*). \]

(ii) \[ L_* z = L^*(LL^* + I)^{-1} z \quad \text{for all } z \in L^2[a, b]. \]

**Lemma 2.3.** \( A_* z = (L^*L + I)^{-1}T^* z \) for all \( z \in \mathcal{D}(T^*) \).

The operator \( L^*L + I \), its Green's function \( \tilde{G}(t, s) \), and [17, Chapter IV] give us the following:

\[
A_* z(t) = \sum_{j=0}^{m} \int_{a}^{b} \left( \frac{\partial}{\partial s} \right)^j \tilde{G}(t, s) \left[ b_j(s) z(s) \right] ds \\
\text{a.e. on } [a, b] \text{ for all } z \in L^2[a, b].
\]

If we combine (2.12) with [17, Chapter III], then we obtain the following results for \( A \) and its adjoint \( A^\# \):

**Theorem 2.4.** Let \( \Delta \in P, 0 \leq j < \infty, \) and \( 0 \leq k < \infty, \) and let \( T \) be an \( m \)th-order differential operator with \( m \leq n. \)

(i) If \( z \in C^{j,k}[a, b] \), then \( A_* z \in C^{2n-m+j,k}[a, b] \), and there exists a positive constant \( \alpha_{jk} \) independent of \( \Delta \) such that

\[ \|A_* z\|_{C^{j,k}} \leq \alpha_{jk} \|z\|_{C^{j,k}} \quad \text{for all } z \in C^{j,k}[a, b]. \]
regularization for boundary value problems

(ii) If \( x \in \mathcal{D}(L) \cap C^{m+j,k}_\Delta[a, b] \), then \( Ax \in C^{l-k}_\Delta[a, b] \) and there exists a positive constant \( \beta_{jk} \) independent of \( \Delta \) such that
\[
\|Ax\|_{C^{l-k}_\Delta} \leq \beta_{jk}\|x\|_{C^{m+j,k}_\Delta} \quad \text{for all } x \in \mathcal{D}(L) \cap C^{m+j,k}_\Delta[a, b].
\]

Recall from Theorem 1.1 that for a given \( y \in L^2[a, b] \) we have \( L^*Lx_\alpha + \alpha^2A^*Ax_\alpha = L^*y \), and hence, for \( u \in \mathcal{D}(L) \),
\[
0 = (L^*Lx_\alpha + \alpha^2A^*Ax_\alpha - L^*y, u) = (L^*Lx_\alpha + \alpha^2A^*Ax_\alpha - L^*y, u)_L.
\]
We conclude that \( x_\alpha \in \mathcal{D}(L) \) also satisfies the equation
\[
(2.13) \quad L^*Lx_\alpha + \alpha^2A^*Ax_\alpha = L^*y.
\]

For the operator \( L^*L + \alpha^2A^*A \) we have the following easily-proven result.

**Lemma 2.5.** For each \( \alpha \neq 0 \) the operator \( L^*L + \alpha^2A^*A \) is a bounded selfadjoint linear operator on \( \mathcal{D}(L) \) under the graph norm structure of \( L \) and it is invertible.

(D) The operator \( K_1 = (L^*L)^+ \), smoothness of the \( x_\alpha \), uniform boundedness of the \( x_\alpha \), and error estimates. Our next objective is to show that the smoothness of the \( x_\alpha \) depends on the smoothness of the given \( y \in L^2[a, b] \), and that the \( x_\alpha \) are uniformly bounded in the \( C^{n-k}_\Delta \)-norm when \( y \in C^{0,k}_\Delta[a, b] \), where \( \Delta \in \mathcal{P} \) and \( 0 \leq k < \infty \). The selfadjoint operator \( L^*_L : \mathcal{D}(L) \to \mathcal{D}(L) \), which is bounded under the graph norm structure of \( L \), will play a key role. Recall that the range \( \mathcal{R}(L) \) is closed in \( L^2[a, b] \) under the \( L^2 \)-structure. Hence, by [3, pp. 481–482] the ranges \( \mathcal{R}(L^*_L) \) and \( \mathcal{R}(L^*_L) \) are closed in \( \mathcal{D}(L) \) under the graph norm structure of \( L \), and \( \mathcal{R}(L^*_L) = \mathcal{R}(L^*_L) \) and \( \mathcal{N}(L^*_L) = \mathcal{N}(L) \). Form the generalized inverse \( K_1 = (L^*L)^+ : \mathcal{D}(L) \to \mathcal{D}(L) \), which is given by
\[
K_1 = (L^*L)^+ = \left[ L^*_L|\mathcal{D}(L) \cap \mathcal{N}(L^*)^1 \right]^{-1}(I - P),
\]
where \( P \) is the \( L^2 \)-orthogonal projection from \( L^2[a, b] \) onto \( \mathcal{N}(L) \). By the Open Mapping Theorem the linear operator \( K_1 \) is bounded under the graph norm structure of \( L \).

Note that \( \mathcal{N}(L^*_L) = \mathcal{R}(L) = \mathcal{N}(L^*) \), and form the generalized inverse of \( L^*_L \), \( (L^*_L)^+ : \mathcal{D}(L) \to L^2[a, b] \), which is given by
\[
(L^*_L)^+ = \left[ L^*_L|\mathcal{N}(L^*)^1 \right]^{-1}(I - P).
\]
By the Open Mapping Theorem \( (L^*_L)^+ \) is bounded from the graph norm structure of \( L \) into the \( L^2 \)-structure.

The following theorem concerns a restriction of the linear operator \( K_1 \). Its proof is based on the fact that
\[
(2.14) \quad K_1 = (L^*L)^+ = L^+(L^*_L)^+ = (I - P) + (L^*L)^+
\]
which follows from Lemma 2.2.

**Theorem 2.6.** If \( \Delta \in \mathcal{P} \), \( 0 \leq j < \infty \), and \( 0 \leq k < \infty \), then \( K_1 : \mathcal{D}(L) \cap C^{n+j,k}_\Delta[a, b] \to \mathcal{D}(L) \cap C^{n+j,k}_\Delta[a, b] \) and there exists a positive constant \( \gamma_{jk} \) independent of \( \Delta \) such that
\[
\|K_1x\|_{C^{n+j,k}_\Delta} \leq \gamma_{jk}\|x\|_{C^{n+j,k}_\Delta} \quad \text{for all } x \in \mathcal{D}(L) \cap C^{n+j,k}_\Delta[a, b].
\]
Applying $K_1$ to the left-hand side of (2.13) produces $(I - P)x_a + \alpha^2 K_1 A_\ast Ax_a$, while applying it to the right-hand side yields $L^+ y$. Thus, equation (2.13) becomes

\[(2.15)\]

\[x_a = Px_a - \alpha^2 K_1 A_\ast Ax_a + L^+ y.\]

It will be used to show how the smoothness of $x_a$ depends on the smoothness of the given $y \in L^2[a, b]$.

**Theorem 2.7.** Assume $T$ is an $m$th-order differential operator with $0 \leq m < n$. If $\Delta \in \mathcal{P}$ and $0 \leq k < \infty$, then $y \in C_\Delta^{0,k}[a, b]$ implies that $x_a \in C_\Delta^{n-k}[a, b]$.

**Proof.** Let $\Delta \in \mathcal{P}$ and assume $0 \leq m < n$, so that $n - m \geq 1$. Then $Px_a \in C^\infty[a, b] \subseteq C_\Delta^{n-k}[a, b]$ for $0 \leq k < \infty$, and $y \in C_\Delta^{0,k}[a, b]$ implies that $L^+ y \in C_\Delta^{n-k}[a, b]$ for $0 \leq k < \infty$. Since $x_a \in \mathcal{D}(L) \subseteq H^m[a, b] \subseteq C_\Delta^{n-1,0}[a, b]$, we have from Theorems 2.4 and 2.6 that $Ax_a \in C_\Delta^{n-m-1,0}[a, b]$, $A_\ast Ax_a \in C_\Delta^{n+2(n-m)-1,0}[a, b]$, and $K_1 A_\ast Ax_a \in C_\Delta^{n+2(n-m)-1,0}[a, b]$. If $0 \leq k \leq 2(n - m) - 1$ and $y \in C_\Delta^{0,k}[a, b]$, then $x_a = Px_a - \alpha^2 K_1 A_\ast Ax_a + L^+ y \in C_\Delta^{n-k}[a, b]$.

If $2(n - m) - 1 < k \leq 4(n - m) - 1$ and $y \in C_\Delta^{0,k}[a, b]$, then $x_a = Px_a - \alpha^2 K_1 A_\ast Ax_a + L^+ y \in C_\Delta^{n,2(n-m)-1}[a, b]$, so that $Ax_a \in C_\Delta^{n-m,2(n-m)-1}[a, b]$, $A_\ast Ax_a \in C_\Delta^{n+2(n-m),2(n-m)-1}[a, b]$, and $K_1 A_\ast Ax_a \in C_\Delta^{n+2(n-m),2(n-m)-1}[a, b] \subseteq C_\Delta^{n,4(n-m)-1}[a, b]$. Hence, $x_a \in C_\Delta^{n,k}[a, b]$.

Proceed by induction. Q.E.D.

Since we have convergence of the $x_a$ to $x_0 = L^+ y$ in the $\ast$-norm by Theorem 1.3, we conjecture from (2.15) that $x_a \rightarrow Px_0 + L^+ y$ as $\alpha \rightarrow 0$, where convergence is in the $C_\Delta^{n,k}$-norm. We also have

\[(2.16)\]

\[x_0 = Px_0 + L^+ y\]

and the following consequence of the Schwarz inequality.

**Lemma 2.8.** For $\Delta \in \mathcal{P}$, $0 \leq j < \infty$, and $0 \leq k < \infty$, there exists a positive constant $\alpha_j$ independent of $\Delta$ such that $\|Pz\|_{C_\Delta^{n,j+k}} \leq \alpha_j \|z\|$ for all $z \in L^2[a, b]$.

Next we examine the uniform boundedness and convergence of the $x_a$. Equation (2.15) will play a key role. Let $\Delta \in \mathcal{P}$ and assume $0 \leq k < \infty$ and $0 \leq m < n$. Assume $y \in C_\Delta^{0,k}[a, b]$, so that $x_a \in C_\Delta^{n,k}[a, b]$ by Theorem 2.7. Also, since $Px_0 \in C^\infty[a, b] \subseteq C_\Delta^{n,k}[a, b]$ and $L^+ y \in C_\Delta^{n,k}[a, b]$, it follows from (2.16) that $x_0 \in C_\Delta^{n,k}[a, b]$. Thus, from (2.15), (2.16), and Lemma 2.8 we get

\[(2.17)\]

\[\|x_a\|_{C_\Delta^{n,k}} \leq \|x_a - x_0\|_{C_\Delta^{n,k}} + \|x_0\|_{C_\Delta^{n,k}} \leq \alpha_k\|x_a - x_0\| + \alpha^2\|K_1 A_\ast Ax_a\|_{C_\Delta^{n,k}} + \|x_0\|_{C_\Delta^{n,k}}.\]

Recall that the $H^n$-norm, the graph norm of $L$, and the $\ast$-norm are all equivalent on $\mathcal{D}(L)$ so that by Theorem 1.3

\[(2.18)\]

\[\|x_a - x_0\| \leq e_n\|x_0\|_{C_\Delta^{n,k}}\alpha^2\]

for $0 \leq |\alpha| \leq \min\{\frac{1}{3}, \sqrt{3} / 2\sqrt{2}\|K\|\}$. 
Finally, from Theorems 2.6 and 2.4 there exists a positive constant $b_k$ independent of $L_l$ such that

\begin{equation}
\|K_1 AAx_\alpha\|_{C^{1-k}} \leq b_k \|x_\alpha\|_{C^{1-k}}.
\end{equation}

Combining (2.17), (2.18), and (2.19) and choosing

$$0 < |\alpha| \leq \min\left\{ \frac{1}{2}, \sqrt{3}/2 \sqrt{2\|K\|}, 1/\sqrt{2b_k} \right\},$$

we get

\begin{equation}
\|x_\alpha\|_{C^{1-k}} \leq 2(\alpha_k e_n + 1)\|x_0\|_{C^{1-k}}
\end{equation}

for $0 < |\alpha| \leq \min\{ \frac{1}{2}, \sqrt{3}/2 \sqrt{2\|K\|}, 1/\sqrt{2b_k} \}$ and for $0 \leq m < n$.

We now show that the $x_\alpha$ converge to $x_0$ in the $C^{n-k}_n$-norm with rate of convergence of order $a^2$. Indeed, by (2.15), (2.16), Lemma 2.8, (2.19), (2.18), and (2.20) we get

\begin{equation}
\|x_\alpha - x_0\|_{C^{n-k}_n} \leq \left[ \alpha_k e_n + 2b_k (\alpha_k e_n + 1) \right]\|x_0\|_{C^{n-k}_n} a^2
\end{equation}

for $0 < |\alpha| \leq \min\{ \frac{1}{2}, \sqrt{3}/2 \sqrt{2\|K\|}, 1/\sqrt{2b_k} \}$ and for $0 \leq m < n$.

We have proved the following

**Theorem 2.9.** Let $\Delta \in \mathcal{D}$, let $0 \leq k < \infty$, and assume $y \in C^{0,k}_n[a,b]$. If $T$ is an $m$th-order differential operator with $0 \leq m < n$, then the $x_\alpha \in C^{n-k}_n[a,b]$, the $x_\alpha$ are uniformly bounded in the $C^{n-k}_n$-norm, and the $x_\alpha$ converge to $x_0 = L^ny = P_{x_0} + L^+ y$ in the $C^{n-k}_n$-norm with rate of convergence of order $a^2$ as $\alpha \to 0$, with estimates (2.20) and (2.21).

**E** $T_\alpha$ and its generalized inverse and associated representor. In this section we work with the linear operator $T_\alpha: \mathcal{D}(L) \to L^2[a,b] \oplus L^2[a,b]$ given by $T_\alpha x = (Lx, \alpha Ax)$, where $\alpha \neq 0$. Recall that $T_\alpha$ is one-to-one, the range $\mathcal{R}(T_\alpha)$ is closed in $L^2[a,b] \oplus L^2[a,b]$ under the $L^2$-product structure, and $T_\alpha$ is bounded from the graph norm structure of $L$ into the $L^2$-product structure. Consequently, there exists an adjoint operator $T_\alpha^*: L^2[a,b] \oplus L^2[a,b] \to \mathcal{D}(L)$, which is bounded from the $L^2$-product structure into the graph norm structure of $L$, given by

$$\langle T_\alpha x, z \rangle = \langle x, T_\alpha^* z \rangle_L \quad \text{for all } x \in \mathcal{D}(L), z \in L^2[a,b] \oplus L^2[a,b].$$

Let $Q_\alpha$ be the orthogonal projection with respect to the $L^2$-product structure from $L^2[a,b] \oplus L^2[a,b]$ onto $N(T_\alpha^*) = \mathcal{R}(T_\alpha)^\perp = \mathcal{N}(T_\alpha^*)$, and consider the generalized inverse $T_\alpha^+: L^2[a,b] \oplus L^2[a,b] \to \mathcal{D}(L)$ given by $T_\alpha^+ = T_\alpha^{-1}(I - Q_\alpha)$. Since the range $\mathcal{R}(T_\alpha)$ is closed in $L^2[a,b] \oplus L^2[a,b]$, we know that $T_\alpha^+$ is bounded from the $L^2$-product structure into the graph norm structure of $L$, or the equivalent $\ast$-structure or $H^n$-structure. Choose constants $a_n$, $b_n$, and $c_n$ such that $\|x\|_L \leq a_n \|x\|_{\ast}$, $\|x\|_{\ast} \leq b_n \|x\|_{H^n}$, and $\|x\|_{H^n} \leq c_n \|x\|_L$ for all $x \in \mathcal{D}(L)$. It follows that

\begin{equation}
\|T_\alpha^+\|_{H^n} \leq c_n \|T_\alpha^+\|_L \leq c_n a_n \|T_\alpha^+\|_{\ast},
\end{equation}

where we take the operator norms of $T_\alpha^+$ from the $L^2$-product structure into the $H^n$-structure, the graph norm structure of $L$, and the $\ast$-structure, resp.
We are now ready to develop the representor theory for $T_{a^+}$. For each integer $j = 0, 1, \ldots, n - 1$ and for each $t \in [a, b]$, define the linear functional $\lambda_{jt}^a$ on $L^2[a, b] \oplus L^2[a, b]$ by

$$\lambda_{jt}^a z = (d/dt)^j T_{a^+} z(t) \quad \text{for } z \in L^2[a, b] \oplus L^2[a, b].$$

Then $|\lambda_{jt}^a z| \leq \|T_{a^+}\|_{HF} |z| \leq c_n a_n |z| / |a|$ for $0 < |a| < 1$ by (2.22) and (1.3), where $c_n$ and $a_n$ are independent of $t$. By the Riesz Representation Theorem there exists a representor $\mathcal{F}_{jt}^a = (l_{jt}^a, a_{jt}^a) \in L^2[a, b] \oplus L^2[a, b]$ with $\lambda_{jt}^a z = \langle \mathcal{F}_{jt}^a, z \rangle$ for all $z \in L^2[a, b] \oplus L^2[a, b]$, and

$$\|\lambda_{jt}^a \| = |\mathcal{F}_{jt}^a| \leq c_n a_n / |a| \quad \text{for } 0 < |a| < 1,$$

where $c_n$ and $a_n$ are independent of $t$. Thus,

$$\frac{d}{dt} T_{a^+} z(t) = \langle \mathcal{F}_{jt}^a, z \rangle = (l_{jt}^a, \xi) + (a_{jt}^a, \eta)$$

for all $z = (\xi, \eta) \in L^2[a, b] \oplus L^2[a, b]$, for $j = 0, 1, \ldots, n - 1$, and for $t \in [a, b]$. We will refer to (2.25) as the generalized Green's function representation of $T_{a^+}$.

Take $x \in \mathcal{D}(L) = \mathcal{D}(T_a)$ and set $T_a x = (Lx, aAx) = (\xi, \eta) \in \mathcal{D}(T_a)$. It follows immediately from (2.25) that

$$x(j)(t) = \langle \mathcal{F}_{jt}^a, T_a x \rangle = (l_{jt}^a, \xi) + (a_{jt}^a, \eta)$$

for $j = 0, 1, \ldots, n - 1$ and for $t \in [a, b]$. Applying $L^+$ to $Lx = \xi$, we get

$$x = L^+ \xi + Px.$$

If we set $u = Px \in N(L)$, then applying (2.26) to $u$ gives

$$\frac{d}{dt} L^+ u(t) = \alpha(a_{jt}^a, APx).$$

To continue this development, we will need to work with the adjoint operator $(L^+)_* : \mathcal{D}(L) \to L^2[a, b]$, which is bounded from the graph norm structure of $L$ into the $L^2$-structure, and is given by

$$(L^+ \omega, x)_L = (\omega, (L^+)_* x) \quad \text{for all } \omega \in L^2[a, b], x \in \mathcal{D}(L).$$

We get immediately that

$$(L^+)_* = (L^*)^+ + L \quad \text{on } \mathcal{D}(L),$$

and

$$(L^+)_* = (L^*)^+ \quad \text{on } \mathcal{D}(L)$$

follows from (2.29) and the fact that $(L_*)^+ = (LL^* + I)(L^*)^+$ on $\mathcal{D}(L)$.

Now let $G(t, s) = G_j(s)$ denote the generalized Green's function of $L$, and for $j = 0, 1, \ldots, n - 1$ set $G_j(t, s) = G_{jt}(s) = (d/dt)^j G(t, s)$. Continuing the above discussion, we have $x(j)(t) - (d/dt)^j Px(t) = (d/dt)^j L^+ Lx(t) = (G_{jt}, Lx)$, while from (2.26), (2.28), and (2.30) we have

$$x(j)(t) - (d/dt)^j Px(t) = (l_{jt}^a + \alpha(L_*)^+ A_{jt} a_{jt}^a, Lx).$$
Upon combining these two results, we get
\[(l^n_j + \alpha(L^\ast)A^n_j a^n_j - G_j, Lx) = 0 \quad \text{for all } x \in \mathcal{D}(L).\]

Since the functions \(l^n_j, (L^\ast)A^n_j a^n_j, \) and \(G_j \) all belong to \(\mathcal{N}(L^\ast)\), we must have
\[(2.31) \quad l^n_j = G_j - \alpha(L^\ast)A^n_j a^n_j \]
for \(j = 0, 1, \ldots, n - 1 \) and for \(t \in [a, b].\)

Finally, for \(j = 0, 1, \ldots, n - 1 \) and for \(t \in [a, b],\) we note that \(G^n_j \in \mathcal{N}(T_a^\ast) = \mathcal{D}(T_a) \) by (2.25). Define \(g^n_j = T_a^{-1}G^n_j \in \mathcal{D}(L),\) so that \(T_ag^n_j = (Lg^n_j, \alpha Ag^n_j) = (l^n_j, a^n_j) = G^n_j.\) Then from (2.27), (2.31), and (2.14) we get
\[(2.32) \quad g^n_j = Pg^n_j - \alpha^2 K_A Ag^n_j + L^+ G_j \]
for \(j = 0, 1, \ldots, n - 1 \) and for \(t \in [a, b].\)

We will use (2.32) to establish global and local smoothness of \(g^n_j\) and to bound various of its derivatives. The global smoothness will be established first. Fix an integer \(j\) with \(0 \leq j \leq n - 1\) and fix \(t \in [a, b].\) We know that \(P g^n_j \in C^\infty[a, b],\) and by [17, Chapter IV] \(G^n_j = (\partial/\partial t)^i G(t, \cdot) \in H^{n-j-1}[a, b],\) so that \(L^+ G_j \in H^{2n-j-1}[a, b].\)

Let \(\Delta_0 \in \mathcal{P}\) be the trivial partition of \([a, b]\) given by \(a = t_0 < t_1 = b.\) Then
\[(*) \quad C_{\Delta_0}^{j,k} = H^j[a, b] \cap C^{j+k}[a, b] = C^{j+k}[a, b]\]
for all \(0 \leq j < \infty \) and \(0 \leq k < \infty.\) Assume \(0 \leq m < n,\) so that \(n - m \geq 1.\) Since \(g^n_j \in \mathcal{D}(L),\) we know that \(g^n_j \in H^n[a, b] \subseteq C^{n-1}[a, b] = C_{\Delta_0}^{-1,0}[a, b]\) by (*). Thus, by Theorems 2.4 and 2.6, \(A g^n_j \in C_{\Delta_0}^{n-m,0}[a, b] \subseteq C_{\Delta_0}^{0,0}[a, b],\) \(A^n_j A g^n_j \in C_{\Delta_0}^{2n-m,0}[a, b] \subseteq C_{\Delta_0}^{0,0}[a, b],\) and \(K_A A g^n_j \in C_{\Delta_0}^{0,0}[a, b] = C^{n+1}[a, b]\) by (*).

If \(2n - j - 1 \leq n + 1,\) then \(C^{n+1}[a, b] \subseteq H^{n+1}[a, b] \subseteq H^{2n-j-1}[a, b],\) so from (2.32) \(g^n_j \in H^{2n-j-1}[a, b].\) Otherwise, \(n + 1 < 2n - j - 1,\) so that \(H^{2n-j-1}[a, b] \subseteq H^{n+1}[a, b] \subseteq C^{n+1}[a, b].\) Thus, from (2.32), \(g^n_j \in H^{n+1}[a, b] \subseteq C^n[a, b] = C_{\Delta_0}^{0,0}[a, b]\) by (*), and hence, by Theorems 2.4 and 2.6, \(A g^n_j \in C_{\Delta_0}^{0-m,0}[a, b] \subseteq C_{\Delta_0}^{0,0}[a, b],\) \(A^n_j A g^n_j \in C_{\Delta_0}^{n-m+1,0}[a, b] \subseteq C_{\Delta_0}^{2n+2,0}[a, b]\) and \(K_A A g^n_j \in C_{\Delta_0}^{n+2,0}[a, b] = C^n[a, b]\) by (*).

If \(2n - j - 1 \leq n + 2,\) then \(C^{n+2}[a, b] \subseteq H^{n+2}[a, b] \subseteq H^{2n-j-1}[a, b],\) so from (2.32), \(g^n_j \in H^{2n-j-1}[a, b].\) Otherwise, \(n + 2 < 2n - j - 1,\) so that \(H^{2n-j-1}[a, b] \subseteq H^{n+2}[a, b] \subseteq C^n[a, b].\) Thus, from (2.32), \(g^n_j \in H^{n+2}[a, b] \subseteq C^{n+1}[a, b] = C_{\Delta_0}^{0,0}[a, b]\) by (*).

Continuing this "bootstrap" argument, we conclude that
\[(2.33) \quad g^n_j \in H^{2n-j-1}[a, b]\]
for \(j = 0, 1, \ldots, n - 1,\) for \(t \in [a, b],\) and for \(0 \leq m < n.\)

We establish the local smoothness of \(g^n_j\) in a similar fashion. Fix an integer \(j\) with \(0 \leq j \leq n - 1,\) fix \(t \in [a, b],\) and let \(\Delta_t \in \mathcal{P}\) be the simple partition of \([a, b]\) given by \(\Delta_t = \{a, t, b\}.\) By [17, Chapter IV] we have \(G_j \in C_{\Delta_t}^{0,\infty}[a, b],\) so that
\[(2.34) \quad L^+ G_j \in C_{\Delta_t}^{0,\infty}[a, b] \subseteq C_{\Delta_t}^{n-1,\infty}[a, b].\]
Assume \( m \leq n \), so that \( m - n \geq 1 \). Since \( g_{jt}^n \in \mathcal{D}(L) \), we know that \( g_{jt}^n \in C_{\Delta_j}^{n-1.0}[a, b] \). By Theorems 2.4 and 2.6, \( A_tg_{jt}^n \in C_{\Delta_j}^{2n-m-0}[a, b] \subseteq C_{\Delta_j}^{0.0}[a, b] \), \( A_tA_tg_{jt}^n \in C_{\Delta_j}^{2n-2m-0}[a, b] \subseteq C_{\Delta_j}^{1.0}[a, b] \), and \( K_1A_tA_tg_{jt}^n \in C_{\Delta_j}^{n+1.0}[a, b] \subseteq C_{\Delta_j}^{n+1.2}[a, b] \). Thus, by (2.32), \( g_{jt}^n \in C_{\Delta_j}^{n-1.2}[a, b] \). Continuing this bootstrap argument, we conclude that \( g_{jt}^n \in C_{\Delta_j}^{n-\infty}[a, b] \) or

(2.35) \[ g_{jt}^n \in C^{\infty}[a, t] \cap C^{\infty}[t, b] \cap \mathcal{D}(L) \]

for \( j = 0, 1, \ldots, n - 1 \), for \( t \in [a, b] \), and for \( 0 \leq m < n \).

Our next objective is to bound various derivatives of \( g_{jt}^n \). Fix an integer \( j \) with \( 0 \leq j \leq n - 1 \), fix \( t \in \Delta \), and assume \( 0 \leq m < n \). From (2.35) it is clear that \( g_{jt}^n \in C_{\Delta_j}^{n,k}[a, b] \) for \( k = 0, 1, 2, \ldots \) and from (2.34) we have that \( L^+G_{jt}^n \in C_{\Delta_j}^{n,k}[a, b] \) for \( k = 0, 1, 2, \ldots \). Fix an integer \( k \) with \( 0 \leq k < \infty \). Then by Theorems 2.4 and 2.6, \( Ag_{jt}^n \in C_{\Delta_j}^{n-m,k}[a, b] \), \( A_tAg_{jt}^n \in C_{\Delta_j}^{3n-2m,k}[a, b] \), \( K_1A_tAg_{jt}^n \in C_{\Delta_j}^{n-2m,k}[a, b] \), and

(2.36) \[ \|K_1A_tAg_{jt}^n\|_{C_{\Delta_j}^{n-2m,k}} \leq b_k\|g_{jt}^n\|_{C_{\Delta_j}^{n,k}} \]

Next, let

(2.37) \[ \|G_{jt}^n\|_{C_{\Delta_j}^{n,k}} \leq \sum_{j=0}^{k} \Omega_{jt} = d_k \]

From Lemma 2.8, equations (1.3) and (2.24), and norm equivalence we have

(2.38) \[ \|Pg_{jt}^n\|_{C_{\Delta_j}^{n,k}} \leq \alpha_k a_n\|g_{jt}^n\|_{C^{n,k}} = \alpha_k a_n\|T^{-1}_aG_{jt}^n\|_{C^{n,k}} \leq e_k/|\alpha|^2 \]

for \( 0 < |\alpha| \leq 1 \). Then by (2.32), (2.38), (2.36), and (2.37) and Lemma 2.1(ii), \( \|g_{jt}^n\|_{C_{\Delta_j}^{n,k}} \leq e_k/|\alpha|^2 + \alpha^2 b_k\|g_{jt}^n\|_{C_{\Delta_j}^{n,k}} + \beta_k d_k \) for \( 0 < |\alpha| \leq 1 \), where \( e_k, b_k, \beta_k \) and \( d_k \) are independent of \( \Delta \). In addition, if \( \alpha \) satisfies \( 0 < |\alpha| < 1/\sqrt{2b_k} \), then

(2.39) \[ \|g_{jt}^n\|_{C_{\Delta_j}^{n,k}} \leq \gamma_k/|\alpha|^2 \]

for \( 0 < |\alpha| \leq \min\{1, 1/\sqrt{2b_k}\} \) for \( j = 0, 1, \ldots, n - 1 \), for \( t \in \Delta \), and for \( k = 0, 1, 2, \ldots \), where \( \gamma_k \) is independent of \( \Delta \).

We have the following theorem.

**Theorem 2.10.** Let \( T \) be an \( m \)-th order differential operator with \( 0 \leq n < m < n \), and let \( g_{jt}^n \in \mathcal{D}(T_a) = \mathcal{D}(L) \) with \( Tg_{jt}^n = G_{jt}^n \) for \( j = 0, 1, \ldots, n - 1 \) and \( t \in [a, b] \). Then \( g_{jt}^n \in H^{2n-j-1}[a, b] \cap C^{n}[a, t] \cap C^{n}[t, b] \) for \( j = 0, 1, \ldots, n - 1 \) and \( t \in [a, b] \). Moreover, if \( \Delta \in \mathcal{P} \) is any partition of \([a, b]\) and \( t \in \Delta \), then \( g_{jt}^n \in C_{\Delta_j}^{n,k}[a, b] \) for \( k = 0, 1, 2, \ldots \), and \( \|g_{jt}^n\|_{C_{\Delta_j}^{n,k}} \leq \gamma_k/|\alpha|^2 \) for \( 0 < |\alpha| \leq \min\{1, 1/\sqrt{2b_k}\} \), for \( j = 0, 1, \ldots, n - 1 \), and for \( k = 0, 1, 2, \ldots \), where \( \gamma_k \) is independent of \( \alpha \) and \( \Delta \).
REMARK 2.11. Fix \( t \in [a, b] \) and let \( \Delta_t \in \mathcal{P} \) be the simple partition of \([a, b]\) given by \( \Delta_t = \{ a, t, b \} \). Then from (2.39) it follows that

\[
(2.40) \quad \left\| \left( \hat{g}_{ji}^a \right)^{(2n)} \right\|_\infty \leq \left\| \hat{g}_{ji}^a \right\|_{C^{\infty}} \leq \gamma_n / |a| \]

for \( 0 < |a| \leq \min \left\{ 1, 1/\sqrt{2(b-a)} \right\} \), for \( j = 0, 1, \ldots, n-1 \), and for \( t \in [a, b] \), where \( \gamma_n \) is independent of \( a \) and \( t \in [a, b] \).

(F) \( L^\infty \)-error estimates for \( x_a - \tilde{x}_a \) and \( x_0 - \tilde{x}_a \). Fix a function \( y \in L^2[a, b] \), and let \( x_0 = L^a y \) be the least squares solution of \( Lx = y \) introduced above. Let \( \tilde{y} = (y, 0) \in L^2[a, b] \oplus L^2[a, b] \), and for each \( \alpha \neq 0 \) let \( x_\alpha = T_\alpha \hat{y} \) be the unique least squares solution of \( T_\alpha x = \tilde{y} \) (see Lemma 1.2).

Let \( \mathcal{P}_0 \subseteq \mathcal{P} \) be a family of uniformly graded partitions of \([a, b]\), i.e., there exists a positive constant \( \sigma \) such that

\[
(2.41) \quad h/h \leq \sigma \quad \text{for all } \Delta \in \mathcal{P}_0.
\]

Throughout, we will assume \( h < 1 \) for all \( \Delta \in \mathcal{P}_0 \).

For \( \alpha \neq 0 \) and for \( \Delta \in \mathcal{P}_0 \), let \( \tilde{x}_a \in \text{Sp}(2n-1, \Delta, n-1) \cap \mathcal{D}(L) \) be the least squares approximate to \( x_a \), which is determined by (2.4) or (2.5).

For \( x \in C^n[a, b] \) and \( \Delta : a = t_0 < t_1 < \cdots < t_N = b \) in \( \mathcal{P}_0 \), it is well known that there exists a unique spline \( \bar{x} \in \text{Sp}(2n-1, \Delta, n-1) \) satisfying

\[
(2.42) \quad \bar{x}(t_j) = x(j)(t_j), \quad 0 \leq j \leq n-1.
\]

The function \( \bar{x} \) is the piecewise Hermite interpolate of degree \( 2n-1 \) to \( x \) [25].

The next theorem establishes superconvergence at the knots for \( x_a - \tilde{x}_a \), where \( \alpha \) is fixed and \( h \to 0 \).

THEOREM 2.12. Assume \( T \) is an \( m \)th-order differential operator with \( 0 < m < n \), assume \( 0 \leq k \leq n \), and let \( \Delta \in \mathcal{P}_0 \). If \( y \in C^{0,k}_\alpha[a, b] \), then

\[
(2.43) \quad \left| x_a(j)(t) - \tilde{x}_a(j)(t) \right| \leq \left( \gamma_n / |a| \right)^2 \left\| x_0 \right\|_{C^{\infty}} h^{n+k}
\]

for \( 0 < |a| \leq \min \left\{ 1, \sqrt[3]{2} / \sqrt[2]{2(b-a)} \right\} \), for \( t \in \Delta \), and for \( j = 0, 1, \ldots, n-1 \), where \( \gamma_n \) is independent of \( a \) and \( \Delta \).

Proof. Since \( x_a \) and \( \tilde{x}_a \) belong to \( \mathcal{D}(L) \), we have by (2.26) that

\[
(2.44) \quad \left| x_a(j)(t) - \tilde{x}_a(j)(t) \right| \leq \| T_a g_{ji}^a \|_\infty \| x_a - \tilde{x}_a \|_H \| g_{ji}^a - \tilde{g}_{ji}^a \|_H
\]

for \( j = 0, 1, \ldots, n-1 \) and \( t \in [a, b] \). Let \( \tilde{x}_a \) and \( g_{ji}^a \in S_N \) be the piecewise Hermite interpolates to \( x_a \) and \( g_{ji}^a \), resp. By (2.6) equation (2.43) becomes \( x_a(j)(t) - \tilde{x}_a(j)(t) = \langle T_a x_a - T_a \tilde{x}_a, T_a g_{ji}^a \rangle \) for \( j = 0, 1, \ldots, n-1 \) and \( t \in [a, b] \), and by the Schwarz inequality, (2.7), and the boundedness of \( T_a \), we have

\[
(2.45) \quad \| T_a \| \leq b_n \quad \text{for } 0 < |a| \leq 1.
\]
Assume \( 0 \leq k \leq n \) and \( y \in C^k_0[a, b] \), so that \( x_a \in C^{n-k}_a[a, b] \) by Theorem 2.7. If \( t \in \Delta \), then from Theorem 2.10 we have \( g_{jt} \in C^m_\Delta[a, b] \). Now by (2.45) and [18, Theorem 2.1] equation (2.44) becomes
\[
| x^{(j)}_a(t) - \tilde{x}^{(j)}_a(t) | 
\leq b_n^2 (1 + \sqrt{b - a})^2 \left\{ \sum_{i=0}^{n} \left\| x_a^{(i)} - \tilde{x}_a^{(i)} \right\|_\infty \right\} \left\{ \sum_{i=0}^{n} \left\| (g_{jt}^{(i)}) - (g_{jt}^n)^{(i)} \right\|_\infty \right\}
\leq b_n^2 (1 + \sqrt{b - a})^2 \gamma \left\{ \sum_{i=0}^{n} \left\| x_a^{(n+k)} \right\|_\infty h^k \right\} \left\{ \sum_{i=0}^{n} \left\| (g_{jt}^{(2n)})^i \right\|_\infty h^n \right\}
\leq \frac{\gamma_n}{| \alpha |^2} \left\| x_a^{(n+k)} \right\|_\infty h^{n+k} \text{ by (2.40)}
\]
for \( 0 < |\alpha| \leq 1 \), for \( j = 0, 1, \ldots, n - 1 \), and for \( t \in \Delta \), where \( \gamma_n \) is independent of \( \alpha \) and \( \Delta \).

From (2.20) we have \( \left\| x_a^{(n+k)} \right\|_\infty \leq 2(\alpha_k e_n + 1)\| x_0 \|_{C^2_{\alpha_k}} \) for \( 0 < |\alpha| \leq \min\left\{ \frac{1}{2}, \sqrt{3} / 2 \sqrt{2} \| K \|, 1 / \sqrt{2} b_k \right\} \), so that (2.46) becomes
\[
| x^{(j)}_a(t) - \tilde{x}^{(j)}_a(t) | \leq \left( \frac{\gamma_n}{| \alpha |^2} \right) \| x_0 \|_{C^2_{\alpha_k}} h^{n+k} \text{ for } 0 < |\alpha| \leq \min\left\{ \frac{1}{2}, \sqrt{3} / 2 \sqrt{2} \| K \|, 1 / \sqrt{2} b_k \right\},
\]
for \( j = 0, 1, \ldots, n - 1 \), and for \( t \in \Delta \), where \( \gamma_n \) is independent of \( \alpha \) and \( \Delta \). Q.E.D.

**THEOREM 2.13.** Assume \( T \) is an \( m \)th-order differential operator with \( 0 < m < n \), assume \( 0 < k < n \), and let \( \Delta \in \mathcal{P}_0 \). If \( y \in C^k_\Delta[a, b] \), then
\[
\left\| x^{(j)}_a - \tilde{x}^{(j)}_a \right\|_\infty \leq \left( \frac{\gamma_n}{| \alpha |^2} \right) \| x_0 \|_{C^2_{\alpha_k}} h^{n+k-j}
\]
for \( 0 < |\alpha| \leq \min\left\{ \frac{1}{2}, \sqrt{3} / 2 \sqrt{2} \| K \|, 1 / \sqrt{2} b_k \right\} \) and for \( j = 0, 1, \ldots, n - 1 \), where \( \gamma_n \) is independent of \( \alpha \) and \( \Delta \).

**PROOF.** Let \( \Delta \in \mathcal{P}_0 \) be given by \( a = t_0 < t_1 < \cdots < t_N = b \), and let \( \tilde{x}_a \) be the piecewise Hermite interpolate to \( x_a \). By Theorem 2.7 we know that \( x_a \in C^{n-k}_a[a, b] \). From (2.1) and (2.42) we get
\[
| \tilde{x}^{(j)}_a(t) - \tilde{x}^{(j)}_a(t) | \leq \sum_{i=0}^{n} \sum_{i=0}^{n-1} | x^{(i)}_a(t_i) - \tilde{x}^{(i)}_a(t_i) | \| \varphi_i^{(j)}(t) |
\]
for \( j = 0, 1, \ldots, n - 1 \) and \( t \in [a, b] \). Since the support of \( \varphi_i \) is the interval \( [t_{i-1}, t_{i+1}] \), it follows from [18, equation 2.5] and Theorem 2.12 that
\[
| \tilde{x}^{(j)}_a(t) - \tilde{x}^{(j)}_a(t) | \leq 2 \sum_{i=0}^{n-1} \frac{\gamma_n}{| \alpha |^2} \| x_0 \|_{C^2_{\alpha_k}} h^{n+k} h^{l-j}
\]
for \( 0 < |\alpha| \leq \min\left\{ \frac{1}{2}, \sqrt{3} / 2 \sqrt{2} \| K \|, 1 / \sqrt{2} b_k \right\} \), for \( j = 0, 1, \ldots, n - 1 \), and for \( t \in [a, b] \), where \( \alpha_n \) is independent of \( \alpha \) and \( \Delta \). Taking the supremum over all \( t \in [a, b] \) in the above inequality, we get
\[
\left\| \tilde{x}^{(j)}_a - \tilde{x}^{(j)}_a \right\|_\infty \leq \left( \frac{\alpha_n}{| \alpha |^2} \right) \| x_0 \|_{C^2_{\alpha_k}} h^{n+k-j}
\]
(2.47)
for $0 < |\alpha| \leq \min\{\frac{1}{3}, \sqrt[3]{\frac{1}{2}} K, 1/\sqrt{2b_k}\}$ and for $j = 0, 1, \ldots, n - 1$ where $\alpha_n$ is independent of $\alpha$ and $\Delta$.

From [18, Theorem 2.1] and (2.20)

\begin{equation}
\left\| x^{(j)} - \tilde{x}_\alpha^{(j)} \right\|_\infty \leq \beta_n \| x_0 \| c_{2.1} h^{n+k-j}
\end{equation}

for $0 < |\alpha| \leq \min\{\frac{1}{3}, \sqrt[3]{\frac{1}{2}} K, 1/\sqrt{2b_k}\}$ and for $j = 0, 1, \ldots, n - 1$, where $\beta_n$ is independent of $\alpha$ and $\Delta$. Let $\gamma_n = \beta_n + \alpha_n$. Then combining (2.47) and (2.48) and using the triangle inequality, we have the desired result. Q.E.D.

The next theorem, which follows immediately from Theorems 2.9 and 2.13, and the triangle inequality, is the main theorem of this section.

**Theorem 2.14.** Let $T$ be an $m$th-order differential operator with $0 \leq m < n$, and let $x_0 = L^n y$. Let $\Delta \in \mathcal{P}_0$, and assume $y \in C_{0,k}^n[a,b]$ so that $x^{(j)} \in C_{0,k}^n[a,b]$, where $0 \leq k \leq n$. Then

\begin{equation}
\left\| x^{(j)}_0 - \tilde{x}_\alpha^{(j)} \right\|_\infty \leq \gamma_n \| x_0 \| c_{2.1} \left[ |\alpha|^2 + h^{n+k-j}/|\alpha|^2 \right]
\end{equation}

for $0 < |\alpha| \leq \min\{\frac{1}{3}, \sqrt[3]{\frac{1}{2}} K, 1/\sqrt{2b_k}\}$ and $j = 0, 1, \ldots, n - 1$, where $\gamma_n$ is independent of $\alpha$ and $\Delta$.

**III. Regularization and discrete least squares.**

(A) Introduction of discrete least squares. In this section we work in the Hilbert space $X = Y = Z = L^2[a,b]$ under the standard inner product and norm. As in §II let $L$ be an $n$th-order differential operator in $L^2[a,b]$, and let $T$ be an $m$th-order differential operator in $L^2[a,b]$ with $m \leq n$. We assume that $\mathcal{D}(L) \subseteq \mathcal{D}(T)$ and $\mathcal{N}(L) = \mathcal{N}(T) = \{0\}$.

For fixed $y \in L^2[a,b]$, for fixed $\alpha \neq 0$, and for $\Delta \in \mathcal{P}_0$, let $\hat{y} = (y,0) \in L^2[a,b] \otimes L^2[a,b]$ and let $\tilde{x}_\alpha \in S_N = \text{Sp}(2n - 1, \Delta, n - 1) \cap \mathcal{D}(L)$ be the continuous least squares approximate to $x_\alpha$, which is determined by (2.4), or equivalently, by (2.5). Let $\varphi_1, \ldots, \varphi_M$ be a basis for $S_N$, where $M = n(N + 1) - k_0$. If we write $\tilde{x}_\alpha = \Sigma_{j=1}^M c_j \varphi_j$, then the coefficients $c_1, \ldots, c_M$ satisfy the linear system (2.8). In the event that the functions in (2.8) are even modestly complicated, exact integration is impossible, and we must resort to numerical quadrature. This transforms the continuous least squares problem to a discrete least squares problem.

To set the problem in a specific context, let $\Delta \in \mathcal{P}_0$ be given by $\Delta: a = t_0 < t_1 < \cdots < t_N = b$, and for a positive integer $l$ introduce points $t_{il} = t_{i-1} < t_{i1} < \cdots < t_{il} = t_i$ for $i = 1, \ldots, N$. For $\varphi \in C[t_{i-1}, t_i]$ let

\begin{equation}
\int_{t_{i-1}}^{t_i} \varphi(t) \, dt = \sum_{j=1}^l \omega_{ij} \varphi(t_{ij})
\end{equation}

be a quadrature rule with positive weights $\omega_{i1}, \ldots, \omega_{il}$ which integrates polynomials of degree $\leq 2n - 2$ exactly on $[t_{i-1}, t_i]$. Define pseudo-inner products and pseudo-norms on $[t_{i-1}, t_i]$ and $[a,b]$ by $(\varphi, \psi)_\Delta = \Sigma_{j=1}^l \omega_{ij} \varphi(t_{ij}) \psi(t_{ij})$, $\| \varphi \|_\Delta = (\varphi, \varphi)_\Delta^{1/2}$, and $(\varphi, \psi)_\Delta = \Sigma_{j=1}^n \varphi(t_j) \psi(t_j)$, $\| \varphi \|_\Delta = (\varphi, \varphi)_\Delta^{1/2}$, resp., for $\varphi, \psi \in C_{0,k}^n[a,b]$. Define a pseudo-inner product and pseudo-norm on $C_{0,k}^n[a,b] \otimes C_{0,k}^n[a,b]$ by $(f, g)_d = (f, g)_\Delta + (g, f)_\Delta$ and $\| (f, g) \|_d = (f, g)_d^{1/2}$, resp., for $(f, g), (\xi, \eta) \in C_{0,k}^n[a,b] \otimes C_{0,k}^n[a,b]$. 

These definitions are used in the next section.
Assume \( y \in C^0_\Delta[a,b] \), so that \( \hat{y} \in C^0_\Delta[a,b] \oplus C^0_\Delta[a,b] \). Define a discrete least squares approximate to \( x_a \) to be any element \( \hat{x}_a \in S_N \) satisfying the equation

\[
|T_a \hat{x}_a - \hat{y}|_d = \inf_{\varphi \in S_N} |T_a \varphi - \hat{y}|_d.
\]

It will be shown that such an element \( \hat{x}_a \) exists, and that conditions exist for which \( \hat{x}_a \) is unique.

It is well known that \( H^n[a,b] \) is a Banach space under the \( H^n \)-norm \( \|x\|_{H^n} = \sum_{i=0}^n \|x^{(i)}\|_{\infty} + \|x^{(n)}\| \) for \( x \in H^n[a,b] \). In this section it will also be convenient to work with \( H^n[a,b] \) under the norm

\[
\|x\|_{H^n} = \left[ \sum_{i=0}^n \|x^{(i)}\|^2 \right]^{1/2} \quad \text{for} \quad x \in H^n[a,b].
\]

It can be shown [17, Chapter II] that these two norms are equivalent on \( H^n[a,b] \).

For each partition \( \Delta \in \mathcal{P}_0 \) we will also work with the Banach space

\[ H^{n,k}_\Delta[a,b] = \{ x \in H^n[a,b] | x \in H^{n+k}[t_{i-1}, t_i], 1 \leq i \leq N \} \]

with norm

\[
\|x\|_{H^{n,k}_\Delta} = \left[ \sum_{i=1}^N \|x^{(i)}\|^2_{S^{n+k}[t_{i-1}, t_i]} \right]^{1/2} \quad \text{for} \quad x \in H^{n,k}_\Delta[a,b].
\]

(B) Existence of \( \hat{x}_a \). We now establish the existence of a discrete least squares approximate \( \hat{x}_a \in S_N \) to \( x_a \), which is given by (3.1). It is based on the fact that (3.1) is true iff

\[
\langle T_a \hat{x}_a - \hat{y}, T_a \varphi \rangle_d = 0 \quad \text{for all} \quad \varphi \in S_N.
\]

If a discrete least squares approximate \( \hat{x}_a \in S_N \) to \( x_a \) exists, then \( \hat{x}_a = \sum_{i=1}^M \alpha_i \varphi_i \), and from (3.2) the coefficients \( \alpha_1, \ldots, \alpha_M \) satisfy the linear system

\[
\sum_{j=1}^M \left[ (L \varphi_i, L \varphi_j)_\Delta + \alpha^2 (A \varphi_i, A \varphi_j)_\Delta \right] \alpha_j = (y, L \varphi_i)_\Delta, \quad 1 \leq i \leq M.
\]

In order to show that (3.3) has a solution \( \alpha = [\alpha_1 \cdots \alpha_M]^T \), relabel the \( IN \) quadrature points as \( s_1 = t_1, s_2 = t_{12}, \ldots, s_i = t_{ij}, s_{i+1} = t_{21}, \ldots, s_{iN} = t_{Ni}, \) and relabel the corresponding weights as \( \omega_1 = \omega_{11}, \omega_2 = \omega_{12}, \ldots, \omega_i = \omega_{1i}, \omega_{i+1} = \omega_{21}, \ldots, \omega_{iN} = \omega_{Ni} \).

Let \( B = [(L \varphi_i, L \varphi_j)_\Delta], \) an \( M \times M \) matrix; \( C = [(A \varphi_i, A \varphi_j)_\Delta], \) an \( M \times M \) matrix; \( \alpha = [\alpha_1 \cdots \alpha_M]^T, \) an \( M \times 1 \) matrix; \( \gamma = [(y, L \varphi_1)_\Delta \cdots (y, L \varphi_M)_\Delta]^T, \) an \( M \times 1 \) matrix, so that (3.3) becomes

\[
B + \alpha^2 C \alpha = \gamma.
\]

Let \( E = [L \varphi_i(s_i)] = [\alpha_i], \) an \( IN \times M \) matrix; let \( F = [A \varphi_i(s_i)] = [\beta_i], \) an \( IN \times M \) matrix; let \( D = [\delta_{ij} \omega_j], \) an \( IN \times IN \) diagonal matrix (\( \delta_{ij} \) is the Kronecker delta); and let \( f = [y(s_1) \cdots y(s_{IN})]^T, \) an \( IN \times 1 \) matrix. Then \( B = E^T D E, C = F^T D F, \) and \( \gamma = E^T D f, \) so that (3.4) becomes

\[
[E^T D E + \alpha^2 F^T D F]\alpha = E^T D f.
\]
Let $G = D^{1/2}E$, let $H = D^{1/2}F$, and let $g = D^{1/2}f$. Then (3.5) becomes

$$[G^T G + \alpha^2 H^T H] \alpha = G^T g. \tag{3.6}$$

It can be shown that the range $\mathcal{R}(G^T) \subseteq \mathcal{R}(G^T G + \alpha^2 H^T H)$. It follows immediately that a solution $\alpha = [\alpha_1 \cdots \alpha_M]^T$ to (3.6) exists, and we have proved the existence of a discrete least squares approximate $\hat{x}_\alpha \in S_N$ to $x_\alpha$.

(C) **Uniqueness of $\hat{x}_\alpha$**. In this section we state conditions under which a discrete least squares approximate $\hat{x}_\alpha \in S_N$ to $x_\alpha$ is unique. Let $\Delta \in \mathcal{P}_0$ be given by $a = t_0 < t_1 < \cdots < t_N = b$, let $\varphi, \psi \in S_N$, and fix $i$ with $1 \leq i \leq N$. Clearly $L \varphi$, $L \psi$ and $L \varphi \cdot L \psi$ belong to $H^{2n-1}[t_{i-1}, t_i]$. Let $(f, g)_i = \int_{t_{i-1}}^{t_i} f(t) g(t) \, dt$ for all $f$, $g \in L^2[a, b]$ and let $\lambda(f) = \int_{t_{i-1}}^{t_i} f(t) \, dt - \sum_{j=1}^{i-1} \omega_j f(t_j)$ for $f \in H^{2n-1}[t_{i-1}, t_i]$.

By the assumptions on our quadrature rule we have $\lambda(P) = 0$ for all polynomials $P$ of degree $\leq 2n - 2$, and hence, applying the Peano Kernel Theorem [7, p. 70].

$$|(L \varphi, L \psi)_i - (L \varphi, L \psi)_\Delta| \leq \frac{1}{(2n - 2)!} \int_{t_{i-1}}^{t_i} \left| \left( \frac{d}{d \xi} \right)^{2n-1} [L \varphi(\xi) L \psi(\xi)] \right| \lambda_i \left| (t - \xi)^{2n-2} \right| \, d \xi,$$

where $(t - \xi)_k^\lambda$ equals $(t - \xi)^k$ for $t \geq \xi$ and 0 for $t < \xi$, and $\lambda_i [(t - \xi)_k^\lambda]$ means that $\lambda$ is applied to $(t - \xi)_k^\lambda$ considered as a function of $t$ with $\xi$ fixed. But

$$|\lambda_i \left[ (t - \xi)^{2n-2} \right]| \leq \int_{t_{i-1}}^{t_i} (t_i - t_{i-1})^{2n-2} \, dt + \sum_{j=1}^{i-1} \omega_j (t_i - t_{i-1})^{2n-2}$$

$$= 2(t_i - t_{i-1})^{2n-1},$$

and hence, in terms of $h = \max_{1 \leq i \leq N} (t_i - t_{i-1})$ we have

$$|(L \varphi, L \psi)_i - (L \varphi, L \psi)_\Delta| \leq \frac{2h^{2n-1}}{(2n - 2)!} \int_{t_{i-1}}^{t_i} \left| \left( \frac{d}{d \xi} \right)^{2n-1} [L \varphi(\xi) L \psi(\xi)] \right| \, d \xi \tag{3.7}$$

for all $\varphi, \psi \in S_N$. Similarly, for $A = T|\mathcal{D}(L)$ we get

$$|(A \varphi, A \psi)_i - (A \varphi, A \psi)_\Delta| \leq \frac{2h^{2n-1}}{(2n - 2)!} \int_{t_{i-1}}^{t_i} \left| \left( \frac{d}{d \xi} \right)^{2n-1} [A \varphi(\xi) A \psi(\xi)] \right| \, d \xi \tag{3.8}$$

for all $\varphi, \psi \in S_N$.

We are now in a position to establish conditions which guarantee that $\hat{x}_\alpha$ is unique. Let $\varphi, \psi \in S_N$. Then by (3.7) and (3.8),

$$|\langle T_\alpha \varphi, T_\alpha \psi \rangle - \langle T_\alpha \varphi, T_\alpha \psi \rangle_\Delta|$$

$$\leq \sum_{i=1}^{N} \frac{2h^{2n-1}}{(2n - 2)!} \int_{t_{i-1}}^{t_i} \left| \left( \frac{d}{d \xi} \right)^{2n-1} [L \varphi(\xi) L \psi(\xi)] \right| \, d \xi$$

$$+ \alpha^2 \sum_{i=1}^{N} \frac{2h^{2n-1}}{(2n - 2)!} \int_{t_{i-1}}^{t_i} \left| \left( \frac{d}{d \xi} \right)^{2n-1} [A \varphi(\xi) A \psi(\xi)] \right| \, d \xi. \tag{3.9}$$
By Leibnitz's rule and the Schwarz inequality,

\[
\int_{t_{i-1}}^{t_i} \left| \left( \frac{d}{d\xi} \right)^{2n-1} [L\varphi(\xi)L\psi(\xi)] \right| d\xi 
\]  
(3.10) \leq \sum_{q=0}^{2n-1} \binom{2n-1}{q} \left[ \int_{t_{i-1}}^{t_i} \left| \left( \frac{d}{d\xi} \right)^{q} L\varphi(\xi) \right|^2 d\xi \right]^{1/2} \cdot \left[ \int_{t_{i-1}}^{t_i} \left| \left( \frac{d}{d\xi} \right)^{2n-1-q} L\psi(\xi) \right|^2 d\xi \right]^{1/2}.

Fix \( q \) with \( 0 \leq q \leq 2n - 1 \). Then by Leibnitz's rule and the Schwarz inequality in finite dimensions,

\[
\left[ \int_{t_{i-1}}^{t_i} \left| \left( \frac{d}{d\xi} \right)^q L\varphi(\xi) \right|^2 d\xi \right]^{1/2} \leq \gamma_1 \|\varphi\|_{S^{2n-1}[t_{i-1}, t_i]},
\]  
(3.11) \[ \] where \( \gamma_1 \) is independent of \( \Delta \) and \( q \). Similarly,

\[
\left[ \int_{t_{i-1}}^{t_i} \left| \left( \frac{d}{d\xi} \right)^{2n-1-q} L\psi(\xi) \right|^2 d\xi \right]^{1/2} \leq \gamma_1 \|\psi\|_{S^{2n-1}[t_{i-1}, t_i]},
\]  
(3.12) \[ \] Using (3.11) and (3.12), (3.10) becomes

\[
\int_{t_{i-1}}^{t_i} \left| \left( \frac{d}{d\xi} \right)^{2n-1} [L\varphi(\xi)L\psi(\xi)] \right| d\xi \leq \gamma_2 \|\varphi\|_{S^{2n-1}[t_{i-1}, t_i]} \|\psi\|_{S^{2n-1}[t_{i-1}, t_i]},
\]  
(3.13) \[ \] where \( \gamma_2 \) is independent of \( \Delta \). Since \( 0 \leq m \leq n \), it can be shown in the same manner that

\[
\int_{t_{i-1}}^{t_i} \left| \left( \frac{d}{d\xi} \right)^{2n-1} [A\varphi(\xi)A\psi(\xi)] \right| d\xi \leq \gamma_3 \|\varphi\|_{S^{2n-1}[t_{i-1}, t_i]} \|\psi\|_{S^{2n-1}[t_{i-1}, t_i]},
\]  
(3.14) \[ \] where \( \gamma_3 \) is independent of \( \Delta \). Now utilizing (3.13) and (3.14), the Schwarz inequality and the fact that \( \varphi, \psi \in \text{Sp}(2n - 1, \Delta, n - 1) \), estimate (3.9) becomes

\[
\langle T_\alpha \varphi, T_\alpha \psi \rangle - \langle T_\alpha \varphi, T_\alpha \psi \rangle_d \leq \frac{1}{(2n-2)!} \gamma_2 (1 + \Delta^2) h^{2n-1} \|\varphi\|_{H^{2n-1}} \|\psi\|_{H^{2n-1}},
\]  
(3.15) \[ \] where \( \gamma_2 \) and \( \gamma_3 \) are independent of \( \alpha \) and \( \Delta \). Finally, invoking [18, Lemma 3.3, p. 1156], which follows from Schmidt's inequality [2, pp. 734–736] and Markov's inequality [27, pp. 507–515], and using the equivalence of the norms \( \| \|_{S^n} \) and \( \| \|_{H^n} \), we get

\[
\langle T_\alpha \varphi, T_\alpha \psi \rangle - \langle T_\alpha \varphi, T_\alpha \psi \rangle_d \leq \gamma_4 (1 + \Delta^2) h \|\varphi\|_{H^n} \|\psi\|_{H^n},
\]  
(3.16) \[ \] for all \( \varphi, \psi \in S_n \), where \( \gamma_4 \) is independent of \( \alpha \) and \( \Delta \).

Recall that \( T_\alpha^{-1} \) is bounded from \( \mathcal{R}(T_\alpha) \) under the \( L^2 \)-product structure into \( \mathcal{D}(L) \) under the \( H^n \)-structure, so that by the norm equivalence and (1.3) we have

\[
\|T_\alpha^{-1}\|_{H^n} \leq \gamma \|z\|/|\alpha| \text{ for all } z \in \mathcal{R}(T_\alpha) \text{ and } 0 < |\alpha| \leq 1, \text{ or } |T_\alpha x| \geq |\alpha||x||_{H^n}/\gamma (\gamma > 0)
\]  
for all \( x \in \mathcal{D}(T_\alpha) = \mathcal{D}(L) \) and for \( 0 < |\alpha| \leq 1 \). Then it follows from (3.16) that

\[
|T_\alpha \varphi|_H^2 \geq |\alpha|^2/\gamma^2 - \gamma_4 (1 + |\alpha|^2) h \|\varphi\|_{H^n}^2 \text{ for all } \varphi \in S_n \text{ and for } 0 < |\alpha| \leq 1, \text{ where }
\]  
(3.16) \[ \] \( \gamma_4 \) is independent of \( \alpha \) and \( \Delta \) (but dependent on the mesh bound \( \sigma \)). For all \( h \)
sufficiently small we obtain

\[ |T_\alpha \varphi|_d \geq (|\alpha|/\gamma) \|\varphi\|_{H^r} \]

for all \( \varphi \in S_N \) and for \( 0 < |\alpha| \leq 1 \), where \( \gamma > 0 \) is independent of \( \alpha \) and \( \Delta \).

We have the following theorem concerning the existence and uniqueness of \( \hat{x}_\alpha \), which follows from (3.2) and (3.17).

**THEOREM 3.1.** Let \( \Delta \in \mathcal{P}_0 \) be given by \( \Delta: a = t_0 < t_1 < \cdots < t_N = b \), let \( y \in C_\Delta^{0,0}[a, b] \), so that \( \hat{y} = (y, 0) \in C_\Delta^{0,0}[a, b] \), and assume that the quadrature rule associated with the pseudo-inner product \((\cdot, \cdot)_\Delta\) integrates polynomials of degree \( \leq 2n - 2 \) exactly on \([t_{i-1}, t_i]\) for \( i = 1, \ldots, N \). Then a discrete least squares approximate \( \hat{x}_\alpha \in S_N \) to \( x_\alpha' \), which is given by (3.1), exists and is unique for all \( h \) sufficiently small.

(D) The quasi-discrete least squares approximate to \( x_\alpha' \). In order to obtain error estimates for \( x_\alpha'^{(j)} - \hat{x}_\alpha'^{(j)} \), we will assume that \( y \in C_\Delta^{0,n+k}[a, b] \), so that \( x_\alpha \in C_\Delta^{n,n+k}[a, b] \). This additional smoothness is also required in \([1, 8, 9, 18, \text{and 26}]\). We begin by introducing a new spline approximate to \( x_\alpha \). Define a quasi-discrete least squares approximate to \( x_\alpha \) to be any element \( \hat{x}_\alpha \in S_N \) satisfying

\[ (3.18) \quad |T_\alpha \hat{x}_\alpha - T_\alpha x_\alpha|_d = \inf_{\varphi \in S_N} |T_\alpha \varphi - T_\alpha x_\alpha|_d. \]

The following equation also characterizes a quasi-discrete least squares approximate to \( x_\alpha \). Let \( \Delta \in \mathcal{P}_0 \), and assume \( y \in C_\Delta^{0,0}[a, b] \), so that \( x_\alpha \in C_\Delta^{n,0}[a, b] \) and \( T_\alpha x_\alpha \in C_\Delta^{0,0}[a, b] \). Then for \( \hat{x}_\alpha \in S_N \), equation (3.18) is satisfied iff

\[ (3.19) \quad \langle T_\alpha \hat{x}_\alpha - T_\alpha x_\alpha, T_\alpha \varphi \rangle_d = 0 \quad \text{for all } \varphi \in S_N. \]

Let \( \varphi_1, \ldots, \varphi_M \) be a basis for \( S_N \), and let \( \hat{x}_\alpha = \sum_{j=1}^{M} \hat{\alpha}_j \varphi_j \) satisfy (3.19). Then

\[ (3.20) \quad \sum_{j=1}^{M} \left[ (L \varphi_i, L \varphi_j)_\Delta + \alpha_2(A \varphi_i, A \varphi_j)_\Delta \right] \hat{\alpha}_j = (L x_\alpha, L \varphi_i)_\Delta + \alpha_2(A x_\alpha, A \varphi_i)_\Delta, \quad 1 \leq i \leq M. \]

In a fashion analogous to that used in establishing the existence of \( \hat{x}_\alpha \), the discrete least squares approximate to \( x_\alpha \), it can be shown that \( \hat{\alpha} = [\hat{\alpha}_1 \cdots \hat{\alpha}_M]^T \) satisfies the matrix equation

\[ (3.21) \quad [G^T G + \alpha^2 H^T H] \hat{\alpha} = G^T h + \alpha^2 H^T k, \]

where \( h = D^{1/2}[L x_\alpha(s_1) \cdots L x_\alpha(s_M)]^T \) and \( k = D^{1/2}[A x_\alpha(s_1) \cdots A x_\alpha(s_M)]^T \).

The existence of \( \hat{x}_\alpha \) follows from (3.21) and the fact that the ranges \( \mathcal{R}(G^T) \) and \( \mathcal{R}(H^T) \) are contained in \( \mathcal{R}(G^T G + \alpha^2 H^T H) \). The following theorem states that it is unique under appropriate conditions. Its proof is based on (3.19) and (3.17).
**THEOREM 3.2.** Let $\Delta \in \mathcal{P}_0$ be given by $\Delta: a = t_0 < t_1 < \cdots < t_N = b$, let $y \in C^{0,0}_\Delta[a,b]$, so that $T_\alpha x_\alpha \in C^{0,0}_\Delta[a,b] \oplus C^{0,0}_\Delta[a,b]$, and assume that the quadrature rule associated with the pseudo-inner product $(\cdot, \cdot)_\Delta$ integrates polynomials of degree $\leq 2n - 2$ exactly on $[t_{i-1}, t_i]$ for $i = 1,\ldots,N$. Then a quasi-discrete least squares approximate $\hat{x}_\alpha \in S_N$ to $x_\alpha$, which is given by (3.18), exists and is unique for all $h$ sufficiently small.

We now proceed to establish $L^\infty$-estimates for the quantities $x_\alpha^{(j)} - \hat{x}_\alpha^{(j)}$ for $j = 0, 1, \ldots, n - 1$. This requires superconvergence estimate at the knots of the partition, which will follow from the following three lemmas.

**LEMMA 3.3.** Let $\Delta \in \mathcal{P}_0$, let $0 \leq k \leq n$, and let $x \in \Omega(L) \cap C^{n+k}_\Delta[a,b]$. If $\tilde{x} \in S_N$ is the piecewise Hermite interpolate to $x$, then $|T_\alpha x - T_\alpha \tilde{x}|_d \leq \gamma \|x^{(n+k)}\|_\infty h^k$ for $0 < |\alpha| \leq 1$, where $\gamma$ is independent of $\alpha$ and $\Delta$.

**PROOF.** We have from the definition of our pseudo-inner product and [18, Theorem 2.1, p. 1154] that

$$|T_\alpha x - T_\alpha \tilde{x}|_d^2 \leq \|Lx - L\tilde{x}\|_\infty^2 + \alpha^2 \|A x - A \tilde{x}\|_\infty^2 \sum_{i=1}^N \sum_{j=1}^i \omega_{ij}$$

$$= (b - a) \left( \sum_{i=0}^n \left\| \sum_{i=0}^n a_i [x^{(i)} - \tilde{x}^{(i)}] \right\|_\infty^2 + \alpha^2 \sum_{i=0}^n b_i \left\| x^{(i)} - \tilde{x}^{(i)} \right\|_\infty^2 \right)$$

$$\leq \gamma^2 \|x^{(n+k)}\|_\infty^2 h^{2k}$$

for $0 < |\alpha| \leq 1$, where $\gamma$ is independent of $\alpha$ and $\Delta$. Q.E.D.

**LEMMA 3.4.** Let $\Delta \in \mathcal{P}_0$, let $0 \leq k \leq n$, and let $x \in C^{n+k}_\Delta[a,b]$, so that $x_\alpha \in C^{n+k}_\Delta[a,b]$. Then $\|Lx_\alpha - L\tilde{x}_\alpha\|_{C^{n+k}_\Delta} \leq (\gamma/|\alpha|) \|x_\alpha\|_{C^{n+k}_\Delta}$ for all $h$ sufficiently small and $0 < |\alpha| \leq 1$, where $\gamma$ is independent of $\alpha$ and $\Delta$.

**PROOF.** Let $\tilde{x}_\alpha \in S_N$ be the piecewise Hermite interpolate to $x_\alpha$. Then by Leibnitz's rule,

(3.22) $\|Lx_\alpha - L\tilde{x}_\alpha\|_{C^{n+k}_\Delta} \leq \gamma_1 \left( \|x_\alpha - \tilde{x}_\alpha\|_{C^{n+k}_\Delta} + \|\tilde{x}_\alpha - \hat{x}_\alpha\|_{C^{n+k}_\Delta} \right)$,

where $\gamma_1$ is independent of $\alpha$ and $\Delta$. Also, by [18, Theorem 2.1, p. 1154]

(3.23) $\|x_\alpha - \tilde{x}_\alpha\|_{C^{n+k}_\Delta} \leq \sum_{i=0}^{2n-1} \gamma_2 \|x_\alpha^{(2n-1)}\|_\infty h^{2n-1-i} + \sum_{i=0}^k \|x_\alpha^{(2n+i)}\|_\infty$

$$\leq \gamma_3 \|x_\alpha\|_{C^{n+k}_\Delta},$$

where $\gamma_3$ is independent of $\alpha$ and $\Delta$, and

(3.24) $\|\tilde{x}_\alpha - \hat{x}_\alpha\|_{C^{n+k}_\Delta} = \|\bar{x}_\alpha - \hat{x}_\alpha\|_{C^{2n-1}_\Delta} \leq \gamma_4 h^{-n} \|\bar{x}_\alpha - \hat{x}_\alpha\|_{C^{n-1}_\Delta}$

by [18, Lemma 3.3, p. 1156], where $\gamma_4$ is independent of $\alpha$ and $\Delta$. 


Finally, by (3.17), (3.18), and Lemma 3.3,

\begin{align}
\|X_a - \hat{X}_a\|_{C^{n-1}} &= \|X_a - \hat{X}_a\|_{H^n} \\
&\leq \left( \gamma_5/|a| \right) \left[ |T_aX_a - T_a\hat{X}_a| + |\hat{T}_aX_a - T_aX_a| \right] \\
&\leq \left( \gamma_6/|a| \right) \|X_a\|_{C^{2n,n+k}} \cdot h^n
\end{align}

for 0 < |a| ≤ 1 and h sufficiently small, where \( \gamma_5 \) is independent of \( a \) and \( \Delta \).

Combining (3.22)–(3.25), we get the desired result. Q.E.D.

The following lemma is proved in a similar fashion.

**Lemma 3.5.** Let \( \Delta \in \mathcal{P}_0 \), let 0 ≤ k ≤ n, and let \( y \in C_{\Delta}^{0,n+k}[a, b] \), so that \( x_a \in C_{\Delta}^{n,n+k}[a, b] \). Then \( \|Ax_a - A\hat{X}_a\|_{C^{2n,n+k}} \leq (\gamma/|a|)\|x_a\|_{C^{2n,n+k}} \) for all h sufficiently small and 0 < |a| ≤ 1, where \( \gamma \) is independent of \( a \) and \( \Delta \).

We can now state the following superconvergence result. It requires additional precision of the quadrature rule.

**Theorem 3.6.** Let \( \Delta \in \mathcal{P}_0 \) be given by \( \Delta: a = t_0 < t_1 < \cdots < t_N = b \), let 0 ≤ k ≤ n, and let \( y \in C_{\Delta}^{0,n+k}[a, b] \), so that \( x_a \in C_{\Delta}^{n,n+k}[a, b] \). Assume that the quadrature rule associated with the pseudo-inner product \( (\cdot, \cdot)_{\Delta} \) integrates polynomials of degree \( \leq 2n - 1 \) exactly on \([t_{i-1}, t_i]\) for \( i = 1, \ldots, N \). Then

\begin{align}
\|X_a^{(j)}(t_i) - \hat{X}_a^{(j)}(t_i)\|_{C^{n,n+k}[t_{i-1}, t_i]} \leq \left( \gamma/|a|^2 \right) \|X_a\|_{C^{n,n+k}[t_{i-1}, t_i]} h^{n+k}
\end{align}

for h sufficiently small, for 0 < |a| ≤ 1, for 0 ≤ i ≤ N, and for 0 ≤ j ≤ n − 1, where \( \gamma \) is independent of \( a \) and \( \Delta \).

**Proof.** Fix integers \( i \) and \( j \) with 0 ≤ i ≤ N and 0 ≤ j ≤ n − 1. By (2.26),

\begin{align}
X_a^{(j)}(t_i) - \hat{X}_a^{(j)}(t_i) &= \langle g_{ji}, T_aX_a - T_a\hat{X}_a \rangle - \langle g_{ji}, T_aX_a - T_a\hat{X}_a \rangle + E(t_i),
\end{align}

where

\begin{align}
E(t_i) &= \langle g_{ji}, T_aX_a - T_a\hat{X}_a \rangle - \langle g_{ji}, T_a\hat{X}_a - T_aX_a \rangle.
\end{align}

Recall that \( g_{ji} = T^{-1}g_{ji} \in C_{\Delta}^{n,n+k}[a, b] \) by Theorem 2.10, and let \( g_{ji} \in S_N \) be the piecewise Hermite interpolate to \( g_{ji} \). Then by (3.19), the Schwarz inequality (in a pseudo-inner product space), Lemma 3.3, and equations (3.18) and (2.40) we have

\begin{align}
\|g_{ji}\|_{C^{n,n+k}[t_{i-1}, t_i]} \leq \gamma_2 \|X_a\|_{C^{n,n+k}[t_{i-1}, t_i]} h^{n+k},
\end{align}

where \( \gamma_2 \) is independent of \( a \) and \( \Delta \).

We next investigate the quadrature error \( E(t_i) \). Since \( x_a \in C_{\Delta}^{n,n+k}[a, b] \), it follows that \( Lx_a \in C_{\Delta}^{0,n+k}[a, b] \), so that \( Lx_a \in C^{n+k}[t_{i-1}, t_i] \subseteq H^{n+k}[t_{i-1}, t_i] \), and \( Ax_a \in C_{\Delta}^{n-m,n+k}[a, b] \subseteq C_{\Delta}^{0,n+k}[a, b] \), so that \( Ax_a \in C^{n+k}_n[t_{i-1}, t_i] \subseteq H^{n+k}[t_{i-1}, t_i] \). If \( \varphi \in S_N \), then \( \varphi, A\varphi \in C^{n+k}[t_{i-1}, t_i] \) so that \( \varphi, A\varphi \in H^{n+k}[t_{i-1}, t_i] \). By assumption, the quadrature rule is exact for polynomials of degree \( \leq n + k - 1 \). Thus, by the Peano Kernel Theorem [7, p. 70], Leibnitz's rule, and
Theorem 2.10 (see the derivation of (3.7))

\[
\left| E(t_i) \right| \leq \sum_{r=1}^{N} \left| \lambda \left( Lg_{jt}^{\alpha} \left[ Lx_{\alpha} - L\hat{x}_{\alpha} \right] \right) \right| + \alpha^{2} \sum_{r=1}^{N} \left| \lambda \left( Ag_{jt}^{\alpha} \left[ Ax_{\alpha} - A\hat{x}_{\alpha} \right] \right) \right|
\]

\[
\leq \frac{(b-a)\gamma_3}{(n+k-1)!} \frac{h^{n+k}}{|\alpha|^2} \left\| Lx_{\alpha} - L\hat{x}_{\alpha} \right\|_{C_{0,n+k}^{\alpha}} + \frac{(b-a)\gamma_4}{(n+k-1)!} h^{n+k} \left\| Ax_{\alpha} - A\hat{x}_{\alpha} \right\|_{C_{0,n+k}^{\alpha}}.
\]

Finally, by Lemmas 3.4 and 3.5, we get

\[
(3.27) \quad \left| E(t_i) \right| \leq \left( \gamma_5/|\alpha|^3 \right) \left\| x_{\alpha} \right\|_{C_{0,n+k}^{\alpha}} h^{n+k},
\]

where \( \gamma_5 \) is independent of \( \alpha \) and \( \Delta \).

Combining (3.26) and (3.27), we get the desired result. Q.E.D.

The principal error estimates for \( x_{\alpha}^{(j)}(t) = x_{\alpha}^{(j)}(t) \), \( 0 \leq j \leq n-1 \), are established in the next theorem.

**Theorem 3.7.** Let \( \Delta \in \mathcal{H}_0 \) be given by \( \Delta: a = t_0 < t_1 < \cdots < t_N = b \), let \( 0 \leq k \leq n \), and let \( y \in C_{a,n+k}^{0,n+k}[a,b] \), so that \( x_{\alpha} \in C_{a,n+k}^{\alpha}[a,b] \). Assume that the quadrature rule associated with the pseudo-inner product \( \langle , \rangle_{\Delta} \) integrates polynomials of degree \( \leq 2n-1 \) exactly on \( [t_{i-1}, t_i] \) for \( i = 1, \ldots, N \). Then

\[
\left\| x_{\alpha}^{(j)}(t) - x_{\alpha}^{(j)}(t) \right\|_{\infty} \leq \left( \gamma_1/|\alpha|^3 \right) \left\| x_{\alpha} \right\|_{C_{0,n+k}^{\alpha}} h^{n+k-j}
\]

for all \( h \) sufficiently small, for \( 0 < |\alpha| \leq 1 \), and for \( 0 \leq j \leq n-1 \), where \( \gamma_1 \) is independent of \( \alpha \) and \( \Delta \).

**Proof.** Fix an integer \( j \) with \( 0 \leq j \leq n-1 \), let \( \bar{x}_{\alpha} \in S_N \) be the piecewise Hermite interpolate to \( x_{\alpha} \), and let \( \{ \varphi_{rs} \}_{0 \leq r \leq N, 0 \leq s \leq n-1} \) be the cardinal basis for \( \text{Sp}(2n-1, \Delta, n-1) \). We know that the support of \( \varphi_{rs} \) is \( [t_{r-1}, t_{r+1}] \), and (2.1) gives us

\[
\bar{x}_{\alpha}(t) = \sum_{r=0}^{N} \sum_{s=0}^{n-1} x_{\alpha}^{(s)}(t_r) \varphi_{rs}(t)
\]

and

\[
\hat{x}_{\alpha}(t) = \sum_{r=0}^{N} \sum_{s=0}^{n-1} \hat{x}_{\alpha}^{(s)}(t_r) \varphi_{rs}(t)
\]

for all \( t \in [a, b] \). Then by Theorem 3.6 and [18, equations 2.5] it follows that

\[
\left| \bar{x}_{\alpha}^{(j)}(t) - \hat{x}_{\alpha}^{(j)}(t) \right| \leq \left( \gamma_1/|\alpha|^3 \right) \left\| x_{\alpha} \right\|_{C_{0,n+k}^{\alpha}} h^{n+k-j}
\]

for all \( t \in [a, b] \), so that

\[
(3.28) \quad \left\| \bar{x}_{\alpha}^{(j)} - \hat{x}_{\alpha}^{(j)} \right\|_{\infty} \leq \left( \gamma_1/|\alpha|^3 \right) \left\| x_{\alpha} \right\|_{C_{0,n+k}^{\alpha}} h^{n+k-j},
\]

where \( \gamma_1 \) is independent of \( \alpha \) and \( \Delta \). The conclusion follows from [18, Theorem 2.1, p. 1154], equation (3.28), and the triangle inequality. Q.E.D.
The following estimates for the higher order derivatives \( x_a^{(j)} - \hat{x}_a^{(j)} \), \( n \leq j \leq n + k \), are indirect consequences of Markov's inequality [27, pp. 507–515].

**Theorem 3.8.** Under the hypotheses of Theorem 3.7,

\[
\| x_a^{(j)} - \hat{x}_a^{(j)} \|_\infty \leq \left( \gamma / |\alpha|^3 \right) \| x_a \|_{C^{0, n+k}} h^{n+k-j}
\]

for all \( h \) sufficiently small, for \( 0 < |\alpha| \leq 1 \), and for \( n \leq j \leq n + k \), where \( \gamma \) is independent of \( \alpha \) and \( \Delta \).

**Proof.** The result follows immediately from [18, Theorem 2.1, p. 1154, and Lemma 3.3, p. 1156], equation (3.28), and the triangle inequality. Q.E.D.

(E) \( L^\infty \)-error estimates for \( \hat{x}_a - x_a \) and \( \hat{x}_a - x_0 \). In this section we first establish \( L^\infty \)-error estimates for \( \hat{x}_a^{(j)} - \hat{x}_a^{(j)} \), where \( \hat{x}_a \) is the discrete least squares approximate to \( x_a \) given by (3.1), and \( \hat{x}_a \) is the quasi-discrete least squares approximate to \( x_a \) given by (3.18). These estimates, combined with those in Theorems 2.9 and 3.7, will give the desired estimates for \( \hat{x}_a^{(j)} - x_0^{(j)} \). As in (D) we require that \( y \in C^{0, n+k}_\Delta [a, b] \). We need the following preliminary result.

**Theorem 3.9.** Under the hypotheses of Theorem 3.7,

\[
|T_a \hat{x}_a - T_a \hat{x}_a|_d \leq \left( \gamma / |\alpha| \right) \left( \| x_a \|_{C^{0, n+k}} + \| y \|_{C^{0, n+k}} \right) h^k
\]

for all \( h \) sufficiently small and \( 0 < |\alpha| \leq 1 \), where \( \gamma \) is independent of \( \alpha \) and \( \Delta \).

**Proof.** We have from equations (3.2), (3.19), and (2.3),

\[
|T_a \hat{x}_a - T_a \hat{x}_a|^2_d = \langle T_a x_a - \hat{y}, T_a \hat{x}_a - T_a \hat{x}_a \rangle_d - \langle T_a x_a - \hat{y}, T_a \hat{x}_a - T_a \hat{x}_a \rangle.
\]

From (3.29), the definition of the pseudo-inner product, and the Peano Kernel Theorem [7, p. 70], it follows that

\[
|T_a \hat{x}_a - T_a \hat{x}_a|^2_d \leq \sum_{r=1}^N \left| \lambda \left( [Lx_a - y] [L\hat{x}_a - L\hat{x}_a] \right) \right| + \alpha^2 \sum_{r=1}^N \left( A x_a \right) \left( A \hat{x}_a - A \hat{x}_a \right) |L\hat{x}_a - L\hat{x}_a|_{C^{0, n+k}_\Delta} + \gamma_2 \alpha^2 h^{n+k} \| x_a \|_{C^{0, n+k}_\Delta} \| A \hat{x}_a - A \hat{x}_a \|_{C^{0, n+k}_\Delta},
\]

where \( \gamma_1 \) and \( \gamma_2 \) are independent of \( \alpha \) and \( \Delta \).
From Lemma 2.1(i), the proof of Lemma 3.4 (see (3.24)), and (3.17) we have

\[(3.31) \quad \left\| L \hat{x}_a - L \tilde{x}_a \right\|_{C_{\Delta}^{0,-1,k}} \leq \left( \gamma_3 / |\alpha| \right) h^{-n} T_a \hat{x}_a - T_a \tilde{x}_a \]

and

\[(3.32) \quad \left\| A \hat{x}_a - A \tilde{x}_a \right\|_{C_{\Delta}^{0,-1,k}} \leq \left( \gamma_4 / |\alpha| \right) h^{-n} T_a \hat{x}_a - T_a \tilde{x}_a \]

for 0 < |\alpha| < 1 and h sufficiently small, where \gamma_3 and \gamma_4 are independent of \alpha and \Delta. Combine (3.30), (3.31), and (3.32) for the desired result. Q.E.D.

**Theorem 3.10.** Under the hypotheses of Theorem 3.7, for all h sufficiently small, for 0 < |\alpha| < 1, and for 0 \leq j \leq n - 1, where \gamma is independent of \alpha and \Delta.

**Proof.** The result follows from (3.17) and Theorem 3.9. Q.E.D.

We now present the main result of §III.

**Theorem 3.11.** Let T be an mth-order differential operator with 0 \leq m < n and let \( x_0 = L^m y \). Let \( \Delta \in \mathcal{P}_0 \) be given by \( \Delta: a = t_0 < t_1 < \cdots < t_N = b, \) let 0 \leq k \leq n, and let \( y \in C_{\Delta}^{0,n+k}[a,b], \) so that \( x_a \in C_{\Delta}^{n,n+k}[a,b]. \) Assume that the quadrature rule associated with the pseudo-inner product \( \left( , \right)_\Delta \) integrates polynomials of degree \leq 2n - 1 exactly on \( [t_{i-1}, t_i] \) for \( i = 1, \ldots, N. \) Let \( \hat{x}_a \in S_N \) be the discrete least squares approximate to \( x_a \) given by (3.1). Then

\[ \left\| \hat{x}_a^{(j)} - x_0^{(j)} \right\|_\infty \leq \gamma \left( \left\| x_0 \right\|_{C_{\Delta}^{0,n+k}} + \left\| y \right\|_{C_{\Delta}^{0,n+k}} \right) \frac{h^k}{|\alpha|^2} + \frac{h^{n+k-j}}{|\alpha|^3} + |\alpha|^2 \]

for all h sufficiently small, for 0 \leq |\alpha| \leq \{ \frac{1}{2}, \sqrt{3} / 2\sqrt{2}\|K\|, 1 / \sqrt{2} b_k \}, and for 0 \leq j \leq n - 1, where \gamma is independent of \alpha and \Delta.

**Proof.** This follows from Theorems 3.10, 3.7, 2.9 and the triangle inequality. Q.E.D.

**Bibliography**


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