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EXTENSIONS OF A THEOREM OF WINTNER ON SYSTEMS WITH ASYMPTOTICALLY CONSTANT SOLUTIONS

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ABSTRACT. A theorem of Wintner concerning sufficient conditions for a system $y' = A(t)y$ to have linear asymptotic equilibrium is extended to a system $x' = A(t)x + f(t,x)$. The integrability conditions imposed on f permit conditional convergence of some of the improper integrals that occur. The results improve on Wintner's even if $f = 0$.

An $n \times n$ system

$$y' = A(t)y, \qquad t > 0, \tag{1}$$

is said to have linear asymptotic equilibrium if for each constant vector c there is a solution of (1) such that $\lim_{t\to\infty} y(t) = c$. It is well known that (1) has this property if A is continuous and

$$\int^{\infty} \|A(t)\|\,dt < \infty. \tag{2}$$

Wintner [5] attributed this result to Bôcher and improved on it as follows.

THEOREM 1 (WINTNER). *Let A be continuous on $[a,\infty)$ and suppose the integrals*

$$A_j(t) = \int_t^{\infty} A_{j-1}(s)A(s)\,ds, \qquad 1 \leqslant j \leqslant k\ (A_0 = I), \tag{3}$$

converge, and

$$\int^{\infty} \|A_k(t)A(t)\|\,dt < \infty. \tag{4}$$

Then (1) *has linear asymptotic equilibrium.*

Notice that (3) is vacuous and (4) reduces to (2) when $k = 0$.

Here we apply Wintner's idea to the system

$$x' = A(t)x + f(t,x). \tag{5}$$

We give sufficient conditions for (5) to have a solution x such that

$$\lim_{t\to\infty} x(t) = c \tag{6}$$

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for a given constant vector c. The assumptions on f in our main theorem apply only "near" the set $\{(t, c) | t \geqslant a\}$. Our integral smallness conditions permit conditional convergence of some of the improper integrals that occur. This continues a theme developed previously in [**3** and **4**]. (See also Hallam [**1**, **2**].) The idea of dealing with the iterated integrals (3) is due to Wintner [**5**].

Throughout the paper k is a nonnegative integer. It is to be understood that conditions stated for $1 \leqslant j \leqslant k$ are vacuous if $k = 0$, and that $\sum_{j=1}^{0} = 0$. Our results apply with x, A, and f real- or complex-valued. Improper integrals occurring in hypotheses are tacitly assumed to converge, and the convergence may be conditional except when the integrand is obviously nonnegative. We use "O" and "o" in the standard way to indicate orders of magnitude as $t \to \infty$.

To indicate the direction of proof of our main theorem, it is convenient to state part of its hypotheses here.

ASSUMPTION A. Let w be continuous and nonincreasing on $[a, \infty)$, $0 < w(t) \leqslant 1$, and suppose that either $\lim_{t \to \infty} w(t) = 0$ or $w = 1$. Suppose A is continuous on $[a, \infty)$ and A_j in (3) exists for $1 \leqslant j \leqslant k$. Let c be a given constant vector, and suppose there is a constant $M > 0$ such that f is continuous and

$$\|f(t, x) - f(t, c)\| \leqslant R(t, \|x - c\|) \tag{7}$$

on

$$S = \{(t, x) | t \geqslant a, \|x - c\| \leqslant Mw(t)\}, \tag{8}$$

where $R(t, \lambda)$ is continuous on $\{(t, \lambda) | t \geqslant a, 0 \leqslant \lambda \leqslant Mw(t)\}$ and nondecreasing in λ for each t.

We define $\Gamma_k = \sum_{j=0}^{k} A_j$, and observe from (3) that

$$\Gamma_k' = -\Gamma_{k-1} A, \qquad k \geqslant 0 \; (\Gamma_{-1} = 0). \tag{9}$$

Moreover, since $\lim_{t \to \infty} \Gamma_k(t) = I$, Γ_k is invertible for large t, and $\lim_{t \to \infty} \Gamma_k^{-1}(t) = I$.

Now let $t_0 \geqslant a$ be such that Γ_k^{-1} exists on $[t_0, \infty)$. For convenience below, we define

$$\mu_k(t) = \sup_{s \geqslant t} \{\|\Gamma_k^{-1}(s)\|\} = 1 + o(1), \tag{10}$$

and

$$\nu_k(t) = \mu_k(t) \sup_{s \geqslant t} \{\|\Gamma_k(s)\|\} = 1 + o(1). \tag{11}$$

Let $C[t_0, \infty)$ be the space of continuous n-vector functions (with real or complex components) on $[t_0, \infty)$, with the topology of uniform convergence on finite intervals. Let $V[t_0, \infty)$ be the closed convex subset of $C[t_0, \infty)$ defined by

$$V[t_0, \infty) = \{x \in C[t_0, \infty) \,|\, \|x(t) - c\| \leqslant Mw(t),\; t \geqslant t_0\}. \tag{12}$$

We obtain our results by applying the Schauder-Tychonov theorem to an appropriate transformation T of $V[t_0, \infty)$ (for sufficiently large t_0) into itself. To motivate the choice of T, we observe that if

$$x(t) = c - \int_t^\infty [A(s)x(s) + f(s, x(s))]\, ds,$$

where the integral is assumed to converge, then x satisfies (5) and (6). Repeated integration by parts, assuming at each step that x satisfies (5), yields the equation

$$\Gamma_k(t)x(t) = c - \int_t^\infty A_k(s)A(s)x(s)\, ds - \int_t^\infty \Gamma_k(s)f(s, x(s))\, ds. \tag{13}$$

Although these manipulations are completely formal, (13) suggests the transformation T defined by

$$(Tx)(t) = \Gamma_k^{-1}(t)\left[c - \int_t^\infty [A_k(s)A(s)x(s) + \Gamma_k(s)f(s, x(s))]\, ds\right]. \tag{14}$$

Assumption A implies that the function $F(t) = f(t, x(t))$ is continuous on $[t_0, \infty)$ if $x \in V[t_0, \infty)$. Hence, if the integrals in (14) converge, differentiation yields

$$(Tx)'(t) = \Gamma_k^{-1}(t)[\Gamma_{k-1}(t)A(t)(Tx)(t) + A_k(t)A(t)x(t)] + f(t, x(t)), \tag{15}$$

where we have used (9) and the fact that $(\Gamma_k^{-1})' = -\Gamma_k^{-1}\Gamma_k'\Gamma_k^{-1}$. Therefore, if T has a fixed point (function) x_0 in $V[t_0, \infty)$, we see on setting $Tx = x = x_0$ in (15) that

$$\begin{aligned} x_0'(t) &= \Gamma_k^{-1}(t)[\Gamma_{k-1}(t) + A_k(t)]A(t)x_0(t) + f(t, x(t)) \\ &= A(t)x_0(t) + f(t, x(t)) \end{aligned}$$

(since $\Gamma_{k-1} + A_k = \Gamma_k$); i.e., x_0 satisfies (5). Moroever, setting $Tx = x = x_0$ in (14) shows that x_0 also satisfies (6).

The following theorem allows the integrals occurring in Tc (the function obtained by setting $x = c$ in (14)) to converge conditionally, and exploits the rapidity with which $Tc - c$ approaches zero for large t to restrict the set $V[t_0, \infty)$ on which T must satisfy the hypotheses of the Schauder-Tychonov theorem. (See Remark 1, below.)

THEOREM 2. *Suppose Assumption* A *holds. Let*

$$h(t) = (\Gamma_k^{-1}(t) - I)c - \Gamma_k^{-1}(t)\int_t^\infty [A_k(s)A(s)c + \Gamma_k(s)f(s, c)]\, ds, \tag{16}$$

and suppose that

$$\overline{\lim_{t\to\infty}} \frac{\|h(t)\|}{w(t)} = \alpha. \tag{17}$$

Suppose also that

$$\overline{\lim_{t\to\infty}} \int_t^\infty \left[\frac{R(s, Mw(s))}{M} + \|A_k(s)A(s)\|w(s)\right] ds = \theta < 1 \tag{18}$$

and

$$\alpha < M(1 - \theta). \tag{19}$$

Then, if t_0 is sufficiently large, there is a solution x_0 of (5) *on* $[t_0, \infty)$ *such that*

$$\|x_0(t) - c\| \leqslant Mw(t), \qquad t \geqslant t_0, \tag{20}$$

and

$$\overline{\lim_{t \to \infty}} \; (w(t))^{-1} \|x_0(t) - c\| \leqslant \alpha + M\theta; \tag{21}$$

or, more precisely,

$$\overline{\lim_{t \to \infty}} \; (w(t))^{-1} \|x_0(t) - c - h(t)\| \leqslant M\theta. \tag{22}$$

PROOF. We can rewrite (14) as

$$(Tx)(t) = c + h(t) - \Gamma_k^{-1}(t) \int_t^\infty \left[A_k(s)A(s)(x(s) - c) + \Gamma_k(s)(f(s, x(s)) - f(s, c)) \right] ds \quad , \tag{23}$$

where the integral converges if $x \in V[t_0, \infty)$, because of (7), (12), and (18); moreover

$$\|(Tx)(t) - c\| \leqslant \|h(t)\| + M\mu_k(t) \int_t^\infty \|A_k(s)A(s)\| w(s)\, ds + \nu_k(t) \int_t^\infty R(s, Mw(s))\, ds. \tag{24}$$

From (10), (11), (18), and (19), we can assume henceforth that t_0 is so large that the right side of (24) is $\leqslant Mw(t)$ if $t \geqslant t_0$. Then

$$T(V[t_0, \infty)) \subset V[t_0, \infty). \tag{25}$$

Now we show that T is continuous on $V[t_0, \infty)$. Let $\{x_j\}$ be a sequence in $V[t_0, \infty)$ which converges to a limit x in $V[t_0, \infty)$. From (10), (11), and (23).

$$\|(Tx_j)(t) - (Tx)(t)\| \leqslant \mu_k(t_0) \int_{t_0}^\infty \|A_k(s)A(s)\| \, \|x_j(s) - x(s)\|\, ds + \nu_k(t_0) \int_{t_0}^\infty \|f(s, x_j(s)) - f(s, x(s))\|\, ds, \qquad t \geqslant t_0.$$

The integrands here converge to zero on $[t_0, \infty)$, and they are dominated by $2M\|A_k(s)A(s)\|w(s)$ and $2R(s, Mw(s))$, respectively. (See (7) and (12).) Therefore, (18) and Lebesgue's dominated convergence theorem imply that the integrals approach zero as $j \to \infty$. Hence, $\{Tx_j\}$ converges to Tx uniformly on $[t_0, \infty)$, and therefore T is continuous on $V[t_0, \infty)$.

From (12) and (25), $T(V[t_0, \infty))$ is equibounded on finite intervals. This, (15), and Assumption A imply that $T(V[t_0, \infty))$ is also equicontinuous on finite intervals. Now we have verified the hypotheses of the Schauder-Tychonov theorem, which implies that $Tx_0 = x_0$ for some x_0 in $V[t_0, \infty)$. Setting $Tx = x_0$ in (24) and invoking (10), (11), (17), and (18) yields (20). Similarly, setting $x = Tx = x_0$ in (23) yields (22). This completes the proof.

COROLLARY 1. *In addition to the assumptions of Theorem* 2, *suppose that*

$$R(t,\lambda_1)/R(t,\lambda_2) \leqslant \lambda_1/\lambda_2, \qquad 0 \leqslant \lambda_1 < \lambda_2. \tag{26}$$

Then (21) *can be replaced by*

$$\varlimsup_{t\to\infty} (w(t))^{-1}\|x_0(t)-c\| \leqslant \alpha/(1-\theta). \tag{27}$$

PROOF. Let

$$\phi(t) = \sup_{s\geqslant t}\left\{(w(s))^{-1}\|x_0(s)-c\|\right\}$$

and

$$\delta = \lim_{t\to\infty}\phi(t) = \varlimsup_{t\to\infty}(w(t))^{-1}\|x_0(t)-c\|. \tag{28}$$

Setting $x = Tx = x_0$ in (23) and using routine estimates yields

$$\begin{aligned}\|x_0(t)-c\| \leqslant \|h(t)\| + \nu_k(t)\int_t^\infty \|A_k(s)A(s)\|\,\|x_0(s)-c\|\,ds \\ +\mu_k(t)\int_t^\infty R\big(s,\|x_0(s)-c\|\big)\,ds.\end{aligned} \tag{29}$$

Applying (26) in the second integral and then dividing (29) by $w(t)$ yields the inequality

$$(w(t))^{-1}\|x_0(t)-c\| \leqslant (w(t))^{-1}\|h(t)\| + P(t)\phi(t), \tag{30}$$

where

$$P(t) = (w(t))^{-1}\left[\nu_k(t)\int_t^\infty \|A_k(s)A(s)\|w(s)\,ds + \frac{\mu_k(t)}{M}\int_t^\infty R(s,Mw(s))\,ds\right],$$

so that

$$\lim_{t\to\infty} P(t) = \theta, \tag{31}$$

from (10), (11), and (18). Letting $t\to\infty$ in (30) and invoking (17), (28), and (31) shows that $\delta \leqslant \alpha + \theta\delta$, which proves (27).

It is worthwhile to state Theorem 2 separately for the case where $w = 1$, so that α and θ are necessarily zero and (19) is automatic.

THEOREM 3. *Suppose Assumption* A *holds with* $w = 1$. *Suppose also that the integral in* (16) *converges and that*

$$\int^\infty R(t,M)\,dt < \infty, \quad \int^\infty \|A_k(t)A(t)\|\,dt < \infty. \tag{32}$$

Then, if t_0 is sufficiently large, there is a solution x_0 of (5) *on* $[t_0,\infty)$ *such that* $\|x_0(t)-c\| \leqslant M$ *for* $t \geqslant t_0$ *and* $\lim_{t\to\infty} x_0(t) = c$.

REMARK 1. The continuity assumption on f is the most stringent when $w = 1$, since the set S in (8) is maximized in that case. More importantly, if R satisfies (26), then (32) implies (18) with $\theta = 0$ for every admissible $w \neq 1$, while the converse is obviously false. Nevertheless, the conclusions of Theorem 2 are *weakest* when

$w = 1$. (See Theorem 3.) The reason for this is that in Theorem 3 it is assumed only that the integral in (16) converges, while Theorem 2 exploits the rapidity of its convergence.

The hypotheses of Theorem 2 may hold for some constant vectors c and fail to hold for others. In the following theorem c may be chosen arbitrarily.

THEOREM 4. *Suppose f is continuous for $t \geqslant a$ and all x, and*

$$\|f(t, x_1) - f(t, x_2)\| \leqslant R\left(t, \|x_1 - x_2\|\right),$$

where $R(t, \lambda)$ is continuous on $[a, \infty) \times [0, \infty)$ and nondecreasing in λ. For some integer $k \geqslant 0$, suppose the integrals $A_1, \ldots, A_{k+1}$ converge, and

$$\|A_j(t)\| = O(w(t)), \quad 1 \leqslant j \leqslant k+1. \tag{33}$$

Suppose also that

$$\left\|\int_t^\infty \Gamma_k(s) f(s, c)\, ds\right\| = O(w(t)) \tag{34}$$

for every constant vector c, and that (18) *holds for all $M > 0$. Then, if c is a given constant vector, there is a solution x_0 of* (5) *which is defined for t sufficiently large and satisfies*

$$\|x_0(t) - c\| = O(w(t)). \tag{35}$$

Moreover, if $\theta = 0$ in (18) *and* (33) *and* (34) *hold with "O" replaced by "o" (which is necessarily true if $w = 1$), then* (35) *holds with "O" replaced by "o."*

PROOF. The hypotheses imply (17) for some α (which may depend upon c, but is zero if (33) and (34) hold with "o"). Simply choose M to satisfy (19) and invoke Theorem 2.

Theorem 4 has the following corollary for the linear system

$$x' = [A(t) + B(t)]x + g(t). \tag{36}$$

COROLLARY 2. *Let A, B, and g be continuous on $[a, \infty)$. Suppose* (33) *holds.*

$$\left\|\int_t^\infty \Gamma_k(s) g(s)\, ds\right\| = O(w(t)), \quad \left\|\int_t^\infty \Gamma_k(s) B(s)\, ds\right\| = O(w(t)). \tag{37}$$

and

$$\overline{\lim_{t \to \infty}}\, (w(t))^{-1} \int_t^\infty \left[\|A_k(s) A(s)\| + \|B(s)\|\right] w(s)\, ds = \theta < 1. \tag{38}$$

Then, for any constant c, (36) *has a solution which satisfies* (35); *moreover, if $\theta = 0$ and* (33) *and* (37) *hold with "O" replaced by "o," then so does* (35).

The following special case of Corollary 2 extends Theorem 1.

COROLLARY 3. *Suppose* (33) *holds and*

$$\overline{\lim_{t \to \infty}}\, (w(t))^{-1} \int_t^\infty \|A_k(s) A(s)\| w(s)\, ds < 1$$

for some w as in Assumption A. *Then* (1) *has linear asymptotic equilibrium.*

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