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BEST RATIONAL APPROXIMATIONS OF ENTIRE FUNCTIONS WHOSE MACLAURIN SERIES COEFFICIENTS DECREASE RAPIDLY AND SMOOTHLY

BY

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ABSTRACT. Let $f = \sum_{j=0}^{\infty} a_j z^j$ be an entire function which satisfies

$$\left| a_{j-1} a_{j+1} / a_j^2 \right| \leqslant \rho^2, \qquad j = 1, 2, 3, \ldots,$$

where $0 < \rho < \rho_0$ and $\rho_0 = 0.4559\ldots$ is the positive root of the equation $2\sum_{j=1}^{\infty} \rho^{j^2} = 1$. Let $r > 0$ be fixed. Let W_{LM} denote the rational function of type (L, M) of best approximation to f in the uniform norm on $|z| \leqslant r$. We show that for any sequence of nonnegative integers $\{M_L\}_{L=1}^{\infty}$ that satisfies $M_L \leqslant 10L$, $L = 1, 2, 3, \ldots$, the rational approximations W_{LM_L} converge to f throughout $\mathbf{C}$ as $L \to \infty$. In particular, convergence takes place for the diagonal sequence and for the row sequences of the Walsh array for f.

1. Introduction. As far as the authors can determine, e^z is the only function for which best rational approximations are known to overconverge throughout $\mathbf{C}$. The known results, all due to Saff [7, 8], include the following. Let $r > 0$ be fixed. For each pair (m, n) of nonnegative integers, let W_{mn} denote a rational function of type (m, n) of best approximation to the function e^z in the uniform norm on $|z| \leqslant r$.

THEOREM A [7]. *As $m + n \to \infty$, $W_{mn}(z) \to e^z$ uniformly on compact subsets of* $\mathbf{C}$.

THEOREM B [8]. *Let $\varepsilon_{mn} = m!n!/((m+n)!(m+n+1)!)$. Then, for each fixed n,*

$$\max_{|z| \leqslant r} \left| e^z - W_{mn}(z) \right| = \varepsilon_{mn} r^{m+n+1}(1 + o(1)) \quad \text{as } m \to \infty.$$

Recently Trefethen [9], using the method of Braess [2], extended Theorem B to non-row sequences of the Walsh array for e^z.

THEOREM C [8]. *For each fixed n,*

$$\lim_{m \to \infty} \frac{e^z - W_{mn}(z)}{(-1)^n \varepsilon_{mn} z^{m+n+1}} = 1,$$

uniformly on each compact subset of $|z| > r$.

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The proof of Saff's theorems, as well as the work of Trefethen and Braess, is based on the fact that analogous results are valid for the Padé approximations to e^z.

Recently one of the authors [3, 4] investigated the Padé tables of entire functions such as

$$f(z) = \sum_{j=0}^{\infty} a_j z^j, \qquad a_j \neq 0, \ j = 0, 1, 2, \ldots, \tag{1}$$

which satisfy

$$\left| a_{j-1} a_{j+1} / a_j^2 \right| \leqslant \rho^2, \qquad j = 1, 2, \ldots, \tag{2}$$

where

$$\rho < \rho_0, \tag{3}$$

and $\rho_0 = 0.4559\ldots$ is the positive root of the equation $2\sum_{j=1}^{\infty}\rho^{j^2} = 1$.

It was shown in [3] that the Padé table of f is normal, and that any sequence of Padé approximants with numerator degrees tending to infinity converges to f uniformly on compact subsets of $\mathbf{C}$. These properties of the Padé approximants (together with their additional properties to be established in §2) enable us to use Saff's argument in a modified form to prove results analogous to Theorems A, B and C for this class of functions.

Throughout this paper we use the following notation. $[L/M]$ will denote the Padé approximant of type (L, M) for f, that is, $[L/M] = P_{LM}/Q_{LM}$, where P_{LM} and Q_{LM} are polynomials of degree $\leqslant L, M$ respectively, $Q_{LM}(0) = 1$, and $Q_{LM}(z)f(z) - P_{LM}(z) = O(z^{L+M+1})$ formally. For any $r > 0$, $W_{LM}(z)$ will denote a rational function of type (L, M) of best approximation to f in the uniform norm on $|z| \leqslant r$. That is,

$$\max_{|z| \leqslant r} |f(z) - W_{LM}(z)| \leqslant \max_{|z| \leqslant r} |f(z) - R(z)|$$

for every rational function R of type (L, M).

We shall use $\|g\|_r$ to denote $\max_{|z| \leqslant r}|g(z)|$. For L, $M \geqslant 0$ we denote by $D(L/M)$ the determinant

$$D(L/M) = \begin{vmatrix} a_L & a_{L+1} & \cdots & a_{L+M-1} \\ \vdots & \vdots & & \vdots \\ a_{L-M+1} & a_{L-M+2} & \cdots & a_L \end{vmatrix} \tag{4}$$

(where $a_j = 0$ if $j < 0$) and we let

$$\varepsilon_{LM} = \frac{D((L+1)/(M+1))}{D(L/M)}. \tag{5}$$

We now formulate our results.

THEOREM 1. *Let f be an entire function that satisfies* (1)–(3). *Let $r > 0$ be fixed. Let W_{LM} be a rational function of type (L, M) of best approximation to f in the uniform norm on $|z| \leqslant r$. Then*:

(i) *For any sequence of nonnegative integers $\{M_L\}_{L=1}^{\infty}$ that satisfies $M_L \leqslant 10L$, $L = 1, 2, 3, \ldots,$ the sequence $\{W_{LM_L}\}_{L=1}^{\infty}$ converges to f uniformly on compact subsets of* $\mathbf{C}$.

(ii) *If in* (2), $\rho < 1/3$, *then the restriction $M_L \leqslant 10L$ can be omitted.*

REMARKS. 1. The number 10 may be replaced by any positive number $< 4|\log \rho_0|/\log(3\rho_0) = 10.0339\ldots$.

2. Note that Saff's Theorem A admits the case $L =$ const, that is the column sequences of the Walsh array for e^z also converge to e^z. For our class of functions the analogous result is not true. Indeed, any entire function f that satisfies (1), (2) is of order 0 and consequently (since it is not rational) has infinitely many zeroes in $\mathbf{C}$. From Hurwitz' theorem follows that no sequence of rational functions with fixed degree of the numerator can converge to f throughout $\mathbf{C}$.

THEOREM 2. *With the notation of Theorem* 1 *and under the restriction $M_L \leqslant 1.4L$, $L = 1, 2, \ldots,$ we have*

(i) $\lim_{L \to \infty} \|f - W_{LM_L}\|_r / (|\varepsilon_{LM_L}| r^{L+M_L+1}) = 1$,

(ii) $\lim_{L \to \infty} (f(z) - W_{LM_L}(z))(-1)^M / (\varepsilon_{LM_L} z^{L+M_L+1}) = 1$,

uniformly on compact subsets of $|z| > r$.

REMARK. The number 1.4 may be replaced by any positive number $< 2|\log \rho_0|/\log 3 = 1.42995\ldots$.

2. Estimations for Padé approximants. We first establish some simple properties of the Maclaurin coefficients of f, which follow from condition (2). This condition can be rewritten in either of two forms:

$$|a_{j+1}/a_j| \leqslant \rho^2 |a_j/a_{j-1}|, \qquad j = 1, 2, 3, \ldots, \tag{6}$$

$$|a_{j-1}/a_j| \leqslant \rho^2 |a_j/a_{j+1}|, \qquad j = 1, 2, 3, \ldots. \tag{7}$$

From (6) it follows by induction that

$$|a_{L+1}/a_L| \leqslant \rho^{2L} |a_1/a_0|, \qquad L = 0, 1, 2, \ldots, \tag{8}$$

and

$$|a_{L+k+1}/a_{L+k}| \leqslant \rho^{2k} |a_{L+1}/a_L|, \qquad L, k = 0, 1, 2, \ldots. \tag{9}$$

From (7) it follows that

$$|a_{L-k-1}/a_{L-k}| \leqslant \rho^{2k} |a_{L-1}/a_L|, \qquad L = 1, 2, \ldots, \; k = 1, 2, \ldots, L-1. \tag{10}$$

If we now write

$$\left|\frac{a_{L+p}}{a_L}\right| = \left|\frac{a_{L+p}}{a_{L+p-1}} \cdot \frac{a_{L+p-1}}{a_{L+p-2}} \cdot \cdots \cdot \frac{a_{L+1}}{a_L}\right|$$

and estimate each factor once using (8) and once using (9), we obtain

$$(11)\qquad |a_{L+p}/a_L| \leqslant \rho^{p(2L+p-1)}|a_1/a_0|^p, \qquad L, p = 0,1,2,\ldots,$$

$$(12)\qquad |a_{L+p}/a_L| \leqslant \rho^{p(p-1)}|a_{L+1}/a_L|^p, \qquad L, p = 0,1,2,\ldots.$$

In a similar way, writing

$$\frac{a_{L-p}}{a_L} = \frac{a_{L-p}}{a_{L-p+1}} \cdot \frac{a_{L-p+1}}{a_{L-p+2}} \cdot \cdots \cdot \frac{a_{L-1}}{a_L}$$

and using (10), we obtain

$$(13)\qquad |a_{L-p}/a_L| \leqslant \rho^{p(p-1)}|a_{L-1}/a_L|^p, \qquad L = 1,2,\ldots,\ p = 1,2,\ldots,L.$$

Finally, from (6) it follows that the sequence $|a_{j+1}/a_j|$ is monotonically decreasing so that $|a_k/a_{k-1}| < |a_l/a_{l-1}|$ for $k > l$, or

$$(14)\qquad |a_k/a_l| < |a_{k-1}/a_{l-1}|, \qquad k > l.$$

We now turn to our first lemma, which deals with the determinant $D(L/M)$ (see (4)).

LEMMA 1.

$$\left(\tfrac{1}{2}\right)^{M-1}|a_L|^M \leqslant |D(L/M)| \leqslant \left(\tfrac{3}{2}\right)^{M-1}|a_L|^M, \qquad L, M = 1,2,\ldots.$$

REMARK. This lemma was proved in [3]. Since we shall need some ideas from the proof in the sequel, we shall reproduce it, but use a different notation from that in [3].

PROOF. Let

$$(15)\qquad \xi_L = |a_{L-1}/a_{L+1}|^{1/2},$$

and let $\Lambda = \operatorname{diag}\{\xi_L, \xi_L^2, \ldots, \xi_L^M\}$. Let $A_{LM} = (a_{L-i+j})_{i,j=1}^M$ and consider the matrix

$$(16)\qquad (b_{ij}) = B = a_L^{-1} \cdot \Lambda^{-1} \cdot A \cdot \Lambda,$$

that is,

$$(17)\qquad b_{ij} = \frac{a_{L-i+j}}{a_L}\xi_L^{j-i}, \qquad i, j = 1,\ldots, M.$$

We shall show that B is a matrix with dominant diagonal, that is,

$$(18)\qquad \sigma = \max_{1 \leqslant i \leqslant M} \sum_{\substack{j=1\\ j\neq i}}^{M} |b_{ij}|/|b_{ii}| < 1.$$

Indeed, for $1 \leqslant i \leqslant M$, since $|b_{ii}| = 1$,

$$\sum_{\substack{j=1 \\ j \neq i}}^{M} \frac{|b_{ij}|}{|b_{ii}|} = \sum_{j=1}^{i-1} |b_{ij}| + \sum_{j=i+1}^{M} |b_{ij}|$$

$$= \sum_{j=1}^{i-1} \left| \frac{a_{L-i+j}}{a_L} \right| \cdot \left| \frac{a_{L-1}}{a_{L+1}} \right|^{(j-i)/2} + \sum_{j=i+1}^{M} \left| \frac{a_{L-i+j}}{a_L} \right| \cdot \left| \frac{a_{L-1}}{a_{L+1}} \right|^{(j-i)/2}$$

(by (15), (17))

$$= \sum_{p=1}^{i-1} \left| \frac{a_{L-p}}{a_L} \right| \cdot \left| \frac{a_{L-1}}{a_{L+1}} \right|^{-p/2} + \sum_{p=1}^{M-i} \left| \frac{a_{L+p}}{a_L} \right| \cdot \left| \frac{a_{L-1}}{a_{L+1}} \right|^{p/2}$$

$$\leqslant \sum_{p=1}^{i-1} \rho^{p(p-1)} \left| \frac{a_{L-1}a_{L+1}}{a_L^2} \right|^{p/2} + \sum_{p=1}^{M-i} \rho^{p(p-1)} \left| \frac{a_{L-1}a_{L+1}}{a_L^2} \right|^{p/2}$$

(by (13), (12))

$$\leqslant \sum_{p=1}^{i-1} \rho^{p^2} + \sum_{p=1}^{M-i} \rho^{p^2} \quad \text{(by condition (2))}$$

$$< 2 \sum_{p=1}^{\infty} \rho^{p^2} < 2 \sum_{p=1}^{\infty} \rho_0^{p^2} = 1,$$

by condition (3) on ρ and by the definition of ρ_0. So we have

$$\sigma < 2 \sum_{p=1}^{\infty} \rho^{p^2} < 1, \tag{19}$$

and (18) is proved.

Applying an inequality of Ostrowski [**5**, formula 8], and noting that the above argument shows that $\sum_{j=1}^{i-1} |b_{ij}| < \frac{1}{2}$, we obtain,

$$\left(\frac{1}{2}\right)^{M-1} < \prod_{i=1}^{M-1} \left(1 - \sigma \sum_{j=1}^{i-1} |b_{ij}|\right) \leqslant |\det B| \leqslant \prod_{i=1}^{M-1} \left(1 + \sigma \sum_{j=1}^{i-1} |b_{ij}|\right) < \left(\frac{3}{2}\right)^{M-1}.$$

Now, the lemma follows from this inequality and from the relation (16) between B and A_{LM}. $\square$

LEMMA 2. *Let L, M be nonnegative integers and $k \geqslant L + M + 1$. Let*

$$A^{(k)}_{L+1,M+1} = \begin{pmatrix} a_{L+1} & \cdots & a_{L+M} & a_k \\ \vdots & & \vdots & \vdots \\ a_{L-M+2} & \cdots & a_{L+1} & a_{k-M+1} \\ a_{L-M+1} & \cdots & a_L & a_{k-M} \end{pmatrix}. \tag{20}$$

Then

$$\left|\det A^{(k)}_{L+1,M+1}\right| \leqslant \left(\tfrac{3}{2}\right)^M |a_{k-M}| \cdot |a_{L+1}|^M.$$

PROOF. Multiplying the last column of $A^{(k)}_{L+1,M+1}$ by a_{L+1}/a_{k-M}, we obtain the matrix

$$\tilde{A}^{(k)}_{L+1,M+1} = \begin{pmatrix} a_{L+1} & \cdots & a_{L+M} & a_k \dfrac{a_{L+1}}{a_{k-M}} \\ \vdots & & & \vdots \\ a_{L-M+2} & \cdots & a_{L+1} & a_{k-M+1}\dfrac{a_{L+1}}{a_{k-M}} \\ a_{L-M+1} & \cdots & a_L & a_{L+1} \end{pmatrix}.$$

For $k = L + M + 1$, this matrix coincides with $A_{L+1,M+1}$, the matrix whose determinant is $D((L+1)/(M+1))$. For $k > L + M + 1$, $\tilde{A}^{(k)}_{L+1,M+1}$ differs from $A_{L+1,M+1}$ only in the first M elements in the last column, which are less in absolute value than the corresponding elements of $A_{L+1,M+1}$. Indeed, by (14),

$$\left|\frac{a_{k-i+1}}{a_{k-M}}\right| < \left|\frac{a_{k-i}}{a_{k-M-1}}\right| < \cdots < \left|\frac{a_{L+M+2-i}}{a_{L+1}}\right|, \qquad i = 1,2,\ldots,M,$$

so that

$$\left|a_{k-i+1}\frac{a_{L+1}}{a_{k-M}}\right| < |a_{L+M+2-i}|, \qquad i = 1,2,\ldots,M.$$

Applying the transformation used in the proof of Lemma 1, with $\Lambda = \operatorname{diag}\{\xi_{L+1},\ldots,\xi_{L+1}^{M+1}\}$, to $A_{L+1,M+1}$ and to $\tilde{A}^{(k)}_{L+1,M+1}$, we obtain the matrices

$$B = a_{L+1}^{-1}\Lambda^{-1}A_{L+1,M+1}\Lambda, \qquad \tilde{B} = a_{L+1}^{-1}\Lambda^{-1}\tilde{A}^{(k)}_{L+1,M+1}\Lambda.$$

From the proof of Lemma 1 we know that B has a dominant diagonal and that $|\det B| < (\frac{3}{2})^M$. But $\tilde{B}$ differs from B only in the first M elements in the last column and these are less in absolute value than the corresponding elements of B. It follows that $\tilde{B}$ is also a matrix with dominant diagonal and that $\det \tilde{B}$ satisfies the same estimate: $|\det \tilde{B}| < (\frac{3}{2})^M$. From this it follows that

$$\left|\det \tilde{A}^{(k)}_{L+1,M+1}\right| < |a_{L+1}|^{M+1}\left(\tfrac{3}{2}\right)^M.$$

Since

$$\det A^{(k)}_{L+1,M+1} = \frac{a_{k-M}}{a_{L+1}} \det \tilde{A}^{(k)}_{L+1,M+1},$$

the lemma follows. □

LEMMA 3. *Let $\tau > 0$ satisfy*

$$\tau < L^{-1}\xi_L, \tag{21}$$

where ξ_L is defined by (15). *Suppose that L is sufficiently large, such that $\xi_L > 1$. Then, for any M complex numbers $c_1,\ldots,c_M$ that satisfy $\max_{1 \leqslant k \leqslant M}|c_k| = 1$, we have*

$$\sum_{j=1}^{M}\left|\sum_{i=1}^{M} c_i a_{L-i+j}\right|\tau^j \geqslant C|a_L|\xi_L^{-(M-1)}\tau^M,$$

where C depends only on ρ.

PROOF. We first prove that for any $\tau > 0$,

$$\sum_{j=1}^{M}\left|\sum_{i=1}^{M} c_i a_{L-i+j}\right|\tau^j \geqslant 1/\max_{1\leqslant j,k\leqslant M}\left(\tau^{-j}\left|A_{jk}^{-1}\right|\right), \tag{22}$$

where $A = (a_{L-i+j})_{i,j=1}^{M}$ and A_{jk}^{-1} denotes the (j,k) element of A^{-1}. Let $\mathbf{c}$ denote the row-vector $(c_1,\ldots,c_M)$. Since $\mathbf{c} = (\mathbf{c}A)A^{-1}$, we obtain

$$|c_k| = \left|\sum_{j=1}^{M}\left(\sum_{i=1}^{M} c_i a_{L-i+j}\right)A_{jk}^{-1}\right| \leqslant \max_{1\leqslant j,k\leqslant M}\left|A_{jk}^{-1}\right|\left(\sum_{j=1}^{M}\left|\sum_{i=1}^{M} c_i a_{L-i+j}\right|\right).$$

Since $\max_{1\leqslant k\leqslant M}|c_k| = 1$, by assumption, (22) follows for $\tau = 1$.

For $\tau \neq 1$, write $\mathbf{c} = (\mathbf{c}A_\tau)A_\tau^{-1}$ with $A_\tau = A \cdot \operatorname{diag}\{\tau, \tau^2, \ldots, \tau^M\}$, and repeat the argument.

REMARK. Estimate (22) is another form of a result of Saff [**6**, Lemma 2], but the above proof is simpler.

We now estimate $|A_{jk}^{-1}|$. As was shown in the proof of Lemma 1, the matrix $B = a_L^{-1}\Lambda^{-1}A\Lambda$ ($\Lambda = \operatorname{diag}\{\xi_L,\ldots,\xi_L^M\}$) has dominant diagonal. Hence (see Ostrowski [**5**, formulas (12)–(14)]), the elements of B^{-1} satisfy

$$\left|B_{jk}^{-1}\right| \leqslant \sigma/(1-\sigma^2), \qquad j,k = 1,\ldots,M,$$

where σ is defined by (18) and $\sigma < 2\sum_{p=1}^{\infty}\rho^{p^2} < 1$ by (19). Since the elements of A^{-1} are related to those of B^{-1}, by

$$A_{jk}^{-1} = a_L^{-1}\xi_L^{j-k}B_{jk}^{-1}, \qquad j,k = 1,\ldots,M,$$

we obtain

$$\left|A_{jk}^{-1}\right| \leqslant C_1|a_L|^{-1}\xi_L^{j-k}, \qquad j,k = 1,\ldots,M,$$

where C_1 depends only on ρ. Hence,

$$\begin{aligned}\max_{1\leqslant j,k\leqslant M}\tau^{-j}\left|A_{jk}^{-1}\right| &\leqslant C_1|a_L|^{-1}\max_{1\leqslant j,k\leqslant M}\tau^{-j}\xi_L^{j-k}\\ &= C_1|a_L|^{-1}\max_{1\leqslant j\leqslant M}\tau^{-j}\xi_L^{j-1} \quad (\text{since } \xi_L > 1)\\ &= C_1|a_L|^{-1}\tau^{-M}\xi_L^{M-1},\end{aligned}$$

since $\tau/\xi_L < 1/L < 1$ by the assumption (21). Substituting the last estimate in (22), we obtain the assertion of the lemma. □

We now establish the properties of the Padé approximants to functions f that satisfy (1)–(3).

LEMMA 4. *For each fixed $r > 0$, there exists a positive integer $L_0 = L_0(r)$, such that for $L \geqslant L_0$ and $M = 1,2,3,\ldots$,*

$$(1 - 2r/\xi_L)^M \leqslant |Q_{LM}(z)| \leqslant (1 + 2r/\xi_L)^M, \qquad |z| \leqslant r,$$

where ξ_L is defined by (15).

PROOF. Let

$$Q_{LM}(z) = \prod_{j=1}^{M}\left(1 - \frac{z}{z_{LMj}}\right),$$

where z_{LMj}, $j = 1, \ldots, M$, are the zeros of Q_{LM} (if some of them lie at ∞, we replace the corresponding factor by 1). It follows from (11) in [3] that z_{LMj} satisfy $|z_{LMj}| > \frac{1}{2}\xi_L$ (in the notation of [3] the bound is $\frac{1}{2}(q_L q_{L+1})^{1/2}$ with $q_j = |a_{j-1}/a_j|$, $j = 1, 2, 3, \ldots$). The lemma follows from this estimate since $\xi_L \to \infty$ as $L \to \infty$. □

LEMMA 5. *Let f satisfy* (1)–(3) *and let $[L/M] = P_{LM}/Q_{LM}$ be the L, M Padé approximant to f. Then,*

$$\|f - [L/M]\|_1 \leqslant C \cdot 3^M |a_{L+1}| \cdot \left|\frac{a_{L+1}}{a_L}\right|^M \Big/ \left(1 - \frac{2}{\xi_L}\right)^M,$$

where C depends only on f.

PROOF. By equation (1.1) in Arms and Edrei [1, p. 8] and using our notation (20), we obtain

$$(fQ_{LM} - P_{LM})(z) = \frac{(-1)^M}{D(L/M)} \sum_{k=L+M+1}^{\infty} \left(\det A^{(k)}_{L+1,M+1}\right) z^k. \tag{23}$$

So, for $|z| \leqslant 1$,

$$\begin{aligned}|(fQ_{LM} - P_{LM})(z)| &\leqslant \frac{1}{|D(L/M)|} \sum_{k=L+M+1}^{\infty} \left|\det A^{(k)}_{L+1,M+1}\right| \\ &\leqslant \frac{1}{|D(L/M)|} \cdot \left(\frac{3}{2}\right)^M \cdot |a_{L+1}|^M \sum_{k=L+M+1}^{\infty} |a_{k-M}| \quad \text{(by Lemma 2).}\end{aligned}$$

Estimating $|D(L/M)|$ from below by Lemma 1 and taking into account that $\sum_{k=L+M+1}^{\infty} |a_{k-M}| \leqslant C|a_{L+1}|$, where C depends on f (by (8), for example), we obtain

$$|(fQ_{LM} - P_{LM})(z)| \leqslant C 3^M |a_{L+1}| \cdot \left|\frac{a_{L+1}}{a_L}\right|^M, \qquad |z| \leqslant 1.$$

Dividing by $Q_{LM}(z)$ and using Lemma 4, we obtain the required estimate. □

LEMMA 6. *Let*

$$0 < c < c_0 = 2|\log \rho_0|/\log 3 = 1.42995\ldots.$$

Let ε_{LM} be defined by (5). *Then for any sequence of nonnegative integers $\{M_L\}_{L=1}^{\infty}$ that satisfies $M_L \leqslant cL$, $L = 1, 2, 3, \ldots$,*

$$\lim_{L \to \infty} \frac{f(z) - [L/M_L](z)}{(-1)^M \varepsilon_{LM_L} z^{L+M_L+1}} = 1,$$

uniformly on compact subsets of $\mathbf{C}$.

PROOF. From the formula (23) and the fact that

$$\det A^{(L+M+1)}_{L+1,M+1} = D((L+1)/(M+1))$$

by (20), we obtain

$$\begin{aligned}(24)\quad &\frac{(fQ_{LM} - P_{LM})(z)}{(-1)^M D((L+1)/(M+1)) z^{L+M+1}/D(L/M)} \\ &= 1 + \sum_{k=L+M+2}^{\infty} \frac{1}{D((L+1)/(M+1))} \left(\det A^{(k)}_{L+1,M+1}\right) z^{k-L-M-1}.\end{aligned}$$

The absolute value of the sum is estimated by Lemmas 1 and 2:

$$\begin{aligned}(25)\quad |\Sigma| &\leqslant \sum_{k=L+M+2}^{\infty} \frac{|a_{k-M}|\cdot|a_{L+1}|^M}{\left(\frac{1}{2}\right)^M |a_{L+1}|^{M+1}} \cdot \left(\frac{3}{2}\right)^M \cdot |z|^{k-L-M-1} \\ &\leqslant 3^M \left|\frac{a_{L+2}}{a_{L+1}}\right| \sum_{k=L+M+2}^{\infty} \left|\frac{a_{k-M}}{a_{L+2}}\right| \cdot |z|^{k-L-M-1} \\ &= 3^M \left|\frac{a_{L+2}}{a_{L+1}}\right| \sum_{p=0}^{\infty} \left|\frac{a_{L+2+p}}{a_{L+2}}\right| \cdot |z|^{p+1} \\ &\leqslant 3^M \rho^{2L} \left\{ \rho^2 \left|\frac{a_1}{a_0}\right| \sum_{p=0}^{\infty} |z| \left(|z| \rho^{2L+p+3} \cdot \left|\frac{a_1}{a_0}\right| \right)^p \right\} \quad \text{(by (11), (8)).}\end{aligned}$$

If $M = M_L \leqslant cL$, we have $(3^M \rho^{2L}) \leqslant (3^c \rho^2)^L \to 0$ as $L \to \infty$, since $3^c\rho^2 < 1$ by choice of c. The term in the braces in (25) is obviously uniformly bounded on each compact subset of $\mathbf{C}$. Hence, the sum on the right-hand side of (24) tends to zero uniformly on compact subsets of $\mathbf{C}$ and we obtain

$$(26)\quad \frac{(fQ_{LM} - P_{LM})(z)}{(-1)^M \varepsilon_{LM} z^{L+M+1}} = 1 + o(1) \quad \text{as } L \to \infty \text{ and } M = M_L \leqslant cL.$$

From (15) and (8), it follows that $1/\xi_L \leqslant C\rho^{2L}$, $L = 1, 2, 3, \ldots$. Then, from Lemma 4, it follows that for $M = M_L \leqslant cL$, $Q_{LM}(z) \to 1$ uniformly on compact subsets of $\mathbf{C}$. Dividing (26) by $Q_{LM}(z)$, we obtain the assertion of the lemma. □

3. Proofs of Theorems 1, 2. As was mentioned in the introduction, the properties of the Padé table that were established in §2 enable us to adapt Saff's arguments to our case. However, for the sake of completeness, we shall give a fairly full proof of Theorem 1.

PROOF OF THEOREM 1. Let f satisfy (1)–(3) and let W_{LM} be a rational function of best approximation to f in the uniform norm on $|z| \leqslant r$. Since condition (2) is invariant under the transformation $f(z) \to f(rz)$, it is sufficient to consider the case $r = 1$. Let L, M be fixed and assume that W_{LM} has a pole $1/\alpha_{LM}$ ($|\alpha_{LM}| < 1$) in the disk $|z| \leqslant \tau/2$ ($\tau > 2$). Assuming L sufficiently large, we shall obtain a lower bound for τ. Write

$$(27)\quad W_{LM}(z) = p_{LM}(z)/[(1 - \alpha_{LM} z) q_{LM}(z)],$$

where p_{LM} is a polynomial of degree $\leqslant L$ and q_{LM} is a polynomial of the form

$$q_{LM}(z) = \sum_{i=1}^{M} c_i z^{i-1}, \tag{28}$$

which is normalized so that $\max_{1 \leqslant i \leqslant M}|c_i| = 1$. Note that then

$$|q_{LM}(z)| \leqslant M, \quad |z| \leqslant 1. \tag{29}$$

By the extremal property of W_{LM} we have

$$\begin{aligned} |W_{LM}(z) - [L/M](z)| &\leqslant |W_{LM}(z) - f(z)| + |f(z) - [L/M](z)| \\ &\leqslant 2\|f - [L/M]\|_1, \qquad |z| \leqslant 1, \end{aligned}$$

where $[L/M] = P_{LM}/Q_{LM}$ is the Padé approximant of type (L, M) for f. Now, (27), (29) and the estimate from above for Q_{LM} (Lemma 4) imply

$$\begin{aligned} &|p_{LM}(z)Q_{LM}(z) - P_{LM}(z)(1 - \alpha_{LM}z)q_{LM}(z)| \\ &\quad \leqslant 4M\|f - [L/M]\|_1(1 + 2/\xi_L)^M, \qquad |z| \leqslant 1. \end{aligned}$$

From this we deduce by Bernstein's Lemma (Walsh [**10**, p. 77]) that

$$\begin{aligned} &|p_{LM}(z)Q_{LM}(z) - P_{LM}(z)(1 - \alpha_{LM}z)q_{LM}(z)| \\ &\quad \leqslant \tau^{L+M} \cdot 4M \cdot \|f - [L/M]\|_1(1 + 2/\xi_L)^M, \qquad |z| \leqslant \tau. \end{aligned}$$

For L sufficiently large, $|Q_{LM}|$ is bounded from below on $|z| \leqslant \tau$ by Lemma 4. Hence,

$$\begin{aligned} &|(1 - \alpha_{LM}z)q_{LM}(z) \cdot [L/M](z) - p_{LM}(z)| \\ &\quad \leqslant \tau^{L+M} \cdot 4M \cdot \|f - [L/M]\|_1 \cdot ((1 + 2/\xi_L)/(1 - 2\tau/\xi_L))^M. \end{aligned} \tag{30}$$

Since this inequality holds for $z = 1/\alpha_{LM}$, it follows that

$$\begin{aligned} &|(1 - \alpha_{LM}z)q_{LM}(z)[L/M](z) - p_{LM}(z) + p_{LM}(1/\alpha_{LM})| \\ &\quad \leqslant 2 \times (\text{right-hand member of } (30)), \qquad |z| \leqslant \tau. \end{aligned}$$

Consequently, noting that $|1 - \alpha_{LM}z| \geqslant (2/\tau)\tau - 1 = 1$ for $|z| \leqslant \tau$, we obtain

$$\begin{aligned} &\|q_{LM}[L/M] - \pi_{LM}\|_\tau \\ &\quad \leqslant \tau^{L+M} \cdot 8M \cdot \|f - [L/M]\|_1 \cdot ((1 + 2/\xi_L)/(1 - 2\tau/\xi_L))^M, \end{aligned} \tag{31}$$

where $\pi_{LM}(z) = [p_{LM}(z) - p_{LM}(1/\alpha_{LM})]/(1 - \alpha_{LM}z)$ is a polynomial of degree $\leqslant L - 1$.

Estimating $\|f - [L/M]\|_1$ by Lemma 5 and assuming that $\tau < L^{-1}\xi_L$, we obtain, from (31),

$$\|q_{LM}[L/M] - \pi_{LM}\|_\tau \leqslant \tau^{L+M}CM3^M|a_{L+1}| \cdot |a_{L+1}/a_L|^M(1 + \delta_L)^M, \tag{32}$$

where $\delta_L \to 0$ as $L \to \infty$.

We now obtain a lower bound for the left-hand member of (32). In a neighbourhood of 0, let

$$q_{LM}(z)[L/M](z) - \pi_{LM}(z) = \sum_{k=0}^{\infty} d_k z^k. \tag{33}$$

Since $\deg \pi_{LM} \leqslant L - 1$, and taking into account (28) and the definition of $[L/M]$, we obtain

$$d_k = \sum_{i=1}^{M} a_{k-i+1} c_i, \qquad k = L, L+1, \ldots, L+M-1. \tag{34}$$

Now, it follows from (33) that

$$\|q_{LM}[L/M] - \pi_{LM}\|_\tau^2 \geqslant (2\pi\tau)^{-1} \int_{|z|=\tau} \left| \sum_{k=0}^{\infty} d_k z^k \right|^2 |dz|$$
$$\geqslant \sum_{k=L}^{L+M-1} |d_k|^2 \tau^{2k} \geqslant \frac{1}{M} \left(\sum_{k=L}^{L+M-1} |d_k| \tau^k \right)^2$$

by the Cauchy-Schwarz inequality. Hence, by (34),

$$\left\| q_{LM} \left[\frac{L}{M} \right] - \pi_{LM} \right\|_\tau \geqslant \frac{1}{\sqrt{M}} \sum_{k=L}^{L+M-1} \left| \sum_{i=1}^{M} c_i a_{k-i+1} \right| \tau^k \tag{35}$$
$$= \frac{\tau^{L-1}}{\sqrt{M}} \sum_{j=1}^{M} \left| \sum_{i=1}^{M} c_i a_{L-i+j} \right| \tau^j$$
$$\geqslant \tau^{L+M-1} \cdot \frac{C}{\sqrt{M}} |a_L| \xi_L^{-(M-1)}$$

by Lemma 3 (remember that we assumed that $\tau < L^{-1}\xi_L$). Now (32), (35) and the definition (15) of ξ_L yield

$$\tau \geqslant C \cdot M^{-3/2} 3^{-M} |a_L/a_{L+1}| \cdot |a_{L-1}/a_{L+1}|^{1/2}$$
$$\cdot \left| a_L^2/(a_{L-1}a_{L+1}) \right|^{M/2} \cdot (1 + \delta_L)^{-M},$$

provided $\tau < L^{-1}|a_{L-1}/a_{L+1}|^{1/2}$. Using (2), (8), we finally obtain,

$$\tau \geqslant C \cdot \min\left\{ L^{-1}\rho^{-2L}, M^{-3/2}(3\rho)^{-M} \rho^{-4L} (1 + \delta_L)^{-M} \right\}, \tag{36}$$

where C depends only on f and $\delta_L = \delta_L(f) \to 0$ as $L \to \infty$.

It is easily verified that for $\rho < \frac{1}{3}$ the right-hand member of (36) tends to infinity as $L \to \infty$ and $M = M_L$ arbitrary, and that for $\frac{1}{3} \leqslant \rho < \rho_0 = 0.4559\ldots$ the same is true provided $M = M_L$ satisfies $M_L \leqslant cL$, $L = 1, 2, \ldots$, where $c < c_1 = 4|\log \rho_0|/\log(3\rho_0) = 10.0339\ldots$. So, the poles of W_{LM_L} tend to infinity as $L \to \infty$. It is well known (see Walsh [**10**, Corollary 5, p. 234]) that this fact implies the uniform convergence of W_{LM_L} to f on compact subsets of **C**. Theorem 1 is proved. □

PROOF OF THEOREM 2. Let $c_0 = 1.42995\ldots$ be the constant that appears in Lemma 6. Let $\{M_L\}_{L=1}^{\infty}$ be any sequence of integers that satisfies $M_L \leqslant cL$, $L = 1, 2, \ldots$, where $c < c_0$. Then Lemma 6 holds.

Furthermore, from (36) it follows that the poles of W_{LM_L} lie outside the disk $|z| < T^L$ $(T > 1)$. Hence, if we normalize the denominator q_{LM_L} of W_{LM_L} by $q_{LM_L}(0) = 1$, we obtain

$$\lim_{L\to\infty} q_{LM_L}(z) = 1, \qquad M_L \leqslant cL,\ L = 1, 2, \ldots,$$

uniformly on compact subsets of **C**. This fact, together with Lemma 6, enables us to use Saff's method of the proof of Theorems B and C without any changes, and Theorem 2 follows. □

We also note that part (i) of Theorem 2 follows immediately from Lemma 6 and Rouché's theorem (cf. the proof in Trefethen [**9**]).

REMARKS. 1. It follows from the proof of Theorem 1, that this theorem is valid not only for the "best" rational functions W_{LM} but for any sequence of rational functions R_{LM} of respective types (L, M) that satisfy for some $r > 0$: $\|f - R_{LM}\|_r \leqslant C\|f - [L/M]\|_r$ (cf. the remark in Saff [**7**, p. 193]).

2. From the previous remark follows via the lemma of Walsh [**10**, p. 101], that Theorem 1 is also true for the case of best rational approximation in the L_p-norm, $p > 0$, on $|z| = r$.

3. In the proof of Theorem 1 we assumed for simplicity of writing that $r = 1$. With a little more effort a stronger result can be obtained, namely:

THEOREM 1′. *Let f satisfy* (1)–(3) *and let $W_{LM}(f; r)$ denote a rational function of type (L, M) of best approximation to f in the uniform norm on $|z| \leqslant r$. Let $\beta_{LM}(f; r)$ denote the pole of $W_{LM}(f; r)$, nearest to the origin. Then for any sequence of indices $\{M_L\}_{L=1}^{\infty}$ such that $M_L \leqslant 10L$, $L = 1, 2, 3, \ldots,$ the following holds*:

$$\inf_{r>0} \left|\beta_{LM_L}(f; r)\right| \to \infty \quad \textit{as } L \to \infty.$$

4. It seems likely that the constant ρ_0 is sharp in the sense that for every $\rho_1 > \rho_0$ there exists an entire function which satisfies (1) and (2) with $\rho = \rho_1$ and such that the set of poles of the functions $W_{LM_L}(f; r)$, $M_L \leqslant 10L$, $L = 1, 2, 3, \ldots,$ has a finite limit point.

Although we cannot prove this fact, we can motivate it by proving that for $\rho = \rho_1 > \rho_0$, the sharpened version of Theorem 1 (Theorem 1′) definitely fails. Indeed, according to a result of Walsh [**11**], if $D(L/M) \neq 0$ for some L, M, then as $r \to 0$, the poles of $W_{LM}(f; r)$ tend to the poles of $[L/M]$. So, to contradict Theorem 1′ it suffices to prove that for any ρ_1, $\rho_1 > \rho_0$, there exists an entire function which satisfies (1) and (2) with $\rho = \rho_1$, and such that some sequence $[L/M_L]$ of its Padé approximations has a finite limit point. This was proved in [**4**] even for the case $M_L =$ const.

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