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SOLVABILITY OF DIFFERENTIAL EQUATIONS WITH LINEAR COEFFICIENTS OF REAL TYPE

BY

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ABSTRACT. Let L be the infinitesimal generator associated with a flow on a manifold M. Regarding L as an operator on a space of testfunctions we deal with the question if L has closed range. (Questions of this kind are investigated in [4, 1, 2].) We provide conditions under which $L + \mu 1: \mathscr{S}(M) \to \mathscr{S}(M)$, $\mu \in \mathbf{C}$, has closed range, where $M = \mathbf{R}^n \times K$, K being a compact manifold; here $\mathscr{S}(M)$ is the Schwartz space of rapidly decreasing smooth functions. As a consequence we show that the differential operator $\sum_{i,j} a_{ij} x_j (\partial/\partial x_i) + b$ defines a surjective mapping of the space $\mathscr{S}'(\mathbf{R}^n)$ of tempered distributions onto itself provided that all eigenvalues of the matrix (a_{ij}) are real. (In the case of imaginary eigenvalues this is not true in general [3].)

1. Preliminaries and notations. Let M be a differentiable manifold. We assume that $\mathbf{R}$ acts on M (on the right) by diffeomorphisms; i.e. we have a one-parameter group $(\rho_t)_{t \in \mathbf{R}}$ of transformations (or a global flow) on M. Let L be the infinitesimal generator associated with this flow. We regard L as a differential operator on M given by

$$L\varphi(m) = \frac{d}{dt}\varphi(m \cdot t)\Big|_{t=0}, \qquad m \in M, \varphi \in C^\infty(M). \tag{1.1}$$

Or, if $\varphi_t := \varphi \circ \rho_t$, $t \in \mathbf{R}$, we have $L\varphi = (d/dt)\varphi_t|_{t=0}$. Furthermore, L is invariant under (ρ_t), i.e.

$$L(\varphi_t) = (L\varphi)_t = \frac{d}{dt}\varphi_t \tag{1.2}$$

for all $t \in \mathbf{R}$. For $\mu \in \mathbf{C}$ we define the first order differential operator $L_\mu := L - \mu 1$.

We denote by $\mathscr{D}(M)$ the space of C^∞-functions with compact support on M. Its dual space $\mathscr{D}'(M)$ is the space of distributions on M. A distribution $T \in \mathscr{D}'(M)$ is called *relatively invariant with weight* μ if

$$\langle T, \varphi_t \rangle = e^{\mu t}\langle T, \varphi \rangle \tag{1.3}$$

for all $\varphi \in \mathscr{D}(M)$, $t \in \mathbf{R}$. We write $\mathscr{D}'_\mu(M)$ for the space of relatively invariant distributions with weight μ.

Clearly, L_μ defines a continuous mapping of $\mathscr{D}(M)$ into itself. The aim of this paper is to provide conditions under which this mapping has closed range. By differentiating equation (1.3) it is seen that the closure $\overline{L_\mu \mathscr{D}(M)}$ of the range of L_μ

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in $\mathscr{D}(M)$ can be characterized as the orthogonal of $\mathscr{D}'_\mu(M)$ in $\mathscr{D}(M)$; we write

$$\overline{L_\mu \mathscr{D}(M)} = \mathscr{D}'_\mu(M)^\perp . \tag{1.4}$$

Let L^t_μ: $\mathscr{D}'(M) \to \mathscr{D}'(M)$ be the transpose of L_μ: $\mathscr{D}(M) \to \mathscr{D}(M)$. Given a distribution $T \in \mathscr{D}'(M)$, by (1.4) we have

$$T \in \mathscr{D}'_\mu(M) \quad \text{iff} \quad L^t_\mu T = 0. \tag{1.5}$$

Let $C^r(M)$ be the space of r-times continuously differentiable functions on M, $r \in \mathbf{N}$. For $\varphi \in C^1(M)$ we have

$$\frac{d}{dt}\left(e^{-\mu t}\varphi_t\right) = e^{-\mu t}\left(L_\mu \varphi\right)_t. \tag{1.6}$$

Therefore, if $L_\mu \varphi = 0$ we have $\varphi_t = e^{\mu t}\varphi$ for all $t \in \mathbf{R}$.

Furthermore, let $L_\mu \varphi = f$, $\varphi \in \mathscr{D}(M)$, and suppose that, if $m \in M$ is given, $e^{-\mu t} f(m \cdot t)$ is integrable over the interval $-\infty < t < 0$ and that

$$\lim_{t \to -\infty} e^{-\mu t}\varphi(m \cdot t) = 0,$$

then from (1.6) we derive the solution formula

$$\varphi(m) = \int_{-\infty}^0 e^{-\mu t} f(m \cdot t)\, dt. \tag{1.7}$$

Moreover, suppose that $e^{-\mu t}\varphi(m \cdot t)$ is integrable over the whole real line $-\infty < t < \infty$ and that $\lim_{t \to \pm\infty} e^{-\mu t}\varphi(m \cdot t) = 0$ for all $\varphi \in \mathscr{D}(M)$. Then the distribution $\lambda_{\mu,m}$: $\varphi \mapsto \int_{-\infty}^{\infty} e^{-\mu t}\varphi(m \cdot t)\, dt$ is relatively invariant with weight μ, i.e.

$$\lambda_{\mu,m} \in \mathscr{D}'_\mu(M). \tag{1.8}$$

Therefore, if $f \in \overline{L_\mu \mathscr{D}(M)}$ we have the equation

$$\int_{-\infty}^0 e^{-\mu t} f(m \cdot t)\, dt = -\int_0^\infty e^{-\mu t} f(m \cdot t)\, dt. \tag{1.9}$$

In this paper we are mainly concerned with the case that our manifold M is a product of $\mathbf{R}^n$ with a d-dimensional compact differentiable manifold K. In this case there is a natural notion of the space $\mathscr{S}(M)$ of Schwartz functions and its dual space $\mathscr{S}'(M)$ of tempered distributions.

Assume that there are d vector fields $Z_1, \ldots, Z_d$ on K such that for every $\tau \in K$ the tangent vectors $Z_1(\tau), \ldots, Z_d(\tau)$ span the tangent space $T_\tau(K)$ to K at τ. Then $\mathscr{S}(\mathbf{R}^n \times K)$ is the space of all smooth functions φ on $\mathbf{R}^n \times K$ such that the term

$$\left(1 + |x|^2\right)^{s/2} \partial_x^\alpha Z_\tau^\beta \varphi(x, \tau) \tag{1.10}$$

is bounded with respect to $(x, \tau) \in \mathbf{R}^n \times K$ for any $s \in \mathbf{N}$ and for any multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_d)$, where α_j, $1 \leqslant j \leqslant n$, and β_k, $1 \leqslant k \leqslant d$, belong to the set $\mathbf{N}_0$ of nonnegative integers and $\partial_x^\alpha := \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$ with $|\alpha| := \sum_{j=1}^n \alpha_j$ and $Z_\tau^\beta := Z_1^{\beta_1} \cdots Z_d^{\beta_d}$. Sometimes it is convenient to write Y_j for $\partial/\partial x_j$, $j = 1, \ldots, n$, and Y_{n+k} for Z_k, $k = 1, \ldots, d$; then we have $\partial_x^\alpha Z_\tau^\beta = Y_1^{\gamma_1} \cdots Y_{n+d}^{\gamma_{n+d}} =: Y^\gamma$ with $\gamma = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_d)$.

A C^∞-function $h(x,\tau,t)$ on $\mathbf{R}^n \times K \times \mathbf{R}$ is called *of type E* (resp. *of type P*) if for any $r \in \mathbf{N}_0$ and any multi-index γ of length $n+d$ there are η, θ, $\sigma \in \mathbf{N}$ such that

$$(1.11)\quad \left|Y^\gamma_{x,\tau}\left(\frac{\partial}{\partial t}\right)^r h(x,\tau,t)\right| \leqslant \theta\left(1+|x|^2\right)^{\sigma/2} e^{\eta|t|}$$

$$\left(\text{resp. } \left|Y^\gamma_{x,\tau}\left(\frac{\partial}{\partial t}\right)^r h(x,\tau,t)\right| \leqslant \theta\left(1+|x|^2\right)^{\sigma/2}(1+t^2)^{\eta/2}\right)$$

for all x, τ, t. (Of course, this definition does not depend on the special chosen vector fields $Z_1,\dots,Z_d$.) It is obvious that sums, products and derivatives of type E functions (resp. type P functions) are of type E (resp. of type P).

Let p and q be the projection of $\mathbf{R}^n \times K$ onto $\mathbf{R}^n$ and K, respectively, and let p_j be the jth component of p. Our one-parameter group (ρ_t) of transformations is called *of type E* (resp. *of type P*) if the functions $p_j((x,\tau)\cdot t)$ and $\psi \circ q((x,\tau)\cdot t)$ are of type E (resp. of type P) for all $j=1,\dots,n$ and for all $\psi \in C^\infty(K)$. In this case we are able to estimate x by $p((x,\tau)\cdot t) =: x'$ for any τ and t. In fact, let $(x',\tau')\cdot(-t) = (x,\tau)$; because

$$\left|p\left((x',\tau')\cdot(-t)\right)\right|^2 \leqslant \theta\left(1+|x'|^2\right)^{\sigma/2} e^{\eta|t|}$$

$$\left(\text{resp. } \left|p\left((x',\tau')\cdot(-t)\right)\right|^2 \leqslant \theta\left(1+|x'|^2\right)^{\sigma/2}(1+t^2)^{\eta/2}\right)$$

for some η, θ, $\sigma \in \mathbf{N}$, we have

$$(1.12)\quad 1+|x|^2 \leqslant (1+\theta)\left(1+\left|p\left((x,\tau)\cdot t\right)\right|^2\right)^{\sigma/2} e^{\eta|t|}$$

$$\left(\text{resp. } 1+|x|^2 \leqslant (1+\theta)\left(1+\left|p\left((x,\tau)\cdot t\right)\right|^2\right)^{\sigma/2}(1+t^2)^{\eta/2}\right)$$

and therefore

$$(1.13)\quad 1+\left|p\left((x,\tau)\cdot t\right)\right|^2 \geqslant \delta\left(1+|x|^2\right)^{\varepsilon/2} e^{-\zeta|t|}$$

$$\left(\text{resp. } 1+\left|p\left((x,\tau)\cdot t\right)\right|^2 \geqslant \delta\left(1+|x|^2\right)^{\varepsilon/2}(1+t^2)^{-\zeta/2}\right)$$

for some δ, ε, $\zeta > 0$.

Clearly, for each $k \in \{1,\dots,n+d\}$ there are C^∞-functions a_{ik} on $\mathbf{R}^n \times K \times \mathbf{R}$, $1 \leqslant i \leqslant n+d$, such that for any $\varphi \in C^\infty(\mathbf{R}^n \times K)$ we have

$$(1.14)\qquad Y_k(\varphi_t)(x,\tau) = \sum_{i=1}^{n+d} a_{ik}(x,\tau,t)(Y_i\varphi)_t(x,\tau)$$

for all x, τ, t. Similarly we have

$$(1.15)\qquad \frac{d}{dt}\varphi_t(x,\tau) = \sum_{i=1}^{n+d} b_i(x,\tau,t)(Y_i\varphi)_t(x,\tau)$$

where b_i, $1 \leqslant i \leqslant n+d$, are C^∞-functions on $\mathbf{R}^n \times K \times \mathbf{R}$.

Now let (ρ_t) be of type E (resp. of type P). Then all the functions a_{ik} and b_i are of type E (resp. of type P). This is evident by inserting p_j and $\psi \circ q$ for φ in (1.14) and (1.15), respectively. Reiterating formula (1.14) we derive that, given $t \in \mathbf{R}$, the function φ_t belongs to $\mathscr{S}(\mathbf{R}^n \times K)$ for any $\varphi \in \mathscr{S}(\mathbf{R}^n \times K)$ and that the mapping $\varphi \mapsto \varphi_t$ is a continuous mapping of $\mathscr{S}(\mathbf{R}^n \times K)$ into itself. Hereby formula (1.13) is used. Together with (1.15) we derive that the infinitesimal generator L defines a continuous mapping of $\mathscr{S}(\mathbf{R}^n \times K)$ into itself, and our previous considerations concerning $\mathscr{D}(M)$ and $\mathscr{D}'(M)$ remain valid with regard to $\mathscr{S}(M)$ and $\mathscr{S}'(M)$, $M = \mathbf{R}^n \times K$.

2. Lemmata. Let $M = \mathbf{R}^n \times K$ and let our one-parameter group (ρ_t) be of type E. In the whole section we assume that there is $\lambda \in \mathbf{R}$ such that

$$p_1(m \cdot t) = e^{-\lambda t} p_1(m) \tag{2.1}$$

for all $t \in \mathbf{R}$, $m = (x, \tau) \in M$. Then we have

$$L_\mu(p_1 \varphi) = p_1 L_{\mu+\lambda} \varphi \tag{2.2}$$

for any continuously differentiable function φ on M.

The submanifold $M^1 := \{m \in M \mid p_1(m) = 0\} \triangleq \mathbf{R}^{n-1} \times K$ is invariant under (ρ_t). Let (ρ_t^1) be the restriction of (ρ_t) to M^1 and let L^1 be the associated infinitesimal generator. If φ is a function on M, let φ^1 be its restriction to M^1. For any continuously differentiable function φ on M we have

$$(L\varphi)^1 = L^1 \varphi^1. \tag{2.3}$$

LEMMA 1. *Suppose that $(L_\mu^1)^t \colon \mathscr{S}'(M^1) \to \mathscr{S}'(M^1)$ is surjective. If $p_1 f \in \overline{L_\mu \mathscr{S}(M)}$ for $f \in \mathscr{S}(M)$, then $f \in \overline{L_{\mu+\lambda} \mathscr{S}(M)}$.*

PROOF. By (1.4), the assertion follows from the inclusion $\mathscr{S}'_{\mu+\lambda}(M) \subseteq p_1 \mathscr{S}'_\mu(M)$, which we are going to prove.

Let $S \in \mathscr{S}'_{\mu+\lambda}(M)$. By division of distributions there is $T_1 \in \mathscr{S}'(M)$ such that $p_1 T_1 = S$. By (2.2) and (1.5) we have

$$p_1 L_\mu^t T_1 = L_{\mu+\lambda}^t S = 0; \tag{2.4}$$

i.e. $L_\mu^t T_1$ is the trivial extension of a distribution $W^1 \in \mathscr{S}'(M^1)$. By assumption, $W^1 = (L_\mu^1)^t R^1$ with $R^1 \in \mathscr{S}'(M^1)$. Let $R \in \mathscr{S}'(M)$ be the trivial extension of R^1 and let $T := T_1 - R$. Then we have $p_1 T = S$, and $T \in \mathscr{S}'_\mu(M)$ since

$$\left\langle L_\mu^t T, \varphi \right\rangle = \left\langle W^1 - \left(L_\mu^1\right)^t R^1, \varphi^1 \right\rangle = 0 \quad \text{for all } \varphi \in \mathscr{S}(M). \qquad \square$$

For convenience, we define the set $(L_\mu \mathscr{S}(M))_k$, $k \in \mathbf{N}_0$, consisting of all functions $f \in \overline{L_\mu \mathscr{S}(M)}$ for which there are functions $\psi_k \in \mathscr{S}(M)$ and $f_k \in \overline{L_{\mu+k\lambda} \mathscr{S}(M)}$ such that $f = L_\mu \psi_k + p_1^k f_k$. Clearly, $(L_\mu \mathscr{S}(M))_{k+1} \subseteq (L_\mu \mathscr{S}(M))_k$ by (2.2). Put

$$\left(L_\mu \mathscr{S}(M)\right)_\infty := \bigcap_{k \in \mathbf{N}} \left(L_\mu \mathscr{S}(M)\right)_k.$$

LEMMA 2. *Suppose that $(L_{\mu+\kappa\lambda}^1)^t \colon \mathscr{S}'(M^1) \to \mathscr{S}'(M^1)$ is surjective for all $\kappa = 0, \ldots, k-1$. Then $\overline{L_\mu \mathscr{S}(M)} = (L_\mu \mathscr{S}(M))_k$.*

PROOF. We prove the lemma by induction on k. For $k = 0$ the assertion is trivial. Now assume $f = L_\mu \psi_k + p_1^k f_k$ with $\psi_k \in \mathscr{S}(M)$ and $f_k \in \overline{L_{\mu+k\lambda}\mathscr{S}(M)}$. Obviously, $f_k^1 \in \overline{L^1_{\mu+k\lambda}\mathscr{S}(M^1)}$. Since $L^1_{\mu+k\lambda}\mathscr{S}(M^1)$ is closed by assumption, $f_k^1 = L^1_{\mu+k\lambda}\psi^1$ for some $\psi^1 \in \mathscr{S}(M^1)$. Select $\psi \in \mathscr{S}(M)$ such that ψ^1 is the restriction of ψ to M^1. Then $f_k - L_{\mu+k\lambda}\psi$ vanishes on M^1. Therefore it can be divided by p_1; i.e. there is a function $f_{k+1} \in \mathscr{S}(M)$ such that $f_k - L_{\mu+k\lambda}\psi = p_1 f_{k+1}$. By Lemma 1, $f_{k+1} \in \overline{L_{\mu+(k+1)\lambda}\mathscr{S}(M)}$. Put $\psi_{k+1} := \psi_k + p_1^k \psi$. Using (2.2) we get the desired equation for $k + 1$.

LEMMA 3. *Let* $\lambda \neq 0$. *Suppose that* $\overline{L_\mu \mathscr{S}(M)} = (L_\mu \mathscr{S}(M))_\infty$. *Then* $L_\mu \mathscr{S}(M)$ *is closed in* $\mathscr{S}(M)$.

PROOF. Replacing (ρ_t) and μ by (ρ_{-t}) and $-\mu$ in case of need, we may assume that $\lambda > 0$.

Let $f \in \overline{L_\mu \mathscr{S}(M)}$. For any $k \in \mathbf{N}$, we take $\psi_k \in \mathscr{S}(M)$ and $f_k \in \overline{L_{\mu+k\lambda}\mathscr{S}(M)}$ such that $f = L_\mu \psi_k + p_1^k f_k$. In the course of the proof we determine $k_0 \in \mathbf{N}$ sufficiently large for our need. First of all we assume that the real part $\nu_k' := \operatorname{Re} \nu_k$ of $\nu_k := \mu + k\lambda$ is positive for $k \geqslant k_0$. For $k \geqslant k_0$ we put

$$\varphi_k(m) := \psi_k(m) - p_1^k(m) \int_0^\infty e^{-\nu_k t} f_k(m \cdot t)\, dt, \qquad m \in M. \tag{2.5}$$

It is easily seen that the distribution $\lambda_{\nu_k, m}$ (see (1.8)) is well defined for all $k \geqslant k_0$ and $m \in M \setminus M^1$. In fact, for $t < 0$ we apply (2.1) and get the estimate

$$|\varphi(m \cdot t)| \leqslant \frac{c(\varphi, r)}{|p_1(m)|^r} e^{r\lambda t}, \qquad \varphi \in \mathscr{S}(M), \tag{2.6}$$

where r is an arbitrary integer $\geqslant 0$ and $c(\varphi, r)$ is constant with respect to m and t. Therefore, by (1.9), for $m \in M \setminus M^1$, equation (2.5) can be written in the following form:

$$\varphi_k(m) = \psi_k(m) + p_1^k(m) \int_{-\infty}^0 e^{-\nu_k t} f_k(m \cdot t)\, dt. \tag{2.7}$$

Now, by induction on $r \in \mathbf{N}$ we see: For any $r \in \mathbf{N}$ there is $k_0 \in \mathbf{N}$ such that $\varphi_k \in C^r(M)$ for $k \geqslant k_0$; in fact, for any multi-index γ with $|\gamma| \leqslant r$ the derivative $Y^\gamma(\varphi_k - \psi_k)(m)$ is a finite sum of terms of the form

$$c p_1^{s'}(m) \int_a^b e^{-\nu_k t} h(m, t) \big(Y^{\gamma'} f_k\big)(m \cdot t)\, dt, \tag{2.8}$$

where $c \in \mathbf{C}$, $s' \in \mathbf{N}$ depend on k, $s' \geqslant k - |\gamma|$, and $h(m,t)$ is a C^∞-function of type E independent of k; $|\gamma'| \leqslant |\gamma|$, $a = 0$, $b = \infty$. Hereby, (1.14) is used.

For $m \in M \setminus M^1$ we can apply (2.6) and, proceeding from (2.7), we can express $Y^\gamma(\varphi_k - \psi_k)(m)$ by a finite sum of terms of the form (2.8) with $a = -\infty$, $b = 0$.

Now let us prove that for any $s \in \mathbf{N}$ and for any multi-index γ there is $k_0 \in \mathbf{N}$ such that for $k \geqslant k_0$ the term

$$\left(1 + |p(m)|^2\right)^{s/2} |Y^\gamma \varphi_k(m)| \tag{2.9}$$

is bounded with respect to $m \in M$. In view of (2.8) and by continuity it is sufficient to prove the boundedness of the terms

$$(2.10) \qquad \left(1+|p(m)|^2\right)^{s/2}|p_1^{s'}(m)|\int_a^b e^{-\nu_k' t}|h(m,t)|\,|Y^{\gamma'}f_k(m\cdot t)|\,dt$$

on the domain $\{0 \leqslant |p_1(m)| \leqslant 1\}$ for $a=0$, $b=\infty$ and on the domain $\{|p_1(m)| \geqslant 1\}$ for $a=-\infty$, $b=0$. For $a=0$, $b=\infty$ we use (1.13); since $f_k \in \mathscr{S}(M)$ we can estimate (2.10) by

$$(2.11) \quad \left(1+|p(m)|^2\right)^{s/2}\int_0^\infty e^{-\nu_k' t}|h(m,t)|\frac{C(f_k,N)}{\delta^N\left(1+|p(m)|^2\right)^{\varepsilon N/2}e^{-\zeta N t}}\,dt,$$

where N is a positive integer which satisfies $\varepsilon N \geqslant s+\sigma$ with σ from (1.11). The boundedness of (2.11) is obvious if k_0 is sufficiently large. For $a=-\infty$, $b=0$ we use (2.1). Let $k \geqslant k_0$ be given, we choose N as above and take a positive integer $r \geqslant s'$; then we get an estimate of (2.10) by the term

$$(2.12) \quad \left(1+|p(m)|^2\right)^{s/2}|p_1^{s'}(m)| \cdot\int_{-\infty}^0 e^{-\nu_k' t}|h(m,t)|\frac{C(f_k,N,r)}{\delta^N\left(1+|p(m)|^2\right)^{\varepsilon N/2}e^{-\zeta N|t|}e^{-\lambda r t}|p_1(m)|^r}\,dt$$

which is obviously bounded if r is sufficiently large.

Now let k_0 be sufficiently large and let $k \geqslant k_0$. From (2.5) and (2.2) we get

$$(2.13) \qquad L_\mu\varphi_k = L_\mu\psi_k - p_1^k L_{\nu_k} g_k,$$

where

$$g_k(m) := \int_0^\infty e^{-\nu_k t} f_k(m\cdot t)\,dt.$$

Applying (1.6) with $\varphi = g_k$, $\mu = \nu_k$ for $t=0$ we get

$$(2.14) \qquad L_{\nu_k} g_k = -f_k$$

and therefore

$$(2.15) \qquad L_\mu \varphi_k = f$$

for any $k \geqslant k_0$.

From (1.6) we derive that $L_\mu\varphi = 0$ implies $\varphi = 0$ for $\varphi \in C^1(M)$ vanishing at infinity; in fact, for $m \in M \setminus M^1$ we have $\varphi(m\cdot t) = e^{\mu t}\varphi(m)$ and therefore $\varphi(m) = 0$ because $m\cdot t \to \infty$ for $t \to -\infty$ by (2.1).

Therefore, looking at (2.15), we see that φ_k does not depend on k; i.e. $\varphi_k =: \varphi$ for all $k \geqslant k_0$. Thus, by (2.9), $\varphi \in \mathscr{S}(M)$.

LEMMA 4. *Let* $\operatorname{Re}\mu \neq 0$. *Suppose that* (ρ_t^1) *is of type P. Then* $(L_\mu^1)^t\colon \mathscr{S}'(M^1) \to \mathscr{S}'(M^1)$ *is surjective*

PROOF. Replacing (ρ_t) and μ by (ρ_{-t}) and $-\mu$ in case of need, we may assume that $\operatorname{Re}\mu < 0$.

By (1.6), $L_\mu^1\colon \mathscr{S}(M^1) \to \mathscr{S}(M^1)$ is injective. To prove that L_μ^1 is also surjective we put

$$\varphi^1(m^1) := \int_{-\infty}^{0} e^{-\mu t} f^1(m^1 \cdot t)\, dt, \qquad m^1 = (x^1, \tau) \in M^1, \tag{2.16}$$

for a given $f^1 \in \mathscr{S}(M^1)$ and show that $\varphi^1 \in \mathscr{S}(M^1)$.

In fact, by equation (1.14), for any $s \in \mathbf{N}$ and for any multi-index γ the term $(1 + |x^1|^2)^{s/2} Y^\gamma \varphi^1(m^1)$ is a finite sum of terms of the form

$$\int_{-\infty}^{0} e^{-\mu t} h^1(m^1, t) g^1(m^1 \cdot t)\, dt$$

where $g^1 \in \mathscr{S}(M^1)$ and $h^1(m^1, t)$ is a C^∞-function of type P. Using (1.13) we see that $|h^1(m^1, t) g^1(m^1 \cdot t)|$ can be estimated by $c(1 + t^2)^{r/2}$ with some $r \in \mathbf{N}$ and some constant $c > 0$.

LEMMA 5. *Let $\lambda \neq 0$. Suppose that (ρ_t^1) is of type P and that $\rho_t(x_1, m^1) = (e^{-\lambda t} x_1, \rho_t^1(m^1))$ for $(x_1, m^1) \in M \triangleq \mathbf{R} \times M^1$. Then $L_\mu \mathscr{S}(M)$ is closed in $\mathscr{S}(M)$.*

PROOF. If $\operatorname{Re}\mu + k\lambda \neq 0$ for all $k \in \mathbf{N}_0$, the assertion follows by Lemmas 4, 2 and 3.

Assume that $\operatorname{Re}\mu + k\lambda = 0$ for some $k \in \mathbf{N}_0$. Given $f \in \overline{L_\mu \mathscr{S}(M)}$, by Lemmas 4 and 2 there are $\psi_k \in \mathscr{S}(M)$ and $f_k \in \overline{L_{\mu + k\lambda} \mathscr{S}(M)}$ such that $f = L_\mu \psi_k + p_1^k f_k$. Therefore, by (2.2), we have only to prove that $L_{\mu + k\lambda} \mathscr{S}(M)$ is closed; i.e. it remains to prove that $L_\mu \mathscr{S}(M)$ is closed for $\mu \in \mathbf{C}$ with $\operatorname{Re}\mu = 0$.

Let $\operatorname{Re}\mu = 0$. Using the assumption we derive

$$\frac{\partial}{\partial x_1} L_\mu \varphi = L_{\mu+\lambda} \frac{\partial \varphi}{\partial x_1} \tag{2.17}$$

for all $\varphi \in \mathscr{S}(M)$. From the previous considerations we know that $L_{\mu+\lambda} \mathscr{S}(M)$ is closed. It follows that $L_{\mu+\lambda}\colon \mathscr{S}(M) \to L_{\mu+\lambda} \mathscr{S}(M)$ is an isomorphism, because $L_{\mu+\lambda}$ is injective by (1.6). Therefore, since

$$\begin{aligned} \mathscr{F}_1 &:= \left\{ \frac{\partial \varphi}{\partial x_1} \,\middle|\, \varphi \in \mathscr{S}(M) \right\} \\ &= \left\{ \psi \in \mathscr{S}(M) \,\middle|\, \int_{\mathbf{R}} \psi(x_1, m^1)\, dx_1 = 0 \text{ for all } m^1 \in M^1 \right\} \end{aligned}$$

is closed, $L_{\mu+\lambda} \mathscr{F}_1$ is closed. Consequently, by (2.17), $(\partial/\partial x_1) L_\mu \mathscr{S}(M)$ is closed. Since $\partial/\partial x_1\colon \mathscr{S}(M) \to \mathscr{F}_1$ is an isomorphism, it follows that $L_\mu \mathscr{S}(M)$ is closed.

3. Main results. Let us briefly sum up our assumptions and notations: We deal with a manifold $M = \mathbf{R}^n \times K$, where K is a d-dimensional compact differentiable manifold with the property that there are d vector fields $Z_1, \ldots, Z_d$ on K such that for each $\tau \in K$ the tangent space to K at τ is spanned by the tangent vectors $Z_1(\tau), \ldots, Z_d(\tau)$. For $(x, \tau) \in \mathbf{R}^n \times K$ we put $p_j(x, \tau) := x_j$ and $q(x, \tau) := \tau$. Let $(\rho_t)_{t \in \mathbf{R}}$ be a one-parameter group of transformations acting on M and let L be the associated infinitesimal transformation (see (1.1)). For $\mu \in \mathbf{C}$ we define the differential operator $L_\mu := L - \mu 1$.

THEOREM. *Let (ρ_t) be of type E. Given $k \in \mathbf{N}$, $1 \leqslant k \leqslant n$, let $M^j := \{(x, \tau) \in M \mid x_1 = \cdots = x_j = 0\}$ be invariant under (ρ_t) for $j = 1, \ldots, k$. We assume that the restriction of (ρ_t) to M^k is of type P and that the projection of $\rho_t(x, \tau)$ onto M^k does not depend on $x_1, \ldots, x_k$. Suppose that there are real numbers λ_j, $1 \leqslant j \leqslant n$, $\lambda_j \neq 0$ for $j = 1, \ldots, k$, $\lambda_j = 0$ for $j = k+1, \ldots, n$, such that $p_j((x, \tau) \cdot t)$ has the form*

$$(3.1) \qquad p_j\big((x, \tau) \cdot t\big) = e^{-\lambda_j t} x_j + w_j\big(x_1, \ldots, x_{j-1}, \tau, t\big), \qquad j = 1, \ldots, n,$$

where w_j are functions independent of $x_j, \ldots, x_n$.

Then L_μ: $\mathscr{S}(M) \to \mathscr{S}(M)$ is injective and its range is closed.

PROOF. First of all, it is easy to see that (2.1) with $\lambda = \lambda_1$ will follow from (3.1). In fact, we have

$$w_1(\tau, t) = p_1\big((0, \tau) \cdot t\big) - e^{-\lambda_1 t} 0$$

and $p_1((0, \tau) \cdot t) = 0$ since M^1 is invariant under (ρ_t) by assumption. From (2.1) we conclude that the orbit $\{(x, \tau) \cdot t \mid t \in \mathbf{R}\}$ is unbounded whenever $x_1 \neq 0$. Together with (1.6) we see that L_μ is injective for any $\mu \in \mathbf{C}$.

Now let us prove by induction on k that $L_\mu \mathscr{S}(M)$ is closed in $\mathscr{S}(M)$ for each $\mu \in \mathbf{C}$. For $k = 1$ the assertion follows by Lemma 5. Let $k > 1$. By induction hypothesis, $L_\mu^1 \mathscr{S}(M^1)$ is closed in $\mathscr{S}(M^1)$ and L_μ^1: $\mathscr{S}(M^1) \to \mathscr{S}(M^1)$ is injective by the consideration above. Since $\mathscr{S}(M^1)$ is a Fréchet space it follows that the transpose $(L_\mu^1)^t$: $\mathscr{S}'(M^1) \to \mathscr{S}'(M^1)$ is surjective for all $\mu \in \mathbf{C}$. Thus, by Lemmas 2 and 3, $L_\mu \mathscr{S}(M)$ is closed.

EXAMPLE. On $M = \mathbf{R}^n \times \mathbf{T}^d$ ($\mathbf{T}^d = d$-dimensional torus), $n, d \in \mathbf{N}_0$, we consider the one-parameter group

$$\rho_t(x, \tau) = \big(x_1 e^{\lambda_1 t}, \ldots, x_n e^{\lambda_n t}, \tau_1 e^{i\alpha_1 t}, \ldots, \tau_d e^{i\alpha_d t}\big),$$

where $\lambda_1, \ldots, \lambda_n, \alpha_1, \ldots, \alpha_d \in \mathbf{R}$. The infinitesimal generator L associated with (ρ_t) is given by

$$L\varphi(x, \tau) = \sum_{j=1}^{n} \lambda_j x_j \frac{\partial \varphi}{\partial x_j}(x, \tau) + \sum_{k=1}^{d} \alpha_k \frac{\partial \varphi}{\partial \tau_k}(x, \tau).$$

By the Theorem, $L_\mu \mathscr{S}(M)$ is closed in $\mathscr{S}(M)$ for any $\mu \in \mathbf{C}$ provided that $n > 0$ and $\lambda_j \neq 0$ at least for one j. (Compare [**4**, Example 2].) In general, $L\mathscr{S}(M)$ is not closed for $n = 0$ [**4**, Example 1]. Particularly, the range of the restriction of L to $\mathbf{T}^d$ may be not closed in spite of the fact that L itself has closed range.

Furthermore, putting $d = 0$ and assuming $\lambda_j \neq 0$ for one j we can conclude that L: $\mathscr{S}'(\mathbf{R}^n) \to \mathscr{S}'(\mathbf{R}^n)$ is surjective. This should be compared with Miwa's result [**5**] affirming that L: $\mathscr{B}(\mathbf{R}^n) \to \mathscr{B}(\mathbf{R}^n)$ is surjective if additionally it is supposed that $|\lambda_j| \leqslant 1$ for all $j = 1, \ldots, n$, where $\mathscr{B}(\mathbf{R}^n)$ is the set of hyperfunctions on $\mathbf{R}^n$.

COROLLARY. *Given a first-order differential operator $\neq 0$ on $\mathbf{R}^n$ with linear coefficients*

$$D = \sum_{i,j=1}^{n} a_{ij} x_j \frac{\partial}{\partial x_i} + b, \qquad a_{ij}, b \in \mathbf{R}.$$

Suppose that all eigenvalues of the matrix (a_{ij}) are real.

Then D: $\mathscr{S}'(\mathbf{R}^n) \to \mathscr{S}'(\mathbf{R}^n)$ is surjective.

PROOF. After change of basis we may assume that the matrix $A = (a_{ij})$ has Jordan form

$$\begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{pmatrix}$$

with Jordan boxes

$$J_\rho = \begin{pmatrix} \lambda_\rho & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & \lambda_\rho \end{pmatrix}, \qquad \lambda_\rho \in \mathbf{R},\ 1 \leqslant \rho \leqslant r,$$

which are arranged in such a manner that $\lambda_\rho \neq 0$ for $\rho = 1, \ldots, k$ and $\lambda_\rho = 0$ for $\rho = k + 1, \ldots, r$, where $0 \leqslant k \leqslant r$. It is easily seen that $D = L^t_\mu$, where L is the infinitesimal generator associated with the one-parameter group $\rho_t(x) = e^{-tA}x$ and $\mu = \operatorname{trace}(A) - b$. Therefore it is sufficient to show that L_μ: $\mathscr{S}(\mathbf{R}^n) \to \mathscr{S}(\mathbf{R}^n)$ is injective and has closed range.

Let $k = 0$. If $\mu \neq 0$, the assertion follows by Lemma 4. If $\mu = 0$, the assertion follows by [3].

Now let $k > 0$. Then we can apply the Theorem, where K is assumed to be trivial.

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