SOLVABILITY OF DIFFERENTIAL EQUATIONS WITH LINEAR COEFFICIENTS OF REAL TYPE

BY
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Abstract. Let $L$ be the infinitesimal generator associated with a flow on a manifold $M$. Regarding $L$ as an operator on a space of test functions we deal with the question if $L$ has closed range. (Questions of this kind are investigated in [4, 1, 2].) We provide conditions under which $L + \mu I : \mathcal{S}(M) \to \mathcal{S}(M)$, $\mu \in \mathbb{C}$, has closed range, where $M = \mathbb{R}^n \times K$, $K$ being a compact manifold; here $\mathcal{S}(M)$ is the Schwartz space of rapidly decreasing smooth functions. As a consequence we show that the differential operator $\sum_{i,j} a_{ij}(\partial / \partial x_j) + b$ defines a surjective mapping of the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions onto itself provided that all eigenvalues of the matrix $(a_{ij})$ are real. (In the case of imaginary eigenvalues this is not true in general [3].)

1. Preliminaries and notations. Let $M$ be a differentiable manifold. We assume that $\mathbb{R}$ acts on $M$ (on the right) by diffeomorphisms; i.e. we have a one-parameter group $(\rho_t)_{t \in \mathbb{R}}$ of transformations (or a global flow) on $M$. Let $L$ be the infinitesimal generator associated with this flow. We regard $L$ as a differential operator on $M$ given by

$$(1.1) \quad L \varphi (m) = \left. \frac{d}{dt} \varphi (m \cdot t) \right|_{t=0}, \quad m \in M, \varphi \in C^\infty (M).$$

Or, if $\varphi_t := \varphi \circ \rho_t$, $t \in \mathbb{R}$, we have $L \varphi = (d/dt) \varphi_t|_{t=0}$. Furthermore, $L$ is invariant under $(\rho_t)$, i.e.

$$(1.2) \quad L(\varphi_t) = (L \varphi)_t = \left. \frac{d}{dt} \varphi_t \right|_{t=0}$$

for all $t \in \mathbb{R}$. For $\mu \in \mathbb{C}$ we define the first order differential operator $L_\mu := L - \mu I$.

We denote by $\mathcal{D}(M)$ the space of $C^\infty$-functions with compact support on $M$. Its dual space $\mathcal{D}'(M)$ is the space of distributions on $M$. A distribution $T \in \mathcal{D}'(M)$ is called relatively invariant with weight $\mu$ if

$$(1.3) \quad \langle T, \varphi_t \rangle = e^{\mu t} \langle T, \varphi \rangle$$

for all $\varphi \in \mathcal{D}(M)$, $t \in \mathbb{R}$. We write $\mathcal{D}'_\mu(M)$ for the space of relatively invariant distributions with weight $\mu$.

Clearly, $L_\mu$ defines a continuous mapping of $\mathcal{D}(M)$ into itself. The aim of this paper is to provide conditions under which this mapping has closed range. By differentiating equation (1.3) it is seen that the closure $L_\mu \mathcal{D}(M)$ of the range of $L_\mu$
in $\mathcal{D}(M)$ can be characterized as the orthogonal of $\mathcal{D}'(M)$ in $\mathcal{D}(M)$; we write

$$L_\mu^* \mathcal{D}(M) = \mathcal{D}'(M)^{\perp}. \quad (1.4)$$

Let $L_\mu^* : \mathcal{D}'(M) \to \mathcal{D}'(M)$ be the transpose of $L_\mu : \mathcal{D}(M) \to \mathcal{D}(M)$. Given a distribution $T \in \mathcal{D}'(M)$, by (1.4) we have

$$T \in \mathcal{D}'(M) \iff L_\mu^* T = 0. \quad (1.5)$$

Let $C'(M)$ be the space of r-times continuously differentiable functions on $M$, $r \in \mathbb{N}$. For $\varphi \in C^1(M)$ we have

$$\frac{d}{dt} \left( e^{-\mu t} \varphi_t \right) = e^{-\mu t} \left( L_\mu \varphi \right)_t. \quad (1.6)$$

Therefore, if $L_\mu \varphi = 0$ we have $\varphi_t = e^{\mu t} \varphi$ for all $t \in \mathbb{R}$.

Furthermore, let $L_\mu \varphi = f$, $\varphi \in \mathcal{D}(M)$, and suppose that, if $m \in M$ is given, $e^{-\mu t} f(m \cdot t)$ is integrable over the interval $-\infty < t < 0$ and that

$$\lim_{t \to -\infty} e^{-\mu t} \varphi(m \cdot t) = 0,$$

then from (1.6) we derive the solution formula

$$\varphi(m) = \int_{-\infty}^{0} e^{-\mu t} f(m \cdot t) \, dt. \quad (1.7)$$

Moreover, suppose that $e^{-\mu t} \varphi(m \cdot t)$ is integrable over the whole real line $-\infty < t < \infty$ and that $\lim_{t \to -\infty} e^{-\mu t} \varphi(m \cdot t) = 0$ for all $\varphi \in \mathcal{D}(M)$. Then the distribution $\lambda_{\mu, m} : \varphi \mapsto \int_{-\infty}^{0} e^{-\mu t} \varphi(m \cdot t) \, dt$ is relatively invariant with weight $\mu$, i.e.

$$\lambda_{\mu, m} \in \mathcal{D}'(M). \quad (1.8)$$

Therefore, if $f \in L_\mu^* \mathcal{D}(M)$ we have the equation

$$\int_{-\infty}^{0} e^{-\mu t} f(m \cdot t) \, dt = - \int_{0}^{\infty} e^{-\mu t} f(m \cdot t) \, dt. \quad (1.9)$$

In this paper we are mainly concerned with the case that our manifold $M$ is a product of $\mathbb{R}^n$ with a $d$-dimensional compact differentiable manifold $K$. In this case there is a natural notion of the space $\mathcal{S}(M)$ of Schwartz functions and its dual space $\mathcal{S}'(M)$ of tempered distributions.

Assume that there are $d$ vector fields $Z_1, \ldots, Z_d$ on $K$ such that for every $\tau \in K$ the tangent vectors $Z_1(\tau), \ldots, Z_d(\tau)$ span the tangent space $T_\tau(K)$ to $K$ at $\tau$. Then $\mathcal{S}(\mathbb{R}^n \times K)$ is the space of all smooth functions $\varphi$ on $\mathbb{R}^n \times K$ such that the term

$$(1 + |x|^2)^{s/2} \partial_{x}^a Z_{\beta}^b \varphi(x, \tau)$$

is bounded with respect to $(x, \tau) \in \mathbb{R}^n \times K$ for any $s \in \mathbb{N}$ and for any multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_d)$, where $\alpha_j$, $1 \leq j \leq n$, and $\beta_k$, $1 \leq k \leq d$, belong to the set $\mathbb{N}_0$ of nonnegative integers and $\partial_x^a := \partial^{[\alpha]} / \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$ with $|\alpha| := \sum_{j=1}^{n} \alpha_j$ and $Z_{\tau}^b := Z_1^{b_1} \cdots Z_d^{b_d}$. Sometimes it is convenient to write $Y_j$ for $\partial / \partial x_j$, $j = 1, \ldots, n$, and $Y_{n+k}$ for $Z_k$, $k = 1, \ldots, d$; then we have $\partial_x^a Z_{\beta}^b = Y_1^{\alpha_1} \cdots Y_n^{\alpha_n} Y_{n+k}^{\beta_k} \cdots Y_d^{\beta_d}$ with $\gamma = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_d)$. License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
A $C^\infty$-function $h(x, \tau, t)$ on $\mathbb{R}^n \times K \times \mathbb{R}$ is called \textit{of type E} (resp. \textit{of type P}) if for any $r \in \mathbb{N}_0$ and any multi-index $\gamma$ of length $n + d$ there are $\eta, \theta, \sigma \in \mathbb{N}$ such that

\begin{equation}
Y_{x, \tau}^{\gamma} \left( \frac{\partial}{\partial t} \right)^r h(x, \tau, t) \leq \theta \left( 1 + |x|^2 \right)^{\sigma/2} e^{\eta |t|}
\end{equation}

(resp. \begin{equation}
Y_{x, \tau}^{\gamma} \left( \frac{\partial}{\partial t} \right)^r h(x, \tau, t) \leq \theta \left( 1 + |x|^2 \right)^{\sigma/2} (1 + t^2)^{\eta/2}
\end{equation}

for all $x, \tau, t$. (Of course, this definition does not depend on the special chosen vector fields $Z_1, \ldots, Z_d$.) It is obvious that sums, products and derivatives of type E functions (resp. type P functions) are of type E (resp. of type P).

Let $p$ and $q$ be the projection of $\mathbb{R}^n \times K$ onto $\mathbb{R}^n$ and $K$, respectively, and let $p_j$ be the $j$th component of $p$. Our one-parameter group $(\rho_t)$ of transformations is called \textit{of type E} (resp. \textit{of type P}) if the functions $p_j((x, \tau) \cdot t)$ and $\psi \circ q((x, \tau) \cdot t)$ are of type E (resp. type P) for all $j = 1, \ldots, n$ and for all $\psi \in C^\infty(K)$. In this case we are able to estimate $x$ by $p((x, t) \cdot t) =: x'$ for any $\tau$ and $t$. In fact, let $(x', \tau') \cdot (-t) = (x, \tau)$; because

\begin{equation}
|p((x', \tau') \cdot (-t))|^2 \leq \theta \left( 1 + |x'|^2 \right)^{\sigma/2} e^{\eta |t|}
\end{equation}

(resp. \begin{equation}
|p((x', \tau') \cdot (-t))|^2 \leq \theta \left( 1 + |x'|^2 \right)^{\sigma/2} (1 + t^2)^{\eta/2}
\end{equation}

for some $\eta, \theta, \sigma \in \mathbb{N}$, we have

\begin{equation}
1 + |x|^2 \leq (1 + \theta) \left( 1 + |p((x, \tau) \cdot t)|^2 \right)^{\sigma/2} e^{\eta |t|}
\end{equation}

(resp. \begin{equation}
1 + |x|^2 \leq (1 + \theta) \left( 1 + |p((x, \tau) \cdot t)|^2 \right)^{\sigma/2} (1 + t^2)^{\eta/2}
\end{equation}

and therefore

\begin{equation}
1 + |p((x, \tau) \cdot t)|^2 \geq \delta \left( 1 + |x|^2 \right)^{\epsilon/2} e^{-\xi |t|}
\end{equation}

(resp. \begin{equation}
1 + |p((x, \tau) \cdot t)|^2 \geq \delta \left( 1 + |x|^2 \right)^{\epsilon/2} (1 + t^2)^{-\xi/2}
\end{equation}

for some $\delta, \epsilon, \zeta > 0$.

Clearly, for each $k \in \{1, \ldots, n + d\}$ there are $C^\infty$-functions $a_{ik}$ on $\mathbb{R}^n \times K \times \mathbb{R}$, $1 \leq i \leq n + d$, such that for any $\varphi \in C^\infty(\mathbb{R}^n \times K)$ we have

\begin{equation}
Y_k(\varphi_t)(x, \tau) = \sum_{i=1}^{n+d} a_{ik}(x, \tau, t)(Y_i \varphi)_t(x, \tau)
\end{equation}

for all $x, \tau, t$. Similarly we have

\begin{equation}
\frac{d}{dt} \varphi_t(x, \tau) = \sum_{i=1}^{n+d} b_i(x, \tau, t)(Y_i \varphi)_t(x, \tau)
\end{equation}

where $b_i$, $1 \leq i \leq n + d$, are $C^\infty$-functions on $\mathbb{R}^n \times K \times \mathbb{R}$.

Now let $(\rho_t)$ be of type E (resp. of type P). Then all the functions $a_{ik}$ and $b_i$ are of type E (resp. of type P). This is evident by inserting $p_j$ and $\psi \circ q$ for $\varphi$ in (1.14) and (1.15), respectively. Reiterating formula (1.14) we derive that, given $t \in \mathbb{R}$, the
function \( \varphi \), belongs to \( \mathcal{S}(\mathbb{R}^n \times K) \) for any \( \varphi \in \mathcal{S}(\mathbb{R}^n \times K) \) and that the mapping \( \varphi \mapsto \varphi_t \) is a continuous mapping of \( \mathcal{S}(\mathbb{R}^n \times K) \) into itself. Hereby formula (1.13) is used. Together with (1.15) we derive that the infinitesimal generator \( L \) defines a continuous mapping of \( \mathcal{S}(\mathbb{R}^n \times K) \) into itself, and our previous considerations concerning \( \mathcal{D}(M) \) and \( \mathcal{D}'(M) \) remain valid with regard to \( \mathcal{S}(M) \) and \( \mathcal{S}'(M) \), \( M = \mathbb{R}^n \times K \).

2. Lemmata. Let \( M = \mathbb{R}^n \times K \) and let our one-parameter group \((\rho_t)\) be of type E. In the whole section we assume that there is \( \lambda \in \mathbb{R} \) such that

\[
(2.1) \quad p_1(m \cdot t) = e^{-\lambda t} p_1(m)
\]

for all \( t \in \mathbb{R} \), \( m = (x, \tau) \in M \). Then we have

\[
(2.2) \quad L_\mu(p_1 \varphi) = p_1 L_{\mu + \lambda} \varphi
\]

for any continuously differentiable function \( \varphi \) on \( M \).

The submanifold \( M^1 := \{m \in M \mid p_1(m) = 0\} = \mathbb{R}^{n-1} \times K \) is invariant under \((\rho_t)\). Let \((\rho_t')\) be the restriction of \((\rho_t)\) to \( M^1 \) and let \( L^1 \) be the associated infinitesimal generator. If \( \varphi \) is a function on \( M \), let \( \varphi^1 \) be its restriction to \( M^1 \). For any continuously differentiable function \( \varphi \) on \( M \) we have

\[
(2.3) \quad (L \varphi)^1 = L^1 \varphi^1.
\]

**Lemma 1.** Suppose that \((L^1_\mu)' : \mathcal{S}'(M^1) \rightarrow \mathcal{S}'(M^1)\) is surjective. If \( p_1 f \in L_\mu \mathcal{S}(M) \) for \( f \in \mathcal{S}(M) \), then \( f \in L_{\mu + \lambda} \mathcal{S}(M) \).

**Proof.** By (1.4), the assertion follows from the inclusion \( \mathcal{S}'(M^1) \subseteq p_1 \mathcal{S}'(M) \), which we are going to prove.

Let \( S \in \mathcal{S}'_{\mu+\lambda}(M) \). By division of distributions there is \( \psi \in \mathcal{S}(M) \) such that

\[
(2.4) \quad p_1 L_\mu \psi T_1 = L_{\mu + \lambda} S = 0;
\]

i.e. \( L_\mu T_1 \) is the trivial extension of a distribution \( W^1 \in \mathcal{S}'(M^1) \). By assumption, \( W^1 = (L^1_\mu)' R^1 \) with \( R^1 \in \mathcal{S}'(M^1) \). Let \( R \in \mathcal{S}'(M) \) be the trivial extension of \( R^1 \) and let \( T := T_1 - R \). Then we have \( p_1 T = S \), and \( T \in \mathcal{S}'(M) \) since

\[
\left\langle L_\mu T, \varphi \right\rangle = \left\langle W^1 - (L^1_\mu)' R^1, \varphi^1 \right\rangle = 0 \quad \text{for all } \varphi \in \mathcal{S}(M) \quad \square
\]

For convenience, we define the set \((L_\mu \mathcal{S}(M))_k, k \in \mathbb{N}_0,\) consisting of all functions \( f \in L_\mu \mathcal{S}(M) \) for which there are functions \( \psi_k \in \mathcal{S}(M) \) and \( f_k \in L_{\mu + k \lambda} \mathcal{S}(M) \) such that \( f = L_\mu \psi_k + p_1 f_k \). Clearly, \((L_\mu \mathcal{S}(M))_{k+1} \subseteq (L_\mu \mathcal{S}(M))_k \) by (2.2). Put

\[
(L_\mu \mathcal{S}(M))_\infty := \bigcap_{k \in \mathbb{N}} (L_\mu \mathcal{S}(M))_k.
\]

**Lemma 2.** Suppose that \((L^1_\mu)' : \mathcal{S}'(M^1) \rightarrow \mathcal{S}'(M^1)\) is surjective for all \( \kappa = 0, \ldots, k - 1 \). Then \( L_\mu \mathcal{S}(M) = (L_\mu \mathcal{S}(M))_{k-1} \).
Proof. We prove the lemma by induction on \( k \). For \( k = 0 \) the assertion is trivial. Now assume \( f = L_{\mu} \psi_k + p_k^t f_k \) with \( \psi_k \in \mathcal{S}(M) \) and \( f_k \in L_{\mu + k \lambda} \mathcal{S}(M) \). Obviously, \( f_k^t \in L_{\mu + k \lambda}^1 \mathcal{S}(M) \). Since \( L_{\mu + k \lambda} \mathcal{S}(M^1) \) is closed by assumption, \( f_k^t = \psi^1_k \in \mathcal{S}(M^1) \) for some \( \psi^1 \in \mathcal{S}(M^1) \). Select \( \psi \in \mathcal{S}(M) \) such that \( \psi^1 \) is the restriction of \( \psi \) to \( M \). Then \( f_k - L_{\mu + k \lambda} \psi \) vanishes on \( M^1 \). Therefore it can be divided by \( p_1 \); i.e. there is a function \( f_{k + 1} \in \mathcal{S}(M) \) such that \( f_k - L_{\mu + k \lambda} \psi = p_1 f_{k + 1} \). By Lemma 1, \( f_{k + 1} \in L_{\mu + (k + 1) \lambda} \mathcal{S}(M) \). Put \( \psi_{k + 1} := \psi_k + p_k^t \psi \). Using (2.2) we get the desired equation for \( k + 1 \).

Lemma 3. Let \( \lambda \neq 0 \). Suppose that \( L_{\mu} \mathcal{S}(M) = (L_{\mu} \mathcal{S}(M))^\infty \). Then \( L_{\mu} \mathcal{S}(M) \) is closed in \( \mathcal{S}(M) \).

Proof. Replacing \((\rho_+), \mu\) by \((\rho_-), -\mu\) in case of need, we may assume that \( \lambda > 0 \).

Let \( f \in L_{\mu} \mathcal{S}(M) \). For any \( k \in \mathbb{N} \), we take \( \psi_k \in \mathcal{S}(M) \) and \( f_k \in L_{\mu + k \lambda} \mathcal{S}(M) \) such that \( f = L_{\mu} \psi_k + p_k^t f_k \). In the course of the proof we determine \( k_0 \in \mathbb{N} \) sufficiently large for our need. First of all we assume that the real part \( \nu_k := \Re \nu_k \) of \( \nu_k := \psi_k + p_k^t \psi \) is positive for \( k \geq k_0 \). For \( k \geq k_0 \) we put

\[
\varphi_k(m) := \psi_k(m) - p_k^t(m) \int_{-\infty}^{\infty} e^{-\nu t} f_k(m \cdot t) \, dt, \quad m \in M.
\]

It is easily seen that the distribution \( \lambda_{\nu_k, m} \) (see (1.8)) is well defined for all \( k \geq k_0 \) and \( m \in M \setminus M^1 \). In fact, for \( t < 0 \) we apply (2.1) and get the estimate

\[
|\varphi(m \cdot t)| \leq \frac{c(\varphi, r)}{|p_1(m)|} e^{r\lambda t}, \quad \varphi \in \mathcal{S}(M),
\]

where \( r \) is an arbitrary integer \( \geq 0 \) and \( c(\varphi, r) \) is constant with respect to \( m \) and \( t \). Therefore, by (1.9), for \( m \in M \setminus M^1 \), equation (2.5) can be written in the following form:

\[
\varphi_k(m) = \psi_k(m) + p_k^t(m) \int_{-\infty}^{0} e^{-\nu t} f_k(m \cdot t) \, dt.
\]

Now, by induction on \( r \in \mathbb{N} \) we see: For any \( r \in \mathbb{N} \) there is \( k_0 \in \mathbb{N} \) such that \( \varphi_k \in C'(M) \) for \( k \geq k_0 \), in fact, for any multi-index \( \gamma \) with \( |\gamma| \leq r \) the derivative \( Y^\gamma(\varphi_k - \psi_k)(m) \) is a finite sum of terms of the form

\[
(2.7) \quad cp_k^t(m) \int_{-\infty}^{0} e^{-\nu t} h(m, t)(Y^\gamma f_k)(m \cdot t) \, dt,
\]

where \( c \in C \), \( s' \in \mathbb{N} \) depend on \( k \), \( s' \geq k - |\gamma| \), and \( h(m, t) \) is a \( C^\infty \)-function of type \( E \) independent of \( k \); \( |\gamma| \leq |\gamma| \), \( a = 0 \), \( b = \infty \). Hereby, (1.14) is used.

For \( m \in M \setminus M^1 \) we can apply (2.6) and, proceeding from (2.7), we can express \( Y^\gamma(\varphi_k - \psi_k)(m) \) by a finite sum of terms of the form (2.8) with \( a = -\infty \), \( b = 0 \).

Now let us prove that for any \( s \in \mathbb{N} \) and for any multi-index \( \gamma \) there is \( k_0 \in \mathbb{N} \) such that for \( k \geq k_0 \) the term

\[
(2.9) \quad (1 + |p(m)|^2)^{s/2} |Y^\gamma \varphi_k(m)|
\]
is bounded with respect to \( m \in M \). In view of (2.8) and by continuity it is sufficient to prove the boundedness of the terms

\[
(2.10) \quad \left(1 + |p(m)|^2\right)^{s/2} |p^*_t(m)| \int_a^b e^{-v_t} |h(m, t)| \left| Y^* f_k(m \cdot t) \right| \, dt
\]
on the domain \( \{0 \leq |p_1(m)| \leq 1\} \) for \( a = 0, b = \infty \) and on the domain \( \{|p_1(m)| \geq 1\} \) for \( a = -\infty, b = 0 \). For \( a = 0, b = \infty \) we use (1.13); since \( f_k \in \mathcal{S}(M) \) we can estimate (2.10) by

\[
(2.11) \quad \left(1 + |p(m)|^2\right)^{s/2} \int_0^\infty e^{-v_t} |h(m, t)| \frac{C(f_k, N)}{\delta N \left(1 + |p(m)|^2\right)^{sN/2} e^{-\alpha N t}} \, dt,
\]
where \( N \) is a positive integer which satisfies \( \epsilon N \geq s + \sigma \) with \( \sigma \) from (1.11). The boundedness of (2.11) is obvious if \( k_0 \) is sufficiently large. For \( a = -\infty, b = 0 \) we use (2.1). Let \( k \geq k_0 \) be given, we choose \( N \) as above and take a positive integer \( r > s' \); then we get an estimate of (2.10) by the term

\[
(2.12) \quad \left(1 + |p(m)|^2\right)^{s/2} |p^*_t(m)| \int_{-\infty}^0 e^{-v_t} |h(m, t)| \frac{C(f_k, N, r)}{\delta N \left(1 + |p(m)|^2\right)^{sN/2} e^{-\alpha N t} e^{-\lambda r} |p_1(m)|} \, dt
\]
which is obviously bounded if \( r \) is sufficiently large.

Now let \( k_0 \) be sufficiently large and let \( k \geq k_0 \). From (2.5) and (2.2) we get

\[
(2.13) \quad L_\mu \varphi_k = L_\mu \varphi_k = p^*_t L_\nu \varphi_k,
\]
where

\[
g_k(m) := \int_0^\infty e^{-v_t} f_k(m \cdot t) \, dt.
\]
Applying (1.6) with \( \varphi = g_k, \mu = \nu_k \) for \( t = 0 \) we get

\[
(2.14) \quad L_\nu \varphi_k = -f_k
\]
and therefore

\[
(2.15) \quad L_\mu \varphi_k = f
\]
for any \( k \geq k_0 \).

From (1.6) we derive that \( L_\mu \varphi = 0 \) implies \( \varphi = 0 \) for \( \varphi \in C^1(M) \) vanishing at infinity; in fact, for \( m \in M \setminus M^1 \) we have \( \varphi(m \cdot t) = e^{\mu t} \varphi(m) \) and therefore \( \varphi(m) = 0 \) because \( m \cdot t \to \infty \) for \( t \to -\infty \) by (2.1).

Therefore, looking at (2.15), we see that \( \varphi_k \) does not depend on \( k \); i.e. \( \varphi_k =: \varphi \) for all \( k \geq k_0 \). Thus, by (2.9), \( \varphi \in \mathcal{S}(M) \).

**Lemma 4.** Let \( \Re \mu \neq 0 \). Suppose that \( (\rho_1) \) is of type \( P \). Then \( (L_\mu)'^* : \mathcal{S}'(M^1) \to \mathcal{S}'(M^1) \) is surjective.

**Proof.** Replacing \( (\rho_t) \) and \( \mu \) by \( (\rho_{-t}) \) and \( -\mu \) in case of need, we may assume that \( \Re \mu < 0 \).
By (1.6), $L^1_\mu : \mathcal{S}(M^1) \to \mathcal{S}(M^1)$ is injective. To prove that $L^1_\mu$ is also surjective we put
\begin{equation}
(2.16)\quad \varphi^1(m^1) := \int_{-\infty}^{0} e^{-\mu t} f^1(m^1 \cdot t) \, dt, \quad m^1 = (x^1, \tau) \in M^1,
\end{equation}
for a given $f^1 \in \mathcal{S}(M^1)$ and show that $\varphi^1 \in \mathcal{S}(M^1)$.

In fact, by equation (1.14), for any $s \in \mathbb{N}$ and for any multi-index $\gamma$ the term $(1 + |x^1|^2)^{s/2} Y^\gamma \varphi^1(m^1)$ is a finite sum of terms of the form
\begin{equation}
\int_{-\infty}^{0} e^{-\mu t} h^1(m^1, t) g^1(m^1 \cdot t) \, dt
\end{equation}
where $g^1 \in \mathcal{S}(M^1)$ and $h^1(m^1, t)$ is a $C^\infty$-function of type $P$. Using (1.13) we see that $|h^1(m^1, t) g^1(m^1 \cdot t)|$ can be estimated by $c(1 + t^2)^{s/2}$ with some $r \in \mathbb{N}$ and some constant $c > 0$.

**Lemma 5.** Let $\lambda \neq 0$. Suppose that $(\rho^1)$ is of type $P$ and that $\rho^1(x^1, m^1) = (e^{-\mu t} x^1, \rho(t^1(m^1)))$ for $(x^1, m^1) \in M \cong \mathbb{R} \times M^1$. Then $L^1_\mu \mathcal{S}(M)$ is closed in $\mathcal{S}(M)$.

**Proof.** If $\text{Re}\, \mu + k\lambda \neq 0$ for all $k \in \mathbb{N}_0$, the assertion follows by Lemmas 4, 2 and 3.

Assume that $\text{Re}\, \mu + k\lambda = 0$ for some $k \in \mathbb{N}_0$. Given $f \in L^1_\mu \mathcal{S}(M)$, by Lemmas 4 and 2 there are $\psi^k \in \mathcal{S}(M)$ and $f_k \in L^1_{\mu + k\lambda} \mathcal{S}(M)$ such that $f = L^1_\mu \psi^k + \rho^1_\mu f_k$. Therefore, by (2.2), we have only to prove that $L^1_{\mu + k\lambda} \mathcal{S}(M)$ is closed; i.e. it remains to prove that $L^1_\mu \mathcal{S}(M)$ is closed for $\mu \in \mathbb{C}$ with $\text{Re}\, \mu = 0$.

Let $\text{Re}\, \mu = 0$. Using the assumption we derive
\begin{equation}
(2.17)\quad \frac{\partial}{\partial x^1} L^1_\mu \varphi = L^1_{\mu + \lambda} \frac{\partial \varphi}{\partial x^1}
\end{equation}
for all $\varphi \in \mathcal{S}(M)$. From the previous considerations we know that $L^1_{\mu + \lambda} \mathcal{S}(M)$ is closed. It follows that $L^1_{\mu + \lambda} \mathcal{S}(M) \to L^1_{\mu + \lambda} \mathcal{S}(M)$ is an isomorphism, because $L^1_{\mu + \lambda}$ is injective by (1.6). Therefore, since
\begin{equation}
\mathcal{F}_1 := \left\{ \frac{\partial \varphi}{\partial x^1} \bigg| \varphi \in \mathcal{S}(M) \right\}
\end{equation}
is closed, $L^1_{\mu + \lambda} \mathcal{F}_1$ is closed. Consequently, by (2.17), $(\partial/\partial x^1) L^1_\mu \mathcal{S}(M)$ is closed. Since $\partial/\partial x^1 : \mathcal{S}(M) \to \mathcal{F}_1$ is an isomorphism, it follows that $L^1_\mu \mathcal{S}(M)$ is closed.

**3. Main results.** Let us briefly sum up our assumptions and notations: We deal with a manifold $M = \mathbb{R}^n \times K$, where $K$ is a $d$-dimensional compact differentiable manifold with the property that there are $d$ vector fields $Z_1, \ldots, Z_d$ on $K$ such that for each $\tau \in K$ the tangent space to $K$ at $\tau$ is spanned by the tangent vectors $Z_1(\tau), \ldots, Z_d(\tau)$. For $(x, \tau) \in \mathbb{R}^n \times K$ we put $p_j(x, \tau) := x_j$ and $q(x, \tau) := \tau$. Let $(\rho_t)_{t \in \mathbb{R}}$ be a one-parameter group of transformations acting on $M$ and let $L$ be the associated infinitesimal transformation (see (1.1)). For $\mu \in \mathbb{C}$ we define the differential operator $L^1_\mu := L - \mu 1$.
THEOREM. Let \((\rho_j)\) be of type E. Given \(k \in \mathbb{N}, 1 \leq k \leq n,\) let \(M^j := \{(x, \tau) \in M|x_1 = \cdots = x_j = 0\}\) be invariant under \((\rho_j)\) for \(j = 1, \ldots, k.\) We assume that the restriction of \((\rho_j)\) to \(M^k\) is of type P and that the projection of \(\rho_j(x, \tau)\) onto \(M^k\) does not depend on \(x_1, \ldots, x_k.\) Suppose that there are real numbers \(\lambda_j, 1 \leq j \leq n, \lambda_j \neq 0\) for \(j = 1, \ldots, k, \lambda_j = 0\) for \(j = k + 1, \ldots, n,\) such that \(p_j((x, \tau) \cdot t)\) has the form
\[
(3.1) \quad p_j((x, \tau) \cdot t) = e^{-\lambda_j x_j} + w_j(x_1, \ldots, x_{j-1}, \tau, t), \quad j = 1, \ldots, n,
\]
where \(w_j\) are functions independent of \(x_j, \ldots, x_n.\)

Then \(L_\mu: \mathcal{S}(M) \to \mathcal{S}(M)\) is injective and its range is closed.

PROOF. First of all, it is easy to see that (2.1) with \(\lambda = \lambda_1\) will follow from (3.1). In fact, we have
\[
w_1(\tau, t) = p_1((0, \tau) \cdot t) - e^{-\lambda_1 t} 0
\]
and \(p_1((0, \tau) \cdot t) = 0\) since \(M^1\) is invariant under \((\rho)\) by assumption. From (2.1) we conclude that the orbit \(\{(x, \tau) \cdot t| t \in \mathbb{R}\}\) is unbounded whenever \(x_1 \neq 0.\) Together with (1.6) we see that \(L_\mu\) is injective for any \(\mu \in \mathbb{C}.\)

Now let us prove by induction on \(k\) that \(L_\mu \mathcal{S}(M)\) is closed in \(\mathcal{S}(M)\) for each \(\mu \in \mathbb{C}.\) For \(k = 1\) the assertion follows by Lemma 5. Let \(k > 1.\) By induction hypothesis, \(L_\mu \mathcal{S}(M)\) is closed in \(\mathcal{S}(M)\) and \(L_\mu: \mathcal{S}(M) \to \mathcal{S}(M)\) is injective by the consideration above. Since \(\mathcal{S}(M)\) is a Fréchet space it follows that the transpose \((L_\mu)^*: \mathcal{S}'(M) \to \mathcal{S}'(M)\) is surjective for all \(\mu \in \mathbb{C}.\) Thus, by Lemmas 2 and 3, \(L_\mu \mathcal{S}(M)\) is closed.

EXAMPLE. On \(M = \mathbb{R}^n \times T^d (T^d = d\)-dimensional torus), \(n, d \in \mathbb{N}_0,\) we consider the one-parameter group
\[
\rho_j(x, \tau) = \left(x_1 e^{\lambda_1 \tau}, \ldots, x_j e^{\lambda_j \tau}, \tau_1 e^{\alpha_1 \tau}, \ldots, \tau_d e^{\alpha_d \tau}\right),
\]
where \(\lambda_1, \ldots, \lambda_n, \alpha_1, \ldots, \alpha_d \in \mathbb{R}.\) The infinitesimal generator \(L\) associated with \((\rho_j)\) is given by
\[
L \psi(x, \tau) = \sum_{j=1}^{n} \lambda_j x_j \frac{\partial \psi}{\partial x_j}(x, \tau) + \sum_{k=1}^{d} \alpha_k \frac{\partial \psi}{\partial \tau_k}(x, \tau).
\]
By the Theorem, \(L_\mu \mathcal{S}(M)\) is closed in \(\mathcal{S}(M)\) for any \(\mu \in \mathbb{C}\) provided that \(n > 0\) and \(\lambda_j \neq 0\) at least for one \(j.\) (Compare [4, Example 2].) In general, \(L_\mu \mathcal{S}(M)\) is not closed for \(n = 0 [4, Example 1].\) Particularly, the range of the restriction of \(L\) to \(T^d\) may be not closed in spite of the fact that \(L\) itself has closed range.

Furthermore, putting \(d = 0\) and assuming \(\lambda_j \neq 0\) for one \(j\) we can conclude that \(L: \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)\) is surjective. This should be compared with Miwa’s result [5] affirming that \(L: \mathcal{B}(\mathbb{R}^n) \to \mathcal{B}(\mathbb{R}^n)\) is surjective if additionally it is supposed that \(|\lambda_j| \leq 1\) for all \(j = 1, \ldots, n,\) where \(\mathcal{B}(\mathbb{R}^n)\) is the set of hyperfunctions on \(\mathbb{R}^n.\)

COROLLARY. Given a first-order differential operator \(\neq 0\) on \(\mathbb{R}^n\) with linear coefficients
\[
D = \sum_{i,j=1}^{n} a_{ij} x_j \frac{\partial}{\partial x_i} + b, \quad a_{ij}, b \in \mathbb{R}.
\]
Suppose that all eigenvalues of the matrix \((a_{ij})\) are real.

Then \(D: \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)\) is surjective.
Proof. After change of basis we may assume that the matrix \( A = (a_{ij}) \) has Jordan form
\[
\begin{pmatrix}
J_1 & & \\
& J_2 & \\
& & \ddots & \\
& & & J_r
\end{pmatrix}
\]
with Jordan boxes
\[
J_\rho = \begin{pmatrix}
\lambda_\rho & & \\
& 1 & & \\
& & \ddots & \\
& & & 1 \\
\end{pmatrix}, \quad \lambda_\rho \in \mathbb{R}, 1 \leq \rho \leq r,
\]
which are arranged in such a manner that \( \lambda_\rho \neq 0 \) for \( \rho = 1, \ldots, k \) and \( \lambda_\rho = 0 \) for \( \rho = k + 1, \ldots, r \), where \( 0 \leq k \leq r \). It is easily seen that \( D = L_\mu' \), where \( L \) is the infinitesimal generator associated with the one-parameter group \( \rho_\mu(x) = e^{-tA}x \) and \( \mu = \text{trace}(A) - b \). Therefore it is sufficient to show that \( L_\mu : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \) is injective and has closed range.

Let \( k = 0 \). If \( \mu \neq 0 \), the assertion follows by Lemma 4. If \( \mu = 0 \), the assertion follows by [3].

Now let \( k > 0 \). Then we can apply the Theorem, where \( K \) is assumed to be trivial.

References
5. T. Miwa, On the existence of hyperfunctions solutions of linear differential equations of the first order

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