

## A SOLUTION OF WARNER'S 3RD PROBLEM FOR REPRESENTATIONS OF HOLOMORPHIC TYPE<sup>1</sup>

BY  
 FLOYD L. WILLIAMS

**ABSTRACT.** In response to one of ten problems posed by G. Warner, we assign (to the extent that it is possible) a geometric or cohomological interpretation—in the sense of Langlands—to the multiplicity in  $L^2(\Gamma \backslash G)$  of an irreducible unitary representation  $\pi$  of a semisimple Lie group  $G$ , where  $\Gamma$  is a discrete subgroup of  $G$ , in the case when  $\pi$  has a highest weight.

**1. Introduction.** Given a discrete subgroup  $\Gamma$  of a semisimple Lie group  $G$  and a special class of nontempered unitary representations  $\pi$  of  $G$ , we express the multiplicity of  $\pi$  in  $L^2(\Gamma \backslash G)$  as the dimension of the sheaf cohomology of a line bundle over  $\Gamma \backslash G$  modulo a compact Cartan subgroup of  $G$ , and we find the best vanishing theorem for this cohomology. These prototype results extend those known in the case when  $\pi$  is a discrete series representation.

In more detail let  $G$  be connected noncompact and linear, and assume  $\Gamma$  is cocompact in  $G$  and torsion free. We assume  $G$  has the rank of a maximal compact subgroup  $K$  and we fix a Cartan subgroup  $H$  of  $G$  contained in  $K$ . Let  $\mathfrak{g}, \mathfrak{k}, \mathfrak{h}$  denote the complexified Lie algebras of  $G, K, H$  respectively, let  $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$  be the set of nonzero roots of  $\mathfrak{g}$  relative to  $\mathfrak{h}$ , let  $\mathfrak{g}_\beta$  be the root space of  $\beta \in \Delta$ , and let  $\Delta_k, \Delta_n$  be the set of compact, noncompact roots respectively in  $\Delta$ . Given a choice of a positive system of roots  $\Delta^+$  in  $\Delta$  one can assign a unique  $G$ -invariant holomorphic structure on the quotient  $D = G/H$  such that the space of antiholomorphic tangent vectors at the origin is given by

$$(1.1) \quad \mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}.$$

A homogeneous line bundle  $\mathcal{L}_\lambda \rightarrow D$  over  $D$  induced by a character  $\lambda$  of  $H$  also has a  $G$ -invariant holomorphic structure and thus we can consider for  $Y = \Gamma \backslash D$  the space  $H^q(Y, \mathcal{L}_\lambda)$  of  $\Gamma$ -invariant  $\mathcal{L}_\lambda$ -valued harmonic  $C^\infty$  forms of type  $(0, q)$  on  $D$ , relative to some  $G$ -invariant Hermitian metrics on  $D, \mathcal{L}_\lambda$ . Alternately  $H^q(Y, \mathcal{L}_\lambda)$  is the  $q$ th-dimensional cohomology of  $Y$  with coefficients in the sheaf  $\mathcal{O}_\lambda$  of local holomorphic sections of  $\Gamma \backslash \mathcal{L}_\lambda$ . Define

$$(1.2) \quad q(\lambda) = |\{\alpha \in \Delta_k^+ : (\lambda + \delta, \alpha) < 0\}| + |\{\alpha \in \Delta_n^+ : (\lambda + \delta, \alpha) > 0\}|$$

where  $\Delta_k^+ = \Delta^+ \cap \Delta_k$ ,  $\Delta_n^+ = \Delta^+ \cap \Delta_n$ ,  $2\delta = \langle \Delta^+ \rangle$  (the sum of roots in  $\Delta^+$ ),  $(\cdot)$  is the Killing form of  $\mathfrak{g}$ , and where  $|S|$  denotes the cardinality of a set  $S$ . Assume that  $\lambda + \delta$  is regular and let  $\pi_{\lambda+\delta} \in \hat{G}$  (the unitary dual of  $G$ ) be the corresponding

---

Received by the editors February 5, 1985. A portion of these results were included in a presentation to the A.M.S. 822nd meeting, October 26-27, 1985.

1980 *Mathematics Subject Classification*. Primary 22E45; Secondary 32L10, 32M15.

<sup>1</sup> Research supported by NSF Grant No. PRM 8205819.

Harish-Chandra discrete series representation of  $G$ . If  $m_{\lambda+\delta}(\Gamma)$  denotes the multiplicity of  $\pi_{\lambda+\delta}$  in  $L^2(\Gamma \backslash G)$  then for a very mild condition on  $\lambda$  (see (2.10) below) we know that

$$(1.3) \quad \begin{aligned} (i) & H^q(Y, \mathcal{L}_\lambda) = 0 \quad \text{for } q \neq q(\lambda) \text{ in (1.2),} \\ (ii) & \dim H^{q(\lambda)}(Y, \mathcal{L}_\lambda) = m_{\lambda+\delta}(\Gamma). \end{aligned}$$

(1.3) is the solution of a conjecture of Langlands [2, 3]—obtained in part in [5] and fully in [9]. It is natural to consider therefore the extent to which analogues of (1.3) are possible for a nondiscrete series  $\pi \in \hat{G}$ . This is the content of problem 3 of Warner [6, p. 139]. Given such a  $\pi$  one of the first steps is to find a candidate  $q(\lambda, \pi)$  for the role of  $q(\lambda)$ .

In this paper we assume that  $G/K$  has a  $G$ -invariant holomorphic structure compatible with the choice  $\Delta^+$  above and we assume that  $\pi \in \hat{G}$  is a highest weight representation with a nonsingular infinitesimal character. Then we find the right candidate  $q(\lambda, \pi)$  and the analogue of (1.3)(ii). Regarding (1.3)(i) our results indicate however that one must expect, in general, the existence of nonzero cohomology in dimensions  $\neq q(\lambda, \pi)$  (which prevents the immediate computation of  $m_\pi(\Gamma)$  by an index formula). We determine all such dimensions. The main result is Theorem 3.9, which appears in sharper form in Theorem 3.13. Theorem 3.9 relies basically on results developed in [4, 7, 8].

**2. More notation.** As before we shall write  $\langle Q \rangle$  for the sum of roots in a subset  $Q$  of  $\Delta$ . We denote by  $W, W_k$  the Weyl groups of  $(\mathfrak{g}, \mathfrak{h}), (\mathfrak{k}, \mathfrak{h})$  respectively, and by  $\mathcal{F}$  the set of integral linear forms on the dual space  $\mathfrak{h}^*$  of  $\mathfrak{h}$ . Let  $\Delta^+ \subset \Delta$  be as before, a positive system which we fix from now on. For  $\Lambda \in \mathfrak{h}^*$  such that  $\Lambda + \delta$  is regular, we set

$$(2.1) \quad \begin{aligned} P^{(\Lambda)} &= \{\alpha \in \Delta : (\Lambda + \delta, \alpha) > 0\}, & 2\delta^{(\Lambda)} &= \langle P^{(\Lambda)} \rangle, \\ Q_\Lambda &= \{\alpha \in \Delta_n^+ : (\Lambda + \delta, \alpha) > 0\}, & Q'_\Lambda &= \Delta_n^+ - Q_\Lambda, \\ \mathfrak{b}_\Lambda &= \mathfrak{h} + \sum_{\alpha \in P^{(\Lambda)}} \mathfrak{g}_\alpha \quad (\text{a Borel subalgebra}). \end{aligned}$$

Define

$$(2.2) \quad \begin{aligned} \mathcal{F}'_0 &= \mathcal{F}'_0(\Delta^+) = \{\Lambda \in \mathcal{F} : \Lambda + \delta \text{ is regular and } \Delta_k^+ \text{-dominant}\}; \\ 2\delta_k &= \langle \Delta_k^+ \rangle, \quad \delta_n = \delta - \delta_k. \end{aligned}$$

If  $\theta = m + u$  is a Levi decomposition of a parabolic subalgebra  $\theta$  of  $\mathfrak{g}$  with unipotent radical  $u$  and a reductive complement  $m$ , we let  $\Delta(m), \Delta(u)$  denote the roots of  $m, u$  respectively and we set

$$(2.3) \quad \theta_{u,n} = \text{the set of noncompact roots in } \Delta(u).$$

Thus if  $P \subset \Delta$  is any positive system and  $\theta$  contains the Borel subalgebra  $\mathfrak{h} + \sum_{\alpha \in P} \mathfrak{g}_\alpha$  then

$$(2.4) \quad m = \mathfrak{h} + \sum_{\alpha \in \Delta(m)} \mathfrak{g}_\alpha, \quad u = \sum_{\alpha \in P - \Delta(m)} \mathfrak{g}_\alpha, \quad \theta_{u,n} = P_n - \Delta(m)$$

where  $P_n = P \cap \Delta_n$ . In particular if  $\theta \supset \mathfrak{b}_\Lambda$  for  $\Lambda \in \mathcal{F}'_0$  we define

$$(2.5) \quad q(\Lambda; \theta) = 2|\theta_{u,n} \cap Q_\Lambda| - |\theta_{u,n}| + |Q'_\Lambda|.$$

Then  $q(\Lambda; \theta)$  is a nonnegative integer. A collection of parabolic subalgebras  $\{\theta_i\}_{i=1}^t$  each containing  $\mathfrak{b}_\Lambda$ ,  $\Lambda \in \mathcal{F}'_0$ , is called *representative* if  $\theta_{u_i, n} \neq \theta_{u_j, n}$  for  $i \neq j$ ,  $1 \leq i, j \leq t$ , and if for any parabolic subalgebra  $\theta \supset \mathfrak{b}_\Lambda$  we have  $\theta_{u, n} = \theta_{u_i, n}$  for some  $i$ . This definition is equivalent to that given in [8]; we thank Professor Kostant for pointing this out to us. Given such a collection we have the following important sets: For an integer  $q \geq 0$

$$(2.6) \quad S(\Lambda, q) \stackrel{\text{def}}{=} \{i: q(\Lambda; \theta_i) = q \text{ and } \theta_{u_i, n} = \theta_{u, n} \text{ for some parabolic subalgebra } \theta = m + u \supset \mathfrak{b}_\Lambda \text{ with } (\Lambda + \delta - \delta^{(\Lambda)}, \Delta(m)) = 0\}.$$

PROPOSITION 2.7. Let  $\{\theta_i\}_{i=1}^t$  be a representative set of parabolics  $\supset \mathfrak{b}_\Lambda$  for  $\Lambda \in \mathcal{F}'_0$ , and let  $\theta = m + u \supset \mathfrak{b}_\Lambda$  be any parabolic such that  $(\Lambda + \delta - \delta^{(\Lambda)}, \Delta(m)) = 0$ . Let  $i_0$  be the unique index,  $1 \leq i_0 \leq t$ , such that  $\theta_{u, n} = \theta_{u_{i_0}, n}$ . Then  $i_0 \in S(\Lambda, q(\Lambda; \theta))$ . If moreover  $\Lambda$  satisfies the condition

$$(2.8) \quad (\Lambda + \delta - \delta^{(\Lambda)}, \alpha) \neq 0 \quad \text{for } \alpha \in Q_\Lambda \cup \theta_{u, n}$$

then in fact  $Q_\Lambda \subset \theta_{u, n}$ ,  $q(\Lambda; \theta) = |Q_\Lambda| + |\Delta_n^+| - |\theta_{u, n}|$  and  $S(\Lambda, q(\Lambda; \theta)) = \{i_0\}$ .

Note: Since  $\Lambda + \delta - \delta^{(\Lambda)}$  is  $P^{(\Lambda)}$ -dominant and  $Q_\Lambda \cup \theta_{u, n} \subset P_n^{(\Lambda)} \subset P^{(\Lambda)}$  (2.8) is equivalent to

$$(2.9) \quad (\Lambda + \delta - \delta^{(\Lambda)}, \alpha) > 0 \quad \text{for } \alpha \in Q_\Lambda \cup \theta_{u, n}.$$

PROOF OF PROPOSITION 2.7. By the definitions involved  $i_0 \in S(\Lambda, q(\Lambda; \theta))$ . Conversely let  $i \in S(\Lambda, q(\Lambda; \theta))$  so  $q(\Lambda; \theta_i) \stackrel{(i)}{=} q(\Lambda; \theta)$  and  $\theta_{u_i, n} \stackrel{(ii)}{=} \theta'_{u', n}$  for some  $\theta' = u' + m' \supset \mathfrak{b}_\Lambda$  for which  $(\Lambda + \delta - \delta^{(\Lambda)}, \Delta(m')) \stackrel{(iii)}{=} 0$ . Let  $N_\Lambda = \{\alpha \in P_n^{(\Lambda)}: (\Lambda + \delta - \delta^{(\Lambda)}, \alpha) = 0\}$ . Since  $\Lambda + \delta - \delta^{(\Lambda)}$  is  $P^{(\Lambda)}$ -dominant (as just noted) and since  $(\Lambda + \delta - \delta^{(\Lambda)}, \Delta(m)) = 0$ ,

$$\begin{aligned} P_n^{(\Lambda)} &= N_\Lambda \cup (P_n^{(\Lambda)} - \Delta(m)) \stackrel{(iv)}{=} N_\Lambda \cup \theta_{u, n} \quad (\text{cf. (2.4)}) \\ &\Rightarrow Q_\Lambda = Q_\Lambda \cap N_\Lambda \cup Q_\Lambda \cap \theta_{u, n}. \end{aligned}$$

By (2.8)

$$N_\Lambda \cap \theta_{u, n} = Q_\Lambda \cap N_\Lambda = \emptyset \Rightarrow n \stackrel{\text{def}}{=} |\Delta_n^+| = |P_n^{(\Lambda)}| \stackrel{(v)}{=} |N_\Lambda| + |\theta_{u, n}|$$

and  $Q_\Lambda \subset \theta_{u, n}$ . Then  $q(\Lambda; \theta) = |Q_\Lambda| + n - |\theta_{u, n}|$  by (2.5). Similarly from (iii)

$$\begin{aligned} P_n^{(\Lambda)} \stackrel{(vi)}{=} (N_\Lambda - \theta'_{u', n}) \cup \theta'_{u', n} &\Rightarrow Q_\Lambda = Q_\Lambda \cap \theta'_{u', n} \quad (\text{again since } Q_\Lambda \cap N_\Lambda = \emptyset) \\ &\Rightarrow q(\Lambda; \theta') = |Q_\Lambda| + n - |\theta'_{u', n}|, \end{aligned}$$

so by (i), (ii)

$$q(\Lambda; \theta) = q(\Lambda; \theta') \Rightarrow |\theta_{u, n}| \stackrel{(vii)}{=} |\theta'_{u', n}|.$$

Then from (v), (vi),

$$\begin{aligned} |N_\Lambda| + |\theta_{u, n}| &= |N_\Lambda - \theta'_{u', n}| + |\theta'_{u', n}| \Rightarrow |N_\Lambda| = |N_\Lambda - \theta'_{u', n}| \\ &\Rightarrow N_\Lambda = N_\Lambda - \theta'_{u', n} \Rightarrow N_\Lambda \cap \theta'_{u', n} = \emptyset, \end{aligned}$$

so by (iv)  $\theta'_{u', n} \subset \theta_{u, n}$ ; i.e. by (ii), (vii)  $\theta_{u_i, n} = \theta_{u, n}$ . But  $\theta_{u, n} = \theta_{u_{i_0}, n}$ ; hence  $i = i_0$ , as desired. Q.E.D.

For the sake of completeness we give the “mild condition” on  $\lambda$  referred to prior to (1.3). It is (cf. (2.9))

$$(2.10) \quad (\lambda + \delta - \delta^{(\lambda)}, \alpha) > 0 \quad \text{for each } \alpha \in P_n^{(\lambda)}.$$

**3. A vanishing theorem and statement of the main result.** We assume now that  $G/K$  has a  $G$ -invariant holomorphic structure so fixed that the space of antiholomorphic tangent vectors at the origin is given by

$$(3.1) \quad \mathfrak{p}^- = \sum_{\alpha \in \Delta_n^+} \mathfrak{g}_{-\alpha}$$

(cf. (1.1)). This is equivalent to requiring that every  $\alpha \in \Delta_n^+$  should be *totally positive*; i.e.  $\alpha + \beta \in \Delta$  for  $\beta \in \Delta_k \Rightarrow \alpha + \beta \in \Delta_n^+$ . Then also

$$(3.2) \quad G/H \rightarrow G/K \text{ and } \Gamma \backslash G/H \rightarrow \Gamma \backslash G/K \text{ are holomorphic fibrations.}$$

As we shall wish to make immediate applications of the results of [7, 8] we assume also that the complexification of  $G$  is simply connected. In general if  $P = P_k \cup P_n \subset \Delta$  is a system of positive roots compatible with a  $G$ -invariant holomorphic structure on  $G/K$  (i.e. every  $\alpha \in P_n$  is totally positive), so is the conjugate system  $P_k \cup -P_n$  which we shall denote by  $\bar{P}$ . If  $\lambda \in \mathcal{F}$  we also denote by  $\lambda$  the character of  $H$  whose differential is  $\lambda$  and thus we consider the cohomology  $H^q(Y, L_\lambda)$  defined in §1. As  $\Lambda \in \mathcal{F}'_0$  is the  $\Delta_k^+$ -highest weight of an irreducible  $K$  module the sheaf  $\mathcal{O}_\Lambda \rightarrow \Gamma \backslash G/K$  and cohomology  $H^q(\Gamma \backslash G/K, \mathcal{O}_\Lambda)$  are defined similarly.

**THEOREM 3.3.** *Let  $\lambda \in \mathfrak{h}^*$  be an integral form. If  $(\lambda + \delta_k, \alpha) = 0$  for some  $\alpha \in \Delta_k^+$  then  $H^q(Y, \mathcal{L}_\lambda) = 0$  for every  $q$ . Assume  $\lambda + \delta$  is regular (so in particular  $(\lambda + \delta_k, \alpha) \neq 0$  for every  $\alpha \in \Delta_k^+$ ) and assume every  $\alpha \in P_n^{(\lambda)}$  is totally positive (cf. (2.1)). Then  $H^q(Y, L_\lambda) = 0$  unless  $q$  has the specific form*

$$(3.4) \quad \begin{aligned} q(\lambda, \theta) &\stackrel{\text{def}}{=} |\{\alpha \in \Delta_k^+ : (\lambda + \delta, \alpha) < 0\}| + q(\lambda; \theta) \\ &= |\{\alpha \in \Delta_k^+ : (\lambda + \delta, \alpha) < 0\}| + 2|\theta_{u,n} \cap Q_\lambda| - |\theta_{u,n}| + |Q'_\lambda| \end{aligned}$$

for some parabolic subalgebra  $\theta = m + u \supset \mathfrak{b}_\lambda$  for which  $(\lambda + \delta - \delta^{(\lambda)}, \Delta(m)) = 0$ ; see (2.1), (2.3), (2.4), (2.5). On the other hand suppose we are given a parabolic subalgebra  $\theta = m + u \supset \mathfrak{b}_\lambda$  for which  $(\lambda + \delta - \delta^{(\lambda)}, \Delta(m)) = 0$  (where we still assume  $\lambda + \delta$  is regular and every  $\alpha \in P_n^{(\lambda)}$  is totally positive). Let  $\{\theta_i = m_i + u_i\}_{i=1}^t$  be a representative collection of parabolic subalgebras containing  $\mathfrak{b}_\lambda$  (as defined earlier) and define for an integer  $q \geq 0$

$$(3.5) \quad \begin{aligned} S(\lambda, q) &= \{i : q(\lambda; \theta_i) = q \text{ and } \theta_{u_i, n} = \theta'_{u', n} \\ &\text{for some parabolic subalgebra } \theta' = m' + u' \supset \mathfrak{b}_\lambda \\ &\text{with } (\lambda + \delta - \delta^{(\lambda)}, \Delta(m')) = 0\}. \end{aligned}$$

Let  $\mu(\lambda, \theta_i)$  be the unique  $\Delta_k^+$ -dominant integral element in the  $W_k$ -orbit of  $\lambda + \delta - \delta^{(\lambda)} + \langle \theta_{u_i, n} \rangle$ . Then for  $i \in S(\lambda, q)$  the irreducible  $G$ -representation  $\pi_{\mu(\lambda, \theta_i)}$  with  $\Delta_k^+ \cup -P_n^{(\lambda)}$ -highest weight  $\mu(\lambda, \theta_i)$  is unitarizable and the multiplicities  $m_{\pi_{\mu(\lambda, \theta_i)}}$  in  $L^2(\Gamma \backslash G)$  satisfy

$$(3.6) \quad \dim H^{q(\lambda, \theta)}(Y, \mathcal{L}_\lambda) = \sum_{i \in S(\lambda, q(\lambda, \theta))} m_{\pi_{\mu(\lambda, \theta_i)}}(\Gamma).$$

PROOF. The fibration  $\Gamma \backslash G/H \rightarrow \Gamma \backslash G/K$  in (3.2) gives rise to a spectral sequence which degenerates, by the Borel-Weil Theorem. If  $\lambda + \delta$  is regular and  $\Lambda(\lambda) \stackrel{\text{def}}{=} w(\lambda + \delta_k) - \delta_k \in \mathcal{F}'_0$ , where  $w \in W_k$  is the unique element such that  $(w(\lambda + \delta_k), \Delta_k^+) > 0$ , then one obtains, consequently,

$$(3.7) \quad H^q(Y, \mathcal{L}_\lambda) = H^{q-q_0}(\Gamma \backslash G/K, \mathcal{O}_{\Lambda(\lambda)}) \quad \text{for } q \geq 0$$

where  $q_0 = |w(-\Delta_k^+) \cap \Delta_k^+| = |\{\alpha \in \Delta_k^+ : (\lambda + \delta, \alpha) < 0\}|$ . Moreover  $H^q(Y, \mathcal{L}_\lambda) = 0$  if  $\lambda + \delta_k$  is  $\Delta_k$ -singular; cf. Corollaries 5.15, 5.16 and Theorem 5.24 of [1]. Note that by our assumptions on  $\Delta^+$ ,  $(\delta_n, \Delta_k^+) = 0 \Rightarrow q_0 = |\{\alpha \in \Delta_k^+ : (\lambda + \delta_k, \alpha) < 0\}|$ , also. From the main result of [7, Theorem 2.3] applied to the right-hand side in (3.7), one obtains the vanishing results stated in Theorem 3.3. Consider the second part of Theorem 3.3.  $\{w\theta_i\}_{i=1}^t$  is a representative collection of parabolic subalgebras containing  $\mathfrak{b}_{\Lambda(\lambda)}$ , where  $P^{(\Lambda(\lambda))} \stackrel{\text{a.}}{=} wP^{(\lambda)}$ ,  $(w\theta_i)_{wu_i, n} \stackrel{\text{(i)}}{=} w\theta_{u_i, n}$  (since  $w \in W_k \Rightarrow wP_n^{(\lambda)} = P_n^{(\lambda)}$ , as every  $\alpha \in P_n^{(\lambda)}$  is totally positive), and  $Q_{\Lambda(\lambda)} = wQ_\lambda$ ; hence  $q(\lambda; \theta_i) \stackrel{\text{(ii)}}{=} q(\Lambda(\lambda); w\theta_i)$ ; see (2.1), (2.3), (2.5). If we put

$$\begin{aligned} \underline{S}(\Lambda(\lambda), q) &= \{i: q(\Lambda(\lambda); w\theta_i) = q \text{ and } (w\theta_i)_{wu_i, n} = \mathfrak{p}_{l, v} \\ &\quad \text{for some parabolic subalgebras } \mathfrak{p} = l + v \supset \mathfrak{b}_{\Lambda(\lambda)} \\ &\quad \text{with } (\Lambda(\lambda) + \delta - \delta^{(\Lambda(\lambda))}, \Delta(l)) = 0\}, \end{aligned}$$

where  $q \geq 0$  is an integer, then by Theorem 3.26 of [8] the irreducible  $G$ -representation  $\pi_{\mu_i}$  with  $\overline{P}(\Lambda(\lambda))$ -highest weight

$$\mu_i \stackrel{\text{def}}{=} \Lambda(\lambda) + \delta - \delta^{(\Lambda(\lambda))} + \langle (w\theta_i)_{wu_i, n} \rangle$$

is unitarizable for  $i \in \underline{S}(\Lambda(\lambda), q)$  and the multiplicities  $m_{\pi_{\mu_i}}(\Gamma)$  in  $L^2(\Gamma \backslash G)$  satisfy

$$(3.8) \quad \dim H^q(\Gamma \backslash G/K, \mathcal{O}_{\Lambda(\lambda)}) = \sum_{i \in \underline{S}(\Lambda(\lambda), q)} m_{\pi_{\mu_i}}(\Gamma)$$

for  $q \geq 0$ . Now (ii) and the fact that  $\Lambda(\lambda) + \delta - \delta^{(\Lambda(\lambda))} \stackrel{\text{(iii)}}{=} w(\lambda + \delta - \delta^{(\lambda)})$  imply that  $S(\lambda, q) = \underline{S}(\Lambda(\lambda), q)$ . Also

$$\overline{P}^{(\Lambda(\lambda))} \stackrel{\text{def}}{=} \Delta_k^+ \cup -P_n^{(\Lambda(\lambda))} = \Delta_k^+ \cup -wP_n^{(\lambda)} \stackrel{\text{(iv)}}{=} \Delta_k^+ \cup -P_n^{(\lambda)},$$

and by (i), (iii)  $\mu_i = w(\lambda + \delta - \delta^{(\lambda)} + \langle \theta_{u_i, n} \rangle)$ ; i.e.  $\mu_i$  is the unique  $\Delta_k^+$ -dominant integral element in the  $W_k$ -orbit of  $\lambda + \delta - \delta^{(\lambda)} + \langle \theta_{u_i, n} \rangle$  so by definition of  $\mu(\lambda, \theta_i)$ ,  $\mu(\lambda, \theta_i) = \mu_i$ . Note that (iv) can also be written as  $\Delta_k^+ \cup -P_n^{(\lambda)} = w\overline{P}^{(\lambda)}$ . Theorem 3.3 now follows (i.e. (3.6) follows) by (3.7), (3.8) if we take  $q$  there equal to  $q(\lambda, \theta), q(\lambda; \theta)$  respectively, since  $q(\lambda, \theta) - q_0 = q(\lambda; \theta)$  by (3.4). Q.E.D.

A corollary of Theorem 3.3 is the following main result.

**THEOREM 3.9.** *Let  $\lambda \in \mathfrak{h}^*$  be an integral form such that  $\lambda + \delta$  is regular and such that every  $\alpha \in P_n^{(\lambda)}$  is totally positive (cf. (2.1)). Let  $\theta = m + u$  be a parabolic subalgebra containing the Borel subalgebra  $\mathfrak{b}_\lambda = \mathfrak{h} + \sum_{\alpha \in P^{(\lambda)}} \mathfrak{g}_\alpha$  for which  $(\lambda + \delta - \delta^{(\lambda)}, \Delta(m)) = 0$  and let  $\mu(\lambda, \theta)$  be the unique  $\Delta_k^+$ -dominant integral element in the  $W_k$ -orbit of  $\lambda + \delta - \delta^{(\lambda)} + \langle \theta_{u, n} \rangle$ ; cf. (2.1), (2.3), (2.4). Then the irreducible*

$G$ -representation  $\pi_{\mu(\lambda,\theta)}$  with  $\Delta_k^+ \cup -P_n^{(\lambda)}$ -highest weight  $\mu(\lambda, \theta)$  is unitarizable and if, in particular,  $\lambda$  satisfies

$$(3.10) \quad (\lambda + \delta - \delta^{(\lambda)}, \alpha) \neq 0 \quad \text{for } \alpha \in Q_\lambda \cup \theta_{u,n}$$

then the multiplicity  $m_{\pi_{\mu(\lambda,\theta)}}(\Gamma)$  of  $\pi_{\mu(\lambda,\theta)}$  in  $L^2(\Gamma \backslash G)$  satisfies

$$(3.11) \quad m_{\pi_{\mu(\lambda,\theta)}}(\Gamma) = \dim H^{q(\lambda)+n-|\theta_{u,n}|}(Y, \mathcal{L}_\lambda)$$

where  $q(\lambda)$  is given by (1.2),  $2n = \dim_R G/K$  (i.e.  $n = |\Delta_n^+|$ );  $q(\lambda) + n - |\theta_{u,n}| = q(\lambda, \theta)$  (see (3.4)) and (3.10) implies in fact that  $Q_\lambda \subset \theta_{u,n}$ . In general, apart from the hypothesis (3.10), (3.11) holds with  $q(\lambda) + n - |\theta_{u,n}|$  replaced by  $q(\lambda, \theta)$  if  $S(\lambda, q(\lambda; \theta))$  in (3.6) reduces to a single element. The cohomology  $H^*(Y, \mathcal{L}_\lambda)$  can survive only in dimensions sharply specified in Theorem 3.3.

PROOF. We shall exploit freely the notation of the preceding proof. As  $\{\theta_i\}_{i=1}^t$  is a representative collection of parabolics  $\supset \mathfrak{b}_\lambda$ ,  $\theta_{u,n} \stackrel{(v)}{=} \theta_{u_{i_0},n}$  for some unique  $i_0$ ,  $1 \leq i_0 \leq t$ ; equivalently, for  $w\theta \supset \mathfrak{b}_{\Lambda(\lambda)}$ ,

$$(w\theta)_{wu,n} = (w\theta_{i_0})_{wu_{i_0},n} = w\theta_{u_{i_0},n}.$$

Also

$$(\Lambda(\lambda) + \delta - \delta^{(\lambda)}, \Delta(wm)) = (w(\lambda + \delta - \delta^{(\lambda)}), w\Delta(m)) = (\lambda + \delta - \delta^{(\lambda)}, \Delta(m)) = 0,$$

by hypothesis, so  $i_0 \in S(\Lambda(\lambda), q(\Lambda(\lambda); w\theta))$  by Proposition 2.7 (applied to  $w\theta$ ), where

$$S(\Lambda(\lambda), q(\Lambda(\lambda); w\theta)) \equiv \underline{S}(\Lambda(\lambda), q(\Lambda(\lambda); w\theta))$$

(in the notation of the preceding proof)

$$\stackrel{(vi)}{\equiv} S(\lambda, q(\lambda; \theta)),$$

and where  $\mu(\lambda, \theta_{i_0}) = \mu_{i_0} = w(\lambda + \delta - \delta^{(\lambda)} + \langle \theta_{u_{i_0},n} \rangle) = w(\lambda + \delta - \delta^{(\lambda)} + \langle \theta_{u,n} \rangle)$ ,

by (v). That is,  $\mu(\lambda, \theta) = \mu_{i_0}$  by definition of  $\mu(\lambda, \theta)$  and thus  $\pi_{\mu(\lambda,\theta)} \stackrel{(viii)}{=} \pi_{\mu_{i_0}} = \pi_{\mu(\lambda,\theta_{i_0})}$  is unitarizable (as we have already observed that  $\pi_{\mu_i}$  is unitarizable for any

$i \in \underline{S}(\Lambda(\lambda), q)$ ,  $q \geq 0$ ) and  $\mu(\lambda, \theta)$  is a  $\overline{P}^{(\Lambda(\lambda))}$ -highest weight;  $\overline{P}^{(\Lambda(\lambda))} = \Delta_k^+ \cup -P_n^{(\lambda)}$ .

It should be noted that the unitarizability of  $\pi_{\mu(\lambda,\theta)}$  also follows directly from [4] of course.

Now if  $\lambda$  satisfies (3.10), then  $\Lambda(\lambda)$  satisfies  $(\Lambda(\lambda) + \delta - \delta^{(\lambda)}, \alpha) \neq 0$  for  $\alpha \in Q_{\Lambda(\lambda)} \cup (w\theta)_{wu,n} = wQ_\lambda \cup w\theta_{u,n} = w(Q_\lambda \cup \theta_{u,n})$  so by Proposition 2.7 (again applied to  $w\theta$ ),  $Q_{\Lambda(\lambda)} \subset (w\theta)_{wu,n}$ ,  $q(\Lambda(\lambda); w\theta) = |Q_{\Lambda(\lambda)}| + |\Delta_n^+| - |(w\theta)_{wu,n}|$ , and

$S(\Lambda(\lambda), q(\Lambda(\lambda); w\theta)) = \{i_0\}$ . That is,  $Q_\lambda \subset \theta_{u,n}$ ,  $q(\lambda; \theta) = |Q_\lambda| + n - |\theta_{u,n}|$ , and

(by (vi))  $S(\lambda, q(\lambda; \theta)) \stackrel{(viii)}{=} \{i_0\}$ . In particular,

$$\begin{aligned} q(\lambda, \theta) &\stackrel{\text{def.}}{=} |\{\alpha \in \Delta_k^+ : (\lambda + \delta, \alpha) < 0\}| + q(\lambda; \theta) \quad (\text{see (3.4)}) \\ &= q(\lambda) + n - |\theta_{u,n}| \end{aligned}$$

(see (1.2) and (2.1)), and by (vii), (viii) equation (3.6) reduces to equation (3.11).

If  $S(\lambda, q(\lambda; \theta))$  reduces to a single element, even without assumption (3.10), then  $S(\lambda, q(\lambda; \theta)) = \{i_0\}$  and again (vii) and (3.6) imply (3.11) provided  $q(\lambda) + n - |\theta_{u,n}|$  in (3.11) is replaced, more generally, by  $q(\lambda, \theta)$ . Q.E.D.

REMARKS. We have seen that  $\mu(\lambda, \theta)$  in Theorem 3.9 is given by  $\mu(\lambda, \theta) = \Lambda(\lambda) + \delta - \delta^{(\Lambda(\lambda))} + w\langle\theta_{u,n}\rangle$  where  $w \in W_k$  is the unique element such that  $(w(\lambda + \delta), \Delta_k^+) > 0$  and  $\Lambda(\lambda) = w(\lambda + \delta_k) - \delta_k \in \mathcal{F}_0^+$  (see (2.2)). Note that  $(w(\lambda + \delta), \Delta_k^+) = (w(\lambda + \delta_k), \Delta_k^+)$  since  $(w\delta_n, \Delta_k^+) = (\delta_n, \Delta_k^+) = 0$ . Also in Theorem 3.9,  $\Delta_k^+ \cup -P_n^{(\lambda)} = w\overline{P}^{(\lambda)} = \overline{P}^{(\Lambda(\lambda))}$ . Now suppose we choose  $\theta = \mathfrak{b}_\lambda$  in Theorem 3.9. Then

$$\theta_{u,n} = P_n^{(\lambda)} \Rightarrow \mu(\lambda, \mathfrak{b}_\lambda) = \Lambda(\lambda) + \delta_n^{(\Lambda(\lambda))} + \delta_n$$

( $P_n^{(\lambda)} = wP_n^{(\lambda)} = P_n^{(\Lambda(\lambda))}$ ). That is,  $\mu(\lambda, \mathfrak{b}_\lambda)$  is the lowest  $K$ -type of the discrete series representation corresponding to the regular element  $\Lambda(\lambda) + \delta$ . The latter representation is unitarily equivalent to that corresponding to the regular element  $\lambda + \delta$  since  $\Lambda(\lambda) + \delta = w(\lambda + \delta)$  with  $w \in W_k$ . Thus  $\pi_{\mu(\lambda, \mathfrak{b}_\lambda)}$  is the holomorphic discrete series representation of  $G$  corresponding to the regular element  $\lambda + \delta$  and the holomorphic Weyl chamber  $\Delta_k^+ \cup -P_n^{(\lambda)}$ . Condition (3.10) then becomes condition (2.10), and (3.11) reduces to (1.3)(ii). Also in (3.4) (again for the choice  $\theta = \mathfrak{b}_\lambda$ )

$$2|\theta_{u,n} \cap Q_\lambda| - |\theta_{u,n}| + |Q'_\lambda| = 2|Q_\lambda| - n + |Q'_\lambda| = |Q_\lambda| \Rightarrow q(\lambda, \mathfrak{b}_\lambda) = q(\lambda)$$

in (1.2). That is, the vanishing result in Theorem 3.3 reduces to (1.3)(i).

A special case of interest is the case  $Q_\lambda = \emptyset$ ; i.e.  $\lambda$  satisfies  $(\lambda + \delta, \Delta_n^+) < 0$  or, equivalently  $(\Lambda(\lambda) + \delta, \Delta_n^+) < 0$  since  $w\Delta_n^+ = \Delta_n^+$ . Then  $P_n^{(\lambda)} = -\Delta_n^+ \Rightarrow$  every  $\alpha \in P_n^{(\lambda)}$  is totally positive, and  $q(\lambda; \theta)$  in (3.4) is  $2 \cdot 0 - |\theta_{u,n}| + n = n - |\theta_{u,n}|$ , so that  $q(\lambda, \theta) = |\{\alpha \in \Delta_k^+ : (\lambda + \delta, \alpha) < 0\}| + n - |\theta_{u,n}|$ . (3.5) becomes for  $q = q(\lambda; \theta) = n - |\theta_{u,n}|$

$$S(\lambda, q(\lambda; \theta)) = \{i : |\theta_{u_i, n}| = |\theta_{u,n}| \text{ and } \theta_{u_i, n} = \theta'_{u', n}\}$$

$$(3.12) \quad \text{for some parabolic subalgebra } \theta' = m' + u' \supset \mathfrak{b}_\lambda \text{ with } (\lambda + \delta - \delta^{(\lambda)}, \Delta(m')) = 0\}$$

for a representative collection  $\{\theta_i = m_i + u_i\}_{i=1}^t \supset \mathfrak{b}_\lambda$ ; cf. (2.3), (2.4). Also

$$P^{(\Lambda(\lambda))} = \Delta_k^+ \cup -\Delta_n^+ = \overline{\Delta}^+ \Rightarrow \delta - \delta^{(\Lambda(\lambda))} = 2\delta_n \Rightarrow \mu(\lambda, \theta) = \Lambda(\lambda) + 2\delta_n + w\langle\theta_{u,n}\rangle.$$

Thus Theorem 3.9 simplifies as follows.

**THEOREM 3.13.** *Let  $\lambda \in \mathfrak{h}^*$  be an integral form such that  $\lambda + \delta$  is regular and such that  $(\lambda + \delta, \Delta_n^+) < 0$ . Let  $\theta = m + u \supset \mathfrak{b}_\lambda$  as in Theorem 3.9 with  $(\lambda + \delta - \delta^{(\lambda)}, \Delta(m)) = 0$  or, equivalently,  $(\Lambda(\lambda) + 2\delta_n, w\Delta(m)) = 0$  in the above notation. Then the irreducible  $G$ -representation  $\pi_{\mu(\lambda, \theta)}$  with  $\Delta^+$ -highest weight  $\mu(\lambda, \theta) = \Lambda(\lambda) + 2\delta_n + w\langle\theta_{u,n}\rangle$  is unitarizable and its multiplicity  $m_{\pi_{\mu(\lambda, \theta)}}(\Gamma)$  in  $L^2(\Gamma \backslash G)$  satisfies*

$$(3.14) \quad m_{\pi_{\mu(\lambda, \theta)}}(\Gamma) = \dim H^{q(\lambda) + n - |\theta_{u,n}|}(Y, \mathcal{L}_\lambda)$$

where  $q(\lambda) = |\{\alpha \in \Delta_k^+ : (\lambda + \delta, \alpha) < 0\}| = |w(-\Delta_k^+) \cap \Delta_k^+|$ , provided that either (i)'  $(\lambda + \delta - \delta^{(\lambda)}, \alpha) \neq 0$  for  $\alpha \in \theta_{u,n}$  (or, equivalently,  $(\Lambda(\lambda) + 2\delta_n, \beta) \neq 0$  for  $\beta \in w\theta_{u,n}$ ) or (ii)' the set  $S(\lambda, q(\lambda; \theta))$  in (3.12) reduces to a single element.  $H^q(Y, \mathcal{L}_\lambda) = 0$  unless  $q = q(\lambda) + n - |\theta'_{u', n}|$  for some parabolic  $\theta = m' + u' \supset \mathfrak{b}_\lambda$  such that  $(\lambda + \delta - \delta^{(\lambda)}, \Delta(m')) = 0$ .

Note. In Theorem (3.13),

$$P^{(\lambda)} = w^{-1}P^{(\Lambda(\lambda))} = w^{-1}\Delta_k^+ \cup -w^{-1}\Delta_n^+ = w^{-1}\Delta_k^+ \cup -\Delta_n^+$$

(see equation (a) in the proof of Theorem 3.3) so that  $\lambda + \delta - \delta^{(\lambda)} = \lambda + 2\delta_n + \delta_k - w^{-1}\delta_k$ .

Finally we note that for  $G = \text{Sp}(m, R)$  or  $\text{SO}^*(2m)$  the set  $S(\lambda, q(\lambda; \theta))$  in (3.12) reduces to a single element, for in these cases

$$\{|\theta_{u,n}|: \theta \supset \mathfrak{b}_\lambda\} = \left\{ \frac{(m+1)m - (r+1)r}{2} : 0 \leq r \leq m \right\}, \\ \left\{ \frac{(m-1)m - (r-1)r}{2} : 1 \leq r \leq m \right\}, \quad \text{respectively;}$$

cf. Table 3.4 of [7]. Thus  $|\theta_{u_r,n}| \neq |\theta_{u_j,n}|$  for  $r \neq j$

COROLLARY 3.15. For  $G = \text{Sp}(m, R)$  ( $m \geq 1$ ) or  $G = \text{SO}^*(2m)$  ( $m \geq 2$ ) condition (ii)' in Theorem 3.13 is always satisfied. Hence (3.14) holds for  $\lambda, \theta$  subject to the conditions of Theorem 3.13 prior to (3.14).

In a future study we hope to extend the results of this paper to more general unitary representations having nontrivial relative  $(\mathfrak{g}, K)$  cohomology.

#### REFERENCES

1. H. Fischer and F. Williams, *Borel-Le Potier diagrams—calculus of their cohomology bundles*, Tôhoku Math. J. **36** (1984), 233–251.
2. P. Griffiths and W. Schmid, *Locally homogeneous complex manifolds*, Acta Math. **123** (1969), 253–302.
3. R. Langlands, *Dimension of spaces of automorphic forms*, Proc. Sympos. in Pure Math., Vol. 9, Amer. Math. Soc., Providence, R.I., 1966, pp. 253–257.
4. R. Parthasarathy, *Criteria for the unitarizability of some highest weight modules*, Proc. Indian Acad. Sci. **89** (1980), 1–24.
5. W. Schmid, *On a conjecture of Langlands*, Ann. of Math. (2) **93** (1971), 1–42.
6. G. Warner, *Selberg's trace formula for nonuniform lattices: The R-rank one case*, Advances in Math., Suppl. Studies, 6, Academic Press, New York, 1979, pp. 1–142.
7. F. Williams, *Vanishing theorems for type  $(0, q)$  cohomology of locally symmetric spaces (I)*, Osaka J. Math. **18** (1981), 147–160.
8. ———, *An alternating sum formula for the multiplicities of unitary highest weight modules in  $L^2(\Gamma \backslash G)$* , unpublished manuscript.
9. ———, *Discrete series multiplicities in  $L^2(\Gamma \backslash G)$  (II): Proof of Langlands' conjecture*, Amer. J. Math. **107** (1985), 367–376.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MASSACHUSETTS, AMHERST, MASSACHUSETTS 01003