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A MARTIN BOUNDARY IN THE PLANE

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ABSTRACT. Let E be an open connected subset of Euclidean space, with a Green function, and let λ be harmonic measure on the Martin boundary Δ of E. We will show that, except for a $\lambda \otimes \lambda$-null set of $(x,y) \in \Delta^2$, x is an entrance point for Brownian motion conditioned to leave E at y. R. S. Martin gave examples in dimension 3 or higher, for which there exist minimal accessible Martin boundary points $x \neq y$ for which this condition fails. We will give a similar example in dimension 2.

1. Introduction. Let E be an open connected subset of Euclidean space, with a Green function, and let λ be harmonic measure on the Martin boundary Δ of E. We will show that except for a $\lambda \otimes \lambda$-null set of $(x,y) \in \Delta^2$, x is an entrance point for Brownian motion conditioned to leave E at y. R. S. Martin gave examples in dimension 3 or higher, for which there exist minimal accessible Martin boundary points $x \neq y$ for which this condition fails. We will give a similar example in dimension 2. The argument uses a recent result of M. Cranston and T. McConnell [**4**], together with Schwarz–Christoffel transformations.

§2 deals with notation, and background material on Martin boundaries, including a characterization due to J. B. Walsh of pairs (x,y) as above. The main results are stated in §3, and are proven in §§4 and 5.

I am grateful to Burgess Davis and to Chris Burdzy for comments that helped shorten/correct the original proof. I would especially like to thank John Walsh for his support and valuable suggestions, as well as for posing the problem to me in the first place.

2. Notation and background. First a word as to sources. We will refer to Walsh [**17**], and occasionally to Meyer, Smythe, and Walsh [**14**] for most results concerning h-transforms. References such as (W 1.1), (W 1.2) for example, refer to Proposition 1.1 and Theorem 1.2 of [**17**]. Many of the proofs may be found in the original source; Doob [**6**]. We will also use Schwarz–Christoffel transformations (see Ahlfors [**1**]), and elliptic integrals. For the latter, we will refer to Gradshteyn and Ryzhik [**8**]; references such as (GR 8.1133) refer to the corresponding formula of [**8**].

Let E be an open connected subset of $\mathbf{R}^d$ possessing a Green function $G(x,y)$, and let $\mathcal{E}$ be the σ-field of its Borel subsets. Let δ be some additional point which will act as a cemetary. Let Ω be the canonical space of functions ω: $[0,\infty) \to E \cup \{\delta\}$ such that there exists $\varsigma > 0$ with ω continuous on $[0,\varsigma)$, and $\{t; \omega(t) = \delta\} = [\varsigma, \infty)$.

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Let X_t, $t \geq 0$, be the coordinate maps on Ω, and let

$$\mathcal{F}_t = \sigma(x_s; s \leq t), \qquad \mathcal{F} = \mathcal{F}_\infty.$$

We will occasionally use the space $\hat{\Omega}$ of functions $\omega: (0, \infty) \to E \cup \{\delta\}$ which are continuous up to their lifetime, as above. By the appropriate abuse of notation, we will use the same symbols $\mathcal{F}_t$, $\mathcal{F}$, X_t, etc... for the corresponding objects on $\hat{\Omega}$. Of course, in this context, X_t is only defined for $t > 0$. We will follow the convention of making all functions vanish at δ unless otherwise specified. We will also use the notation

$$\begin{aligned}&(\theta_t\omega)(s) = \omega(t+s),\\ &T_A = \inf\{t > 0; X_t \in A\},\\ &L_A = \sup\{t > 0; X_t \in A\} \qquad (\sup(\phi) = 0).\end{aligned}$$

Let $P_t(x, dy)$ be the transition function of Brownian motion on $\mathbf{R}^d$, killed upon leaving E. If h is excessive for (P_t), we have Doob's h-transformed transition function;

$${}_hP_t(x, dy) = \begin{cases} (1/h(x))P_t(x, dy)h(y), & 0 < h(x) < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Both transition functions are defined as usual on δ, so as to be Markov. The corresponding measures on $(\Omega, \mathcal{F})$ are written P^x and ${}_hP^x$. In general, if we are given a measurable space, a positive σ-finite measure P thereon, and a random variable Y with values in $(\Omega, \mathcal{F})$, we say that Y is an *h-transform* under P if it is strong Markov with respect to P and the filtration $Y^{-1}(\mathcal{F}_{t+})$, with transition function ${}_hP_t$. We use the same notation if Y takes values in $\hat{\Omega}$, but now the strong Markov property is at strictly positive times. By (W 1.2), (X_t) is an h-transform under each ${}_hP^x$.

Now, following Helms [**9**], let $t_0 > 0$ and let ϕ be bounded, increasing, and concave, with $\phi(t) = t$ for $t \in [0, t_0]$. For $y_0 \in E$, define the Martin function (with respect to the base point y_0) as

$$K(x, y) = G(x, y)/\phi(G(x, y_0)).$$

From these, one obtains, as in Martin [**13**], the so-called Martin metric; the completion $\overline{E}$ of E in this metric is compact, and $\Delta = \overline{E} \backslash E$ is the *Martin boundary* of E. K extends to a continuous function from $\overline{E} \times E$ to $(0, \infty]$, taking on the value ∞ only on points (x, y) with $x = y$. $K(x, \cdot)$ is harmonic for $x \in \Delta$. If it is minimal, we call x a minimal point. Let Δ_0 be the set of minimal $x \in \Delta$. Then Δ_0 is Borel, and we have the Martin integral representation: For each positive excessive h, there is a unique measure ν on $\Delta_0 \cup E$ such that

$$h = \int K(x, \cdot)\nu(dx).$$

Moreover, h is harmonic iff $\nu(E) = 0$. If h is harmonic on E and continuous on an open subset U of $\overline{E}$, then in fact

$$\nu(dy) = h(y)\mu_0(dy) \quad \text{on } U; \tag{2.1}$$

where μ_0 represents the function identically equal to 1. For $y \in \Delta_0$ and $x \in E$, let ${}_yP^x = {}_{K(y,\cdot)}P^x$. Recall that $y \in \Delta_0$ is *accessible* if ${}_yP^x(\varsigma < \infty) = 1$ for some (and hence every) $x \in E$. Let Acc denote the set of accessible points. Now let

$$\tilde{X}_t = \begin{cases} X_{\varsigma - t}, & t \in (0, \varsigma), \\ \delta, & t \geq \varsigma, \end{cases}$$

be the reverse of (X_t) from its lifetime. One can show the following by time reversal and Doob's martingale convergence theorem (in their present form, the statements of (2.2) are taken from an as yet unpublished set of notes [**18**] by John Walsh on time reversal and Martin boundaries).

THEOREM (2.2). (a) *Let h be excessive, and assume that ${}_hP^x(\varsigma < \infty) = 1$ for every $x \in E$. Let μ be a probability on E. Then $(\tilde{X}_t)_{t>0}$ is a v-transform under ${}_hP^\mu$, where*

$$v(x) = \int \frac{G(z,x)}{h(z)} \mu(dz).$$

(b) *If P is any probability on $(\hat{\Omega}, \mathcal{F})$ under which $(X_t)_{t>0}$ is an h-transform, then there is some excessive v such that $(\tilde{X}_t)_{t>0}$ is a v-transform under P.*

(c) *Let μ represent the excessive function h. If $h(x) < \infty$ then $X_{\varsigma-}$ exists in $\overline{E}$, ${}_hP^x$-a.s., and has distribution*

$${}_hP^x(X_{\varsigma-} \in dy) = (1/h(x))K(y,x)\mu(dy).$$

(d) *Let $u(x) = \int G(x,z)\nu(dz)$ be an a.e. finite potential, and let $y \in$ Acc. Suppose that $\int K(y,z)\nu(dz) < \infty$. Then there is a unique probability ${}_uP^y$ on $\hat{\Omega}$ such that (X_t) is a u-transform under ${}_uP^y$, and*

$${}_uP^y(X_t \to y \text{ in } \overline{E} \text{ as } t \downarrow 0) = 1.$$

We will be interested in the set $D = \{(x,y) \in (\text{Acc} \cup E) \times (\text{Acc} \cup E)$; there exists a probability ${}_yP^x$ on $\hat{\Omega}$ under which (X_t) is a $K(y,\cdot)$-transform and $X_{0+} = x$ a.s.$\}$. Theorem (2.2) gives that

$$D \supset (\text{Acc} \times E) \cup (E \times \text{Acc}),$$

and one can further show that if $(x,y) \in D$ then the law ${}_yP^x$ is unique, $(y,x) \in D$, and for $\Lambda \in (\hat{\Omega}, \mathcal{F})$,

$${}_xP^y(\Lambda) = {}_yP^x(\tilde{X} \in \Lambda).$$

Let $R_A h(x) = E^x[h(X_{T_A})]$. One can show that, for A open in $\overline{E}$ and $y \in A \cap \Delta_0$,

$$R_{E \setminus A} K(y,\cdot) = \int_E G(z,\cdot)\nu(dz)$$

for some measure ν on E (that is, this function is the potential of some measure ν). Then the following gives a characterization of D:

THEOREM (2.3) (J. B. WALSH [**18**]). *Let $x, y \in$ Acc, and let A be an open subset of $\overline{E}$ containing y, yet not containing x in its closure. Let $z \in A \cap E$ and set $f(w) = K(y,w)/G(z,w)$. Then the following conditions are equivalent:*

(a) $(x,y) \in D$.

(b) *Let $R_{E \setminus A}K(y,\cdot)$ be the potential of ν. Then*

$$\int K(x,z)\nu(dz) < \infty.$$

(c) $f(X_t)$ *converges to a finite limit as* $t \uparrow \varsigma$, ${}_xP^w$*-a.s., for every* $w \in E$.
(d) $\liminf_{t\uparrow\varsigma} f(X_t) < \infty\, {}_xP^w$*-a.s. for some* $w \in E$.

Naïm [**15**] showed that the limit in (c) exists (possibly with value ∞). Thus for $x, y \in \mathrm{Acc}$, $(x, y) \in D$ iff the so called Naïm kernel is finite at (x, y).

Finally, Mike Cranston has pointed out to me that the boundary Harnack principle for Lipschitz domains (cf. Jerison and Kenig [**11**]) together with results of Hunt and Wheeden [**10**] and Cranston [**5**], shows the following

THEOREM (2.4). *Let E be a bounded Lipschitz domain. Then $\overline{E}$ is homeomorphic to the Euclidean closure of E, all boundary points are minimal and accessible, and* $D = \{(x, y) \in \overline{E} \times \overline{E}; x \neq y\}$.

(The same is true for bounded 'N.T.A.-domains' (see Jerison and Kenig [**11**]); one now uses results of [**11**], together with the fact that boundary points are accessible here as well (M. Cranston—private communication).)

3. Description of results. Let

$$D' = \{(x, y); x, y \in \mathrm{Acc} \cup E, x \neq y\},$$

and let λ be harmonic measure on Δ_0 (that is, $\lambda(dy) = P^{y_0}(X_{\varsigma-} \in dy)$). We will use Theorem (2.3) to give a probabilistic proof of the following result (a consequence of Theorem 9.2 of Doob [**7**]).

THEOREM (3.1). $\lambda \otimes \lambda(D' \backslash D) = 0$ (*actually, we will give a proof only for E bounded*).

It is not in general true that $D = D'$. An example of a domain D for which this fails may be found in Example 1 of Martin [**13**]. Martin constructed a certain bounded domain E in $\mathbf{R}^3$, together with a subset E_0 of its Martin boundary, and showed that

$$\limsup_{\substack{z \to y \\ z \in E}} K(x, z) = \infty$$

for $x, y \in E_0$. One can modify the parameters in his construction slightly, and obtain (as in (b) of Theorem (3.3), below) that also

$$\lim_{\substack{w \to y \\ w \in E}} K(x, w)/G(z, w) = \infty$$

for $z \in E$ and $x, y \in E_0$. Thus condition (d) of Theorem (2.3) fails, so that $(x, y) \notin D$ for any $x, y \in E_0$. Modulo showing that E_0 contains at least two distinct accessible points, this gives us our counterexample. This latter property is in fact not difficult to show. One embeds E in a product $E' = \mathbf{R} \times E''$, where $E'' \subset \mathbf{R}^2$ is a bounded domain. By a result of M. Cranston and T. McConnell ([**4**]; Theorem (3.2) below), all minimal Martin boundary points of E'' are accessible.

One can therefore construct a harmonic h on E' such that the h-transform $X_t = (Y_t, Z_t)$ has finite lifetime ς; (Y_t) is a Brownian motion up till time ς, independent of (Z_t) given ς; and with positive probability, $X_t \in E$ for every $t < \varsigma$, and X_t converges in E as $t \uparrow \varsigma$, with limit in E_0. From this, one sees immediately that $E_0 \cap \mathrm{Acc}$ has many points.

Rather than dwelling on this, we will consider an analogous example in $\mathbf{R}^2$, in detail. The two-dimensional situation is distinguished from that of higher dimensions, by the availability of the Riemann mapping theorem. That is, if we are given a domain $E_1 \subset \mathbf{R}^2$ whose complement (in the Riemann sphere) has n components ($n \geq 1$), none of which is a single point, then there is a conformal equivalence Φ mapping E_1 onto a bounded domain E_2 with smooth boundaries.

Being conformal, Φ preserves Green functions. That is, if G_1 and G_2 are the Green functions of E_1 and E_2, respectively, then $G_2(\Phi(x), \Phi(z)) = G_1(x, z)$. Thus, Φ will also preserve the Martin function, and hence extends to a homeomorphism of the Martin compactifications of E_1 and E_2, taking minimal points to minimal points. There is, of course, no reason why Φ should behave so nicely on the Euclidean boundary of E_1. In our case, all the Martin boundary points of E_2 are accessible (and minimal) by Theorem (2.4), so that by the criteria of Theorem (2.3), it follows that $D = D'$. In dimension ≥ 3, Martin's example is homeomorphic to the unit ball, but this shows that to obtain the same pathology in dimension 2, we must consider domains of infinite connectivity.

Unlike the case of minimality, the accessibilty or inaccessibility of a Martin boundary point will not in general be preserved under a conformal map. This problem is resolved by the following result of Cranston and McConnell [**4**].

THEOREM (3.2). *There exists a constant C such that, for each domain E in $\mathbf{R}^2$ and each excessive function h on E, we have that*

$$ {}_hE^x[\varsigma] \leq C \cdot \textit{Area of } E, \quad \textit{for } x \in E. \quad \square $$

Thus, all minimal Martin boundary points of bounded domains in $\mathbf{R}^2$, are accessible.

We will now construct a domain $E \subset \mathbf{R}^2$ for which $D \neq D'$. Let $a(n), b(n) \in (0, 1)$, $n = 1, 2, \ldots$. Set $s(0) = 0$ and

$$ s(n) = \sum_{i=1}^{n} b(n), \qquad n \geq 1. $$

Assume that $a(n)$ and $b(n)$ are decreasing in n, and that $s(\infty) = 1$. Let

$$ E = (-1, 1) \times (0, 1) \setminus \bigcup_{n=1}^{\infty} [a(n) - 1, 1 - a(n)] \times \{s(n)\}. $$

We will show that, for every fixed sequence $(b(n))$ as above, the $a(n)$ may be chosen to converge to zero sufficiently fast that $D \neq D'$. Let

$$ \begin{aligned} c(n) &= (0, (s(n) + s(n-1))/2), \qquad n \geq 1, \\ A(n) &= \begin{cases} (1 - a(n), 1) \times \{s(n)\}, & n \geq 1, \\ (-1, a(-n) - 1) \times \{s(-n)\}, & n \leq -1, \end{cases} \\ A+ &= \bigcup_{n \geq 1} A(n), \qquad A- = \bigcup_{n \leq -1} A(n), \\ A &= A + \bigcup A-, \quad \text{and} \quad B(n) = A(n) \cup A(-n). \end{aligned} $$

(See Figure 1.)

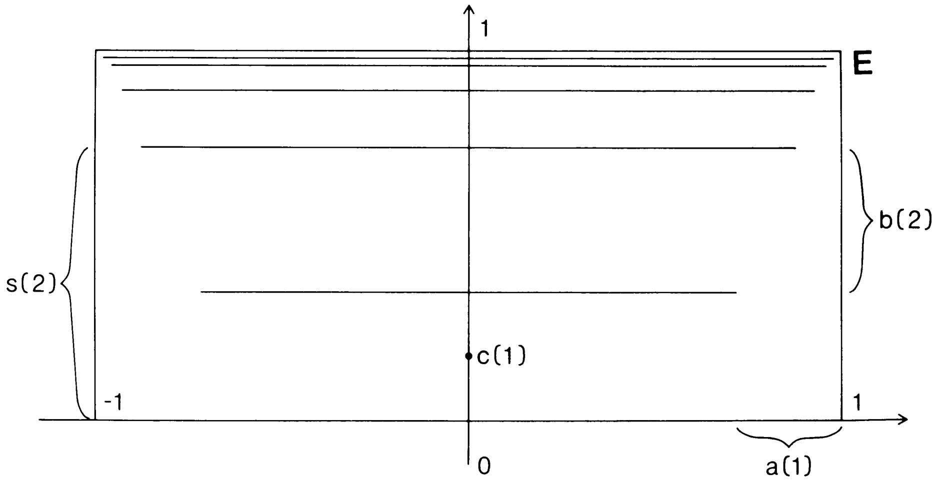

FIGURE 1

Let x_0 be a limit point in $\overline{E}$ of the sequence $(c(n))$. Define the Martin function relative to the point $c(1)$, so that $h = K(x_0, \cdot)$, is symmetric on E (that is, $h(-\overline{z}) = h(z)$; as the reader will have gathered, we are allowing ourselves to pass freely between notations used for $\mathbf{R}^2$ and for $\mathbf{C}$). We will write P^x for the law of Brownian motion, started at $x \in E$ and killed upon contact with ∂E. For ease of notation, we will write Q for the probability ${}_hP^{c(1)}$.

Martin's argument will give us the first part of the following:

THEOREM (3.3). *Let g be positive and harmonic on E, with boundary limit zero at any $y \in \partial E$ with* $\operatorname{Im} y < 1$. *Suppose that*

$$\limsup_{n\to\infty} b(n) \log(1/a(n-1)) > 4\pi.$$

Then

(a) $g(c(n)) \to \infty$ *as* $n \to \infty$;

(b) *let* $z \in E$; *then* $g(w)/G(z,w) \to \infty$ *as* $\operatorname{Im}(w) \uparrow 1$.

The only reason this result does not immediately give us that $D \neq D'$ is that it is conceivable that the speed at which $a(n) \to 0$ forces that part $[-1,1] \times \{1\}$ of the Euclidean boundary of E to collapse to a single point of the Martin boundary. By definition of h, $\operatorname{Im}(X_t) \to 1$ as $t \uparrow \varsigma$, ${}_hP^x$-a.s. Thus, if we could show that h was not minimal, then there would have to be at least two minimal Martin boundary points x_1 and x_2 such that $w \to x_i$ implies that $\operatorname{Im}(w) \to 1$, $i = 1, 2$. In this case, (b) of Theorem (3.3) together with criterion (d) of Theorem (2.3) would show that $D \neq D'$. Our main result will be

THEOREM (3.4). *Suppose that* $a(n-1) = O(b(n))$. *Then* $Q(T_{A-} = \infty) > 0$.

By symmetry, also $Q(T_{A+} = \infty) > 0$, so that also $Q(T_{A(n)} < \infty$ for infinitely many $n \geq 1) \in (0,1)$, and hence we conclude from (W 1.5), that h cannot be minimal. Thus we have

COROLLARY (3.5). *Suppose that*

$$\limsup_{n\to\infty} b(n) \log(1/a(n-1)) > 4\pi.$$

Then $D \neq D'$. $\square$

[*Note*: K. Burdzy [**3**] has recently improved on Theorem (3.4), using different methods. He obtains the same conclusion, assuming only that $\liminf_{n\to\infty} a(n) < 1$.]

4. Proof of Theorems (3.3) and (3.4). We start by estimating the parameters of a simple Schwarz–Christoffel transformation.

Let $b > 0$ and $a \in (0,1)$. Let Φ be the 'symmetric' conformal map of the unit disk onto $W = (-1,1)\times(0,b)$ (that is, it maps the points $-1, 0$, and 1 to $(-1,b/2)$, $(0,b/2)$, and $(1,b/2)$). Let

$$e^{i\theta} = \Phi^{-1}((1,b)); \qquad \theta \in (0,\pi/2),$$
$$e^{i\psi} = \Phi^{-1}((1-a,b)); \qquad \psi \in (\theta,\pi/2).$$

Write $\mu(x,dy)$ for harmonic measure on ∂W, based at x.

LEMMA (4.1). *Let $b \downarrow 0$, with $a = O(b)$. Then*
(a) $\theta \to 0$,
(b) $b \sim \pi/\log(1/\theta)$,
(c) $\psi \sim \theta$,
(d) *there is a constant C such that, if $x \in W$ with* $\operatorname{Re} x \ge 0$, *then*

$$\mu(x,\{-1\}\times[0,b]) \le e^{-c/b}.$$

PROOF. The Schwarz–Christoffel formula is (see Ahlfors [**1**])

$$\Phi(w) = \phi\int_0^w [(z-e^{i\theta})(z+e^{i\theta})(z-e^{-i\theta})(z+e^{-i\theta})]^{-1/2}\,dz + \phi'$$

for some constants ϕ and ϕ'. By definition,

$$\frac{b}{2} = \int_{-\theta}^{\theta}\frac{d}{dt}|\Phi(e^{it})|\,dt\Big/\int_{\theta}^{\pi-\theta}\frac{d}{dt}|\Phi(e^{it})|\,dt,$$

and

$$\begin{aligned}\left|\frac{d}{dt}\Phi(e^{it})\right| &= |\phi i e^{it}[(e^{it}-e^{i\theta})(e^{it}+e^{i\theta})(e^{it}-e^{-i\theta})(e^{it}+e^{-i\theta})]^{-1/2}| \\ &= 2|\phi|\,|\cos 2t - \cos 2\theta|^{-1/2}.\end{aligned}$$

Let $f(t) = |\cos 2t - \cos 2\theta|^{-1/2}$. Recall that elliptic integrals of the first kind are defined as (GR 8.1112)

$$F(\tau,k) = \int_0^{\sin\tau}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \qquad k\in[0,1],\ \tau\in[0,\pi/2].$$

Then

$$\begin{aligned}\int_{-\theta}^{\theta} f(t)\,dt &= 2\int_0^{\theta}(\cos 2t - \cos 2\theta)^{-1/2}\,dt \\ &= \sqrt{2}\int_{\cos\theta}^{1}\frac{dx}{\sqrt{(1-x^2)(x^2-\cos^2\theta)}} \\ &= \sqrt{2}F(\pi/2,\sin\theta) \qquad \text{(GR 3.1529)},\end{aligned}$$

and similarly

$$\int_\theta^{\pi-\theta} f(t)\,dt = \sqrt{2}\int_0^{\cos\theta} \frac{dx}{\sqrt{(1-x^2)(\cos^2\theta - x^2)}}$$
$$= \sqrt{2}F(\pi/2, \cos\theta) \qquad \text{(GR 3.1528)}.$$

We therefore obtain (a) and (b) from the formulae

$$F(\pi/2, 0) = \pi/2, \qquad F(\pi/2, k) \sim -\tfrac{1}{2}\log(1-k^2) \quad \text{as } k \uparrow 1 \text{ (GR 8.1133)}.$$

Let

$$\sin\rho = \frac{1}{\cos\theta}\left(\frac{\cos^2\theta - \cos^2\psi}{1-\cos^2\psi}\right)^{1/2}.$$

As above, we have

$$a = 2\int_\theta^\psi f(t)\,dt \Big/ \int_\theta^{\pi-\theta} f(t)\,dt,$$

and

$$\int_\theta^\psi f(t)\,dt = \sqrt{2}\int_{\cos\psi}^{\cos\theta} \frac{dx}{\sqrt{(1-x^2)(\cos^2\theta - x^2)}} = \sqrt{2}F(\rho, \cos\theta).$$

Since $a = O(b)$, we must have $\rho \to 0$, so that also $\psi \to 0$. Thus

$$\rho \sim \left[\frac{(1-\theta^2/2) - (1-\psi^2/2)}{\psi^2/2}\right]^{1/2} = (1-(\theta/\psi)^2)^{1/2},$$

showing (c).

Finally, write $w = \Phi^{-1}(x)$. Then

$$\mu(x, \{-1\}\times[0,b]) = \frac{1}{2\pi}\int_{\pi-\theta}^{\pi+\theta} \frac{1-|w|^2}{|w-e^{it}|^2}\,dt \le \frac{1}{2\pi}\int_{\pi-\theta}^{\pi+\theta} \frac{dt}{\cos^2\theta},$$

which, together with (a) and (b), yields the desired estimate. □

PROOF OF THEOREM (3.3). Fix n, and consider $E' = E \cap (-1,1)\times(0, s(n))$. For $x \in E'$, let P'^x be the law of Brownian motion started at x and stopped upon leaving E'. Also, let

$$W(n) = (-1,1)\times(s(n-1), s(n)) \subset E',$$
$$C(n) = (-1,1)\times\{(s(n)+s(n-1))/2\}, \text{ and}$$
$$\mu(x, dy) = P'^x(X_{T_{\partial W(n)}} \in dy).$$

Let Φ be the Schwarz–Christoffel transformation mapping the unit disk to $W(n)$ and taking the points -1, 0, and 1, to $(-1, (s(n)+s(n-1))/2)$, $c(n)$, and $(1, (s(n)+s(n-1))/2)$. Since composing Φ with a Brownian motion on the disk produces a time changed Brownian motion on $W(n)$, it follows that Φ preserves hitting distributions. Thus, by the corresponding result for the disk, we see that, for each $x \in W(n)$, there exists a density $u(x,y)$, continuous in y, for $\mu(x,dy)$ with respect to $\mu(c(n), dy)$, and that further, there exists a constant $k(n)$ with

$$\frac{u(x,y)}{u(x,y')} \le k(n) \quad \text{for } x \in C(n) \quad \text{and } y, y' \in (-1,1)\times\{s(n)\}.$$

Let

$$m(n,w) = k(n)^2/\mu(w, (-1,1)\times\{s(n)\}).$$

Thus, for $x, w \in C(n)$ and $y \in (-1,1) \times \{s(n)\}$, we have that

$$\begin{aligned}\mu(x, dy) &= \frac{u(x,y)}{u(w,y)}\mu(w,dy) \\ &\leq \frac{u(x,y)}{u(w,y)}\mu(w,dy) \\ &\quad \cdot \frac{\int_{(-1,1)\times\{s(n)\}} \frac{u(x,y')}{u(w,y')}\mu(w,dy')}{\inf\{\frac{u(x,y')}{u(w,y')}; y' \in (-1,1)\times\{s(n)\}\}\mu(w,(-1,1)\times\{s(n)\})} \\ &\leq m(n,w)\mu(x,(-1,1)\times\{s(n)\})\mu(w,dy).\end{aligned}$$

Let $R(0) = 0$, and

$$R(i+1) = R(i) + (T_{B(n-1)} + T_{C(n)} \circ \theta_{T_{B(n-1)}}) \circ \theta_{R(i)}, \qquad i \geq 0.$$

Then for $x, w \in C(n)$ and $y \in (-1,1) \times \{s(n)\}$,

$$\begin{aligned}P'^x(X_{T_{\partial E'}} \in dy) &= \sum_{i=0}^{\infty} E'^x\left[R(i) < \infty, P'^{X_{R(i)}}\left(R(1) = \infty, X_{T_{\partial E'}} \in dy\right)\right] \\ &= \sum_{i=0}^{\infty} E'^x[R(i) < \infty, \mu(X_{R(i)}, dy)] \\ &\leq \sum_{i=0}^{\infty} m(n,w)\mu(w,dy)E'^x[R(i) < \infty, \mu(X_{R(i)}, (-1,1)\times\{s(n)\})] \\ &= m(n,w)\mu(w,dy)P'^x(X_{T_{\partial E'}} \in (-1,1)\times\{s(n)\}) \\ &\leq m(n,w)\mu(w,dy).\end{aligned}$$

Also, one easily shows by the appropriate Schwarz–Christoffel transformation that there is a constant M (independent of j) such that, for each j, the first hitting distribution of Brownian motion started at $c(1)$, on the boundary of $(-1,1) \times (0, s(j))$, has a density with respect to the arc length measure, which is bounded by M.

Now let g be as in the theorem. Then for $w \in C(n)$,

$$\begin{aligned}g(c(1)) &= \iint g(y)P'^x(T_{B(n)} < \infty, X_{T_{B(n)}} \in dy)P'^{c(1)}(T_{C(n)} < \infty, X_{T_{C(n)}} \in dx) \\ &\leq P'^{c(1)}(T_{B(n-1)} < \infty)m(n,w)\int g(y)\mu(w,dy) \\ &\leq 2Ma(n-1)m(n,w)\int g(y)P'^w(X_{T_{\partial E'}} \in dy) \\ &= 2Ma(n-1)m(n,w)g(w).\end{aligned}$$

Thus, provided $a(n-1)m(n,c(n)) \to 0$ as $n \to \infty$, it follows that $g(c(n)) \to \infty$.

Now, consider $G(z,\cdot)$. Let z belong to the closure of $W(j-1)$, and suppose that the n used above satisfies $n > j$. $G(z,x)$ is bounded on $B(j)$ (say by M'), and converges to zero as x approaches ∂E. Thus, since

$$\begin{aligned}\mu(w,(-1,1)\times\{s(n)\}) &= \mu(w,(-1,1)\times\{s(n-1)\}) \\ &\geq P'^w(T_{B(j)} < T_{\partial E}) \quad \text{for } w \in C(n),\end{aligned}$$

also

$$G(z,w) \le M'\mu(w,(-1,1)\times\{s(n)\}) \quad \text{for } w\in C(n).$$

Combining our two estimates, we see that

$$\begin{aligned}\frac{g(w)}{G(z,w)} &\ge \frac{g(c(1))}{2MM'a(n-1)m(n,w)\mu(w,(-1,1)\times\{s(n)\})}\\ &= \frac{g(c(1))}{2MM'}\cdot\frac{1}{a(n-1)k(n)^2} \quad \text{for } w\in C(n).\end{aligned}$$

Let $i(n)$ be the expression on the right-hand side. Then if $a(n)\to 0$ sufficiently fast, also $i(n)\to\infty$.

Let

$$E(n) = (-1,1)\times((s(n-1)+s(n))/2,(s(n)+s(n+1))/2)\cap E.$$

Then $E(n)$ is regular for the Dirichlet problem, so that if $\nu(x,dy)$ is the hitting distribution of $\partial E(n)$ by Brownian motion started at x, then whenever f is harmonic on $E(n)$, with a continuous extension to its closure, we have that

$$f(x) = \int_{\partial E(n)} f(y)\nu(x,dy).$$

These conditions hold for both g and $G(z,\cdot)$. Both functions vanish on $\partial E(n)\cap\partial E$, so that we have $g(\cdot)\ge (i(n)\wedge i(n+1))G(z,\cdot)$ on $\partial E(n)$. By the integral representation, this inequality holds throughout $E(n)$.

We have therefore shown that conditions (a) and (b) hold, provided that both $m(n,c(n))a(n-1)$ and $k(n)^2a(n-1)$ converge to zero. Since

$$\mu(c(n),(-1,1)\times\{s(n)\})\to\tfrac{1}{2},$$

the first of these is irrelevant, and it just remains to estimate $k(n)$.

Let Φ be the 'symmetric' conformal map of Lemma (4.1), with $b=b(n)$. Write dy for normalized arc length on the boundary of the unit disk. The first hitting distribution of this boundary by Brownian motion started at x is then $((1-|x|^2)/(|x-y|^2))\,dy$, so that since Φ preserves hitting distributions, we have that

$$k(n) = \sup\left\{\frac{1-|x|^2}{|x-e^{it}|^2}\Big/\frac{1-|x|^2}{|x-e^{is}|^2};\ \operatorname{Im}(x)=0, s,t\in[\theta,\pi-\theta]\right\}.$$

By simple calculus, this becomes $k(n)=(1+\cos\theta)/(1-\cos\theta)$. By (a) of Lemma (4.1), $\theta\to 0$ and hence $k(n)\sim(2/\theta)^2$ as $n\to\infty$. Thus, $b(n)\log k(n)\to 2\pi$ by (b) of Lemma (4.1). By hypothesis, there is some $\xi>2\pi$ such that

$$a(n-1) < \exp(-2\xi/b(n)) \quad \text{for large } n.$$

Thus

$$a(n-1)k(n)^2 < \exp(-2[\xi - b(n)\cdot\log k(n)]/b(n)) \to 0,$$

as required. □

The proof of Theorem (3.4) will be given along the following lines: Because we know little about h (other than its 'symmetry'), we will reduce the problem to that of estimating certain quantities not depending on h. (An essential ingredient in this is Theorem (3.2), which estimates the expected lifetime of an h-transform, but in a manner independent of h.) These quantities are local in that they involve

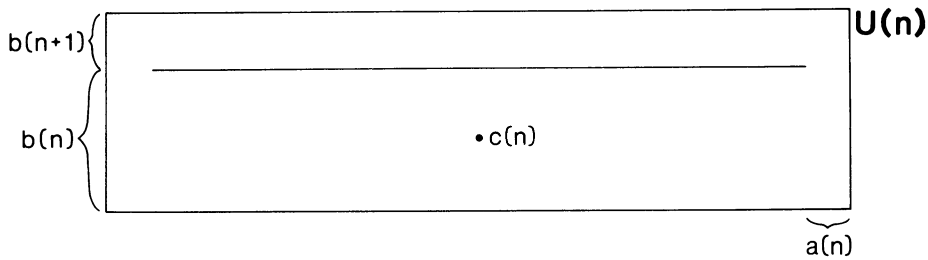

FIGURE 2

potential theoretic properties of certain subsets of E. In each case, we will reduce our problem to one involving an appropriate rectangle; that is, to one of estimating the parameters of its Schwarz–Christoffel transformation. Lemma (4.1) lets us do this.

Before proceeding with the proof of Theorem (3.4), we will fix some notation. Let $S(0) = T(0) = T_A$. For $k \geq 0$, define

$$\begin{aligned} N(k) &= n \qquad \text{if } X_{T(k)} \in A(n), \\ T(k+1) &= T(k) + T_{A\setminus B(N(k))} \circ \theta_{T(k)}, \\ S(k+1) &= T(k) + T_{A\setminus A(N(k))} \circ \theta_{T(k)}. \end{aligned}$$

Let $U(n) = E \cap (-1,1) \times (s(n-1), s(n+1))$, and write P_n^x for the law of Brownian motion started at x and stopped upon leaving $U(n)$. Define the Martin function $K_n(x,y)$ on $U(n)$, relative to the base point $c(n)$, so that K_n is symmetric (that is, $K_n(-\overline{x}, -\overline{y}) = K_n(x,y)$). As usual, ${}_hP_n^x$ and ${}_yP_n^x$ will denote the laws of the transformations by the functions $h|_{U(n)}$, $K_n(y,\cdot)$ respectively, but we will also introduce the notation ${}_y\overline{P}_n^x$ for the transformation by the function $K_n(y,\cdot) + K_n(-\overline{y},\cdot)$. Note that this object would be changed if we had used a base point other than $c(n)$. Set

$$\alpha(x,y) = \begin{cases} {}_y\overline{P}_n^x(\operatorname{Sgn} N(1) = \operatorname{Sgn} N(0), S(1) = T(1)) \\ \qquad\qquad \text{if } x \in B(n) \text{ and } y \in B(n-1) \cup B(n+1) \text{ for some } n, \\ 0, \quad \text{otherwise.} \end{cases}$$

$$\beta(n,m) = \inf\{\alpha(x,y); x \in B(n), y \in B(m)\} \qquad n,m \geq 1,$$

$$\gamma(x,y) = \begin{cases} {}_y\overline{E}_n^x[T(1)] & \text{if } x \in B(n) \text{ and } y \in B(n-1) \cup B(n+1) \text{ for some } n, \\ 0, & \text{otherwise,} \end{cases}$$

$$\eta(n,m) = \inf\{\gamma(x,y); x \in B(n), y \in B(m)\}; \qquad n,m \geq 1.$$

Note that $\beta(n,m) = \eta(n,m) = 0$ unless $|m-n| = 1$. The estimates of these quantities that we will need are contained in the following result:

LEMMA (4.2). *Let $a(n-1) = O(b(n))$. Then there exist constants $C_1 > 0$ and $C_2 > 0$ such that, for every $n \geq 1$,*
(a) $(1 - \beta(n,n+1)) \vee (1 - \beta(n,n-1)) \leq \exp(-C_1/b(n))$,
(b) $\eta(n,n+1) \wedge \eta(n,n-1) \geq C_2 b(n)^2$.

PROOF. By hypothesis, there is some $\varepsilon > 0$ such that $\varepsilon a(n-1) < b(n)$ for $n > 1$. We may further assume, without loss of generality, that $a(n) < 1/2$ for each n, and that $4b(n) < \varepsilon$ for $n > 1$.

(a) We must show that there is a constant $C_1 > 0$ such that

$$ {}_y\overline{P}^x_n(X_{S(1)} \in A-) \le \exp(-C_1/b(n)) $$

for $x \in A(n)$ and $y \in B(n-1) \cup B(n+1)$, $n \ge 1$. We will obtain this in two parts; first exhibiting such a bound for

$$ I_1(x,y) = {}_y\overline{P}^x_n(X_{S(1)} \in A(-n)), $$

and then for

$$ I_2(x,y) = {}_y\overline{P}^x_n(X_{S(1)} \in A(-(n-1)) \cup A(-(n+1))). $$

Let $V(n) = [(-1,1) \times (s(n-1), s(n+1))] \backslash [(-1, 1-a(n)) \times \{s(n)\}]$. Write $\Delta(n)$ for the Martin boundary of $V(n)$, and write $\hat{P}^x_n$ for the law of Brownian motion started at x and killed upon leaving $V(n)$. Let L_n be the Martin function on $V(n)$ with respect to the basepoint $c = c(n)$. Also note that points of $A(-n) \subset \partial V(n)$ split into two points of the Martin boundary $\Delta(n)$ of $V(n)$; one being the limit of points above $A(-n)$, and the other from below. We let $A'(-n)$ be the collection of all points of the Martin boundary of $V(n)$ that are associated to a point of $A(-n)$ in this way.

Let

$$ \mu(\Lambda) = \hat{P}^c_n(X_{\varsigma-} \in \Lambda) \quad \text{for } \Lambda \subset \Delta(n) \text{ measurable.} $$

Fix a point $y \in B(n-1) \cup B(n+1)$, and let $e(z) = K_n(y,z) + K_n(-\overline{y}, z)$. Let ν be the measure representing e on $\Delta(n)$ so that $e = \int_{\Delta(n)} L_n(w,\cdot)\nu(dw)$. By (2.1) and (c) of Theorem (2.2), we have that $\nu(dw)$ is the sum of the measure $e(w)\mu(dw \cap A'(-n))$, and of two point masses, one at y and the other at $-\overline{y}$. Since killing an e-transform upon first leaving $V(n)$ produces an e-transform (now on the set $V(n)$, of course), we have that

$$ \begin{aligned} I_1(x,y) &= {}_eP^x_n(X_{S(1)} \in A(-n)) = {}_e\hat{P}^x_n(X_{\varsigma-} \in A'(-n)) \\ &= \frac{1}{e(x)} \int_{A'(-n)} L_n(w,x)\,{}_w\hat{P}^x_n(X_{\varsigma-} \in A'(-n))\nu(dw) \\ &= \int_{A'(-n)} \frac{L_n(w,x)}{e(x)} e(w)\mu(dw). \end{aligned} $$

Now apply the Harnack inequality for

$$ [(1-2a(n),1) \times (s(n)-\varepsilon a(n), s(n)+\varepsilon a(n))] \backslash [(1-2a(n), 1-a(n))] \times \{s(n)\}] $$

(which lies in $V(n)$ by choice of ε) to obtain a constant C (which, by scale invariance, will not depend on n) such that

$$ \frac{L_n(w,x)}{e(x)} \le C\frac{L_n(w,x')}{e(x')} \quad \text{for } x, x' \in A(n),\ w \in A'(-n). $$

Since $e(x') = e(-\overline{x}')$, we can choose $x' \in A(n)$ to maximize e over $A(n)$, and have $e(w)/e(x') \le 1$ for $w \in A'(-n)$. Thus

$$ I_1(x,y) \le C\int_{A'(-n)} \frac{L_n(w,x')}{e(x')} e(w)\mu(dw) \le C\hat{P}^{x'}_n(X_{\varsigma-} \in A'(-n)). $$

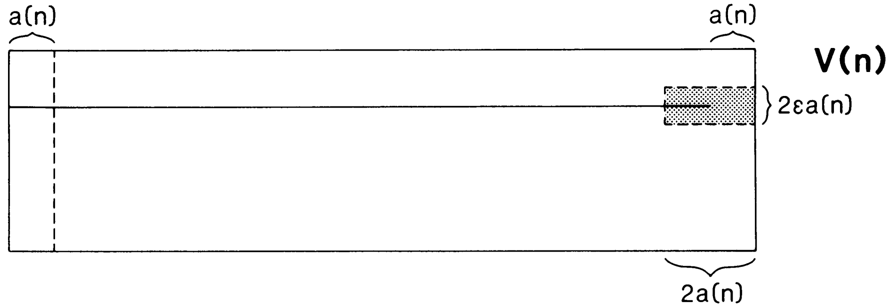

FIGURE 3

This is bounded by C times the probability that Brownian motion started at x' leaves the box $(a(n)-1,1)\times(s(n-1),s(n+1))$ on its left-hand side. Applying (d) of Lemma (4.1) yields a bound on $I_1(x,y)$ of the desired form. (See Figure 3.)

Now consider $I_2(x,y)$. Suppose for now that $y\in B(n-1)$, so that we may assume without loss of generality that $y\in A(-(n-1))$. Recall that $W(n)=(-1,1)\times(s(n-1),s(n))$. Let $\tilde{P}^x_n$ be the law of Brownian motion started at x and killed upon leaving $W(n)$, and define the Martin function $\tilde{L}_n$ on $W(n)$ using the base point $c(n)$. Consider X after its last exit from $B(n)$; $Y_t=X_{t+L_{B(n)}}$. Let $v(x)=e(x)_eP^x(T_{B(n)}=\infty)$. Then by Theorem (5.1) of Meyer, Smythe and Walsh [**14**], (Y_t) is a v-transform on $W(n)$ under ${}_eP^x_n$. Because e is symmetric (that is, $e(z)=e(-\overline{z})$ for $z\in W(n)$), the same will be true of v. By (c) of Theorem (2.2) and our choice of base point, we therefore have that v is a multiple of $\tilde{L}_n(y,\cdot)+\tilde{L}_n(-\overline{y},\cdot)$. Thus

$$I_2(x,y)\le {}_eP^x_n(Y_0\in A(n),Y_{\varsigma-}\in A(-(n-1)))$$
$$\le \sup_{z\in A(n)} {}_v\tilde{P}^z_n(X_{\varsigma-}=y)=\sup_{z\in A(n)}\frac{\tilde{L}_n(y,z)}{\tilde{L}_n(y,z)+\tilde{L}_n(-\overline{y},z)}.$$

Resume the notation of Lemma (4.1) with $b=b(n)$. Letting $a=a(n)$ and $a(n-1)$ gives us two values for ψ; call them ψ_1 and ψ_2, respectively. The 'symmetric' conformal map from the unit disk to $W(n)$ preserves the Martin function (by our choice of basepoints), so that the above inequality becomes

$$I_2(x,y)\le \sup_{\substack{s\in[\psi_2,-\theta]\\ t\in[\theta,\psi_1]}}\frac{|e^{is}-e^{it}|^2}{|e^{is}-e^{it}|^2+|e^{i(\pi-s)}-e^{it}|^2}.$$

Because $\theta\to 0$, this is $\le C(\psi_1-\psi_2)^2\le 2C(\psi_1^2+\psi_2^2)$. By (b) and (c) of Lemma (4.1),

$$b(n-1)\log(1/\psi_j)\sim b(n-1)(\log(1/\psi_j)+\log(\psi_j/\theta))\sim\pi,$$

which yields a bound of the desired form. The case $y\in B(n+1)$ is handled in the same manner, showing (a).

(b) By symmetry,

$$\eta(n,n+1)=\inf\{{}_yE^x_n[T(1)];\ x\in B(n),\ y\in A(n+1)\}.$$

Consider the rectangle $Z(n)=(1-2b(n+1)/\varepsilon,1)\times(s(n),s(n+1))$. (See Figure 4.) Let

$$Y_t=X_{t+L_{U(n)\setminus Z(n)}}.$$

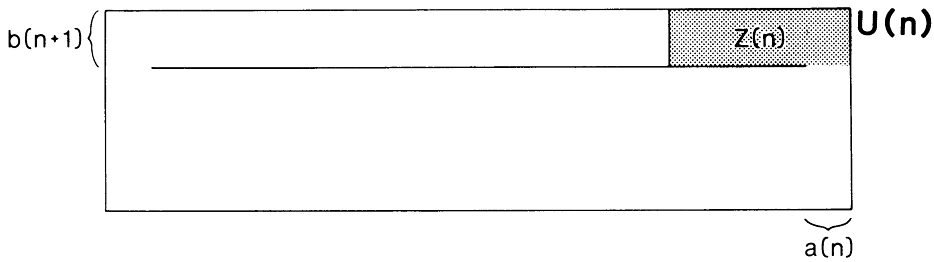

FIGURE 4

Under ${}_yP_n^x$, (Y_t) will be a v-transform for some v on $Z(n)$ (by Theorem 5.1 of Meyer, Smythe and Walsh [**14**]), $Y_{\varsigma-} = y$, and Y_0 lies on the lower or left sides of $Z(n)$. Let $Z = (0, 2/\varepsilon) \times (0,1)$, and define $\Phi\colon Z(n) \to Z$ by

$$\Phi(z) = [z - (1 - 2b(n+1)/\varepsilon, s(n))]/b(n+1).$$

Let $Y_t' = \Phi(Y_{tb(n+1)^2}) \in Z$. Because this scaling preserves Brownian motion, (Y_t') will be a transform by $v \circ \Phi^{-1}$, under ${}_yP_n^x$. Let P_Z^z be the law of Brownian motion started at z and killed upon leaving Z. Thus

$$\begin{aligned}\eta(n, n+1) &\geq \inf\{{}_yE_n^x[\text{lifetime of } (Y_t)]; y \in A(n+1), x \in B(n)\} \\ &\geq b(n+1)^2 \inf\{{}_zE_Z^w[\varsigma]; z \in (1 - a(n)/b(n+1), 1) \times \{1\}, \\ &\qquad\qquad w \in [0, 2/\varepsilon] \times \{0\} \cup \{0\} \times [0,1]\}.\end{aligned}$$

Since $a(n) < b(n+1)/\varepsilon$, this infimum remains bounded away from 0 as $n \to \infty$. The corresponding estimate on $\eta(n, n-1)$ follows similarly, showing (b). $\square$

Now, let I be the function that replaces the x-coordinate by its absolute value. That is,

$$I(z) = \begin{cases} z, & \operatorname{Re}(z) \geq 0, \\ -\bar{z}, & \operatorname{Re}(z) \leq 0. \end{cases}$$

Recall that $\mathcal{F}_t = \sigma(X_s; s \leq t)$. We let

$$\mathcal{G} = \sigma(I(X_{T(k)}); k \geq 0).$$

LEMMA (4.3). (a) *Let $H(x,y)$ be positive and jointly measurable in x and y. Let T and S be $(\mathcal{F}_{t+})$ stopping times, and set $H'(x) = {}_hE^x[H(x, X_T)]$. Then*

$$E_Q[H(X_S, X_{S+T\circ\theta_S}) \mid \mathcal{F}_{S+}] = H'(X_S) \qquad Q\text{-a.s.}$$

(b) *$I(X_t)$ is strong Markov with respect to Q and $(\mathcal{F}_t)$.*

(c) *Let $Z \in \mathcal{F}$ be positive, and set*

$$H(x,y) = {}_y\overline{E}_n^x[Z], \quad \text{for } x \in B(n),\ y \in B(n-1) \cup B(n+1).$$

Then ${}_hE_n^x[Z \mid I(X_{T(1)})] = H(x, I(X_{T(1)}))$ a.s.

(d)

$$\begin{aligned}Q(\operatorname{Sgn} N(n) = \operatorname{Sgn} N(n-1), T(n) = S(n) \mid \mathcal{F}_{T(n-1)+} \vee \mathcal{G}) \\ = \alpha(X_{T(n-1)}, I(X_{T(n)})) \quad \text{a.s., for every } n \geq 1.\end{aligned}$$

PROOF. (a) If $H_n \uparrow H$, then $H_n(x,\cdot) \uparrow H(x,\cdot)$, so that $H_n' \uparrow H'$. The class of functions H for which (a) holds is therefore a monotone class, and it contains functions $H(x,y) = H_1(x)H_2(y)$ by the strong Markov property.

(b) By symmetry of E and of Brownian motion, we have that

$$P^x(I(X.) \in \Lambda) = P^{I(x)}(I(X.) \in \Lambda)$$

for every $x \in E$ and $\Lambda \in \mathcal{F}$. We have chosen h so that $h(x) = h(I(x))$ for every x. Thus, if $f_1, \dots, f_n \in \mathcal{E}$ are positive, T is an $(\mathcal{F}_{t+})$ stopping time, and $t_1 < \cdots < t_n$, then

$$\begin{aligned} E_Q\left[\prod f_i(I(X_{T+t_i})) \mid \mathcal{F}_{T+}\right] &= {}_hE^{X_T}\left[\prod f_i(I(X_{t_i}))\right] \\ &= \frac{1}{h(X_T)} E^{X_T}\left[h(X_{t_n}) \prod f_i(I(X_{t_i}))\right] \qquad \text{(W 1.2)} \\ &= \frac{1}{h(I(X_T))} E^{I(X_T)}\left[h(I(X_{t_n})) \prod f_i(I(X_{t_i}))\right], \end{aligned}$$

so that $I(X_t)$ is strong Markov, with semigroup ${}_hP_t(x, dy) + {}_hP_t(x, -\overline{dy})$.

(c) Let the integral representation of h on $U(n)$ be

$$h = \int K_n(y, \cdot)\mu(dy).$$

By uniqueness of the representation, and the fact that h and K_n are preserved by the transformation $x \to -\overline{x}$ of $U(n)$, it follows that $\mu(dy) = \mu(-\overline{dy})$. Thus, for $\Lambda \subset A(n+1) \cup A(n-1)$ and $Z \in \mathcal{F}$ positive,

$$\begin{aligned} {}_hE_n^x[Z, I(X_{T(1)}) \in \Lambda] &= {}_hE_n^x[Z, X_{T(1)} \in \Lambda \cup -\overline{\Lambda}] \\ &= \frac{1}{h(x)} \int_{\Lambda \cup -\overline{\Lambda}} K_n(y,x)\, {}_yE_n^x[Z]\mu(dy) \qquad \text{(W 1.3)} \\ &= \frac{1}{h(x)} \int_{\Lambda} (K_n(y,x)\, {}_yE_n^x[Z] + K_n(-\overline{y},x)\, {}_{-\overline{y}}E_n^x[Z])\mu(dy) \\ &= \frac{1}{h(x)} \int_{\Lambda} (K_n(y,x) + K_n(-\overline{y},x))\, {}_y\overline{E}_n^x[Z]\mu(dy) \\ &= \frac{1}{h(x)} \int_{\Lambda \cup -\overline{\Lambda}} K_n(y,x)\, {}_y\overline{E}_n^x[Z]\mu(dy) \quad \text{(symmetry of } \mu) \\ &= {}_hE_n^x[H(I(X_{T(1)})), I(X_{T(1)}) \in \Lambda], \end{aligned}$$

as required.

(d) First, observe that if $Z \in \mathcal{F}_{T(n)}$, then

$$E_Q[Z \mid \mathcal{F}_{T(n-1)+} \vee \mathcal{G}] = E_Q[Z \mid \mathcal{F}_{T(n-1)+} \vee \sigma(I(X_{T(n)}))],$$

by part (b). Now let $f \in \mathcal{E}$ be positive, and set $H(x,y) = f(I(y))\alpha(x,y)$. Also put $H'(x) = {}_hE^x[H(x, X_{T(1)})]$, and

$$Z = 1_{\{\mathrm{Sgn} N(1) = \mathrm{Sgn} N(0), S(0) = T(0)\}}.$$

By (c), we have that

$$H'(x) = {}_hE^x[f(I(X_{T(1)}))\alpha(x, I(X_{T(1)}))] = {}_hE^x[f(I(X_{T(1)}))Z].$$

Thus, if also $\Lambda \in \mathcal{F}_{T(n-1)+}$, then

$$\begin{aligned}
&E_Q[\Lambda, f(I(X_{T(n)})), \operatorname{Sgn} N(n) = \operatorname{Sgn} N(n-1), T(n) = S(n)] \\
&\quad = E_Q[\Lambda, f(I(X_{T(n)})), Z \circ \theta_{T(n-1)}] \\
&\quad = E_Q[\Lambda, {}_hE^{X_{T(n-1)}}[f(I(X_{T(1)}))Z]] = E_Q[\Lambda, H'(X_{T(n-1)})] \\
&\quad = E_Q[\Lambda, H(X_{T(n-1)}, X_{T(n)})] \quad \text{(by (a))} \\
&\quad = E_Q[\Lambda, f(I(X_{T(n)}))\alpha(X_{T(n-1)}, I(X_{T(n)}))],
\end{aligned}$$

as required. $\square$

PROOF OF THEOREM (3.4). For $n \geq 1$, and $m = n+1$ or $n-1$, let $N(n,m)$ be the number of $k \geq 0$ such that $|N(k)| = n$ and $|N(k+1)| = m$. Let $H(x,y) = \gamma(x,y)$, and apply (a) of Lemma (4.3). We have that

$$H'(x) = {}_hE^x[\gamma(x, X_{T(1)})] = {}_hE^x[T(1)] \qquad \text{(W 1.3)},$$

so that

$$\begin{aligned}
E_Q[T(k+1) - T(k)] &= E_Q[H'(X_{T(k)})] \\
&= E_Q[\gamma(X_{T(k)}, X_{T(k+1)})] \geq E_Q[\eta(|N(k)|, |N(k+1)|)].
\end{aligned}$$

By Theorem (3.2), we therefore have that

$$\begin{aligned}
\infty > E_Q[\varsigma] &= \sum_{k=0}^{\infty} E_Q[T(k+1) - T(k)] + E_Q[T(0)] \\
&\geq \sum_{n=1}^{\infty} [\eta(n, n+1) E_Q[N(n, n+1)] + \eta(n, n-1) E_Q[N(n, n-1)]].
\end{aligned}$$

Since $\log(1/(1-x)) \leq 2x$ for $x \geq 0$ small, we have by Lemma (4.2) that there exists a constant C with

$$\log(1/\beta(n,m)) \leq C\eta(n,m) \quad \text{for } n \geq 1,\ m = n \pm 1.$$

Thus also

$$\begin{aligned}
\infty > \sum_{n=1}^{\infty} [&\log(1/\beta(n, n+1)) E_Q[N(n, n+1)] \\
&+ \log(1/\beta(n, n-1)) E_Q[N(n, n-1)]] \\
= E_Q &\left[-\log\left(\prod_{n=1}^{\infty} \beta(n, n+1)^{N(n,n+1)} \beta(n, n-1)^{N(n,n-1)} \right) \right].
\end{aligned}$$

The integrand is therefore finite almost surely, so that also

$$\begin{aligned}
&E_Q\left[N(0) = 1, \prod_{k=0}^{\infty} \beta(|N(k)|, |N(k+1)|) \right] \\
&\quad = E_Q\left[N(0) = 1, \prod_{n=1}^{\infty} \beta(n, n+1)^{N(n,n+1)} \beta(n, n-1)^{N(n,n-1)} \right] \\
&\quad > 0.
\end{aligned}$$

But by (d) of Lemma (4.3),

$$
\begin{aligned}
E_Q&\left[N(0) = 1, \prod_{k=0}^{j} \beta(|N(k)|, |N(k+1)|)\right]\\
&\leq E_Q[N(0) = 1, E_Q[\operatorname{Sgn} N(1) = \operatorname{Sgn} N(0), S(1) = T(1),\\
&\qquad E_Q[\cdots E_Q[\operatorname{Sgn} N(j) = \operatorname{Sgn} N(j-1),\\
&\qquad\qquad S(0) = T(0) \mid \mathcal{F}_{T(j-1)+} \vee \mathcal{G}] \cdots] | \mathcal{F}_{T(0)+} \vee \mathcal{G}]]\\
&= Q(N(0) = 1, \operatorname{Sgn} N(k) = \operatorname{Sgn} N(k-1) \text{ and } S(k) = T(k), k = 1, \dots, j)\\
&= Q(T_{A-} > T(j)).
\end{aligned}
$$

Letting $j \to \infty$, we see that $Q(T_{A-} = \infty) > 0$, as required. □

5. Proof of Theorem (3.1). This section is independent of the last one, and we will feel free to recycle letters used as notation there.

Let E be a bounded domain in $\mathbf{R}^d$, and let U be a large open ball containing the closure of E. We retain the notation $(\hat{\Omega}, \mathcal{F})$ for paths with values in $E \cup \{\delta\}$, and write $(V, \mathcal{V})$ for the corresponding object, where now paths take values in U. The coordinate process on $\hat{\Omega}$ will still be (X_t). We let the coordinate process on V be (W_t). Write $G(x, y)$ for the Green function of U, and write Q^x and P^x for the laws of Brownian motion, started at x and killed upon leaving U in the first case, and E in the second. Let μ and ν be probabilities on U such that a set is μ-null if and only if it is ν-null. Assume that neither one charges some neighbourhood of the closure of E. Write

$$h = \int G(z, \cdot)\nu(dz),$$

and $P = {}_hQ^\mu$. Let $\tilde{P}$ be the law under P of the reverse of the coordinate process (X_t) from its lifetime. Then by (W 1.3) and a remark following (2.2),

$$
\begin{aligned}
P(\Lambda) &= \iint \frac{G(x, y)}{h(x)}\, {}_yQ^x(\Lambda)\mu(dx)\nu(dy),\\
\tilde{P}(\Lambda) &= \iint \frac{G(x, y)}{h(y)}\, {}_yQ^x(\Lambda)\nu(dx)\mu(dy)
\end{aligned}
$$

for $\Lambda \in \mathcal{V}$. Thus, since μ and ν share the same null sets, the same will be true for P and $\tilde{P}$.

Let $M = \{t > 0; W_t \notin E \cup \{\delta\}\}$, and let M_0 be the set of points of M which are isolated on the right. Let

$$Y_t(s) = \begin{cases} W_{t+s}, & s < T_{U\setminus E} \circ \theta_t,\\ \delta, & s \geq T_{U\setminus E} \circ \theta_t, \end{cases}$$

and let

$$\rho(A) = E\left[\sum_{t \in M_0} 1_A(Y_t)\right], \quad \text{for } A \in (\hat{\Omega}, \mathcal{F}).$$

Since every excessive function for Brownian motion is regular (see Blumenthal and Getoor [**2**]), the hypotheses of Maisonneuve [**12**, Proposition (9.2)] are met. Thus, there exists a kernel $n(x, du)$ and an additive functional (L_t) such that

$$\rho(A) = E\left[\int_0^\zeta n(W_t, A)\, dL_t\right]$$

for every measurable subset A of $\hat{\Omega}$, and (X_t) is an h-transform under each $n(x,\cdot)$. Thus (X_t) is also an h-transform under ρ. It follows from (c) of Theorem (2.2) and a time reversal argument that

$$Z = \lim_{t\uparrow\varsigma} X_t \quad \text{and} \quad Z' = \lim_{t\downarrow 0} X_t$$

exists in the topology of the Martin compactification of E, ρ-a.s.. Because ν does not charge E, the limit actually lies in Acc $\subset \Delta$. Let η be the measure on $\Delta \times \Delta$ defined by

$$\eta(dx, dy) = \rho(Z \in dx, Z' \in dy).$$

THEOREM (5.1). $\eta(D'\backslash D) = 0$.

Thus, with probability one, every excursion into E starts and finishes at points z, z' of the Martin boundary for which $(z, z') \in D$.

LEMMA (5.2). *η is σ-finite.*

PROOF. By minimality, we have that for every $y \in \Delta_0$ and $x \in E$, either there exists a point $e(y) \in \partial E$ such that

$$_yP^x(X_t \to e(y) \text{ in the Euclidean metric, as } t \uparrow \varsigma) = 1, \quad \text{or}$$
$$_yP^x(X_t \text{ converges in the Euclidean metric, as } t \uparrow \varsigma) = 0.$$

In the latter case, we write $e(y) = \delta$.

Since the Euclidean limits of X_t, as $t \uparrow \varsigma$ and $t \downarrow 0$, both exist ρ-a.s., it follows that $\rho(e(Z) = \delta$ or $e(Z') = \delta) = 0$. Since Brownian motion does not return to points, we have by the strong Markov property that also $\rho(e(Z) = e(Z')) = 0$. Thus, it will suffice to show that

$$\rho(|e(Z) - e(Z')| > 2\varepsilon) < \infty \quad \text{for every } \varepsilon > 0.$$

Let $T(0) = 0$, $T(k+1) = \inf\{t > T(k);\ |X_t - X_{T(k)}| > \varepsilon\}$. Because h is bounded and bounded away from zero on a neighbourhood of the closure of E, we obtain from (W 1.3) a constant $m > 0$ such that $_hE^x[T(1)] \geq m$ whenever x lies in E. Thus

$$\infty > E[\varsigma] \geq mE\left[\sum_{k=0}^{\infty} 1_E(X_{T(k)})\right]$$
$$\geq mE\left[\sum_{t\in M_0} 1_{\{|X_t - X_{T_{\partial E}\circ\theta_t}| > 2\varepsilon\}}\right] = m\rho(|e(Z) - e(Z')| > 2\varepsilon),$$

as required. □

PROOF OF THEOREM (5.1). Since η is σ-finite, we may use the 'classical argument' of the last section of Maisonneuve [**12**] to obtain a kernel $\rho(x, y; du)$ such that

$$\rho = \int \rho(x, y; \cdot)\eta(dx, dy).$$

Now, let K be the Martin function on E, and suppose that $A \in \mathcal{F}$. It is easily checked (by (W 1.3)) that $_hE^x_P[A \mid Z] = a(x, Z)$, where $a(x, y) = {_yE^x_P}[A]$. Suppose in addition that f_0 and f_∞ are positive and measurable on Δ, and that $B \in \mathcal{F}_s$. Then

$$E_\rho[f_0(Z')f_\infty(Z), A \circ \theta_s, B] = E_\rho[f_0(Z')_h E_Q^{X_s}[A, f_\infty(Z)], B]$$
$$= E_\rho[f_0(Z')f_\infty(Z)a(X_s, Z'), B],$$

from which we see that (X_t) is a $K(y, \cdot)$-transform under $\rho(x, y; \cdot)$ for η-a.e. (x, y). Thus $(x, y) \in D$ for η-a.e. (x, y), as required. □

Now fix $y_0 \in E$, and recall that harmonic measure λ on Δ is defined by $\lambda(dz) = P^{y_0}(Z \in dz)$. By (W 1.5) and (2.1), we have that for every $y \in E$ the measure ${}_hP^y(Z \in dz)$ shares the same null sets as λ. The more potential theoretic Theorem (3.1) now follows immediately from

LEMMA (5.3). *$\lambda \otimes \lambda$ is absolutely continuous with respect to η.*

PROOF. Let A be a compact subset of E with nonempty interior, and set

$$\eta_k(dx, dy) = \rho(Z' \in dx, Z \in dy, T_A < \infty).$$

We will actually show that $\lambda \otimes \lambda$ is absolutely continuous with respect to η_k.

We can write

$$\eta_k(\Lambda) = \int_{\{T_A<\infty\}} \int_\Delta 1_\Lambda(Z', y)_h P^{X_{T_A}}(Z \in dy)\, d\rho.$$

If $\rho(Z' \in B) = 0$, then $\tilde{P}(Z \in B) = P(Z' \in B) = 0$, so that since P is absolutely continuous with respect to $\tilde{P}$, also $\eta(\Delta \times B) = P(Z \in B) = 0$ (this is the reason for our choice of h). Thus $\eta_k(\Delta \times B) = 0$, and hence ${}_hP^{X_{T_A}}(Z \in B) = 0$ ρ-a.s. Because A has nonempty interior, we have $\rho(T_A < \infty) > 0$, so that by our above remark on the null sets of λ, also $\lambda(B) = 0$. That is, $\lambda(dz)$ is absolutely continuous with respect to $\rho(Z' \in dz)$.

If now $\eta_k(\Lambda) = 0$, then

$$\int 1_\Lambda(Z', y)_h P^{X_{T_A}}(Z \in dy) = 0 \quad \rho\text{-a.s.}$$

Thus

$$\int 1_\Lambda(Z', y)\lambda(dy) = 0 \quad \rho\text{-a.s.},$$

and so also

$$\int 1_\Lambda(x, y)\lambda(dy) = 0 \quad \text{for } \lambda\text{-a.e. } x.$$

This gives immediately that $\lambda \otimes \lambda(\Lambda) = 0$. □

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