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ENTIRE FUNCTIONS WHICH ARE INFINITELY INTEGER VALUED AT A FINITE NUMBER OF POINTS

BY

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ABSTRACT. This paper determines arithmetic limits for the growth rates of entire functions which are infinitely integer valued on a finite set S. The characterization of such functions with growth rate less than the arithmetic limit is complete if there exist exponential polynomials which are infinitely integer valued on S.

1. Introduction. Let K be the rational field $\mathbf{Q}$ or an imaginary quadratic number field. An analytic function $f(z)$ is *infinitely integer valued* at a point z_0 if for all $n \geq 0$, $f^{(n)}(z_0)$ is an integer of K. In this note we continue the study of the family $\mathcal{F}$ of entire functions which are infinitely integer valued at a fixed set of points $z_1, z_2, \ldots, z_s$ of the complex plane.

Let $M(R, f)$ be the maximum modulus of $f(z)$ on $|z| = R$. Let $\rho, \sigma_0, \sigma_1, \ldots, \sigma_k$ be real numbers with $\rho > 0$, $\sigma_0 > 0$. We say that $f(z)$ *has growth rate* $(\rho; \sigma_0, \sigma_1, \ldots, \sigma_k)$ if for every $\varepsilon > 0$,

$$\text{(i)} \qquad \log M(R, f) < R^\rho \left(\sigma_0 + \frac{\sigma_1}{\log R} + \cdots + \frac{\sigma_k + \varepsilon}{(\log R)^k}\right)$$

for all sufficiently large R, and

$$\text{(ii)} \qquad \log M(R, f) > R^\rho \left(\sigma_0 + \frac{\sigma_1}{\log R} + \cdots + \frac{\sigma_k - \varepsilon}{(\log R)^k}\right)$$

for arbitrarily large R.

In this definition we allow the possibility that $k = -1$; then $f(z)$ has growth rate (ρ) if and only if $f(z)$ has order ρ.

For each fixed k, we order the growth rates $(\rho; \sigma_0, \ldots, \sigma_k)$ lexicographically; we do not compare growth rates with different values of k.

We call $(\rho; \sigma_0, \ldots, \sigma_k)$ an *arithmetic limit for* $\mathcal{F}$ if

(i) $\mathcal{F}$ contains only countably many functions with growth rate less than $(\rho; \sigma_0, \ldots, \sigma_k)$, and

(ii) $\mathcal{F}$ contains uncountably many functions with growth rate greater than $(\rho; \sigma_0, \ldots, \sigma_k)$.

The work of E. G. Straus, D. Sato and A. M. Cayford [**2,3,7,9**] leads to the following.

THEOREM 1.1. *For any $\{z_1, \ldots, z_s\}$ and any $\rho \geq s$, there exist $2^{\aleph_0}$ entire functions of order ρ which are infinitely integer valued at $z_1, \ldots, z_s$. If there exists an entire function $f(z)$ of order $\rho < s$ which is infinitely integer valued at $z_1, \ldots, z_s$,*

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then $f(z)$ satisfies a linear differential equation with integral coefficients. In fact, $\rho = 0$ or 1 and $f(z)$ is either a polynomial or an exponential polynomial, i.e. a Laurent polynomial in an exponential $e^{\lambda z}$.

Hence the arithmetic limit for the orders of functions in $\mathcal{F}$ is equal to s; in particular, it depends only on the number of points and not on their location.

The existence of a nontrivial function in $\mathcal{F}$ of order $< s$ places a severe restriction on the differences $z_2 - z_1, \dots, z_s - z_1$; In fact, the number of such $(s-1)$-tuples is denumerable and readily characterized.

In [**7**] the arithmetic limit $(\rho; \sigma_0)$ for $\mathcal{F}$ is also determined. The result is as follows.

THEOREM 1.2. *Let*

$$\sigma_0 = \prod_{1 \le i < j \le s} |z_i - z_j|^{-2/s},$$

where $\sigma_0 = 1$ if $s = 1$. Then there are $2^{\aleph_0}$ functions in $\mathcal{F}$ with growth rate $> (s; \sigma_0)$, but only countably many functions in $\mathcal{F}$ with growth rate $< (s; \sigma_0)$.

The characterization of functions in $\mathcal{F}$ with growth rate $< (s; \sigma_0)$ is made more difficult by the fact that type limitations are not preserved under multiplication.

In §2 and §3 we determine the arithmetic limits $(\rho; \sigma_0, \dots, \sigma_k)$ for $\mathcal{F}$.

In §4 and §5 we discuss the case where $\mathcal{F}$ contains nonconstant functions of order $< s$.

2. Generalized Taylor series and growth rates of entire functions. Given a sequence $\varsigma_1, \varsigma_2, \dots$, with $|\varsigma_n| \le r$ for all n, we can, as in [**2**], expand any entire function in the generalized Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - \varsigma_1)(z - \varsigma_2) \cdots (z - \varsigma_n), \tag{2.1}$$

$$a_n = \frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)}{(z - \varsigma_1) \cdots (z - \varsigma_{n+1})}\, dz, \qquad R > r.$$

We define

$$M(R, f) = \max_{|z|=R} |f(z)|,$$

$$\mu(R, f) = \max_{n} \max_{|z|=R} |a_n||z - \varsigma_1||z - \varsigma_2| \cdots |z - \varsigma_n|.$$

If $f(z)$ has finite order, then

$$\log \mu(R, f) \le \log M(R, f) \le \log \mu(R, f) + O(\log R) \tag{2.2}$$

(see [**4**, 1.11]). This enables us to use $\mu(R, f)$ and $M(R, f)$ interchangeably in what follows.

LEMMA 2.1. *Suppose $\phi(x)$ is a twice differentiable function such that $x\phi'(x)$ increases to ∞ for large x. Suppose further that $\lim_{x\to\infty} (\log \phi(x))/(\log x) = \rho$. Define $p(t)$ by $p(t)\phi'(p(t)) = t$ for all large t. Then if $f(z)$ is entire of order ρ and*

$$\log M(R, f) \le \phi(R), \tag{2.3}$$

there exists a constant $\delta_1 < 1$ *so that*

$$\log|a_n| \le \phi(p(n)) - n\log p(n) + O(n^{\delta_1}). \tag{2.4}$$

PROOF. L'Hospital's Rule implies $\log p(t)/\log t \to 1/\rho$. From (2.1) we have

$$|a_n| \le \frac{1}{2\pi}\left\|\int_{|z|=R}\frac{f(z)}{(z-\varsigma_1)\cdots(z-\varsigma_{n+1})}\,dz\right\| \le RM(R,f)(R-r)^{-(n+1)}.$$

Hence

$$\log|a_n| \le \phi(R) - n\log R + O(n/R). \tag{2.5}$$

Setting $R = p(n)$, we get (2.4).

LEMMA 2.2. *Suppose* $\psi(x)$ *is a twice differentiable function such that* $\psi'(x)$ *decreases to* $-\infty$ *for large* x. *Suppose also that* $\lim_{x\to\infty}(\psi(x))/(x\log x) = -1/\rho$. *Let* $q(t)$ *satisfy* $\psi'(q(t)) = -\log t$. *Then if* $f(z)$ *is entire of order* ρ *and*

$$\log|a_n| \le \psi(n), \tag{2.6}$$

there exists a constant $\delta_2 < \rho$ *so that*

$$\log M(R,f) \le \psi(q(R)) + q(R)\log R + O(R^{\delta_2}). \tag{2.7}$$

PROOF. The conditions on ψ imply that $\log q(t)/\log t \to \rho$. We have

$$\log\mu(R,f) \le \log\max_n|a_n|(R+r)^n \le \max_n\left(\psi(n) + n\log R + \frac{nr}{R}\right).$$

A straightforward computation now yields

$$\log M(R,f) \le \psi(q(R)) + q(R)\log R + O\left(\frac{q(R)\log R}{R}\right)$$

which implies (2.7).

We now combine Lemmas 2.1 and 2.2.

LEMMA 2.3. (i) *Let* ϕ *satisfy the hypotheses of Lemma* 2.1. *Define* $\psi(t) = \phi(p(t)) - t\log p(t)$. *Then there exists a constant* $\delta_2 < \rho$ *so that*

$$\log M(R,f) \le \phi(R) + O(R^{\delta_2}) \tag{2.8}$$

if and only if there exists a $\delta_1 < 1$ *so that*

$$\log|a_n| \le \psi(n) + O(n^{\delta_1}). \tag{2.9}$$

(ii) *On the other hand, let* ψ *satisfy the hypotheses of Lemma* 2.2. *Define* $\phi(t) = \psi(q(t)) + q(t)\log t$. *Then there exists a* $\delta_1 < 1$ *so that*

$$\log|a_n| \le \psi(n) + O(n^{\delta_1}) \tag{2.10}$$

if and only if there exists a $\delta_2 < \rho$ *so that*

$$\log M(R,f) \le \phi(R) + O(R^{\delta_2}). \tag{2.11}$$

PROOF. To prove (i), observe that ψ satisfies the conditions in Lemma 2.2 and

$$\psi'(t) = (\phi'(p(t)) - t/p(t))p'(t) - \log p(t) = -\log p(t). \tag{2.12}$$

Thus $p(q(t)) = t$, and $\psi(q(t)) + q(t)\log t = \phi(t)$. Therefore, by Lemmas 2.1 and 2.2, we are done. The proof of (ii) is analogous.

We say that the coefficients a_n in (2.1) have growth rate $[1/\rho;\tau_0,\dots,\tau_k]$ if for every $\varepsilon>0$, we have

$$\log|a_n| < -\frac{1}{\rho}n\log n + n\left(\tau_0+\frac{\tau_1}{\log n}+\cdots+\frac{\tau_k+\varepsilon}{(\log n)^k}\right) \tag{2.13}$$

for all large n, and

$$\log|a_n| > -\frac{1}{\rho}n\log n + n\left(\tau_0+\frac{\tau_1}{\log n}+\cdots+\frac{\tau_k-\varepsilon}{(\log n)^k}\right) \tag{2.14}$$

for arbitrarily large n.

We order these growth rates lexicographically, and define arithmetic limits in the obvious way.

THEOREM 2.4. *The entire function $f(z)$ given by* (2.1) *has growth rate* $(\rho;\sigma_0,\sigma_1,\dots,\sigma_k)$ *if and only if the coefficients a_n have growth rate* $[1/\rho;\tau_0,\tau_1,\dots,\tau_k]$, *where the relation between $\sigma_0,\sigma_1,\dots,\sigma_k$ and $\tau_0,\tau_1,\dots,\tau_k$ is determined by the relation between the functions $\phi(R)$ and $\psi(t)$ in Lemma* 2.3.

PROOF. We apply the first half of Lemma 2.3 to the function

$$\phi(R)=R^\rho(\sigma_0+\sigma_1/\log R+\cdots+\sigma_k/(\log R)^k).$$

Let $R_t=p(t)$. Using this function in Lemma 2.1, we easily see that $R_t^\rho/(\log R_t)^{k+1} = O(t/(\log t)^{k+1})$. Hence,

$$R_t^\rho(\lambda_0+\lambda_1/\log R_t+\cdots+\lambda_k/(\log R_t)^k) = t+O(t/(\log t)^{k+1}), \tag{2.15}$$

where $\lambda_i=\rho\sigma_i-(i-1)\sigma_{i-1}$; $i=0,1,\dots,k$ $(\sigma_{-1}=0)$. Inverting (2.15) we get constants $\mu_0,\dots,\mu_k$ such that

$$R_t^\rho = t(\mu_0+\mu_1/\log t+\cdots+\mu_k/(\log t)^k)+O(t/(\log t)^{k+1}). \tag{2.16}$$

Taking logarithms in (2.16) we get constants $\nu_0,\dots,\nu_k$ such that

$$\log R_t = \frac{1}{\rho}\log t+\nu_0+\nu_1/\log t+\cdots+\nu_k/(\log t)^k+O(1/(\log t)^{k+1}). \tag{2.17}$$

Using Lemma 2.3, we get the desired form for $\psi(t)$. The converse computation of $\phi(R)$ from $\psi(t)$ is entirely analogous.

The actual computation of $\tau_0,\dots,\tau_k$ in terms of $\rho,\sigma_0,\dots,\sigma_k$ is rather cumbersome. We will carry it one step beyond the known fact

$$\tau_0=\frac{1}{\rho}\log(e\rho\sigma_0). \tag{2.18}$$

Setting $\phi(R)=R^\rho(\sigma_0+\sigma_1/\log R)$ we get

$$t=R_t\phi'(R_t)=\rho\phi(R_t)+O(R_t^\rho/(\log R_t)^2).$$

Hence

$$\log t=\rho\log R_t+\log(\sigma_0\rho)+\frac{\sigma_1/\sigma_0}{\log R_t}+O\left(\frac{1}{(\log R_t)^2}\right),$$

$$\log R_t=\frac{1}{\rho}\log t-\frac{1}{\rho}\log(\rho\sigma_0)-\frac{\sigma_1/\sigma_0}{\log t}+O\left(\frac{1}{(\log t)^2}\right),$$

and

$$\psi(t) = \phi(R_t) - t\log R_t$$
$$= -\frac{1}{\rho} t\log t + t\left(\frac{1}{\rho}\log(\sigma_0\rho e) + \frac{\sigma_1/\sigma_0}{\log t} + O\left(\frac{1}{(\log t)^2}\right)\right).$$

Thus

$$\tau_1 = \sigma_1/\sigma_0. \tag{2.19}$$

3. The arithmetic limits. We expand the function $f \in \mathcal{F}$ in a generalized Taylor series at the points $\varsigma_1, \varsigma_2, \dots$, where all $\varsigma_n \in \{z_1, z_2, \dots, z_s\}$. Thus

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_1)^{n_1} \cdots (z - z_s)^{n_s},$$

where $n_i = n_i(n)$ is a nondecreasing function of n and $n_1 + n_2 + \cdots + n_s = n$. We further choose the exponents n_i so that the quantities

$$\Psi_i(n) = n_i! \prod_{\substack{j=1 \\ j\neq i}}^{s} (z_i - z_j)^{n_j}, \qquad i = 1, 2, \dots, s, \tag{3.1}$$

are nearly equal in absolute value.

We define $\psi_i(t)$ by

$$-\psi_i(t) = \log|\Psi_i(t)| = \log\Gamma(t_i + 1) + \sum_{j\neq i} t_j \log|z_i - z_j|,$$

for $i = 1, 2, \dots, s$, where $t = t_1 + \cdots + t_s$. For a given t, we want to choose $t_1, \dots, t_s$ in such a way that $\psi_i(t)$ does not depend on i, that is

$$\psi_1(t) = \cdots = \psi_s(t), \qquad t_1 + t_2 + \cdots + t_s = t.$$

Let us call $\psi(t)$ the common value of $\psi_i(t)$ so obtained; thus

$$-\psi(t) = \log\Gamma(t_i + 1) + \sum_{j\neq i} t_j \log|z_i - z_j|, \qquad i = 1, 2, \dots, s. \tag{3.2}$$

The solution of (3.2) leads to asymptotic expansions of the form

$$t_i = \frac{1}{s} t\left(1 + \frac{\gamma_{i1}}{\log t} + \frac{\gamma_{i2}}{(\log t)^2} + \cdots + \frac{\gamma_{ik}}{(\log t)^k}\right) + O\left(\frac{t}{(\log t)^{k+1}}\right), \tag{3.3}$$

where $\sum_{i=1}^{s} \gamma_{ij} = 0$, $j = 1, 2, \dots, k$, and

$$\psi(t) = -\frac{1}{s} t\log t + \frac{1}{s} t\left(\gamma_0 + \frac{\gamma_1}{\log t} + \cdots + \frac{\gamma_k}{(\log t)^k}\right) + O\left(\frac{t}{(\log t)^{k+1}}\right). \tag{3.4}$$

Let $P(z) = (z - z_1)\cdots(z - z_s)$. If we compare terms of order t, then (3.2) yields

$$\begin{aligned} \gamma_0 &= 1 + \log s - \gamma_{i1} - \sum_{j\neq i} \log|z_i - z_j| \\ &= 1 + \log s - \gamma_{i1} - \log|P'(z_i)|. \end{aligned} \tag{3.5}$$

Summing over i, we get

$$\gamma_0 = 1 + \log s - 2\log \prod_{1\le i<j\le s} |z_i - z_j| = 1 + \log s - 2\log V, \tag{3.6}$$

where V is the absolute value of the Vandermonde of $z_1, \dots, z_s$. Thus (3.5) yields

$$\gamma_{i1} = \log |V^2/P'(z_i)|. \tag{3.7}$$

We now compare terms of order $t/\log t$ to get

$$\begin{aligned}\gamma_1 = &- \gamma_{i2} + \gamma_{i1}\log s - \sum_{j\neq i} \gamma_{j1} \log |z_i - z_j| \\ = &- \gamma_{i2} + 2\log V \log s - \log |P'(z_i)| \log s - 2\log V \log |P'(z_i)| \\ &+ \sum_{j\neq i} \log |z_i - z_j| \log |P'(z_i)|.\end{aligned} \tag{3.8}$$

Summing over i, we obtain

$$\begin{aligned}\gamma_1 = &\left(2 - \frac{2}{s}\right) \log V \log s - \frac{4}{s} (\log V)^2 \\ &+ \frac{1}{s} \sum_{i=1}^{s} \sum_{j\neq i} \log |z_i - z_j| \log |P'(z_i)|.\end{aligned} \tag{3.9}$$

Hence

$$\begin{aligned}\gamma_{i2} = &\frac{2}{s} \log V \log s + \frac{4}{s}(\log V)^2 - \frac{1}{s} \sum_{i=1}^{s} \sum_{j\neq i} \log |z_i - z_j| \log |P'(z_i)| \\ &- \log |P'(z_i)| \log(sV^2) + \sum_{j\neq i} \log |z_i - z_j| \log |P'(z_i)|.\end{aligned} \tag{3.10}$$

In an analogous manner we can compute the constants γ_{ij} for $i = 1, 2, \dots, s$, $j = 3, \dots, k$.

We now choose the integers n_i to satisfy

$$n_i(n) = t_i(n) + O(1). \tag{3.11}$$

To do this, we let ς_n be the z_i of lowest index for which $t_i(n) - n_i(n-1)$ is maximal.

THEOREM 3.1. *The arithmetic limit for the generalized Taylor coefficients of functions in* $\mathcal{F}$ *is* $[1/s; \gamma_0/s, \dots, \gamma_k/s]$, *and the corresponding arithmetic limit for the growth rates of these functions is* $(s; \sigma_0, \sigma_1, \dots, \sigma_k)$, *where the relation between the* σ_i *and the* $\tau_i = \gamma_i/s$ *is as in Theorem* 2.4.

PROOF. Let $f(z) \in \mathcal{F}$ have generalized Taylor expansion

$$f(z) = \sum_{k=0}^{\infty} a_n (z - z_1)^{n_1} \cdots (z - z_s)^{n_s}, \tag{3.12}$$

where the n_i satisfy (3.11). Assume that $\varsigma_{n+1} = z_i$, that is $n_i(n+1) = n_i(n) + 1$. Then (3.12) yields

$$f^{(n_i)}(z_i) = a_n \Psi_i(n) + \mathcal{R}_n, \tag{3.13}$$

where the remainder $\mathcal{R}_n$ depends only on the coefficients $a_0, \dots, a_{n-1}$.

If the growth rate of the a_n is less than $[1/s; \gamma_0/s, \dots, \gamma_k/s]$, then we must have

$$\log |f^{(n_i)}(z_i) - \mathcal{R}_n| = \log |a_n| - \psi_i(n) \to -\infty \quad \text{as } n \to \infty. \tag{3.14}$$

Since any two distinct integers of K have difference at least 1 in absolute value, there is at most one integral value $f^{(n_i)}(z_i)$ which makes the left side of (3.14) less than $\log \frac{1}{2}$ for a given $\mathcal{R}_n$. Thus when n is sufficiently large, the coefficients $a_0, a_1, \ldots, a_{n-1}$ determine $f^{(n_i)}(z_i)$, and hence a_n, uniquely. This shows that there can only be a countable number of such $f(z)$.

On the other hand, there is a constant c such that every circle of radius c contains at least two integers of K. Thus if we permit $|f^{(n_i)}(z_i) - \mathcal{R}_n| \geq c$ for all n, we get at least two possible choices for $f^{(n_i)}(z_i)$. Hence we get an uncountable set of functions in $\mathcal{F}$.

REMARK. If $s = 1$, the remainder $\mathcal{R}_n$ in (3.13) is 0 for all n. Hence if the growth of $f(z)$ is below the arithmetic limit $(1, \sigma_0, \sigma_1, \ldots, \sigma_k)$, then $f(z)$ is a polynomial.

4. Functions in $\mathcal{F}$ with growth rate below the arithmetic limit. It is known [**3**] that a function $f(z) \in \mathcal{F}$ of order $\rho < s$ satisfies a linear differential equation with K-integral coefficients. For more general growth rates, the situation is as follows.

THEOREM 4.1. *Let $F(z) \in \mathcal{F}$ have growth rate less than the arithmetic limit $(s; \sigma_0, \ldots, \sigma_k)$. If there exists a nonconstant function $f(z) \in \mathcal{F}$ of order $< s$, then $F(z)$ satisfies a linear differential equation with coefficients in $K[f]$, the ring of polynomials in f with coefficients in K.*

REMARK. If $s = 1$, $F(z)$ is a polynomial by the remark following Theorem 3.1. Hence it satisfies the equation $F^{(n)} \equiv 0$ for some n. Thus we may assume from now on that $s \geq 2$.

For the proof of Theorem 4.1 we need an estimate for the derivatives of entire functions.

LEMMA 4.2. *For any entire function f and any $r > 0$, we have*

$$M(R, f^{(n)}) \leq r^{-n} M(R + nr, f). \tag{4.1}$$

PROOF. By Cauchy's formula we have

$$f'(z) = \frac{1}{2\pi i} \int_{|\varsigma - z| = r} \frac{f(\varsigma) d\varsigma}{(\varsigma - z)^2},$$

and hence

$$M(R, f') \leq r^{-1} M(R + r, f). \tag{4.2}$$

Iterating (4.2) n times, we get (4.1).

PROOF OF THEOREM 4.1. Suppose $F(z) \in \mathcal{F}$ and

$$M(R, F) < R^s \left(\sigma_0 + \frac{\sigma_1}{\log R} + \cdots + \frac{\sigma_k - \delta}{(\log R)^k} \right) \tag{4.3}$$

for some $\delta > 0$ and R sufficiently large. We construct a function

$$\Phi(z) = \sum_{u=0}^{U} \sum_{v=0}^{V} \lambda_{uv} f(z)^u F^{(v)}(z), \tag{4.4}$$

where the λ_{uv} are rational integers, not all zero.

Let m be a positive integer to be chosen sufficiently large later, and set

$$U = [m^{1-1/s}/(\log m)^{k+2}], \quad V = [m^{1/s}(\log m)^{2k+4}].$$

We expand Φ in the generalized Taylor series given by (3.11), (3.12), and choose the λ_{uv} so that

$$\Phi^{(\nu_i)}(z_i) = 0, \qquad \nu_i = 0, 1, \dots, n_i(m);\ i = 1, 2, \dots, s. \tag{4.5}$$

This is a system of $(n_1(m) + 1) + (n_2(m) + 1) + \cdots + (n_s(m) + 1) = m + s$ linear equations with K-integral coefficients for the $(U + 1)(V + 1)$ unknowns λ_{uv}. If $K \neq \mathbf{Q}$ we take rational and irrational parts, getting $2m + 2s$ equations with coefficients in $\frac{1}{2}\mathbf{Z}$. These coefficients are rational or irrational parts of

$$\begin{aligned} C_{i,\nu_i,u,v} &= \frac{d^{\nu_i}}{dz^{\nu_i}}(f(z)^u F^{(v)}(z))|_{z=z_i} \\ &= \sum_{k=0}^{\nu_i} \binom{\nu_i}{k} \frac{d^k}{dz^k}(f(z)^u)|_{z=z_i} F^{(v+\nu_i-k)}(z_i). \end{aligned}$$

By Lemma 4.2, with $R = \max_{1\le i\le s}|z_i|$ and $r = 1/m$, we have

$$\begin{aligned} |C_{i,\nu_i,u,v}| \le{}& 2^{n_i(m)} m^{n_i(m)} M(R + n_i(m)/m, f^U) \\ &\times m^{V+n_i(m)} M(R + (V + n_i(m))/m, F). \end{aligned} \tag{4.6}$$

From (3.10) and (3.3) we see that $n_i(m) < m(1/s + \varepsilon)$ for large m. Hence for some fixed $C > 1$,

$$m^{n_i(m)} \le m^{V+n_i(m)} \le C^{m\log m},$$

$$M(R + n_i(m)/m, f^U) < C^m,$$

$$M(R + (V + n_i(m)/m), F) < C.$$

Thus (4.6) yields

$$|C_{i,\nu_i,u,v}| < C^{m\log m}. \tag{4.7}$$

The number of unknowns is $(U + 1)(V + 1) > m(\log m)^{k+2} > 2m + 2s$. Hence by Siegel's lemma (see [**1**]), we can find rational integers λ_{uv}, not all zero, satisfying (4.5) and

$$|\lambda_{uv}| \le (C_1^{m\log m})^{(2m+2s)/(m(\log m)^{k+2}-(2m+2s))} < C_2^{m/(\log m)^{k+1}}. \tag{4.8}$$

We now show that if m is sufficiently large, then $\Phi(z)$ vanishes identically. From (4.5) we have $a_0 = a_1 = \cdots = a_m = 0$.

Let i be the index with $n_i(m + 2) = n_i(m + 1) + 1$. It suffices to show that $\Phi^{n_i(m+1)}(z_i) = 0$ if m is sufficiently large. For this implies that $a_{m+1} = 0$, and the result follows by induction.

We have

$$\begin{aligned} \Phi^{n_i(m+1)}(z_i) &= a_{m+1}\Psi_i(m + 1) \\ &= \Psi_i(m + 1)\frac{1}{2\pi i}\int_{|z|=R} \frac{\Phi(z)\,dz}{\prod_{j=1}^s (z - z_j)^{n_j(m+1)}(z - z_i)}. \end{aligned}$$

Thus

$$\log|\Phi^{n_i(m+1)}(z_i)| \leq -\psi_i(m+1) + \log M(R,\phi) - (m+1)\log R + (Cm/R). \tag{4.9}$$

We need only show that for sufficiently large M the right side of (4.9) is negative. Now

$$\begin{aligned} \log M(R,\Phi) &\leq \log(U+1) + \log(V+1) + \log\max|\lambda_{uv}| \\ &\quad + U\log M(R,f) + \max_{v\leq V}\log M(R,F^{(v)}) \\ &\leq Cm/(\log m)^{k+1} + CRm^{1-1/s}/(\log m)^{k+2} \\ &\quad - V\log r + \log M(R+Vr,F). \end{aligned} \tag{4.10}$$

Setting $r = 1/V$ and $L = \log R$, we obtain

$$\begin{aligned} \log M(R,\Phi) &\leq Cm(\log m)^{k+1} + CRm^{1-1/s}(\log m)^{k+2} \\ &\quad + R^s\left(\sigma_0 + \frac{\sigma_1}{L} + \cdots + \frac{\sigma_k - \delta}{L^k}\right) + CR^{s-1}. \end{aligned} \tag{4.11}$$

Substituting (4.11) in (4.9), we get

$$\begin{aligned} \log|\Phi^{n_i(m+1)}(z_i)| \leq &\left\{\frac{1}{s}m\log m - \frac{1}{s}m\left(\gamma_0 + \frac{\gamma_1}{\log m} + \cdots + \frac{\gamma_k}{(\log m)^k}\right)\right. \\ &\quad \left. + R^s\left(\sigma_0 + \frac{\sigma_1}{L} + \cdots + \frac{\sigma_k}{L^k}\right) - m\log R\right\} \\ &+ Cm/R - \log R + CR^{s-1} + Cm/(\log m)^{k+1} \\ &+ CRm^{1-1/s}/(\log m)^{k+2} - \frac{\delta R^s}{(\log R)^k} + O\left(\frac{m}{(\log m)^{k+1}}\right). \end{aligned} \tag{4.12}$$

We now choose $R = p(m)$ as in Lemma 2.1. By Theorem 2.4, the quantity in braces is $O(m/(\log m)^{k+1})$. Since $R = O(m^{1/s})$, the term $-\delta R^s/(\log R)^k$ dominates, and the limit is $-\infty$. This completes the proof.

5. Solutions of differential equations with periodic coefficients. We continue to assume in this section that there is a nonconstant function $f \in \mathcal{F}$ of order $< s$. It was shown in [**3**] and [**11**] that f is either a polynomial or an exponential polynomial. In the latter case f is clearly periodic.

By Theorem 4.1, every function $F \in \mathcal{F}$ with order $s > 1$ and growth rate below the arithmetic limit $(s, \sigma_0, \sigma_1, \ldots, \sigma_k)$ satisfies a linear differential equation with coefficients in $K[f]$.

We now discuss the case where $f(z)$ is periodic, say with period ω. The function $F(z)$ cannot be periodic, since it would then be infinitely integer valued at more than s points—in fact, at infinitely many points $z_i + n\omega$. This would imply that $F(z)$ is of order ≤ 1, contrary to assumption.

LEMMA 5.1. *Let $F(z)$ be entire and satisfy the differential equation*

$$A_0(z)F + A_1(z)F' + \cdots + A_n(z)F^{(n)} = 0, \tag{5.1}$$

where $A_i(z+\omega) = A_i(z)$, $0 \leq i \leq n$. Then $F(z)$ satisfies a linear difference equation

$$C_0F(z) + C_1F(z+\omega) + \cdots + C_pF(z+p\omega) = 0 \tag{5.2}$$

with constant coefficients $C_0, C_1, \ldots, C_p$; $C_p \neq 0$.

PROOF. The functions $F(z+k\omega)$, $0 \le k \le n$, are all solutions of (5.1); hence they are linearly dependent.

Let the zeros of the characteristic polynomial $C_0 + C_1 t + \cdots + C_p t^p$ of (5.2) be $e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_k}$, with multiplicities $M_1 + 1, \ldots, M_k + 1$. Then for all integers q, we have

$$F(z+q\omega) = \sum_{i=1}^{k} (a_{i0}(z) + a_{i1}(z)q + \cdots + a_{iM_i}(z)q^{M_i})e^{\lambda_i q} \tag{5.3}$$

(see, for example, [**5**, p. 384]). Here the $a_{ij}(z)$ are entire functions which satisfy

$$\sum_{j=0}^{M_i} a_{ij}(z+\omega)q^j = e^{\lambda_i} \sum_{j=0}^{M_i} a_{ij}(z)(q+1)^j. \tag{5.4}$$

If we set $b_{ij}(z) = a_{ij}(z)e^{-\lambda_i z/\omega}$, then (5.4) becomes

$$\sum_{j=0}^{M_i} b_{ij}(z+\omega)q^j = \sum_{j=0}^{M_i} b_{ij}(z)(q+1)^j. \tag{5.5}$$

In particular, this shows that $b_{i0}(z) = \sum_{j=0}^{M_i} c_{ij}(z)z^j$, where $c_{ij}(z+\omega) = c_{ij}(z)$. From (5.3) with $q = 0$ we get

$$F(z) = \sum_{i=1}^{k} \sum_{j=0}^{M_i} c_{ij}(z)z^i e^{\lambda_i z/\omega}. \tag{5.6}$$

LEMMA 5.2. *Let e^{λ_i}, $i = 1,2,3,\ldots$, be distinct. Then the functions $z^j e^{\lambda_i z/\omega}$, $i = 1,2,3,\ldots$; $j = 0,1,2,\ldots$, are linearly independent over the set of functions of period ω.*

PROOF. Assume not. Then there exist functions $F_{ij}(z)$ of period ω, not all zero, with

$$\sum_{i=1}^{n} \sum_{j=0}^{m} F_{ij}(z)z^j e^{\lambda_i z/\omega} = 0. \tag{5.7}$$

We may assume that m is minimal and that the number of nonzero $F_{im}(z)$ is minimal. Collecting the nonzero terms of (5.7) with $j = m$, we obtain

$$z^m(F_{n_1 m}(z)e^{\lambda_{n_1} z/\omega} + \cdots + F_{n_k m}(z)e^{\lambda_{n_k} z/\omega}),$$

where the $F_{n_i m}(z)$ are nonzero. Substitution of $z+\omega$ into (5.7) gives an equation with corresponding term

$$z^m(F_{n_1 m}(z)e^{\lambda_{n_1} z/\omega}e^{\lambda_{n_1}} + \cdots + F_{n_k m}(z)e^{\lambda_{n_k} z/\omega}e^{\lambda_{n_k}}).$$

Since the $e^{\lambda_{n_i}}$ are distinct and k is minimal, this implies that $k = 1$. Thus we may assume that the highest terms of (5.7) are

$$z^{m-1}(F_{p_1 m-1}(z)e^{\lambda_{p_1} z/\omega} + \cdots + F_{p_r m-1}(z)e^{\lambda_{p_r} z/\omega}) + z^m F(z)e^{\lambda z/\omega},$$

where the $F_{p_k m-1}(z)$ and $F(z)$ are nonzero, and $\lambda = \lambda_i$ for some i.

Substituting $z+\omega$ into (5.7), we get an equation with highest terms

$$z^{m-1}(F_{p_1 m-1}(z)e^{\lambda_{p_1} z/\omega}e^{\lambda_{p_1}} + \cdots + F_{p_r m-1}(z)e^{\lambda_{p_r} z/\omega}e^{\lambda_{p_r}} + m\omega F(z)e^{\lambda z/\omega}e^{\lambda}) + z^m F(z)e^{\lambda z/\omega}e^{\lambda}.$$

Since m is minimal we must have $m\omega F(z)e^{\lambda z/\omega}e^{\lambda} = 0$, which implies $m=0$. Thus (5.7) reduces to $F(z)e^{\lambda z/\omega} = 0$, which is impossible.

LEMMA 5.3. *Let $F(z)$ be analytic in $0<|z|<\infty$ with Laurent expansion*

$$F(z) = \sum_{n=-\infty}^{\infty} a_n z^n. \tag{5.8}$$

Suppose that both $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} a_{-n} z^n$ have order 0. If $F(z)$ satisfies a linear differential equation with rational functions as coefficients, then it is a Laurent polynomial.

PROOF. By clearing denominators, we may assure that the coefficients of the differential equation are polynomials. We wish to show that $a_n = 0$ for all large $|n|$ in (5.8). Since the conditions are symmetric in z and $1/z$, it suffices to prove the result for large positive n. The hypothesis implies that

$$\log|a_n|/n\log n \to -\infty,$$

where we take $\log 0$ to mean $-\infty$. Thus either $a_n = 0$ for all sufficiently large n, or for every $M>0$ there is an infinite sequence of indices $n_1, n_2, \ldots$ with

$$|a_{n_i+p}| \le |a_{n_i}|/n_i^{pM}, \qquad p = 0,1,2,\ldots. \tag{5.9}$$

Now consider the differential equation satisfied by $F(z)$:

$$(b_{00} + b_{01}z + \cdots + b_{0k_0}z^{k_0})F + \cdots + (b_{l0} + b_{l1}z + \cdots + b_{lk_l}z^{k_l})F^{(l)} = 0. \tag{5.10}$$

We substitute the Laurent series (5.8) into (5.10) and set the coefficient of z^n equal to 0. This gives a system of linear equations E_n for the a_i. If $k = \mathrm{Max}_j(k_j - j)$ and $n = k+\nu$, then $a_0, a_1, \ldots, a_{\nu-1}$ do not appear in equation E_n. Consider E_{k+n_i}, where n_i satisfies (5.9). The coefficient of a_{n_i} in E_{k+n_i} is

$$B_{n_i} = \sum_j{}' b_{jk_j} n_i!/(n_i - j)!, \tag{5.11}$$

where the summation is extended over all j for which $k_j - j = k$. It is clear that B_{n_i} is nonzero for all but a finite number of indices i. Moreover there exists a positive B such that $|B_{n_i}| > B$ for all large i. On the other hand, the coefficients of all terms a_{n_i+p} in E_{k+n_i} are bounded by $C_{n_i}^l$. We choose $M > l$ in (5.9). Then for all large n_i, the contribution of the terms a_{n_i+p} $(p>0)$ is too small to cancel the term $B_{n_i}a_{n_i}$. This contradiction shows that $f(z)$ and $g(z)$ are polynomials, so $F(z)$ is a Laurent polynomial.

We now apply these lemmas to the case where the $A_i(z)$ in (5.1) are exponential polynomials, i.e. Laurent polynomials in a single exponential. Substituting (5.6) into (5.1) and using Lemma 5.2, we get a system of equations

$$L_{ij}c_{ij}(z) + L_{ij+1}c_{ij+1}(z) + \cdots + L_{iM_i}c_{iM_i}(z) = 0, \qquad i = 1,2,\ldots,k;\ j = 0,1,\ldots,M_i, \tag{5.12}$$

where the L_{ij} are linear differential operators with exponential polynomial coefficients. Since the $C_{ij}(z)$ are entire of period ω, they can be expressed as Laurent series in $w = e^{2\pi iz/\omega}$ which satisfy the hypotheses of Lemma 5.3. Thus (5.12) with $j = M_i$ implies that the C_{iM_i} satisfy linear differential equations with coefficients which are Laurent polynomials in w. Lemma 5.3 implies that the C_{iM_i} are themselves exponential polynomials. Substituting these into (5.12) and using Lemma 5.3 again, we find that the C_{iM_i-1} are exponential polynomials. Continuing in this way, we find that all the C_{ij} are exponential polynomials. Hence, by (5.6) the order of $F(z)$ is ≤ 1. Thus we have the following

THEOREM 5.4. *If $\mathcal{F}$ contains nonconstant periodic functions of order $< s$, then all functions in $\mathcal{F}$ with growth rate below the arithmetic limit $(s; \sigma_0, \ldots, \sigma_k)$ have order ≤ 1.*

REFERENCES

1. A. Baker, *Transcendental number theory*, Cambridge Univ. Press, London, 1975.
2. A. H. Cayford, *A class of integer valued entire functions*, Trans. Amer. Math. Soc. **141** (1969), 415–432.
3. A. Cayford and E. G. Straus, *On differential rings of entire functions*, Trans. Amer. Math. Soc. **209** (1975), 283–293.
4. B. Ja. Levin, *Distribution of zeros of entire functions*, Transl. Math. Monos., vol. 5, Amer. Math. Soc., Providence, R. I., 1964.
5. L. M. Milne-Thomson, *The calculus of finite differences*, Macmillan, New York, 1933.
6. L. D. Neidleman and E. G. Straus, *Functions whose derivatives at a point form a finite set*, Trans. Amer. Math. Soc. **140** (1969), 411–414.
7. D. Sato, *Integer valued entire functions*, Thesis, UCLA, 1961; see also Sugaku **14** (1962/63), 95–98 and 99–100.
8. D. Sato and E. G. Straus, *On the rate of growth of Hurwitz functions of a complex or p-adic variable*, J. Math. Soc. Japan **171** (1965), 17–29.
9. E. G. Straus, *On entire functions with algebraic derivatives at certain algebraic points*, Amer. J. Math. **52** (1950),188–198.
10. ——, *On polynomials whose derivatives have integer values at the integers*, Proc. Amer. Math. Soc. **2** (1951), 24–27.
11. ——, *Differential rings of meromorphic functions*, Acta Arith. **21** (1972), 271–284.
12. ——, *Differential rings of meromorphic functions of a non-Archimedean variable, Diophantine approximation and its applications* (Proc. Conf. Washington, D. C., 1972), Academic Press, New York, 1973, pp. 295–308.

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