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ON THE DICHOTOMY PROBLEM FOR TENSOR ALGEBRAS

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ABSTRACT. Let I, J be discrete spaces and $E \subset I \times J$. Then either E is a V-Sidon set (in the sense of [2, §11]), or the restriction algebra $A(E)$ is analytic. The proof is based on probabilistic methods, involving Slépian's lemma.

1. Introduction and definitions. A subset E of $I \times J$ is called a V-Sidon set provided the restriction of $l^\infty(I) \hat{\otimes} l^\infty(J)$ coincides with $l^\infty(E)$. It is known then that E is obtained as the finite union of "sections" $F \subset I \times J$, meaning that either $\pi_1 |_F$ or $\pi_2 |_F$ is one-to-one (π_1, π_2 respective coordinate projections). Our purpose is to show that the algebra $A(E)$, obtained by restricting $l^\infty(I) \hat{\otimes} l^\infty(J)$ to E, is either $l^\infty(E)$ or analytic. Recall that an algebra is analytic provided that only analytic functions operate on it (see [2] for more details). In view of Malliavin's characterization of analytic algebras, it amounts to showing the following (see [2, p. 102]).

THEOREM. *If $E \subset I \times J$ is not a V-Sidon set, then for some $c > 0$*

$$\sup_{\substack{\|\phi\|_{A(E)} \leq 1 \\ \phi \text{ real}}} \left\| e^{it\phi} \right\|_{A(E)} > e^{ct}, \qquad t > 0.$$

In fact, c will be an absolute constant.

2. A condition for analyticity. In this section, a criterion is explained which permits us to minorize $\|e^{it\phi}\|_{A(E)}$. Let f_z stand for the translate of f by z.

LEMMA 1. *Let G be a compact Abelian group and E be a subset of the dual group Γ of G. Denote by C_E the space of continuous functions with Fourier transform supported by E. Fix a positive integer and assume the existence of a function f in C_E and a sequence of points $x_1, \ldots, x_l$ in G satisfying*

$$f(0) = \|f\|_\infty = 1, \tag{1}$$

$$\sum_{S \subset \{1, \ldots, l\}} \left| f_{\Sigma_{k \in S} x_k} \right| \leq B \text{ pointwise on } G. \tag{2}$$

($\Sigma_S x_k$ *refers to the group operation in G.) Then (c = numerical)*

$$\sup_{\substack{\|\phi\|_{A(E)} \leq 1 \\ \phi \text{ real}}} \left\| e^{it\phi} \right\|_{A(E)} \geq e^{ct} \quad \text{if } B < t < l. \tag{3}$$

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PROOF. Define $\sigma(t)$ to be the left member of (3). From the simple estimation, valid in any Banach algebra A,

$$\Big\|\prod_k (1+u_k)\Big\|_A \leqslant e^{\Sigma\|u_k\|^2} \sup_S \|e^{\Sigma_S u_k}\|, \qquad \|u_k\| < 1/2;$$

applied to the elements

$$u_k = \frac{it}{4l}\varepsilon_k\left[(1-i\varepsilon_k')\hat{\delta}_{x_k}|_E + (1+i\varepsilon_k')\hat{\delta}_{-x_k}|_E\right] \qquad (i^2=-1),$$

it follows that

$$\Big\|\prod_{1\leqslant k\leqslant l}\Big[1+\frac{it}{4l}\varepsilon_k\left[(1-i\varepsilon_k')\hat{\delta}_{x_k}|_E + (1+i\varepsilon_k')\hat{\delta}_{-x_k}|_E\right]\Big]\Big\|_{A(E)} \tag{4}$$
$$\leqslant 2e^{t^2/l}\sigma(t).$$

Here $\varepsilon \in \{1,-1\}^l$, $\varepsilon' \in \{1,-1\}^l$ will be used in an averaging argument. Let $\{d_S \mid S \subset \{1,\dots,l\}\}$ be elements of the unit disc. From the $C_E - A(E)$ norm duality and (2), the following minoration for the left member of (4) is valid (w_S refers to the usual Walsh system):

$$\frac{1}{B}\int \left|\left\langle \prod_{1\leqslant k\leqslant l}\Big[1+\frac{it}{4l}\varepsilon_k(\cdots)\Big], \sum_S d_S w_S(\varepsilon) f_{\Sigma_S x_k}\right\rangle\right| d\varepsilon$$
$$\geqslant \frac{1}{B}\left|\sum_{S\subset\{1,\dots,l\}} d_S\left(\frac{it}{4l}\right)^{|S|}\left\langle \mathop{*}_{k\in S}\left[(1-i\varepsilon_k')\delta_{x_k} + (1+i\varepsilon_k')\delta_{-x_k}\right], f_{\Sigma_S x_k}\right\rangle\right|.$$

For an appropriate choice of the $d_S = d_S(\varepsilon')$, the identity

$$\int\left\{\mathop{*}_{k\in S}\left[(1-i\varepsilon_k')\delta_{x_k} + (1+i\varepsilon_k')\delta_{-x_k}\right]\right\}\left(\prod_{k\in S}\frac{1+i\varepsilon_k'}{2}\right) d\varepsilon' = \delta_{\Sigma_S x_k}$$

and integration in ε' lead to the minoration

$$\sum_{S\subset\{1,\dots,l\}} B^{-1}\left(\frac{t}{2\sqrt{2}\,l}\right)^{|S|}\left|\left\langle f_{\Sigma_S x_k}, \delta_{\Sigma_S x_k}\right\rangle\right| = \left(1+\frac{t}{2\sqrt{2}\,l}\right)^l\frac{1}{B}$$

as a consequence of (1). Hence $\sigma(t) \geqslant (1/B)e^{-t^2/l}\cdot e^{c_1 t}$, and the result easily follows. □

REMARK. To satisfy (1), (2) is possible only if C_E contains l_k^∞-subspaces of arbitrary large dimension k (in the Banach space sense). Hence, a natural question is the "cotype-dichotomy" problem (explained in [**4**]). This conjecture was recently solved in the affirmative (see [**1**]), and implies that if E is not a Sidon set, then

$$\sup_{\substack{\|\phi\|_{A(E)}\leqslant 1\\ \phi \text{ real}}} \left\|e^{it\phi}\right\| > ct, \quad \forall t > 0.$$

3. Verification of the condition in the tensor algebra case. It remains to prove that if $E \subset I \times J$ is not a V-Sidon set, then (1), (2) of Lemma 1 can be realized. In this case, let G be a Cantor-group $\{1,-1\}^{\mathbf{N}} \times \{1,-1\}^{\mathbf{N}}$ and identify I (resp. J) with the

Rademacher sequence $\alpha_i(x)$ (resp. $\beta_j(y)$) on the first (resp. second) factor ($i, j = 1, l, \dots$). The following well-known (and easy) combinatorial lemma is applied to E (see [**2**, 11.8.1]).

LEMMA 2. *If $E \subset I \times J$ is not a V-Sidon set, then for arbitrary K there are finite subsets $I_1 \subset I$ and $J_1 \subset J$ (say $|I_1| \geqslant |J_1|$), and for each $i \in I_1$ a subset $A_i \subset J_1$, $|A_i| = K$, satisfying $\bigcup_{i \in I_1}(\{i\} \times A_i) \subset E$.*

With those notations, let

$$f = \sum_{i \in I_1} \sum_{j \in A_1} \alpha_i \otimes \beta_j.$$

Thus $f(0) = K|I_1| = \|f\|_\infty$. The realization of (1), (2) above will be clear from

LEMMA 3. *Let $2^l < K^{1/4}$. Then, as for an absolute constant C,*

$$\int_G \left\| \sum_{S \subset \{1, \dots, l\}} |f_{\Sigma_S z_k}| \right\|_\infty dz_1 \cdots dz_l \leqslant CK|I_1|. \tag{5}$$

(*G^l is the l-fold product $G \times \cdots \times G$ with normalized measure.*)

PROOF. Write $z \in G \equiv \{1, -1\}^{\mathbf{N}} \times \{1, -1\}^{\mathbf{N}}$ as $z = (u, v)$. Thus

$$f_z = \sum_i \sum_{j \in A_i} \alpha_i(u) \beta_j(v) \alpha_i \otimes \beta_j.$$

For fixed $(x, y) \in G$, there are 1-bounded scalars $\{c_S | S \subset \{1, \dots, l\}\}$ satisfying

$$\sum_S |f_{\Sigma_S z_k}(x, y)| = \left| \sum_{i \in I_1} \alpha_i(x) \sum_{S \subset \{1, \dots, l\}} c_S \alpha_i\Big(\sum_S u_k\Big) \sum_{j \in A_i} \beta_j\Big(\sum_S v_k\Big) \beta_j(y) \right|$$

$$\leqslant |I_1|^{1/2} \left\{ \sum_{i \in I_1} \left| \sum_S c_S \alpha_i\Big(\sum_S u_k\Big) \left\{ \sum_{j \in A_i} \beta_j\Big(\sum_S v_k\Big) \beta_j(y) \right\} \right|^2 \right\}^{1/2}.$$

The second factor may be estimated by expanding the inner square as

(6)

$$\left\{ \sum_{i \in I_1} \sum_S \left| \sum_{j \in A_i} \beta_j\Big(\sum_S v_k\Big) \cdot \beta_j(y) \right|^2 \right\}^{1/2}$$

$$+ \left\{ \sum_{S \neq S'} \left| \sum_{i \in I_1} \alpha_i\Big(\sum_{S \triangle S'} u_k\Big) \left\{ \sum_{j \in A_i} \beta_j\Big(\sum_S v_k\Big) \beta_j(y) \right\} \left\{ \sum_{j \in A_i} \beta_j\Big(\sum_{S'} v_k\Big) \beta_j(y) \right\} \right| \right\}^{1/2}$$

It remains to take the supremum over y. The first and second terms will be treated separately.

First term in (6). Fix $i \in I_1$. Linearize the square function by considering scalars $\{a_S \mid S \subset \{1, \dots, l\}\}, \Sigma |a_S|^2 = 1$ so that

$$\left(\sum_S \left| \sum_{j \in A_i} \beta_j\Big(\sum_{k \in S} v_k\Big)\beta_j(y)\right|^2\right)^{1/2} = \left|\sum_{j \in A_i} \beta_j(y) \sum_S a_S \beta_j\Big(\sum_{k \in S} v_k\Big)\right|$$
$$\leqslant |A_i|^{1/2}\left\{\sum_{j \in A_i}\left|\sum_S a_S \beta_j\Big(\sum_{k \in S} v_k\Big)\right|^2\right\}^{1/2}$$

by the Cauchy-Schwarz inequality.

For fixed $j \in A_i$, expand the square. Reversing the order of summation yields the estimation

$$(7)\quad \begin{aligned} &|A_i|^{1/2}\left\{|A_i| + \sum_{S \neq S'} |a_S||a_{S'}|\left|\sum_{j \in A_i} \beta_j\Big(\sum_{k \in S \Delta S'} v_k\Big)\right|\right\}^{1/2} \\ &\leqslant |A_i|^{1/2}\left\{|A_i| + \left(\sum_{S \neq S'}\left|\sum_{j \in A_i} \beta_j\Big(\sum_{k \in S \Delta S'} v_k\Big)\right|^2\right)^{1/2}\right\}^{1/2}. \end{aligned}$$

This estimation is uniform on G and depends only on $z_1, \dots, z_l$. Since for $S \neq S'$, $(v_1, \dots, v_l) \mapsto \Sigma_{k \in S \Delta S'} v_k$ gives the Haar measure on $\{1, -1\}^{\mathbf{N}}$ as the image measure, the integration w.r.t. $z_1, \dots, z_l$ appearing in (5) yields the estimation for (7)

$$|A_i|^{1/2}\left(|A_i| + 2^l\left\|\sum_{j \in A_i} \beta_j\right\|_2\right)^{1/2} \leqslant K^{1/2}(K + 2^l K^{1/2})^{1/2} < 2K,$$

by hypothesis on l and $|A_i| = K$.

Second term in (6). Let $\{g_i(\omega) \mid i \in I_1\}$ denote a sequence of independent Gaussian variables on some probability space Ω, and let $X_i(y) = \Sigma_{j \in A_i} \beta_j(y)$ be defined on $\{1, -1\}^{\mathbf{N}}$. The expressions

$$\int \sup_y \left|\sum_{i \in I_1} \alpha_i\Big(\sum_{S \Delta S'} u_k\Big) X_i\Big(y \sum_S v_k\Big) X_i\Big(y \sum_{S'} v_k\Big)\right| du_1 \cdots du_l$$

are dominated by

$$(8)\qquad \int \sup_{y, y' \in \{1, -1\}^{\mathbf{N}}} \left|\sum_{i \in I_1} g_i(\omega) X_i(y) X_i(y')\right| d\omega.$$

It follows from the inequality

$$\left(\sum_{i \in I_1} |X_i(y)X_i(y') - X_i(y_1)X_i(y_1')|^2\right)^{1/2}$$
$$\leqslant K\left\{\sum_{I_1} |X_i(y) - X_i(y_1)|^2 + \sum_{I_1} |X_i(y') - X_i(y_1')|^2\right\}^{1/2}$$

and Slépian's comparison lemma for Gaussian processes [3] that (8) may be estimated by

$$CK\int \sup_{y\in\{1,-1\}^{\mathbf{N}}}\Big|\sum_{I_1} g_i(\omega)X_i(y)\Big|\,d\omega. \tag{9}$$

Since

$$\Big|\sum_{I_1} g_i(\omega)X_i(y)\Big| \leqslant \sum_{j\in J_1}\Big|\sum_{i\,|\,j\in A_i} g_i(\omega)\Big|,$$

(9) is less than

$$CK\Big\{\sum_{j\in J_1}|\{i\,|\,j\in A_i\}|^{1/2}\Big\} \leqslant CK|J_1|^{1/2}\Big(\sum_{i\in I_1}|A_i|\Big)^{1/2} \leqslant CK^{3/2}|I_1|.$$

Therefore, (6) contributes to $C2^lK^{3/4}|I_1|^{1/2}$. Collecting estimations, one concludes

$$\int_{G^l}\Big\|\sum_S|f_{\sum_{k\in S}z_k}|\Big\|_\infty dz_1\cdots dz_l \leqslant C|i_1|^{1/2}\big(K|I_1|^{1/2}+2^lK^{3/4}|I_1|^{1/2}\big) < CK|i_1|$$

since l was chosen small enough. Hence (5) is proved.

4. Further remarks. (1) The result stated in the abstract can be generalized as follows: Let k be a positive integer, $I_1,\ldots,I_k$ discrete spaces and $E\subset(I_1\times\cdots\times I_k)$. Then either E is a V-Sidon set, or the restriction algebra

$$[l^\infty(I_1)\hat{\otimes}\cdots\hat{\otimes}l^\infty(I_k)]/E^\perp$$

is analytic. The argument presented above can indeed be adapted to the case of several factors. This adaptation, however, requires some additional work. (Notice that the role of the factors I and J in the previous computation is different.)

(2) Let F be a finite subset of the dual Γ of a compact abelian group G. According to [5], call the arithmetical diameter $d(F)$ of F the smallest number d for which there exists a subset P of the unit ball of $PM(F)$, $|P|=d$, such that

$$\|f\|_\infty \leqslant 2\sup_{\mu\in P}|\langle f,\mu\rangle| \quad \text{if } f\in C_F.$$

The method presented in this note permits us to show that for $E\subset\Gamma$ the restriction algebra $A(E)$ is analytic as soon as

$$\overline{\lim_k}\sup_{\substack{F\subset E\\|F|=k}}\frac{(\log|F|)^2}{\log d(F)}=\infty,$$

improving on the sufficient condition obtained in [5]. Details will appear elsewhere.

(3) The verification "at random" of the dichotomy conjecture for Sidon sets [6] is possible by using the criterion presented in §2 of this note.

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