THE DISTRIBUTION OF SOLUTIONS TO EQUATIONS OVER FINITE FIELDS

BY

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Abstract. Let $\mathbb{F}_q$ be the finite field with $q = p^f$ elements, $F(\mathbf{x})$ be a $k$-tuple of polynomials in $\mathbb{F}_q[x_1, \ldots, x_n]$, $V$ be the set of points in $\mathbb{F}_q^n$ satisfying $F(\mathbf{x}) = 0$ and $S, T$ be any subsets of $\mathbb{F}_q^n$. Set $\phi(V, 0) = |V| - q^{-k}$,

$$\phi(V, y) = \sum_{\mathbf{x} \in V} e\left(\frac{2\pi i}{p} \text{Tr}(\mathbf{x} \cdot y)\right) \quad \text{for } y \neq 0,$$

and $\Phi(V) = \max_{y} \phi(V, y)$. We use finite Fourier series to show that $(S + T) \cap V$ is nonempty if $|S| |T| > \Phi^2(V)q^k$. In case $q = p$ we deduce from this, for example, that if $C$ is a convex subset of $\mathbb{R}^n$ symmetric about a point in $\mathbb{Z}^n$, of diameter $< 2p$ (with respect to the sup norm), and $\text{Vol}(C) > 2^{2n}\Phi(V)p^k$, then $C$ contains a solution of $F(\mathbf{x}) \equiv 0 \pmod{p}$.

We also show that if $B$ is a box of points in $\mathbb{F}_q^n$ not contained in any $(n - 1)$-dimensional subspace and $|B| > 4 \cdot 2^{n/2}\Phi(V)q^k$, then $B \cap V$ contains $n$ linearly independent points.

1. Introduction. Let $\mathbb{F}_q$ be the finite field with $q = p^f$ elements where $p$ is a prime. Let $F(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_k(\mathbf{x}))$ be a $k$-tuple of polynomials in $\mathbb{F}_q[x_1, \ldots, x_n]$ and $V = V(F)$ be the algebraic subset of $\mathbb{F}_q^n$ defined by the equations

$$f_1(\mathbf{x}) = \cdots = f_k(\mathbf{x}) = 0.$$  

Considerable attention has been given to the problem of finding solutions of (1.1) in which the variables are restricted to a box of points of the type

$$B = \left\{ \mathbf{x} \in \mathbb{F}_q^n : x_i = \sum_{j=1}^f x_{ij}\xi_j, a_{ij} < x_{ij} < a_{ij} + m_{ij}, \quad 1 \leq i \leq n, 1 \leq j \leq f \right\},$$

where $\xi_1, \ldots, \xi_f$ is a basis for $\mathbb{F}_q$ over $\mathbb{F}_p$ and $a_{ij}, m_{ij}$ are integers such that $1 \leq m_{ij} \leq p$ for $1 \leq i \leq n, 1 \leq j \leq f$. (Here we have identified $\mathbb{F}_p$ with the set of integers $\{0, 1, \ldots, p - 1\}$.) See for example Mordell [Mo1, Mo2], Chalk [Ch1, Ch2], Chalk and Williams [CW], Tietäväinen [Ti], R. Smith [Sm], Spackman [Sp] and Myerson [My].
In this work we extend the method of Tietäväinen [Ti] by viewing it in a new way, in terms of the convolution of finite Fourier series. In so doing we obtain solutions of (1.1) in sets of the form \( S + T = \{ s + t: \ s \in S, t \in T \} \) where \( S \) and \( T \) are subsets of \( \mathbb{F}_q^n \); see Theorem 1.1. We also obtain linearly independent solutions of (1.1) in boxes of sufficiently large cardinality; see Theorem 1.4.

The key ingredient in the investigations mentioned above is a uniform upper bound on the function

\[
\phi(V, y) = \begin{cases} 
\sum_{x \in V} e(x \cdot y), & \text{for } y \neq \emptyset, \\
|V|^{-q^{n-k}}, & \text{for } y = \emptyset,
\end{cases}
\]

where \( e(\alpha) = e^{(2\pi i/p)\operatorname{Tr}(\alpha)} \) for any \( \alpha \in \mathbb{F}_q \), \( x \cdot y = \sum_{i=1}^n x_i y_i \), \( \operatorname{Tr} \alpha \) is the trace of \( \alpha \) from \( \mathbb{F}_q \) to \( \mathbb{F}_p \) and \( |V| \) denotes the cardinality of \( V \). Set \( \Phi(V) = \max_{y \in \mathbb{F}_p} |\phi(V, y)| \).

From Deligne's work on the Riemann Hypothesis, a good bound for \( \Phi(V) \) is available if \( V \) is suitably nonsingular. To be precise we shall say that a polynomial \( f(x) \) over \( \mathbb{F}_q \) is nonsingular at infinity over \( \mathbb{F}_q \) if its maximal homogeneous part is nonsingular as a form over the algebraic closure of \( \mathbb{F}_q \) and that a \( k \)-tuple \( F(x) = (f_1(x), \ldots, f_k(x)) \) is "nonsingular" at infinity over \( \mathbb{F}_q \) if every polynomial in the pencil \( \{ \lambda \cdot f = \sum_{i=1}^k \lambda_i f_i: \ \lambda \in \mathbb{F}_q^k, \lambda \neq \emptyset \} \) is of degree \( d \geq 2 \), \( p + d \), and is nonsingular at infinity.

If \( F(x) \) is "nonsingular" at infinity then it follows from Theorem 8.4 of Deligne [De] and the observation

\[
\phi(V, y) = q^{-k} \sum_{\lambda \in \mathbb{F}_q^k} \sum_{x \in \mathbb{F}_q^n} e(\lambda \cdot F(x) + x \cdot y)
\]

for all \( y \) in \( \mathbb{F}_q^n \), that

\[
\Phi(V) \leq (d - 1)^n q^{n/2},
\]

where \( d \) is the maximum degree of the polynomials in \( F(x) \). In the special case that \( g(x) \) is a quadratic polynomial in an odd number of variables over \( \mathbb{F}_q \) and nonsingular at infinity, one can use estimates for Salie sums to improve on (1.4). In this case \( \Phi(V(g)) \leq 2q^{n/2 - 1/2} \); see e.g. Carlitz [Car].

We can now state our main results.

**Theorem 1.1.** Let \( S \) and \( T \) be subsets of \( \mathbb{F}_q^n \) and \( V \) be an algebraic subset of \( \mathbb{F}_q^n \) as defined by (1.1). Then \( (S + T) \cap V \) is nonempty provided that \( |S||T| > \Phi^2(V)q^{2k} \).

This theorem has interesting geometric consequences. For example if we let \( q = p \), then (1.1) can be viewed as the system of congruences

\[
f_1(x) \equiv \cdots \equiv f_k(x) \equiv 0 \pmod{p},
\]

where now the \( f_i \) are taken as polynomials in \( \mathbb{Z}[x_1, \ldots, x_n] \). Let \( B_p \) be the box in \( \mathbb{R}^n \) given by \( B_p = \{ x \in \mathbb{R}^n: 0 \leq x_i < p, 1 \leq i \leq n \} \), and again let \( V \) be the set of points in \( \mathbb{F}_p^n \) satisfying (1.5). We then have

**Theorem 1.2.** If \( C \) is a convex subset of \( B_p \) containing the origin and the projections of \( C \) onto the coordinate planes and \( \operatorname{Vol}(C) > 2^n \Phi(V) p^k \), then \( C \) contains an integral solution of (1.5).
Of course, since $\Phi(V)$ is invariant under translations and nonsingular linear transformations (mod $p$), Theorem 1.2 can be applied to a wider class of subsets of $\mathbb{R}^n$. In Corollary 4.1 we state a similar result for any convex subset of $\mathbb{R}^n$ symmetric about a point in $\mathbb{Z}^n$.

Another consequence of Theorem 1.1 is the following

**Corollary 1.3.** Let $B$ be a box of points in $\mathbb{F}_q^n$ as given by (1.2) and $V$ be the set of solutions of (1.1). Then $B \cap V$ is nonempty provided that

$$|B| > 2^{n/\phi(V)}q^k.$$  

(1.6)

The corollary follows by applying Theorem 1.1 with

$$S = \left\{ x \in \mathbb{F}_q^n : x_i = \sum_{j=1}^{f} x_{ij}\xi_j, 0 \leq x_{ij} < \left(\frac{(m_{ij} + 1)/2}{}, \right., \quad 1 \leq i \leq n, 1 \leq j \leq f \right\}$$

and $T = S + a$, where $a = (\sum_{j=1}^{f} a_{ij}\xi_j, \ldots, \sum_{j=1}^{f} a_{n,j}\xi_j)$, observing that $S + T \subseteq B$ and that $|S| = |T| \geq 2^{-n/|B|}$. When $V$ is defined by a set of polynomials “nonsingular” at infinity this corollary is essentially Myerson’s Theorem 2 [My]. However, we have eliminated the hypotheses of his theorem that $p$ be sufficiently large and that $V$ be absolutely irreducible over $\mathbb{F}_p$. R. C. Baker [Ba, Theorem 2] can improve on Corollary 1.3 in the case that $B$ is centered at the origin, $p$ is sufficiently large and $V = V(f)$, where $f$ is a nonsingular form of degree $\geq 3$. He obtains a nontrivial zero $x$ of $f$ with $0 < \max_i |x_i| \leq p^{1/2+\delta_n+\epsilon}$ where $\delta_n = 1/(2n - 2)$ for $n \geq 4$ and $\delta_3 = \frac{1}{6}$. We shall say that the points $x_1, \ldots, x_n$ in $\mathbb{F}_q^n$ are linearly independent if they are linearly independent as vectors over the field $\mathbb{F}_q$. In order for a subset of $\mathbb{F}_q^n$ to contain $n$ linearly independent points it is necessary that it not be contained in any $(n - 1)$-dimensional subspace of $\mathbb{F}_q^n$. On the other hand, if the set is a box we have

**Theorem 1.4.** Let $B$ be a box of points in $\mathbb{F}_q^n$ as given by (1.2) and $V$ be the set of solutions of (1.1). If $B$ is not contained in any $(n - 1)$-dimensional subspace of $\mathbb{F}_q^n$ and $|B| > 4 \cdot 2^{n/\Phi(V)}q^k$, then $B \cap V$ contains $n$ linearly independent points.

Thus, by increasing the cardinality of $B$ by a factor of 4 we are ensured not only of a solution of (1.1) in $B$ (see (1.6)) but of $n$ linearly independent solutions of (1.1) in $B$. In particular, if $F(x)$ is “nonsingular” at infinity then there exist $n$ linearly independent solutions $x = (x_1, \ldots, x_n)$ of $F(x) = 0$ with $x_i = \sum_{j=1}^{f} x_{ij}\xi_j$ and

$$\max_{i,j} |x_{ij}| \leq 4^{1/\phi}(d - 1)^{1/2} p^{1/2 + k/n},$$

provided the latter quantity is $< p/2$, where $d$ is the maximum degree of the polynomials in $F$.

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2. Method of proof, finite Fourier series. Throughout the paper we shall abbreviate “complete” sums $\sum_{\mathbf{x} \in \mathbb{F}_q^n} (\cdot)$ by just $\sum_{\mathbf{x}} (\cdot)$. Let $S$ be a subset of $\mathbb{F}_q^n$ and $V$ be an algebraic subset of $\mathbb{F}_q^n$ as defined by (1.1). Let $\alpha(\mathbf{x})$ be a real valued function on $\mathbb{F}_q^n$ such that $\alpha(\mathbf{x}) \leq 0$ for all $\mathbf{x}$ not in $S$, and $\sum_{\mathbf{x}} \alpha(\mathbf{x}) > 0$. In order to show $S \cap V$ is nonempty it suffices to choose $\alpha(\mathbf{x})$ so that $\sum_{\mathbf{x} \in V} \alpha(\mathbf{x}) > 0$. Now $\alpha(\mathbf{x})$ has a finite Fourier expansion $\alpha(\mathbf{x}) = \sum_{\mathbf{y}} \alpha(\mathbf{y}) e(\mathbf{y} \cdot \mathbf{x})$, where $\alpha(\mathbf{y}) = q^{-n} \sum_{\mathbf{x} \in V} \alpha(\mathbf{x}) e(-\mathbf{y} \cdot \mathbf{x})$ for $\mathbf{y} \in \mathbb{F}_q^n$. Thus

$$
\sum_{\mathbf{x} \in V} \alpha(\mathbf{x}) = \sum_{\mathbf{x} \in V} \sum_{\mathbf{y}} \alpha(\mathbf{y}) e(\mathbf{y} \cdot \mathbf{x})
$$

$$
= a(0)|V| + \sum_{\mathbf{y} \neq 0} \alpha(\mathbf{y}) \sum_{\mathbf{x} \in V} e(\mathbf{y} \cdot \mathbf{x})
$$

$$
= a(0)q^{-k} + a(0)(|V| - q^{-k}) + \sum_{\mathbf{y} \neq 0} \alpha(\mathbf{y}) \phi(V, \mathbf{y})
$$

and so,

$$
(2.1) \sum_{\mathbf{x} \in V} \alpha(\mathbf{x}) = q^{-k} \sum_{\mathbf{x}} \alpha(\mathbf{x}) + \sum_{\mathbf{y} \neq 0} \alpha(\mathbf{y}) \phi(V, \mathbf{y}).
$$

Equation (2.1) expresses the “incomplete” sum $\sum_{\mathbf{x} \in V} \alpha(\mathbf{x})$ as a fraction of the “complete” sum $\sum_{\mathbf{x}} \alpha(\mathbf{x})$ plus an error term. In §5 we consider the problem of making an optimal choice of $\alpha(\mathbf{x})$ in order to minimize the error term.

The idea of Tietäväinen [Ti] which has since been used by Chalk [Ch2] and Myerson [My] was to count the number of ways of expressing points in $V$ as the sum of points from subsets $S$ and $T$ of $\mathbb{F}_q^n$. This can be viewed as a special case of (2.1), taking $\alpha(\mathbf{x})$ as the convolution of $\chi_S$ and $\chi_T$, the characteristic functions of $S$ and $T$ respectively. Chalk’s equation (15) [Ch2] is a variation of (2.1) for this choice of $\alpha(\mathbf{x})$. We recall that if $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ are complex valued functions on $\mathbb{F}_q^n$, then their convolution, written $\alpha \ast \beta$, is defined by

$$
\alpha \ast \beta(\mathbf{x}) = \sum_{\mathbf{u}} \alpha(\mathbf{u}) \beta(\mathbf{x} - \mathbf{u}) = \sum_{\mathbf{u} + \mathbf{v} = \mathbf{x}} \alpha(\mathbf{u}) \beta(\mathbf{v}) \quad \text{for} \quad \mathbf{x} \in \mathbb{F}_q^n.
$$

If $H$ is an additive subgroup of $\mathbb{F}_q^n$ we define its orthogonal space $H^\perp$ as follows:

$$
H^\perp = \{ \mathbf{x} \in \mathbb{F}_q^n : \text{Tr}(\mathbf{x} \cdot \mathbf{y}) = 0 \text{ for all } \mathbf{y} \in H \}.
$$

Using the fact that $\mathbb{F}_q^n = H \oplus H^\perp$ one can easily deduce that the Fourier coefficients $a_H(\mathbf{y})$ of $\chi_H$ are given by

$$
a_H(\mathbf{y}) = \begin{cases} 
q^{-n}|H| & \text{if } \mathbf{y} \in H^\perp, \\
0 & \text{if } \mathbf{y} \notin H^\perp.
\end{cases}
$$

Thus

$$
(2.2) \sum_{\mathbf{y}} |a_H(\mathbf{y})| = 1.
$$
3. Proofs of Theorems 1.1 and 1.4. Let $S$ and $T$ be subsets of $\mathbb{F}_q^n$ and $H$ be an additive subgroup of $\mathbb{F}_q^n$. The proofs of Theorems 1.1 and 1.4 are based on the following identity:

\[
\sum_{x \in S \cap V} \chi_S(x) \chi_T(x) = q^{-k} \sum_{x \in H} \chi_S(x) \chi_T(x) + \theta \Phi(V)|S|^{1/2}|T|^{1/2}
\]

for some $\theta$ with $|\theta| \leq 1$. To obtain (3.1) we use equation (2.1) with $a(x) = (\chi_S(x) \chi_T(x))$. It suffices to show that the error term in (2.1) is less than $\Phi(V)|S|^{1/2}|T|^{1/2}$ in absolute value, and so it is enough to show that $\sum_y |a(y)| \leq |S|^{1/2}|T|^{1/2}$. Let $a_H(y), a_S(y)$ and $a_T(y)$ be the Fourier coefficients of $\chi_H, \chi_S$ and $\chi_T$ respectively. Then by elementary properties of Fourier coefficients, $a(y) = q^n((a_S \cdot a_T) \ast a_H)(y)$, and so by (2.2) we have

\[
\sum_y |a(y)| \leq q^n \left| \sum_y (a_S \cdot a_T)(y) \right| \cdot |S|^{1/2} \cdot |T|^{1/2}
\]

Using Parseval's identity we deduce that

\[
\sum_y |a(y)| \leq q^n \left( q^{-n} \sum_x |\chi_S(x)|^2 \right)^{1/2} \left( q^{-n} \sum_x |\chi_T(x)|^2 \right)^{1/2}
\]

\[
= |S|^{1/2} |T|^{1/2}.
\]

To prove Theorem 1.1 we apply (3.1) with $H = \mathbb{F}_q^n$, yielding

\[
\sum_{x \in V} \chi_S(x) \chi_T(x) \geq q^{-k} |S| |T| \Phi(V)|S|^{1/2} |T|^{1/2}.
\]

The left-hand side of (3.2) is positive provided that $|S||T| > \Phi^2(V)q^{2k}$.

Theorem 1.4 follows from the following proposition. For any subsets $S$, $T$ and $H$ of $\mathbb{F}_q^n$ we set

\[
N(H, S, T) = \sum_{x \in H} \chi_S(x) \chi_T(x) = \# \{(s, t) \in S \times T: s + t \in H\}.
\]

**Proposition 3.1.** Let $S$ and $T$ be subsets of $\mathbb{F}_q^n$ and $V$ be an algebraic subset of $\mathbb{F}_q^n$ as given by (1.1). Suppose $k$ is a number less than one such that for every $(n - 1)$-dimensional subspace $H$ of $\mathbb{F}_q^n$, $N(H, S, T) \leq k|S||T|$. Then $(S + T) \cap V$ contains $n$ linearly independent points provided that

\[
|S||T| > \left( \frac{2}{1 - k} \right)^2 \Phi^2(V)q^{2k}.
\]

**Proof.** Suppose that $(S + T) \cap V$ contains no more than $(n - 1)$ linearly independent points. Then there exists an $(n - 1)$-dimensional subspace $H$ such that $(S + T) \cap V \subset H$, which implies that

\[
\sum_{x \in H \cap V} \chi_S(x) \chi_T(x) = \sum_{x \in V} \chi_S(x) \chi_T(x).
\]
Therefore, by (3.1) and our assumption on $N(H, S, T)$,

$$\sum_{\delta \in \nu} \chi_S \ast \chi_T (\delta) \leq \kappa q^{-k} |S||T| + \Phi(V)|S|^{1/2}|T|^{1/2}.$$ 

Hence, by (3.2) we conclude that

$$|S||T| \leq \left( \frac{2}{1-\kappa} \right)^2 \Phi^2(V)q^{2k}.$$ 

**Proof of Theorem 1.4.** We simply apply the proposition to the boxes $S$ and $T$ as defined by (1.7). It suffices to show that $\kappa$ can be taken as $\frac{1}{2}$. Let $H$ be an $(n-1)$-dimensional subset of $F_q^n$ without loss of generality we may assume that $H$ is the zero set of a linear equation $\sum_{r=1}^{r} a_r x_r = 0$, where $1 \leq r \leq n$ and $a_r \neq 0$ for $1 \leq i \leq r$. The quantity $N(H, S, T)$ is the number of $(s, t)$ in $S \times T$ such that $\sum_{r=1}^{r} a_r (s_r + t_r) = 0$. Now, $S$ and $T$ can be written as $S = S_1 \times \cdots \times S_n$ and $T = T_1 \times \cdots \times T_n$. If $|S_i| > 1$ or $|T_i| > 1$ for some $i$ with $1 \leq i \leq r$, then on solving for $s_i$ or $t_i$ respectively in the above equation we see that $N(H, S, T) \leq \frac{1}{2} |S||T|$. Thus we may suppose that $S_i = \{\sigma_i\}$ and $T_i = \{\tau_i\}$ for some $\sigma_i, \tau_i \in F_q$, $1 \leq i \leq r$. Since $(S + T) \not\subset H$ it follows that $N(H, S, T, 0) = 0$ in this case.

**4. Geometric consequences of Theorem 1.1.** Let $F(x)$ be a $k$-tuple of polynomials in $\mathbb{Z}[x_1, \ldots, x_n]$ and $p$ be a prime. We define $V = V(F)$ and $\Phi(V)$ as in §1, reading the polynomials in $F(x)$ modulo $p$. For any subset $S$ of $\mathbb{Z}^n$ let $|\hat{S}|$ denote the number of distinct points in $S$ (mod $p$), that is $|\hat{S}| = |(S + p\mathbb{Z}^n)/p\mathbb{Z}^n|$. Theorem 1.1 now says that for any subsets $S$ and $T$ of $\mathbb{Z}^n$, $S + T$ contains a solution of (1.5) provided that $|\hat{S}| |\hat{T}| > \Phi^2(V)p^{2k}$. In particular if we let $C$ be any convex subset of $\mathbb{R}^n$ and let $S = \frac{1}{2} C \cap \mathbb{Z}^n = \{ x \in \mathbb{Z}^n : 0 < x_i < 1, 1 \leq i \leq n \}$, then $C$ contains an integral solution of (1.5) provided that $|\hat{S}| > \Phi(V)p^k$. This follows by taking $T = S$ and observing that $S + T \subset \frac{1}{2} C + \frac{1}{2} C \subset C$.

**Proof of Theorem 1.2.** Let $C$ be a convex subset of $B_p$ containing the origin and the projections of $C$ onto the coordinate planes. It is easy to see that for any $x$ in $C$, $C$ contains the set of $y$ in $\mathbb{R}^n$ such that $0 < y_i < x_i$ for $1 \leq i \leq n$. Let $S = \frac{1}{2} C \cap \mathbb{Z}^n$ and let $D$ be the unit box $D = \{ x \in \mathbb{R}^n : 0 \leq x_i < 1, 1 \leq i \leq n \}$. We know $\frac{1}{2} C \subset \cup_{s \in S}(s + D)$, for if $x \in \frac{1}{2} C$ then $y = ([x_1], [x_2], \ldots, [x_n]) \in \frac{1}{2} C \cap \mathbb{Z}^n = S$ and $x \in y + D$. Thus $\text{Vol}(\frac{1}{2} C) \leq |S||\hat{S}|$ and so it suffices to take $\text{Vol}(C) \geq 2^n \Phi(V)p^k$ in order for $C$ to contain a solution of (1.5).

For any $x \in \mathbb{R}^n$ let $||x|| = \max_{i=1,\ldots,n}|x_i|$, and for any subset $S$ of $\mathbb{R}^n$ let $||S|| = \sup_{x,y \in S}|x - y|$.

**Corollary 4.1.** Let $C$ be a convex subset of $\mathbb{R}^n$ symmetric about a point $z$ in $\mathbb{Z}^n$ such that $||C|| < 2p$ and

$$\text{Vol}(C) > 2^{n-1}(\Phi(V)p^k + 1).$$ 

Then $C$ contains a solution of (1.5).

**Proof.** Since $\Phi(V)$ is invariant under translations, we may assume that $z = 0$. Let $S = \frac{1}{2} C \cap \mathbb{Z}^n$ and suppose $\text{Vol}(C)$ satisfies (4.1). Then

$$\text{Vol}(\frac{1}{2} C) > 2^n(\frac{1}{2}\Phi(V)p^k + \frac{1}{2}).$$
and so by a generalized version of Minkowski’s fundamental theorem (see [Cas, Theorem II, p. 71]), \( \frac{1}{2} C \) contains at least \( \Phi(V) p^k \) distinct lattice points. But as \( \| \frac{1}{2} C \| < p \), this implies that \( |\hat{S}| > \Phi(V) p^k \) and so \( C \) contains a solution of (1.5).

5. **Best possible choices for** \( a(\chi) \). Let \( S, V \) and \( a(\chi) \) be as defined in §2, where without loss of generality \( a(\chi) \) is taken so that \( \sum a(\chi) = 1 \). We now seek the optimal choice of \( a(\chi) \) in order to show \( S \cap V \) is nonempty, that is, \( \sum_{\gamma} a(\chi) > 0 \). This amounts to minimizing the error term

\[
E(V, a) = \sum_{\gamma} a(\gamma) \phi(V, \gamma)
\]

in equation (2.1). If we bound \( E(V, a) \) by saying

\[
|E(V, a)| \leq \Phi(V) \sum_{\gamma} |a(\gamma)|,
\]

then the problem becomes one of minimizing \( \sum_{\gamma} |a(\gamma)| \), a quantity which depends only on the pair \( S, a(\chi) \) and not on \( V \). The following lemma gives us a lower bound on this quantity.

**Lemma 5.1.** Let \( a(\chi) \) be a real valued function on \( \mathbb{F}_q^n \) such that \( a(\chi) \leq 0 \) for \( \chi \not\in S \), \( \sum a(\chi) = 1 \) and \( a(\chi) = \sum_{y} a(y) e(\chi \cdot y) \). Then \( \sum_{\gamma} |a(\gamma)| \geq |S|^{-1} \).

**Proof.** For any subset \( W \) of \( \mathbb{F}_q^n \) it follows from the assumption \( \sum a(\chi) = 1 \) that

\[
\sum_{\chi \in W} a(\chi) = q^{-n} |W| + \sum_{\gamma \neq 0} a(\gamma) \phi(W, \gamma),
\]

where as before \( \phi(W, y) = \sum_{\chi \in W} e(\chi \cdot y) \) for \( y \neq 0 \). If we take \( W \) to be the complement of \( S \) in \( \mathbb{F}_q^n \), then for \( y \neq 0 \), \( \phi(W, y) = \sum_{\chi} e(\chi \cdot y) - \sum_{\chi \in S} e(\chi \cdot y) = -\sum_{\gamma} e(\chi \cdot y) \), and so \( |\phi(W, \gamma)| \leq |S| \). Since \( W \cap S = \emptyset \), we deduce from (5.2) that

\[
0 \geq \sum_{\chi \in W} a(\chi) \geq q^{-n} |W| - |S| \sum_{\gamma \neq 0} |a(\gamma)| = 1 - |S| \sum_{\gamma} |a(\gamma)|,
\]

and the conclusion follows.

If \( S \) is a box of points as given by (1.2), then the lower bound in Lemma 5.1 can be obtained, up to a factor of \( 2^{nf} \). This is seen by taking \( a(\chi) = |T|^{-1} |U|^{-1/2} e_T(\chi_U(\chi)) \), where \( U \) and \( T \) are boxes as given by (1.7). As we saw in deriving equation (3.1), \( \sum_{\gamma} |a(\gamma)| \leq |T|^{-1/2} |U|^{-1/2} \leq 2^{nf} |B|^{-1} \). Thus the only improvement that can be made in Corollary 1.3 if we use a bound of the type (5.1) is a savings of a factor of \( 2^{nf} \) in (1.6).

**References**


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