JOINT SPECTRA AND ANALYTIC SET-VALUED FUNCTIONS

BY

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ABSTRACT. We investigate analyticity of joint spectra of A^m-valued holomorphic mappings, where A denotes a complex Banach algebra. We show also that if K is an analytic set-valued function whose values are compact subsets of $\mathbf{C}^n$ and d is the transfinite diameter in $\mathbf{C}^n$, then the upper-semicontinuous regularization of $\log d(K)$ is plurisubharmonic. Moreover, we give higher dimensional extensions of Aupetit's Scarcity Theorem.

1. Introduction. Analytic functions whose values are compact subsets of the plane were introduced by Oka [9] in 1934. He defined them by means of pseudo-concavity. Afterwards they were practically forgotten for nearly fifty years. In 1980, Słodkowski [15] proposed a different definition of analytic set-valued functions using plurisubharmonic functions. This new definition was equivalent to the old one in the case of functions with values contained in $\mathbf{C}$ and was suitable for higher dimensions. Moreover it was much more flexible. The publication of [15] seems to have been the turning point in the history of analytic set-valued (or multivalued) functions. In the 1980's numerous articles dealing with this topic have already appeared. They were written by various authors including Alexander, Aupetit, Ransford, Słodkowski, Vesentini, Wermer, Zemanek and Zraibi. Ransford's dissertation [10] is an excellent survey article that also contains plenty of new results.

Analytic set-valued functions are very useful in spectral theory. A deep result proved in [15] says that if f is a holomorphic function with values in a Banach algebra, then the spectrum of $f(z)$ is an analytic set-valued function with respect to z. (Moreover every analytic set-valued function with values contained in $\mathbf{C}$ is locally of this form (see [15] for details).)

In our first theorem we prove a similar property for joint spectra. Let A be a complex Banach algebra with unit e. Following Harte [4] we define for $a = (a_1, \ldots, a_m) \in A^m$ the left joint spectrum of a,

$$\sigma_A^L(a) = \left\{ z \in \mathbf{C}^m : e \notin \sum_j A(a_j - z_j e) \right\},$$

the right joint spectrum of a,

$$\sigma_A^R(a) = \left\{ z \in \mathbf{C}^m : e \notin \sum_j (a_j - z_j e) A \right\},$$

and the joint spectrum of a,

$$\sigma_A(a) = \sigma_A^L(a) \cup \sigma_A^R(a).$$

Throughout the paper ^ will denote the polynomially convex hull.

THEOREM 1. *Let Ω be an open subset of $\mathbf{C}^n$ and let A be a complex Banach algebra with unit. Let $f = (f_1, \ldots, f_m): \Omega \to A^m$ be a holomorphic mapping such that $\{f_1(z), \ldots, f_m(z)\}$ is a commuting system of elements for each $z \in \Omega$. Then $z \to \hat{\sigma}_A(f(z))$ is an analytic set-valued function on Ω. Furthermore, if A is commutative, then $z \to \sigma_A(f(z))$ is analytic.*

This theorem together with the maximum principle for analytic set-valued functions (see e.g. [10]) implies the maximum principle for the polynomially convex hulls of joint spectra proved by Vesentini [18, Theorem I].

Yamaguchi [20] has proved that if $z \to K(z)$ is an analytic set-valued function with values in Comp $\mathbf{C}$, then $z \to \log c(K(z))$ is plurisubharmonic (where $c =$logarithmic capacity=Fekete's transfinite diameter). In the next theorem we show that this result can be generalized for analytic functions with values in Comp $\mathbf{C}^m$ provided that Fekete's transfinite diameter is replaced by the $\mathbf{C}^m$-transfinite diameter (for $m = 1$ they coincide). It should be noted that Yamaguchi's theorem can also be extended in a different direction (see [18]).

We shall prove the following

THEOREM 2. *Let $\Omega \subset \mathbf{C}^n$ be open and let $K: \Omega \to \text{Comp } \mathbf{C}^m$ be an analytic set-valued function. If d is the transfinite diameter in $\mathbf{C}^m$, then the function $u: \Omega \to [-\infty, \infty)$ defined by the formula $u(z) = (\log d(K(z)))^*$ is plurisubharmonic (* stands for the upper semicontinuous regularization). Moreover if Ω is connected, then either $K(z)$ is pluripolar for all z or the set $\{z \in \Omega: K(z)$ is pluripolar$\}$ is itself pluripolar.*

Aupetit [1, 2] has proved the so-called 'Scarcity Theorem' that gives a classification of analytic functions of one variable whose values are finite subsets of the complex plane (see also [10, Theorem 3.11]). The next two results are higher dimensional extensions of Aupetit's result. Before formulating our theorems, recall that a set $K \subset \mathbf{C}^m$ is said to be unisolvent for polynomials of degree k if the following implication is true: If $f: \mathbf{C}^m \to \mathbf{C}$ is a polynomial of degree $\leq k$ and $f = 0$ on K, then $f = 0$ in $\mathbf{C}^n$.

THEOREM 3. *Let Ω be an open connected subset of $\mathbf{C}^n$ and let $K: \Omega \to \text{Comp } \mathbf{C}^m$ be an analytic set-valued function. Define $\text{ord } K(z) = \sup\{k \geq 0: K(z)$ is unisolvent for polynomials of degree $k\}$ for $z \in \Omega$.*

Then either $\{z \in \Omega: \text{ord } K(z) < \infty\}$ is pluripolar or there exists a positive integer p and a pluripolar subset F of Ω such that for every $z \in \Omega \setminus F$, $\text{ord } K(z) = p$ and for every $z \in F$, $\text{ord } K(z) < p$.

THEOREM 4. *Let Ω and K be as in Theorem 3. Then either $\{z \in \Omega: \#K(z) < \infty\}$ is pluripolar or there is a positive integer p and an analytic subvariety F of Ω such that for each $z \in \Omega \setminus F$, $\#K(z) = p$ and for each $z \in F$, $\#K(z) < p$. In the latter case there exists a holomorphic mapping $H: \Omega \times \mathbf{C}^m \to \mathbf{C}^t$ (where $t = (m-1)p + 1$) such that $w \in K(z)$ if and only if $H(z, w) = 0$.*

It should be emphasized that even for $m = 1$ the functions $z \to \sigma_A^L(f(z))$ and $z \to \sigma_A^R(f(z))$ may not be analytic (see [10, p. 83]). In spite of this, Theorems 2, 3 and 4 are true for $z \to \sigma_A^L(f(z))$, $z \to \sigma_A^R(f(z))$ and $z \to \sigma_A(f(z))$ provided that f satisfies the assumptions of Theorem 1. The reason for this is quite simple: For any compact subset K of $\mathbf{C}^m$, $d(K) = d(\hat{K})$ (see [21]), $\text{ord } K = \text{ord } \hat{K}$ and, if K is finite, $K = \hat{K}$.

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2. Set-valued mappings. Let X and Y be topological spaces. A mapping F that assigns to each point x of X a closed subset $F(x)$ of Y is said to be upper (lower) semicontinuous if for every open (closed) set $S \subset Y$ the set $\{x \in X \colon F(x) \subset S\}$ is open (closed). In particular a function $u \colon X \to [-\infty, +\infty]$ is upper (lower) semicontinuous in the usual sense if and only if the mapping F_u defined by $F_u(x) = [-\infty, u(x)]$ is upper (lower) semicontinuous in the sense of the above definition (for more details see [5, 6, 3]). Let $\text{Comp } Y$ denote the space of all nonempty compact subsets of Y. If Y is a metric space, one can define the Hausdorff metric on $\text{Comp } Y$ by the formula

$$\rho(A, B) = \max \left\{ \sup_{a \in A} \text{dist}(a, B), \sup_{b \in B} \text{dist}(b, A) \right\}$$

for $A, B \in \text{Comp } Y$. The pair $(\text{Comp } Y, \rho)$ is a metric space and a mapping $K \colon X \to \text{Comp } Y$ is continuous if and only if it is both upper and lower semicontinuous.

Let Ω be an open subset of $\mathbf{C}^n$. A mapping $K \colon \Omega \to \text{Comp } \mathbf{C}^m$ is said to be analytic set-valued (ASV) if it is upper semicontinuous and the following condition is satisfied:

(ASV)   For each open subset Ω_1 of Ω and for each plurisubharmonic function u defined on an open neighbourhood of the set $\{(z, w) \in \Omega_1 \times \mathbf{C}^m \colon w \in K(z)\}$ the function $v \colon \Omega_1 \to [-\infty, +\infty)$ is plurisubharmonic, where

$$v(z) = \sup\{u(z, w) \colon w \in K(z)\} \quad \text{for } z \in \Omega_1.$$

It is clear that (ASV) is a local property.

EXAMPLE 2.1 (RANSFORD [10]). Let $u \colon \Omega \to [-\infty, +\infty)$ be an upper semicontinuous function on an open set $\Omega \subset \mathbf{C}^n$. Define $K(z) = \{w \in \mathbf{C} \colon |w| \leq \exp(u(z))\}$. Then K is ASV if and only if u is plurisubharmonic.

EXAMPLE 2.2 (RANSFORD [10]). Let $f \colon \Omega \to \mathbf{C}^m$ be a function on an open set $\Omega \subset \mathbf{C}^n$. Then $K(z) = \{f(z)\}$, $z \in \Omega$, is ASV if and only if f is holomorphic.

The following generalization of the last example is due to Ransford [10] and Zraibi [22].

THEOREM 2.3. *Let $\Omega \subset \mathbf{C}^n$ be connected and $K \colon \Omega \to \text{Comp } \mathbf{C}^m$ be an analytic set-valued function. If U and V are disjoint open sets in $\mathbf{C}^m$ such that $K(z) \subset U \cup V$ for all $z \in \Omega$, then either $K(z) \cap U = \varnothing$ for all $z \in \Omega$ or $K(z) \cap U \neq \varnothing$ for all $z \in \Omega$. In the latter case $z \to K(z) \cap U$ is an analytic set-valued function.*

The next result is due to Słodkowski [15].

THEOREM 2.4. (a) *If K and L are analytic set-valued functions such that the values of K are contained in the domain of L, then $L \circ K$ is an analytic set-valued function where $L \circ K(z) = \bigcup\{L(w) \colon w \in K(z)\}$ for any z in the domain of K.*

(b) *If K and L are analytic set-valued functions defined on the same open set Ω, then $K \times L$ is an analtyic set-valued function, where $(K \times L)(z) = K(z) \times L(z)$, $z \in \Omega$.*

3. Transfinite diameter in $\mathbf{C}^n$.

Let $x \colon \mathbf{N} \to \mathbf{Z}_+^n$ be a bijection such that $|x(j)| \leq |x(j+1)|$ for $j \geq 1$, where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}_+^n$. Define $p_j = |x(1)| + |x(2)| + \cdots + |x(m_j)|$, where m_j is the number of monomials in $\mathbf{C}^n$ of degree $\leq j$. Clearly $m_j = \binom{n+j}{j}$. Notice that p_j does not depend on our choice of x.

Let $\xi_1, \ldots, \xi_m$ be points in $\mathbf{C}^n$. Define $e_j(\xi) = \xi^{x(j)}$ for all $\xi \in \mathbf{C}^n$ and $V(\xi_1, \ldots, \xi_m) = \det[e_i(\xi_j)]_{1 \leq i,j \leq n}$. For a compact set $K \subset \mathbf{C}^n$ define

$$V_m(K) = \sup\{|V(\xi_1, \ldots, \xi_m)| \colon \{\xi_1, \ldots, \xi_m\} \subset K\}.$$

We call the number

$$d(K) = \limsup_{j \to \infty} (V_{m_j}(K))^{1/p_j}$$

the transfinite diameter of K. In the one-dimensional case d coincides with Fekete's transfinite diameter. Answering a question asked by Siciak in [12] (see also [7]), Zaharjuta [21] has shown that for any compact set in $\mathbf{C}^n$ the sequence $(V_{m_j}(K))^{1/p_j}$ converges as $j \to \infty$ and hence "lim sup" in the definition of $d(K)$ can be replaced by "lim". The transfinite diameter can also be characterized in terms of Tchebysheff constants (see [21]).

We will need the following lemma.

LEMMA 3.1. *Let Ω be an open set in $\mathbf{C}^n$ and let $K \colon \Omega \to \operatorname{Comp} \mathbf{C}^m$ be an upper semicontinuous mapping. Define $u(z) = \log d(K(z))$, $z \in \Omega$. Then u is locally bounded from above and $u^*(z) = -\infty$ if and only if $K(z)$ is pluripolar.*

PROOF. Fix $z_0 \in \Omega$. It is clear that if $\lambda > 0$ and E is a compact set in $\mathbf{C}^m$, then $d(\lambda E) = \lambda^m d(E)$. Hence we may assume without loss of generality that $K(z_0)$ is contained in the open unit ball $B \subset \mathbf{C}^m$. Upper semicontinuity of K implies that there exists a neighborhood U of z_0 such that for each $z \in U$, $K(z) \subset B$. Consequently $u \leq 0$ in U.

For any compact set $E \subset B$ define

$$T(E) = (\sup\{\Phi_E(z) \colon \|z\| = 1\})^{-1},$$

where Φ_E is the Siciak extremal function (see e.g. [12, 13]), that is

$$\Phi_E(z) = \sup\{|p(z)|^{1/\deg p}\},$$

where the supremum is taken over all polynomials p such that $|p| \leq 1$ on E and $\deg p \geq 1$. It can be shown that

(a) $T(E) = 0$ if and only if E is pluripolar (see [13]).

(b) If $E_1 \supset E_2 \supset E_3 \supset \cdots$ is a sequence of compact sets, then

$$T\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j \to \infty} T(E_j) \qquad (\text{see } [13]).$$

(c) There are positive constants A and δ such that for any compact set E contained in B, $n^{-1/2}T(E) \leq d(E) \leq AT(E)^{\delta}$ (see [**8**]).

The properties (b) and (c) imply that if $d(K(z)) = 0$, $u^{*}(z) = u(z) = -\infty$. Combining this with (a) and (c) we obtain the lemma.

In the proof of Theorem 3 we will use the following characterization of unisolvent sets (see [**12**]).

LEMMA 3.2. *For every compact set $K \subset \mathbf{C}^n$ the following conditions are equivalent*:
   (a) *K is unisolvent for polynomials of degree $\leq j$.*
   (b) *$V_i(K) \neq 0$ for $i = 1, 2, \ldots, m_j$.*
   (c) *$V_{m_j}(K) \neq 0$.*

4. Approximation lemmata for plurisubharmonic functions. Let Ω be an open subset of $\mathbf{C}^n$. By $\mathrm{PSH}(\Omega)$ we will denote the family of all plurisubharmonic functions on Ω (including $-\infty$) and by $P(\mathbf{C}^n)$ the family of all polynomials of n complex variables.

Define $\mathrm{PSH}_0(\Omega)$ as the family of all functions $u \in \mathrm{PSH}(\Omega)$ which can be written in the form $u = \sup\{\alpha_f \mathrm{Log}|f|\colon f \in \mathcal{F}\}$ for some $\mathcal{F} \subset P(\mathbf{C}^n)$ and $(\alpha_f)_{f \in \mathcal{F}} \subset \{t \in \mathbf{R}\colon t > 0\}$. In view of the following lemma we may restrict ourselves to countable families $\mathcal{F}$ in the definition of PSH_0.

LEMMA 4.1. *If $\mathcal{F} \subset \mathcal{C}(\Omega, \mathbf{R})$, then there is a countable family $\mathcal{F}_0 \subset \mathcal{F}$ such that for all $z \in \Omega$*
$$\sup\{f(z)\colon f \in \mathcal{F}\} = \sup\{f(z)\colon f \in \mathcal{F}_0\}.$$

PROOF. We have
$$\left\{(z,t) \in \Omega \times \mathbf{R}\colon \sup_{f \in \mathcal{F}} f(z) > t\right\} = \bigcup_{f \in \mathcal{F}} \{(z,t) \in \Omega \times \mathbf{R}\colon f(z) > t\}.$$

All sets that appear in the equality are open, so in view of the Lindelöf property there exists a countable family $\mathcal{F}_0 \subset \mathcal{F}$ such that
$$\left\{(z,t) \in \Omega \times \mathbf{R}\colon \sup_{f \in \mathcal{F}} f(z) > t\right\} = \bigcup_{f \in \mathcal{F}_0} \{(z,t) \in \Omega \times \mathbf{R}\colon f(z) > t\}$$
which proves the lemma.

LEMMA 4.2 (SEE [**13**, THEOREM 2.13]). *Let $g \in \mathrm{PSH}(\mathbf{C}^n)$ such that*
$$(*) \qquad \sup_{z \in \mathbf{C}^n}\{g(z) - \log(1 + \|z\|)\} < +\infty.$$
Then there is a locally bounded from above sequence $\{g_m\}_{m \geq 1} \subset \mathcal{C}^{\infty} \cap \mathrm{PSH}_0(\mathbf{C}^n)$ such that $g(z) = \lim_{m \to \infty} g_m(z)$, $z \in \mathbf{C}^n$.

LEMMA 4.3. *Let P_0 and P_1 be closed polynomial polyhedra in $\mathbf{C}^n$ such that $P_0 \subset \mathrm{int}\, P_1$. If $u \in L^{\infty} \cap \mathrm{PSH}(\mathrm{int}\, P_1)$, then there is a sequence $\{u_m\}_{m \geq 1} \subset \mathrm{PSH}_0(\mathrm{int}\, P_0)$ which is locally bounded from above and such that*
$$u(z) = \lim_{m \to \infty} u_m(z), \qquad z \in \mathrm{int}\, P_0.$$

PROOF. Take $f_1, \ldots, f_k \in P(\mathbf{C}^n)$ such that
$$P_1 = \{z \in \mathbf{C}^n\colon |f_j(z)| \leq 1, j = 1, \ldots, k\}.$$

We can choose $\theta \in (0,1)$ such that $P_0 \subset \text{int } P_\theta$ where $P_\theta = \{z \in \mathbf{C}^n : |f_j(z)| \leq \theta, \ j = 1, \ldots, k\}$. Clearly it is enough to show the lemma for P_θ instead of P_0. There are positive numbers A and B such that $v = A(u - B) < 0$ on P_1 and $dv > \log \theta$ on P_θ where $d = \max\{\deg f_1, \ldots, \deg f_n\}$. Define $w(z) = \max\{(1/d) \log |f_j(z)| : j = 1, \ldots, k\}$ and

$$g(z) = \begin{cases} \max\{v(z), w(z)\}, & z \in \text{int } P_1, \\ w(z), & z \in \mathbf{C}^n \setminus \text{int } P_1. \end{cases}$$

Then $g \in \text{PSH}(\mathbf{C}^n)$ satisfies $(\ast)$. Moreover $g = v$ on P_θ. Let $\{g_m\}$ be the sequence from Lemma 2 and let $g_m = \sup_{f \in \mathcal{F}_m}\{\alpha_f \log |f|\}$, where $\mathcal{F}_m \subset P(\mathbf{C}^n)$ and $\alpha_f > 0$ for each $f \in \mathcal{F}_m$. Define

$$u_m = \sup_{f \in \mathcal{F}_m} \frac{\alpha_f}{A} \log \left| f \exp \frac{AB}{\alpha_f} \right|.$$

By Lemma 2 $u = \lim u_m$ on P_θ.

LEMMA 4.4. *Let $(u_m)_{m \geq 1}$ be a locally bounded from above sequence in $\text{PSH}(\Omega)$ such that $u = \limsup_{m \to \infty} u_m \in \text{PSH}(\Omega)$. If K is a compact subset of Ω, then $\sup u(K) = \limsup_{m \to \infty}(\sup u_m(K))$.*

PROOF. Since u is upper semicontinuous there exists $z \in K$ such that

$$\sup u(K) = u(z) = \limsup_{m \to \infty} u_m(z) \leq \limsup_{m \to \infty}(\sup u_m(K)).$$

The opposite inequality follows from Hartogs' Lemma (see e.g. [**19**, p. 76]). (If $\varepsilon > 0$, the set $\{u < \sup u(K) + \varepsilon/2\}$ is open and contains K, so—in view of Hartogs' Lemma—there exists j_0 such that for all $j > j_0$ we have $\sup u_j(K) \leq \sup u(K) + \varepsilon$.)

5. Joint spectra and ASV functions. In this section we shall prove Theorem 1. First of all notice that the second conclusion of the theorem is quite easy to verify. For if A is commutative, then

$$\sigma_A(a_1, \ldots, a_m) = \sigma_A^L(a_1, \ldots, a_m) = \sigma_A^R(a_1, \ldots, a_m)$$
$$= \{(\phi(a_1), \ldots, \phi(a_m)) : \phi \in \mathcal{M}\},$$

where $\mathcal{M}$ is the maximal ideal space of A.

Let Ω_1 be an open subset of Ω and let u be a plurisubharmonic function on an open neighbourhood of the set $\{(z, w) \in \Omega_1 \times \mathbf{C}^m : w \in \sigma_A(f(z))\}$. Then

$$v(z) = \sup\{u(z, w) : w \in \sigma_A(f(z))\}$$
$$= \sup\{u(z, \phi(f_1(z)), \ldots, \phi(f_m(z))) : \phi \in \mathcal{M}\}.$$

The mapping $z \to (z, \phi(f_1(z)), \ldots, \phi(f_m(z)))$ is holomorphic, hence v is the supremum of a family of plurisubharmonic functions. Furthermore, as $\sigma_A : A^m \to \text{Comp } \mathbf{C}^m$ is continuous (see e.g. [**11**]) the function v is upper semicontinuous and hence plurisubharmonic.

The noncommutative case has to be dealt with separately. If we prove the first statement of Theorem 1 for the joint left spectrum, the result will follow because $\sigma_A = \sigma_A^L \cup \sigma_A^R$ and $\sigma_A^R = \sigma_{\tilde{A}}^L$, where $\tilde{A}$ is obtained from A by 'reversing' multiplication (i.e. by defining a new product of two elements a and b as ba). Let $B_m = \{a \in A^m : \sigma_A^L(a) \neq \varnothing\}$. By modifying the proof of Newburgh's Theorem (see e.g. [**11**]) we show the following (see also [**4**, p. 93]).

LEMMA 5.1. *For each positive integer m the mapping $\sigma_A^L : B_m \to \operatorname{Comp} \mathbf{C}^m$ is upper semicontinuous.*

PROOF. Suppose the lemma is false. There exist a neighbourhood V of 0 in $\mathbf{C}^m$, a point $x = (x_1, \ldots, x_m) \in B_m$ and sequences $\{a_n\} \subset B_m$ and $\{z_n\} \subset \mathbf{C}^m$ such that $\|a_n - x\| < 1/n$, $z_n \in \sigma_A^L(a_n) \setminus (\sigma_A^L(x) + V)$ and $\|a_n\| \leq \|x\| + 1/n$. Since $\sigma_A^L(x) \subset \sigma_A(x_1) \times \cdots \times \sigma_A(x_m)$ and $(y_1, \ldots, y_m) \to \sigma(y_1) \times \cdots \times \sigma(y_m)$ is upper semicontinuous by Newburgh's Theorem, the sequence $\{z_n\}$ is bounded. Passing to a subsequence, if necessary, we may assume that $\{z_n\}$ is convergent. Let $\xi = \lim z_n$. Clearly $\xi \notin (\sigma_A^L(x) + V)$. Hence $\xi \notin \sigma_A^L(x)$. Consequently there are $y_1, \ldots, y_m \in A$ such that

$$e = \sum_{i=1}^{n} y_i (x_i - \xi_i e),$$

where $\xi = (\xi_1, \ldots, \xi_m)$. Let $a_n = (a_{n1}, \ldots, a_{nm})$ and $z_n = (z_{n1}, \ldots, z_{nm})$. The set of invertible elements of A is open, so replacing $(x_i - \xi_i e)$ by $(a_{ni} - z_{ni} e)$ (for large n) we get an invertible element $\sum_{i=1}^{m} y_i (a_{ni} - z_{ni} e)$ which is impossible as $z_n \in \sigma_A^L(a_n)$.

LEMMA 5.2. *Let A be a Banach algebra with unit and let $S_m = \{x \in A^m : \sigma_A^L(g(x)) = g(\sigma_A^L(x))$ for all polynomials g of n complex variables$\}$.*

Let Ω be an open subset of $\mathbf{C}^n$ and let $f : \Omega \to S_m$ be holomorphic. If P is a closed polynomial polyhedron in $\mathbf{C}^n$ such that $\sigma_A^L(f(z)) \subset \operatorname{int} P$ for all $z \in \Omega$ and $u \in \operatorname{PSH}(\operatorname{int} P)$, then the function

$$v(z) = \sup\{u(w) : w \in \sigma_A^L(f(z))\}$$

is plurisubharmonic in Ω. (Note that $S_m \subset B_m$.)

PROOF. Fix $z_0 \in \Omega$. The polynomially convex hull of $\sigma_A^L(f(z_0))$ is a compact subset of $\operatorname{int} P$. Take closed polynomial polyhedra P_0 and P_1 such that

$$\sigma_A^L(f(z_0)) \subset \operatorname{int} P_0 \subset P_0 \subset \operatorname{int} P_1 \subset P_1 \subset \operatorname{int} P.$$

By Lemma 5.1 there is a neighbourhood W of z_0 such that $\sigma_A^L(f(z)) \subset \operatorname{int} P_0$ for all $z \in W$. Moreover, upper semicontinuity of u implies that u is upper bounded on P_1. By Lemma 4.4 and the fact that $u = \lim_{j \to \infty} \max(u, -j)$ we may assume that $u \in L^\infty(P_1)$. Lemmata 4.1, 4.3 and 4.4 imply that it is enough to consider u of the form $u = \alpha \log |g|$, where $\alpha > 0$ and $g \in P(\mathbf{C}^m)$. Note that the values of f lie in S_m. Thus by Vesentini's Theorem [16, 17] the function $z \to r(h(z))$ (where r denotes the spectral radius and $h = g \circ f$) is logarithmically plurisubharmonic. Hence $v \in \operatorname{PSH}(W)$.

PROOF OF THEOREM 1. Let Ω_1 be an open subset of Ω and let G be an open set in $\mathbf{C}^n \times \mathbf{C}^m$ such that

$$\{z\} \times \hat{\sigma}_A^L(f(z)) \subset G \quad \text{for all } z \in \Omega_1.$$

Suppose $u \in \operatorname{PSH}(G)$. If $g(z_1, \ldots, z_n) = (z_1 e, \ldots, z_n e)$, then

$$\{z\} \times \sigma_A^L(f(z)) = \sigma_A^L(g(z), f(z)) \stackrel{\text{def}}{=} s(z)$$

and

$$\{z\} \times \hat{\sigma}_A^L(f(z)) = \hat{s}(z).$$

Fix $z_0 \in \Omega_1$. Take a closed polynomial polyhedron $P \subset G$ such that $\hat{s}(z_0) \subset \operatorname{int} P$. By Lemma 5.1 there is a neighbourhood W of z_0 such that for each $z \in W$, $s(z) \subset \operatorname{int} P$ and hence $\hat{s}(z) \subset \operatorname{int} P$. Define

$$v(z) = \sup\{u(z,w) \colon w \in \hat{\sigma}_A^L(f(z))\}, \qquad z \in \Omega_1.$$

Then

$$v(z) = \sup\{u(t) \colon t \in \hat{s}(z)\}, \qquad z \in \Omega_1.$$

But $\operatorname{int} P$ is a Runge domain, so $\hat{s}(z)$ coincides with the $\operatorname{PSH}(\operatorname{int} P)$-hull of $s(z)$ for each $z \in W$. Therefore for $z \in W$ we have

$$v(z) = \sup\{u(t) \colon t \in s(z)\}.$$

According to Harte's Spectral Mapping Theorem (see [4, pp. 105–106]) if $x \in A^m$ is a commuting system of elements, then $x \in S_m$. Thus Lemma 5.2 implies that $v \in \operatorname{PSH}(W)$. Since plurisubharmonicity is a local property, the above shows that $v \in \operatorname{PSH}(\Omega_1)$.

6. Pluripolar sets and ASV functions.

In this section we prove Theorems 2, 3 and 4.

PROOF OF THEOREM 2. Observe that

$$
\begin{aligned}
\log d(K(z)) &= \limsup_{j \to \infty} (1/p_j) \log V_{m_j}(K(z)) \\
&= \limsup_{j \to \infty} \sup\{(1/p_j) \log |V(\xi)| \colon \xi \in [K(z)]^{m_j}\} \\
&= \limsup_{j \to \infty} u_j(z).
\end{aligned}
$$

From Theorem 2.4(b) and the definition of ASV-functions it follows that $u_j \in \operatorname{PSH}(\Omega)$. Therefore our theorem follows from Lemma 3.1.

PROOF OF THEOREM 3. Suppose that the set $A = \{z \in \Omega \colon \operatorname{ord} K(z) < +\infty\}$ is not pluripolar. As in the proof of the previous theorem we can show that the function $u_j(z) = \log V_{m_j}(K(z))$, $z \in \Omega$, is plurisubharmonic. Lemma 3.2 implies that $A = \bigcup_j \{z \in \Omega \colon u_j(z) = -\infty\}$. Countable unions of pluripolar sets are pluripolar, and hence there is an integer p_0 such that $u_{p_0} \equiv -\infty$. By Lemma 3.2 we have $u_k \equiv -\infty$ for all $k \geq p_0$. Hence p_0 can be chosen so that $u_p \not\equiv -\infty$, where $p = p_0 - 1$. Define $F = \{z \in \Omega \colon u_p(z) = -\infty\}$. Obviously p and F have the required properties.

PROOF OF THEOREM 4. As in the definition of Fekete's transfinite diameter we may define for any compact set $K \subset \mathbf{C}^m$

$$\delta_j(K) = \sup\left\{\prod_{p<q} \|z_p - z_q\|^{2/(j-1)j} \colon z_1, \ldots, z_j \in K\right\}$$

for $j = 1, 2, \ldots$. Exactly as in the one-dimensional case it follows from Theorem 2.4 that $z \to \log \delta_j(K(z))$ is plurisubharmonic. Using the same reasoning as for $m = 1$ (see [1, p. 66, Theorem 1; 2; 10, Theorem 3.11]) we can prove that either $\{z \in \Omega \colon \#K(z) < \infty\}$ is pluripolar or there is a positive integer p such that $\#K(z) \leq p$ for all $z \in \Omega$. Let us take the smallest p which has this property. By Theorem 2.3 (see also [10, Corollary 3.10]) K is continuous and hence $F = \{z \in \Omega \colon \#K(z) < p\}$

is a closed subset of Ω. We have to show that F is an analytic subset of Ω and we have to construct the function H.

Let $\alpha\colon \mathbf{C}^m \to \mathbf{C}$ be a $\mathbf{C}$-linear mapping. For $z \in \Omega \setminus F$ define

$$f_\alpha(z) = \prod_{i \neq j} \alpha(v_i(z) - v_j(z)),$$

where $K(z) = \{v_1(z), \ldots, v_p(z)\}$. If $z \in F$, let $f_\alpha(z) = 0$. Since K is continuous, the function f_α is continuous at every point of F. Moreover, in view of Theorem 2.3 (and Example 2.2) for each $z_0 \in \Omega \setminus F$ there exists a neighbourhood W of z_0 and holomorphic functions $v_1, v_2, \ldots, v_p\colon W \to \mathbf{C}$ such that $K(z) = \{v_1(z), \ldots, v_p(z)\}$ for all $z \in W$. Hence f_α is holomorphic on $\Omega \setminus F$. Rado's Theorem implies that f_α is holomorphic on Ω.

Let us take a system of linear functionals $\alpha_1, \alpha_2, \ldots, \alpha_r$ on $\mathbf{C}^m$ such that each subsystem of m elements forms a linearly independent set and

$$r = \tfrac{1}{2}(m-1)p(p-1) + 1.$$

We claim that

$$F = \bigcap_{j=1}^{r} \{z \in \Omega \colon f_{\alpha_j}(z) = 0\}.$$

There are only $\frac{1}{2}p(p-1)$ distinct pairs (i,j) such that $1 \leq i < j \leq p$. Suppose $z \in \Omega \setminus F$ and $f_{\alpha_1}(z) = \cdots = f_{\alpha_r}(z) = 0$. Then there exists a pair (i,j) and m distinct values of k for which $\alpha_k(v_i(z) - v_j(z)) = 0$. Hence $v_i(z) = v_j(z)$. This contradicts our assumption that $z \notin F$. Therefore $f_{\alpha_1}(z) = \cdots = f_{\alpha_r}(z) = 0$ if and only if $z \in F$.

In a similar way we may prove the existence of H. For $z \in \Omega \setminus F$ and $w \in \mathbf{C}^m$ define

$$h_k(z,w) = \prod_{j=1}^{p} \beta_k(w - v_j(z)),$$

where $k = 1, 2, \ldots, (m-1)p + 1$ and the β_k's are $\mathbf{C}$-linear functionals on $\mathbf{C}^m$ chosen so that every set of m of these functionals is linearly independent. Theorem 2.3 implies analyticity of h_k on $(\Omega \setminus F) \times \mathbf{C}^m$. As a consequence of Riemann's extension theorem, for each h_k there is a holomorphic function $H_k\colon \Omega \times \mathbf{C}^m \to \mathbf{C}$ that coincides with h_k on $(\Omega \setminus F) \times \mathbf{C}^m$. Clearly if $z \in \Omega \setminus F$, then $w \in K(z)$ if and only if $H_k(z,w) = 0$ for all $k = 1, 2, \ldots, (m-1)p + 1$. Indeed, there are only p distinct points in $K(z)$. So, if $H_k(z,w) = 0$ for $k = 1, 2, \ldots, (m-1)p + 1$, then there exists a point $w_0 \in K(z)$ and m distinct values of k for which $\beta_k(w - w_0) = 0$. Hence $w = w_0$.

Now assume that $z_0 \in F$. We may proceed as in [10, p. 40]. Take a sequence $\{z_j\} \subset \Omega \setminus F$ such that $z_j \to z_0$. Then $H(z_0, w_0) = 0$ if and only if there exists a sequence $w_j \to w_0$ with $w_j \in K(z_j)$. This, in turn, is equivalent to the fact that $w_0 \in K(z_0)$, since K is continuous.

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