A RIGIDITY PROPERTY FOR THE SET OF ALL CHARACTERS INDUCED BY VALUATIONS

BY

ROBERT BIERI AND JOHN R. J. GROVES

ABSTRACT. If $K$ is a field and $G$ a finitely generated multiplicative subgroup of $K$ then every real valuation on $K$ induces a character $G \to \mathbb{R}$. It is known that the set $\Delta(G) \subseteq \mathbb{R}^n$ of all characters induced by valuations is polyhedral. We prove that $\Delta(G)$ satisfies a certain rigidity property and apply this to give a new and conceptual proof of the Brewster-Roseblade result [4] on the group of automorphisms of $K$ stabilizing $G$.

1. Introduction.

1.1. Let $K$ be a field and $k \subseteq K$ a subfield. By a real valuation on $K$ over $k$ we mean a homomorphism $w: K^x \to \mathbb{R}$ of the multiplicative group of $K$ into the additive group of the reals which is trivial on $k^x$ and satisfies $w(a + b) \geq \min\{w(a), w(b)\}$ for all $a, b \in K^x$ with $a + b \neq 0$. Throughout the paper $G$ shall denote a finitely generated multiplicative subgroup of $K$. Then $w$ induces a character $\chi = w|_G: G \to \mathbb{R}$ on $G$ and so, following G. M. Bergman [1], we consider the set $\Delta(G)$ of all characters of $G$ induced by real valuations on $K$ over $k$. Thus $\Delta(G)$ is a subset of the real vector space $G^* = \text{Hom}(G, \mathbb{R}) \cong \mathbb{R}^n$, where $n = \text{rk}_k G$ is the torsion free rank of $G$. Since valuations on the subfield $k(G)$ can always be extended to the field $K$, $\Delta(G)$ depends only on $k(G)$ and not on $K$.

In [2] we have shown that the subset $\Delta(G) \subseteq \mathbb{R}^n$ has some special geometric features. In particular, we established Bergman’s conjecture that $\Delta(G)$ is a rational spherical polyhedron, i.e., a finite union of finite intersections of closed half spaces given by inequalities with integer coefficients. Here we continue the investigation of the geometry of $\Delta(G)$ by showing that it decomposes into a cartesian product $\Delta(G) = \mathbb{R}^m \times \Delta_1$ of an affine space $\mathbb{R}^m$ and a polyhedron $\Delta_1 \subseteq \mathbb{R}^{n-m}$ satisfying a certain rigidity property.

1.2. We sketch our main result. In the situation above the group $G$ contains two interesting subgroups $A \leq B \leq G$, which are in some sense dual to one another. On the one hand we have the unique maximal subgroup $A$ subject to the condition that $k(A)$ is algebraic over $k$ (thus $A$ only consists of all elements of $G$ which are algebraic over $k$); and on the other hand $G$ contains a unique minimal subgroup $B$ subject to the condition that $k(G)$ is purely transcendental over $k(B)$ (equivalently, the number of generators of $G/B$ coincides with $\text{tr deg}_{k(B)} k(G)$, the transcendence degree of $k(G)$ over $k(B)$). Note that the factors $G/B$ and $B/A$ are torsion free.

Received by the editors November 1, 1983.
1980 Mathematics Subject Classification. Primary 13A18, 12F99.

©1986 American Mathematical Society
0002-9947/86 $1.00 + $.25 per page
Any subgroup \( H \leq G \) gives rise to a short exact sequence of \( \mathbb{R} \)-vector spaces
\[(G/H)^* \to G^* \to H^*,\]
so that we can identify \((G/H)^*\) with the kernel of the restriction map \( \text{res} : G^* \to H^* \). Moreover, we have \( \text{res} \Delta(G) = \Delta(H) \). Our first result, then, exhibits the subspaces \((G/A)^*\) and \((G/B)^*\) in terms of the polyhedron \( \Delta(G) \). The following terminology will be convenient: We say that the vector subspace \( V \leq \mathbb{R}^n \) is an affine cartesian factor of the polyhedron \( \Delta \subseteq \mathbb{R}^n \) if \( \Delta = \pi^{-1}(\pi(\Delta)) \), where \( \pi \) is the canonical projection \( \mathbb{R}^n \to \mathbb{R}^n / V \) (so that \( \Delta \cong V \times \pi(\Delta) \)). Every polyhedron has a unique maximal affine cartesian factor.

**Theorem A.** If \( A \leq B \leq G \) as above, then \((G/B)^* \subseteq \Delta(G) \subseteq (G/A)^*\). In fact we have
(a) \((G/A)^*\) is the subspace spanned by \( \Delta(G) \),
(b) \((G/B)^*\) is the maximal affine cartesian factor of \( \Delta(G) \).

By the Bergman carrier \( \mathcal{C}(\Delta(G)) \) of \( \Delta(G) \) we mean the uniquely determined (finite) set of subspaces \( X \leq G^* \) such that the union \( \bigcup \{ X \mid X \in \mathcal{C}(\Delta(G)) \} \) contains \( \Delta(G) \) and is minimal with respect to that property. By [2, Theorem 1.1], the dimension of each \( X \in \mathcal{C}(\Delta(G)) \) is equal to the transcendence degree \( m \) of \( k(G) \) over \( k \). We stress that nontriviality of \( A \) and \( G/B \) above should be considered as a degenerate situation. In the nondegenerate, “generic”, case the polyhedron \( \Delta(G) \) has a certain rigidity property which is the main result of this paper.

**Theorem B.** If \( A \) and \( G/B \) are finite, then \( G^* \) is spanned by the one-dimensional intersections \( X_1 \cap X_2 \cap \cdots \cap X_m \) of spaces \( X_i \) in the Bergman carrier \( \mathcal{C}(\Delta(G)) \).

In the general case, Theorem B applies, of course, to the polyhedron \( \Delta(B) = \Delta(B/A) \) in \((B/A)^*\).

1.3. As an immediate consequence we obtain a new and more conceptual proof of Roseblade’s key result [4, Theorem D]. This application is in the spirit of, but much stronger than, Bergman’s proof of Zalesskii’s conjecture [1].

Let \( \Gamma \) be the group of all automorphisms of \( k(G) \) over \( k \) stabilizing \( G \). Then both \( A \) and \( B \) are \( \Gamma \)-invariant so that \( \Gamma \) acts on \( A, B/A \) and \( G/B \). The action on \( A \) and \( G/B \) is easy to understand: the automorphism group induced on \( A \) is a subgroup of the Galois group of \( k(A) \) over \( k \) and hence is finite; and the automorphism group induced on \( G/B \) is the full group \( \text{GL}_r(\mathbb{Z}), r = \text{rk}(G/B) \). Thus the interesting part is the action on \( B/A \).

The action of \( \Gamma \) on \( B \) induces an action on its character group \( B^* \). Since \( \Gamma \) permutes the valuations on \( K \) over \( k \), the subset \( \Delta(B) \subseteq B^* \) is invariant under this action. Hence \( \Gamma \) permutes the subspaces of the Bergman carrier \( \mathcal{C}(\Delta(B)) \). Bergman knew that if \( A \neq B \), then \( \mathcal{C}(\Delta(B)) \) is nonempty and consists of proper subspaces of \( G^* \), whence his conclusion that \( \Gamma \) contains a subgroup of finite index stabilizing a proper subspace of \( G^* \) and therefore a subgroup of infinite index in \( G \). But now, with Theorem B available, we know that \( \Gamma \) permutes a set of one-dimensional subspaces of \( B^* \) spanning \((B/A)^*\). Hence \( \Gamma \) contains a subgroup of finite index fixing \((B/A)^*\), and therefore \( B/A \), pointwise. In other words we have

**Corollary.** The automorphism group induced by \( \Gamma \) on \( B/A \) is finite.
This is the core of the Brewster-Roseblade result [4, Theorem D]. Their full result, asserting that, in fact, the automorphism group induced by $\Gamma$ on $B$ is finite, can be deduced from the Corollary by an elementary argument which we sketch in the Appendix.

We observe that in the definition of $\Delta(G)$, $G$ could be a torsion free subgroup in $K^\times$ of finite rank, in which case $\Delta(G)$ coincides with $\Delta(G_1)$ for any finitely generated subgroup $G_1$ of maximum rank in $G$. The Corollary holds as before.

1.4. The proof of Theorem A is given in §2. As to Theorem B, the only ingredients in its proof, apart from Theorem A, are two geometric properties of $\Delta(G)$ which we established in [2]: homogeneity and concavity. §§3, 4 and 5 contain a purely elementary-geometric analysis of these two conditions leading eventually to the rigidity result.

Finally, although we have followed Bergman in assuming that $k$ is a field, similar results hold in the more general case of $k$ being a Dedekind domain; we briefly sketch the necessary changes in §6.

The second author thanks the Deutsche Forschungsgemeinschaft for its financial support and the University of Frankfurt for its hospitality during the preparation of this paper.

2. Proof of Theorem A.

2.1. Let us first substantiate the remark we made immediately after stating Theorem A: that nontriviality of $G/B$ is to be considered a degenerate situation. As above let $K$ be a field, $k \subseteq K$ a subfield and $G \subseteq K$ a finitely generated multiplicative subgroup of $K$. Let $kG$ denote the group algebra of $G$, $k[G]$ the subalgebra of $K$ generated by $G$ over $k$, and $I$ the kernel of the canonical projection $kG \rightarrow k[G]$.

**Proposition 2.1.** If $H \subseteq G$ is a subgroup with torsion free factor $G/H$, then the following statements are equivalent:

(i) $I$ is controlled by $H$, that is, $I = (I \cap kH)kG$.

(ii) $k[G]$ is induced from $k[H]$, that is, the canonical map $kH \rightarrow k[H]$ induces an isomorphism $k[G] \cong k[H] \otimes_{kH} kG$.

(iii) $\text{trdeg}_{k(H)}k(G) = \text{rk}(G/H)$.

(iv) $\Delta(G) = \text{res}^{-1}\Delta(H)$.

**Proof.** (i) $\Rightarrow$ (ii) Tensoring the short exact sequence $I \cap kH \rightarrow kH \rightarrow k[H]$ with $kG$ over $kH$ yields the short exact sequence

$$(I \cap kH)kG \rightarrow kG \rightarrow k[H] \otimes_{kH} kG,$$

whence the assertion.

(ii) $\Rightarrow$ (iii) Let $X$ be a complement of $H$ in $G$. Then $kG \cong kH \otimes_k kX$, whence

$$k[H] \otimes_{kH} kG \cong k[H] \otimes_k kX.$$
(iii) \( \Rightarrow \) (iv) Let \( X \) be a complement of \( H \) in \( G \) and \( \chi = (\chi_H, \chi_X) \in G^* \) with \( \chi_H \in \Delta(H) \) and \( \chi_X \in X^* \). Let \( v: k(H)^{\times} \to R \) be a valuation on \( k(H) \) inducing \( \chi_H \). Any basis of \( X \) generates \( \chi_X \) over \( k(H) \) and so must be algebraically independent. Hence \( k(G) \) is just the field of fractions of the group ring \( k(H)X \) and \( v \) can be extended to a valuation \( w: k(G)^{\times} \to R \) inducing \( \chi_X \). This shows that \( \chi = (\chi_H, \chi_X) \in \Delta(G) \), as asserted.

(iv) \( \Rightarrow \) (iii) By [2, Theorem 1.1] the dimension of \( \Delta(G) \) coincides with the transcendence degree of \( k(G) \) over \( k \) and similarly for \( \Delta(H) \). Hence \( \text{rk}(G/H) = \dim \Delta(G) - \dim \Delta(H) = \text{trdeg}_{k(H)} k(G) \), as asserted.

2.2. The proof of Theorem A is now easily completed. The elements \( x \in G \) which are algebraic over \( K \) are characterized by the property that \( v(x) = 0 \) for all valuations on \( K \) over \( K \)—in other words, by the property that \( \Delta(G) \subseteq (G/\text{gp}(x))^* \), where \( \text{gp}(x) \) stands for the subgroup generated by \( x \). This shows that \( (G/A)^* \) is the intersection of all rational hyperspaces of \( \mathbb{R}^n \) containing \( \Delta(G) \)—a subspace of \( \mathbb{R}^n \) is said to be rational if it is induced by a subspace of \( Q^n \). But by [2, Theorem 1.1] we know that \( \Delta(G) \) is a rational polyhedron, and so the intersection of all rational subspaces containing \( \Delta(G) \) is, in fact, the subspace spanned by \( \Delta(G) \). This proves assertion (a). As to (b), note that if \( H \) is a subgroup of \( G \), then \( \Delta(G) \) is the cartesian product of \( (G/H)^* \) and \( \Delta(H) \) if and only if \( \Delta(G) = \text{res}^{-1} \Delta(H) \), and so, in view of Proposition 2.1, \( (G/B)^* \) is the maximal rational subspace of \( G^* \) with \( \Delta(G) = \text{res}^{-1} \Delta(H) \). Since \( \Delta(G) \) is a rational polyhedron the assertion follows.

3. Polyhedrons.

3.1. We have to recall some affine geometric notation from [2]. A subset \( S \) of the affine space \( \mathbb{R}^n \) is said to be a convex polyhedron if it can be written as the intersection of finitely many closed affine half spaces in \( \mathbb{R}^n \). The dimension of the convex polyhedron \( S \) is defined to be the dimension of the affine subspace of \( \mathbb{R}^n \) spanned by \( S \) and denoted \( \dim S \). A subset \( \Delta \subseteq \mathbb{R}^n \) is said to be a polyhedron if \( \Delta \) can be written as the union

\[
\Delta = C_1 \cup C_2 \cup \cdots \cup C_s
\]

of a finite number of convex polyhedrons \( C_i \). The dimension of \( \Delta \), denoted \( \dim \Delta \), is the maximum number \( \dim C_i \) as \( 1 \leq i \leq s \). The polyhedron \( \Delta \) is said to be homogenous if the decomposition (3.1) can be chosen such that \( \dim \Delta = \dim C_i \) for all \( 1 \leq i \leq s \).

In order to describe the local behaviour of a polyhedron \( \Delta \) at a point \( x \in \Delta \) it is convenient to introduce the local cone of \( \Delta \) at \( x \), denoted \( \text{LC}_x(\Delta) \). It consists of all points \( y \in \mathbb{R}^n \) with the property that the segment \( \{ x + \rho(y - x) \mid 0 \leq \rho \leq \varepsilon \} \) joining \( x \) and \( x + \varepsilon(y - x) \) is contained in \( \Delta \) for some \( \varepsilon > 0 \). Note that \( \text{LC}_x(\Delta) \) is a polyhedron which contains and is homeomorphic to an open neighbourhood of \( x \) in \( \Delta \). A point \( x \in \Delta \) is said to be regular if its local cone \( \text{LC}_x(\Delta) \) is an affine subspace of dimension \( \dim \Delta = m \). A nonregular point is said to be singular. By reg \( \Delta \) and sing \( \Delta \) we denote the set of all regular and singular points of \( \Delta \), respectively. By the essential part of \( \Delta \), denoted \( \text{ess} \Delta \), we mean the closure of reg \( \Delta \) in \( \mathbb{R}^n \). The essential part could also be described as the union of all \( m \)-dimensional \( C_i \)'s occurring in (3.1). Although
this second description does not make it apparent that ess $\Delta$ is independent of the decomposition (3.1), it does show that this is a homogenous polyhedron of dimension $m$. A similar argument shows that the set $\mathcal{C}(\Delta)$ of all affine subspaces of $\mathbb{R}^n$ which are the local cone of some regular point of $\Delta$ is finite. This is the carrier of $\Delta$. The union of all affine subspaces in $\mathcal{C}(\Delta)$ contains $\Delta$ and is minimal with respect to this property.

3.2. Let us consider the special case of an $n$-dimensional polyhedron in $\mathbb{R}^n$ in somewhat more detail. We prove

**Lemma 3.1.** If $\Delta$ is an $n$-dimensional homogenous polyhedron in $\mathbb{R}^n$, but not equal to $\mathbb{R}^n$, then $\text{sing} \Delta$ is an $(n - 1)$-dimensional homogenous polyhedron.

**Proof.** As $\Delta$ is a finite union of finite intersections of closed half spaces, the set theoretic complement $\Delta'$ is a finite union of finite intersections of open half spaces. But the intersection of finitely many open half spaces in $\mathbb{R}^n$ is either empty or $n$-dimensional, whence the closure $\overline{\Delta'}$ is an $n$-dimensional homogenous polyhedron. A point $x \in \Delta$ is singular if and only if $\text{LC}_x(\Delta) \neq \mathbb{R}^n$ which means that every neighbourhood of $x$ in $\mathbb{R}^n$ contains points not in $\Delta$. In other words we have $\text{sing} \Delta = \Delta \cap \Delta'$, and thus, in particular, $\text{sing} \Delta$ is a polyhedron. Clearly, $\dim(\text{sing} \Delta) \leq n - 1$. On the other hand $\mathbb{R}^n$ is the disjoint union of $\Delta'$, $\text{reg} \Delta$ and $\text{sing} \Delta$, and both $\text{reg} \Delta$ and $\Delta'$ are nonempty and open in $\mathbb{R}^n$. Hence the complement of $\text{sing} \Delta$ in $\mathbb{R}^n$ is not connected and so $\dim(\text{sing} \Delta) \geq n - 1$. Finally, let $x \in \text{sing} \Delta$, and let $U$ be an open ball in $\mathbb{R}^n$ with centre $x$. As $\Delta$ is homogenous, the set $\text{reg} \Delta$ of regular points is dense in $\Delta$ and so $U \cap \text{reg} \Delta \neq \emptyset$. The argument above shows that, in fact, $\dim(U \cap \text{sing} \Delta) = n - 1$. Hence the regular points of $\text{sing} \Delta$ are dense in $\text{sing} \Delta$, which amounts to saying that $\text{sing} \Delta$ is homogenous.

3.3. A polyhedron $\Delta$ has, of course, many decompositions of the form (3.1). For later applications, however, it will be crucial that there is a canonical one in favourable cases. Let us say that the union (3.1) is a convex cell decomposition of $\Delta$, if the intersection $C_i \cap C_j$ is empty or a common face of both $C_i$ and $C_j$, for all $i \neq j$. Unfortunately not every polyhedron has a canonical convex cell decomposition (try, e.g., the union of a plane $P$ and a line $L$ intersecting $P$ in a single point). But we can prove

**Lemma 3.2.** Every $n$-dimensional homogenous polyhedron $\Delta \subseteq \mathbb{R}^n$ has a canonical convex cell decomposition $\Delta = \bigcup_{i=1}^r C_i$, where each $C_i$ is an $n$-dimensional convex polyhedron and

$$\mathcal{C}(\text{sing} \Delta) = \bigcup_{i=1}^r \mathcal{C}(\text{sing} C_i).$$

Note that $\mathcal{C}(\text{sing} C_i)$ is, of course, just the set of all affine subspaces spanned by the $(n - 1)$-dimensional faces of the convex polyhedron $C_i$. Any face of $C_i$ contained in $\text{sing} \Delta$ will be called a face of $\Delta$. The faces of $\Delta$ form a convex cell decomposition of the homogenous polyhedron $\text{sing} \Delta$. 
PROOF (OF LEMMA 3.2). \( C(\text{sing } \Delta) \) is a finite set of affine hyperspaces of \( \mathbb{R}^n \) and so defines a convex cell decomposition of the affine space \( \mathbb{R}^n \),

\[
\mathbb{R}^n = \bigcup_{j=1}^{s} D_j,
\]

where \( D_1, D_2, \ldots, D_s \) are \( n \)-dimensional convex polyhedrons. Let \( \hat{D}_j \) denote the interior of the convex cell \( D_j \). Clearly, \( \hat{D}_j \) has empty intersection with \( \text{sing } \Delta \), whence

\[
\hat{D}_j = (\hat{D}_j \cap \text{reg } \Delta) \cup (\hat{D}_j \cap \Delta^c).
\]

Thus \( \hat{D}_j \) is the union of two disjoint open sets. As \( \hat{D}_j \) is connected, one of these must be empty, that is, either \( \hat{D}_j \subseteq \Delta \) or \( \hat{D}_j \subseteq \Delta^c \). It follows that \( \Delta \) is the union of all convex cells \( D_j \) contained in \( \Delta \), whence the lemma.

**Corollary 3.3.** Let \( \Delta \subseteq \mathbb{R}^n \) be an \( n \)-dimensional homogenous polyhedron and \( L \subseteq \mathbb{R}^n \) a line which intersects but is not contained in \( \Delta \). Then the carrier \( C(\text{sing } \Delta) \) of \( \text{sing } \Delta \) contains a hyperspace \( X \) which intersects \( L \) in a single point.

**Proof.** By Lemma 3.2 we may assume that \( \Delta \) is convex. Then we proceed by induction on \( n \). As \( \Delta \cap L \) is neither empty nor all of \( L \), there is a point \( x \in L \cap \text{sing } \Delta \). Hence there is an \((n-1)\)-dimensional face \( F \) of \( \Delta \) with \( x \in L \cap F \). Let \( Y \) be the affine subspace spanned by \( F \). Then \( Y \in C(\text{sing } \Delta) \), and if \( L \cap Y \) is the singleton set \( \{ x \} \), we put \( X = Y \) and are done. Otherwise, \( L \subseteq Y \) and, by induction, there is an \((n-2)\)-dimensional face \( F_1 \) of \( F \) such that the intersection of \( L \) with the subspace \( Z_1 \) spanned by \( F_1 \) is a singleton. But \( F_1 \) is the face of exactly two \((n-1)\)-dimensional faces, one of which is \( F \) (see e.g. [3, Satz 5.2]). The subspace \( X \) spanned by the other one cannot contain \( L \), whence the corollary.

**4. The effect of total concavity.**

4.1. Let \( \Delta \subseteq \mathbb{R}^n \) be a polyhedron. Recall from [2] that \( \Delta \) is said to be **concave at a point** \( x \in \Delta \) if the convex hull of the local cone \( L\text{C}_x(\Delta) \) is an affine subspace of \( \mathbb{R}^n \). \( \Delta \) is **totally concave** if it is concave at all points \( x \in \Delta \).

**Lemma 4.1.** If \( \Delta \) is a homogenous \( n \)-dimensional totally concave polyhedron in \( \mathbb{R}^n \), then \( \Delta = \mathbb{R}^n \).

**Proof.** According to the proof of Lemma 3.2 the affine space \( \mathbb{R}^n \) has a (finite) convex cell decomposition \( \mathbb{R}^n = C_1 \cup C_2 \cup \cdots \cup C_r \) with each \( C_i \) \( n \)-dimensional, such that \( \Delta \) is the union of a subset of the \( C_i \)'s. If \( \Delta \neq \mathbb{R}^n \), then there is a pair of indices \( i, j \) such that \( C_i \) and \( C_j \) have a common \((n-1)\)-dimensional face \( F \), \( C_i \subseteq \Delta \) and \( C_j \not\subseteq \Delta \). Then \( \Delta \) is not concave at the regular points of \( F \).

4.2. Let \( \Delta \subseteq \mathbb{R}^n \) be an arbitrary homogenous \( m \)-dimensional polyhedron. Then the set \( \text{reg } \Delta \) of all regular points of \( \Delta \) is dense in \( \Delta \), hence every point \( x \in \Delta \) is contained in the essential part of \( \Delta \cap X \) for some \( X \in C(\Delta) \); that is, one has the canonical decomposition

\[
\Delta = \bigcup_{X \in C(\Delta)} \text{ess}(\Delta \cap X).
\]
Note that each of the polyhedrons \( \text{ess}(\Delta \cap X) \) is \( m \)-dimensional, homogenous and contained in some \( X = \mathbb{R}^n \), whence has a canonical convex cell decomposition by Lemma 3.2.

**Proposition 4.2.** Let \( \Delta \subseteq \mathbb{R}^n \) be a homogenous polyhedron and \( X \in \mathcal{C}(\Delta) \). If \( \Delta \) is totally concave, then every space \( Z \) in the carrier \( \mathcal{C}(\text{sing}(\text{ess}(\Delta \cap X))) \) is the intersection \( Z = X \cap Y \) of \( X \) with some other \( Y \in \mathcal{C}(\Delta) \).

**Proof.** \( Z \) is the local cone of some regular point \( z \) of the polyhedron \( \text{sing}(\text{ess}(\Delta \cap X)) \). If \( z \notin \text{ess}(\Delta \cap Y) \) for all \( Y \in \mathcal{C}(\Delta) \) other than \( X \), then there is an open neighbourhood \( U \) of \( z \) in \( \mathbb{R}^n \) with \( \text{ess}(\Delta \cap Y) \cap U = \emptyset \), all \( X \neq Y \in \mathcal{C}(\Delta) \).

Hence \( \Delta \cap U = \text{ess}(\Delta \cap X) \cap U \) by (4.1). This shows that the behaviour of \( \Delta \) in a neighbourhood of \( z \) is given by \( \text{ess}(\Delta \cap X) \). From that we infer that the local cones of \( \Delta \) and \( \text{ess}(\Delta \cap X) \) at \( z \) coincide and are totally concave. But since \( \text{LC}_{z}(\text{ess}(\Delta \cap X)) \) is homogenous and of dimension \( = \dim X \), Lemma 4.1 applies and \( \text{LC}_{z}(\text{ess}(\Delta \cap X)) = X \), contradicting the assumption that \( z \) be a singular point of \( \text{ess}(\Delta \cap X) \). This shows that there is some \( Y \in \mathcal{G}(\Delta), Y \neq X \), with \( z \in X \cap Y \). The same arguments apply for all points in a neighbourhood of \( z \) in \( \text{sing}(\text{ess}(\Delta \cap X)) \); and so, as \( \mathcal{G}(\Delta) \) is finite, one of the subspaces \( Y \in \mathcal{G}(\Delta), Y \neq X \), must contain such a neighbourhood and hence \( Z \). Since \( \dim Z = \dim(\text{sing}(\text{ess}(\Delta \cap X))) = \dim \Delta - 1 \geq \dim(X \cap Y) \), we have \( Z = X \cap Y \), as asserted.

4.3. As a consequence of Proposition 4.2 we obtain that homogenous totally concave polyhedrons behave somewhat similarly to the set of all singular points of an \( n \)-dimensional convex polyhedron in \( \mathbb{R}^n \) (compare Corollary 3.3).

**Proposition 4.3.** Let \( \Delta \) be a homogenous and totally concave polyhedron in \( \mathbb{R}^n \), and \( L \subseteq \mathbb{R}^n \) a line which intersects but is not contained in \( \Delta \). Then there is some \( X \in \mathcal{G}(\Delta) \) which intersects \( L \) in a single point.

**Proof.** As \( \Delta \) is homogenous, there is some \( X \in \mathcal{G}(\Delta) \) such that \( L \) intersects but is not contained in \( \text{ess}(\Delta \cap X) \). If \( L \subseteq X \), then we are done; otherwise Corollary 3.3 applies for \( \text{ess}(\Delta \cap X) \) and yields \( Z \in \mathcal{G}(\text{sing}(\text{ess}(\Delta \cap X))) \) such that \( Z \cap L \) is a singleton. By Proposition 4.2 there is \( Y \in \mathcal{G}(\Delta) \) with \( X \cap Y = Z \) and \( Y \) cannot contain \( L \) since \( X \neq Y \). Hence \( Y \cap L \) consists of a single point.

4.4. Define the vector carrier \( \mathbb{B}\mathcal{G}(\Delta) \) to be the set of vector subspaces \( V \subseteq \mathbb{R}^n \) such that some affine translate of \( V \) lies in \( \mathcal{G}(\Delta) \). This, of course, is equal to \( \mathcal{G}(\Delta) \) in case \( \Delta \) is a cone.

**Corollary 4.4.** Let \( \Delta \subseteq \mathbb{R}^n \) be a polyhedron which is homogeneous and totally concave. Then the intersection \( V \) of all spaces in \( \mathbb{B}\mathcal{G}(\Delta) \) is an affine cartesian factor of \( \Delta \).

**Proof.** Let \( \pi \) be the projection \( \pi: \mathbb{R}^n \rightarrow \mathbb{R}^n/V \). We must show that \( \pi^{-1}(\pi(\Delta)) = \Delta \) or, equivalently, that \( \Delta + V = \Delta \). Now \( \Delta + V = \Delta \) if and only if \( z + \mathbb{R}v \subseteq \Delta \) for all \( z \in \Delta, v \in V \); that is, if and only if every line parallel to \( V \) which meets \( \Delta \) lies...
wholly in $\Delta$. But a line is parallel to $V$ if and only if it is parallel to each $X \in \mathcal{E}(\Delta)$ or, equivalently, each $X \in \mathcal{E}(\Delta)$. Thus Proposition 4.3 implies that $V$ is an affine cartesian factor of $\Delta$.

**Proof of Theorem B.**

5.1. Let $K$ be a field, $k \subseteq K$ a subfield and $G \leq K^\times$ a finitely generated multiplicative subgroup of $K$. Since positive multiples of valuations over $k$ are again valuations over $k$, the polyhedron $\Delta(G) \subseteq G^*$ is a cone. Moreover, we have shown in [2] that $\Delta(G)$ is homogenous and totally concave.

As to homogeneity, we can do better than this. If $H \trianglelefteq G$ is a subgroup, then the restriction map $\text{res}: G^* \to H^*$ is $R$-linear and maps $\Delta(G)$ onto $\Delta(H)$. This shows that not only $\Delta(G)$ is homogenous but also every image of $\Delta(G)$ under linear maps induced by embeddings of subgroups. We leave it to the reader to extend this observation to arbitrary linear maps, that is, to prove

**Theorem 5.1.** The polyhedron $\Delta(G) \subseteq G^* = R^n$ has the property that all its images under $R$-linear maps $\pi: R^n \to R^m$ are homogenous.

We shall really need the result only for the case when $\pi$ is induced by the embedding of a subgroup of corank 1.

5.2. It is now clear that Theorem B is an immediate consequence of Theorem A and the following purely geometric result. (Observing that, as $\Delta$ is a cone, the carrier and vector carrier of $\Delta$ coincide.)

**Proposition 5.2.** Let $\Delta \subseteq R^n$ be a polyhedron with the following properties.

(i) $\Delta$ is totally concave.

(ii) All images of $\Delta$ under linear maps $R^n \to R^m$ are homogeneous.

(iii) $\Delta$ has no nontrivial affine cartesian factor.

Then the subspace $R^n$ spanned by $\mathcal{E}(\Delta)$ is spanned by one-dimensional subspaces of the form $X_1 \cap X_2 \cap \cdots \cap X_n$, $X_i \in \mathcal{E}(\Delta)$.

5.3. The proof of Proposition 5.2 relies on

**Lemma 5.3.** Let $V$ be an $n$-dimensional vector space over any field $K$ and $\mathcal{C}$ a finite family of $(n-1)$-dimensional subspaces. If $\mathcal{C}$ has the property that it contains complements to every one-dimensional subspace $L \leq V$ in $V$, then $V$ is spanned by the one-dimensional subspaces of the form $X_1 \cap X_2 \cap \cdots \cap X_{n-1}$, $X_i \in \mathcal{C}$.

**Proof (by induction on $n$).** If $n = 2$, the spaces in $\mathcal{C}$ are one-dimensional themselves and there are at least two different ones. So let $n > 2$, pick $X \in \mathcal{C}$, and consider

$$\mathcal{C}_X = \{ X \cap Y | X \neq Y \in \mathcal{C} \}$$

which is a finite family of $(n-2)$-dimensional subspaces of $X$. For every one-dimensional subspace $L \leq X$ there is $Y \in \mathcal{C}$ with $V = L \oplus Y$. Note $L \not\leq Y$ so that $Y \neq X$, whence $X = L \oplus (X \cap Y)$. By induction $X$ is spanned by one-dimensional subspaces of the form $X \cap Y_i \cap \cdots \cap Y_{n-2}$, $Y_i \in \mathcal{C}$. This holds for all $X \in \mathcal{C}$ and $\mathcal{C}$ contains more than one space.
5.4. Proof of Proposition 5.2. Let \( X \in \mathfrak{C}(\Delta) \), \( L \subseteq X \) a one-dimensional subspace and \( \pi: \mathbb{R}^n \to \mathbb{R}^{n-1} \) a linear map with kernel \( L \). Let \( m = \dim \Delta \). If \( \dim \pi(\Delta) = m - 1 \), then \( L \subseteq Y \) for every \( Y \) in the carrier \( \mathfrak{C}(\Delta) \). But that would mean, by Corollary 4.4, that \( L \) is a cartesian factor of \( \Delta \). Therefore \( \dim \pi(\Delta) = m \). But certainly the dimension of \( \pi(X) \) is equal to \( m - 1 \), and so, as \( \pi(\Delta) \) is homogenous, \( \pi(X) \) must be contained in \( \pi(Y) \) for some \( Y \in \mathfrak{C}(\Delta) \), \( X \neq Y \). In other words \( X \subseteq Y + L \) and hence \( X = (Y \cap X) \oplus L \). This shows that the family

\[
\mathfrak{C}_X = \{ X \cap Y | X \neq Y \in \mathfrak{C}(\Delta) \}
\]

of subspaces of \( X \) satisfies the assumption of Lemma 5.3. Hence \( X \) is spanned by one-dimensional intersections of subspaces in the carrier of \( \Delta \). This holds for all \( X \in \mathfrak{C}(\Delta) \), whence the assertion.

This completes the proof of Proposition 5.2 and hence that of Theorem B.

6. Note on a generalization. The assumption that the valuations considered on the field \( K \) are trivial when restricted to the base field \( k \) is unnecessarily strong. One can take an arbitrary discrete valuation \( v: k^\times \to \mathbb{Z} \) and consider the set \( \Delta'(G) \) of all characters of \( G \) induced by a real valuation on \( K \) extending \( v \). Then \( \Delta'(G) \), although not necessarily a cone, is homogeneous and totally concave and does have the property in Proposition 5.1 (see [2]).

The proof of all the intermediate results in §§2–5 now carry through for \( \Delta'(G) \) (as well as for the global sets \( \Delta^k/\partial'(G) \) for \( D \subseteq k \) a Dedekind domain; see [2]) yielding appropriate versions of Theorems A and B. More precisely, in Theorem A, we must replace “the subspace spanned by \( \Delta(G) \)” by “the subspace spanned by the vector carrier of \( \Delta'(G) \)” and in Theorem B we must replace “\( \mathfrak{C}(\Delta(G)) \)” by “\( \mathfrak{C}(\Delta'(G)) \)”.

However, it should be noticed that by [2, Theorem C1], \( \Delta(G) = \Delta^0(G) \) is the local cone of \( \Delta'(G) \) at infinity, and so the vector carrier of \( \Delta'(G) \) contains the Bergman carrier of \( \Delta^0(G) \). Moreover, it can be shown (by using Proposition 2.1 or a geometric argument involving total concavity) that the maximal affine cartesian factors of \( \Delta^0(G) \) and \( \Delta'(G) \) coincide. Hence the rigidity result for \( \Delta'(G) \) is no stronger than the rigidity result for \( \Delta^0(G) \)—and this is the reason why we restricted attention to the field case.

Appendix. We sketch the proof that our Corollary in the introduction implies the full result of Theorem D in [4]. We show that, in fact, the automorphism group induced by \( \Gamma \) on \( B \) is finite.

We have shown that the automorphism groups induced by \( \Gamma \) on both \( B/A \) and \( A \) are finite. Thus it suffices to show that any automorphism \( \rho \) which is trivial on both \( B/A \) and \( A \) has finite order.

Let \( F \subseteq k(B) \) be the fixed field of \( \rho \) and let \( \overline{F} \) be its algebraic closure in \( k(B) \). Then \( |\overline{F}:F| \) is finite and so, after passing to a suitable power of \( \rho \), we may assume that \( \overline{F} = F \). We claim now that \( k(B) = F \); we suppose not and derive a contradiction.

Let \( k(B) = F(x_1, \ldots, x_t) \) with \( \{x_i\}_{i=1}^t \subseteq B \) multiplicatively independent. By the definition of \( B \) the \( x_i \) cannot be algebraically independent over \( F \); choose \( x_1, \ldots, x_t \),
so that \( x_1, \ldots, x_{t-1} \) are algebraically independent over \( F \) and \( x_t \) is algebraic over \( F(x_1, \ldots, x_{t-1}) \). Let \( P(X_1, \ldots, X_t) \in F[X_1, \ldots, X_t] \) be the minimal polynomial of \( x_t \) over \( F[x_1, \ldots, x_{t-1}] \).

Since \( \rho \) fixes both \( B/A \) and \( A \), we have \( \rho(x_t) = x_t a_t \) with \( a_t \in A \subseteq F \). Also, as \( F \) is the fixed field of \( \rho \) and is algebraically closed in \( k(B) \), the \( a_t \) are multiplicatively independent. Now, \( P(x_1, a_1, \ldots, x_t, a_t) = 0 \) and so \( P(x_1 a_1, \ldots, x_t a_t) \) is a multiple, clearly a \( F \)-scalar multiple, of \( P(x_1, \ldots, x_t) \). Comparing coefficients of monomials and recalling that the \( a_t \) are multiplicatively independent, we observe that \( P \) is a monomial in the \( X_t \)—an evident contradiction.

Thus \( F = k(B) \) and so \( \rho \) (or, in the original statement, some power of \( \rho \)) is trivial.

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FRANKFURT, 6000 FRANKFURT AM MAIN, WEST GERMANY

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MELBOURNE, PARKVILLE, VICTORIA 3052, AUSTRALIA