

## MARTINGALE TRANSFORMS AND COMPLEX UNIFORM CONVEXITY

BY

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ABSTRACT. Martingale transforms and Calderon-Zygmund singular integral operators are bounded as operators from  $L_2(L_1)$  to  $L_2(L_q)$  when  $0 < q < 1$ . If  $Y$  is a reflexive subspace of  $L_1$  then  $L_1/Y$  can be renormed to be 2-complex uniformly convex. A new proof of the cotype 2 property of  $L_1/H_1$  is given.

**Introduction.** Recently, vector-valued versions of several martingale transform inequalities have appeared. Burkholder [10] has shown that, for example, martingale transforms are bounded on  $L_2(X)$ , where  $X$  is a Banach space, if and only if  $X$  is a UMD space. That is, whenever  $(d_k)$  is an  $X$ -valued martingale difference sequence with values in  $L_2(X)$ , it is an unconditional basic sequence there. This also leads to a study of the boundedness of singular integral transforms, such as the Hilbert transform, on  $L_2(X)$ . Burkholder and McConnell [9] and Bourgain [2] proved that the Hilbert transform is bounded on  $L_2(X)$  if and only if  $X$  is UMD. Here we give an analogous result, namely the boundedness of martingale transforms and singular integral transforms as operators from  $L_2(L_1)$  to  $L_2(L_q)$  when  $0 < q < 1$ . One application of that result is a new proof of the cotype 2 inequality for the space  $L_1/H_1$  [3].

Complex uniformly convex spaces were studied in [12]. There martingales of a special form were utilized to provide renorming theorems analogous to those of Pisier [18] in the (real) uniformly convex case. In analogy with the real case, for cotype it is shown here that  $L_1/Y$  is isomorphically complex uniformly convex whenever  $Y$  is a reflexive subspace of  $L_1$ . This uses the Hilbert transform result cited. This contrasts with the cotype 2 property for  $L_1/H_1$  since  $L_1/H_1$  cannot be renormed to be complex uniformly convex (e.g., Pisier's example in [12]).

The final section of this paper contains a small result concerning the convergence in  $L_1(T^{\mathbb{N}})$  and a.e. of series of the form  $\sum f_k(\theta_1, \dots, \theta_{k-1})e^{i\theta_k} = f(\Theta)$  in the case  $\|f\|_{L_1} < \infty$ . This contrasts with the failure in general of martingales to converge in  $L_1$ .

The authors are grateful to D. J. H. Garling for reading the manuscript and

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Received by the editors July 12, 1984 and, in revised form, February 15, 1985.

1980 *Mathematics Subject Classification*. Primary 46B20, 46E40, 60G46.

*Key words and phrases*. Martingale transforms, UMD spaces, complex uniform convexity.

<sup>1</sup>Supported in part by NSF grant number BMS-8403218.

suggesting substantial changes in the proof of Theorem 1.1. The authors also appreciate the very careful review of the paper by the referee who suggested many changes to enhance the readability of the paper.

**1. MT operators.** D. Burkholder has been studying the class of Banach spaces in which martingale difference sequences are unconditional basic sequences. Among other things, he has shown that these are the spaces in which martingale transforms are bounded, and in which the vector-valued Hilbert transform is bounded the same way it is in the scalar case. In §§2 and 3 of this paper, we shall want to know that the vector Hilbert transform is bounded as an operator from  $L_2(L_1)$  to  $L_2(L_q)$  for some  $q \in (0, 1)$ . For this, we can prove a more general result following very closely the lines of Burkholder's arguments in [9 and 10]. Since  $L_q$  is only quasinormed in the range  $0 < q < 1$ , some care must be taken in claiming the equivalences in Theorem 1.1.

In what follows,  $(\Omega, \mathcal{F}, P)$  is a probability space,  $(\mathcal{F}_k)$  is an increasing sequence of sub  $\sigma$ -algebras of  $\mathcal{F}$ ,  $f = (f_n)$  is an  $X$ -valued martingale adapted to  $(\mathcal{F}_k)$  and  $d_n = f_n - f_{n-1}$  is the corresponding martingale difference sequence. Further,  $u = (u_k)$  is a (scalar-valued)  $L_\infty$ -bounded predictable sequence (i.e.,  $u_k$  is  $\mathcal{F}_{k-1}$ -measurable), and  $g_n = \sum_{k=1}^n u_k d_k$  is the transform of  $f$  by  $u$ , and  $f^*(\omega) = \sup \|f_n(\omega)\|$ .

**DEFINITION.** Let  $X$  be a Banach space,  $Y$  a continuously quasinormed linear space and  $A$  a bounded operator from  $X$  to  $Y$ . We say that  $A$  is an MT operator (martingale transform) if for any  $p \in (1, \infty)$ , there exists a constant  $C$  such that for any martingale  $f$  and any predictable sequence as above, the transform  $g$  of  $f$  by  $v$  satisfies

$$(1.1) \quad \|Ag\|_{L_p(Y)} \leq C \|f\|_{L_p(X)}.$$

**REMARK.** In the definition, the expression  $\|Ag\|_{L_p(Y)}$  is, as usual,  $\sup \|Ag_n\|_{L_p(Y)}$ . However, since  $Y$  is only assumed to be quasinormed, we have lost Jensen's inequality and cannot, therefore, claim that  $\|Ag\| = \lim \|Ag_n\|$ . In Burkholder's works cited, it is shown that MT spaces are those for which martingale transforms by constant sequences are bounded. That is,  $\|\sum \varepsilon_k d_k\|_p \leq C \|\sum d_k\|_p$  is adequate to derive (1.1). That would be obvious here as well if  $Y$  were assumed to be normed, but it is not apparent a priori. Since the first version of this paper was written, D. Trautman [20] has shown that an operator is an MT operator if and only if it is a UMD operator. That is, in what follows, it is sufficient to consider nonrandom martingale transforms of the form  $\sum \varepsilon_k A d_k$  valued in  $Y$  where  $\varepsilon_k = \pm 1$  for all  $k$ .

The following result follows almost exactly the reasoning of Burkholder in [10]. Details are included for completeness, and to indicate the changes in Burkholder's arguments demanded by the fact that  $Y$  is only quasinormed, and not normed.

**THEOREM 1.1.** *Let  $A: X \rightarrow Y$  with  $\|A\| \leq 1$  (for convenience). The validity of each statement below for every  $f$ ,  $g$  and  $v$  as above is equivalent to the validity of each of the others.*

(a) *There is a constant  $C_a$  such that*

$$Ag^* \geq 1 \text{ a.e. implies } \|f\|_1 \geq C_a.$$

(b) *There is a constant  $C_b$  such that*

$$\lambda P[Ag^* > \lambda] \leq C_b \|f\|_1.$$

(c) *For each  $p \in (1, \infty)$  there is a constant  $C_p$  such that*

$$\|Ag\|_p \leq C_p \|f\|_p.$$

(d) *For some  $p \in (1, \infty)$  there is a constant  $C_p$  such that*

$$\|Ag\|_p \leq C_p \|f\|_p.$$

(e) *For each  $q \in (0, 1)$  there is a constant  $C_q$  such that*

$$\|Ag\|_q \leq C_q \|f\|_\infty.$$

(f) *For some  $q \in (0, 1)$  there is a constant  $C_q$  such that*

$$\|Ag\|_q \leq C_q \|f\|_\infty.$$

PROOF. All of the implications (c)  $\Rightarrow$  (d)  $\Rightarrow$  (e)  $\Rightarrow$  (f) are trivial. (a)  $\Rightarrow$  (b): Let  $f, v$  and  $g$  be as above with  $f$  starting at 0 as in [10]. We shall show, with  $\|y_1 + y_2\|_Y \leq K\|y_1\|_Y + K\|y_2\|_Y$ , that

$$(1.2) \quad C_a P[Ag_n^* > 2K] \leq \|f\|_1.$$

Then  $P[Ag_n^* > \lambda] = P[(2K/\lambda)Ag_n^* > 2K] \leq (2K/C_a\lambda)\|f\|_1$  proving (b) with  $C_b = 2K/\lambda$ .

Take independent copies of  $(f, v, g)$  on some space  $\Omega$  and call these copies  $(f_j, v_j, g_j)$ . Assume that for some  $n$ ,  $P[Ag_n^* > 2K] > 0$ , or rescale so that this occurs. Define  $u_j = 1_{[Ag_{j,n}^* \leq 2K]}$  and a new martingale difference sequence by  $D = (d_{1,1}, d_{1,2}, \dots, d_{1,n}, u_1 d_{2,1}, \dots, u_1 u_2 d_{3,1}, \dots)$  and a new predictable sequence  $V = (v_{1,1}, v_{1,2}, \dots, v_{1,n}, v_{2,1}, \dots)$ . Notice that

$$F_{kn} = d_{1,1} + \dots + u_1 u_2 \dots u_{k-1} (d_{k,1} + \dots + d_{k,n})$$

so that

$$\begin{aligned} \|F\|_1 &\leq \|f\|_1 \mathbf{E}(1 + u_1 + u_1 u_2 + \dots) \\ &= \|f\|_1 (1 + P[Ag_n^* \leq 2K] + \dots + (P[Ag_n^* \leq 2K])^k + \dots), \end{aligned}$$

so

$$(1.3) \quad \|F\|_1 \leq \frac{\|f\|_1}{P[Ag_n^* > 2K]}.$$

We need to see that  $P[AG^* > 1] = 1$ . For this, notice

$$(1.4) \quad [AG_{(k+1)n}^* > 1] \supset [AG_{kn}^* > 1] \cup [AG_{kn}^* \leq 1, u_1 u_2 \dots u_k Ag_{k+1,n}^* > 2K]$$

since  $\|y_1 + y_2\|_Y \geq (1/K)\|y_1\|_Y - \|y_2\|_Y$ . Notice that  $AG_{kn}^* \leq 1$  forces  $g_{1,n}^* \leq 2K, g_{2,n}^* \leq 2K, \dots, g_{k,n}^* \leq 2K$ , so that

$$(1.5) \quad [AG_{kn}^* \leq 1, u_1 \dots u_k Ag_{k+1,n}^* > 2K] = [AG_{kn}^* \leq 1, Ag_{k+1,n}^* > 2K].$$

Therefore,

$$P[AG_{(k+1)n}^* > 1] \geq P[AG_{kn}^* > 1] + P[AG_{kn}^* \leq 1]P[Ag_{k+1,n}^* > 2K].$$

This gives  $P[AG^* > 1] = 1$  as desired. From (1), then, and (1.3), we have

$$C_a P[Ag_n^* > 2K] \leq \|f\|_1,$$

proving (b).

We now want to show that (b)  $\lambda P[Ag^* > \lambda] \leq c\|f\|_1$  implies (c'). If  $\Phi$  is convex, increasing satisfying a  $\Delta_2$  condition, then there is  $c > 0$  such that

$$\mathbf{E}\Phi(Ag) \leq c\mathbf{E}\Phi(f).$$

$\Phi$  satisfies  $\Delta_2$  means there is  $\alpha > 0$  such that  $\Phi(2\lambda) \leq \alpha\Phi(\lambda)$ . Let  $f$  be an  $X$ -valued martingale,  $f_n = \sum_{k=1}^n d_k$ , and  $\{v_k\}$  a predictable sequence with  $|v_k(\omega)| \leq 1$  for all  $\omega$ , and  $Ag_n = \sum_{k=1}^n v_k(\omega)Ad_k(\omega)$ . We first perform a Burgess-Davis decomposition [11] on  $f$ : Let  $y_k = d_k 1_{\|d_k\| \leq 2d_{k-1}^*}$  and  $z_k = d_k - y_k$ . Let  $a_k = y_k - \mathbf{E}(y_k | \mathcal{F}_{k-1})$  and  $b_k = z_k - \mathbf{E}(z_k | \mathcal{F}_{k-1})$ , so that as in [11 or 8] one has  $\|a_k(\omega)\|_X \leq 4d_{k-1}^*$ , and  $\sum \|z_k\| \leq 2d^*$ . Using Lemma 16.1 in [8] one sees that

$$(1.6) \quad \mathbf{E}\Phi\left(\sum_{k=1}^{\infty} \|b_k\|_X\right) \leq c_1 \mathbf{E}\Phi(d^*) \leq c_2 \mathbf{E}\Phi(f^*)$$

since  $\|b_k\| \leq \|z_k\| + \mathbf{E}(\|z_k\| | A_{k-1})$  and  $d^* \leq 2f^*$ . Then split  $f$  by  $f_n = \varphi_n + \Psi_n$  where  $\varphi_n = \sum_{k=1}^n a_k$  and  $\Psi_n = \sum_{k=1}^n b_k$ . Now we return to the argument of Burkholder in [10, pp. 1000–1001]: Let  $h_n(\omega) = \sum_{k=1}^n v_k(\omega)Aa_k(\omega)$ ,  $\delta > 0$ ,  $\beta > \delta + 1$  and define stopping times

$$\mu(\omega) = \inf\{n | \|h_n(\omega)\|_Y > \lambda\}, \quad \nu(\omega) = \inf\{n | \|h_n(\omega)\|_Y > K\beta\lambda\}$$

and

$$\sigma(\omega) = \inf\{n | \|\varphi_n(\omega)\| > \delta\lambda \text{ or } d_n^*(\omega) > \frac{1}{4}\delta\lambda\}.$$

Let  $u_k = 1_{[\mu < k \leq \nu \wedge \sigma]}$  which is  $A_{k-1}$ -measurable. Define  $\tilde{\varphi}_n$  to be the transform of  $\varphi_n$  by  $u_k$ . It is clear that  $\tilde{\varphi}^* \leq 3\delta\lambda 1_{[h^* > \lambda]}$ , so that

$$(1.7) \quad \|\tilde{\varphi}\|_1 \leq 3\delta\lambda P[h^* > \lambda].$$

Let  $\tilde{h}_n$  be the transform of  $h_n$  by  $\{u_k\}$  and use  $\|\alpha + \beta\|_Y \geq (1/K)\|\alpha\|_Y - \|\beta\|_Y$  to see that

$$P[h^* > K\beta\lambda, \varphi^* \vee (4d^*) \leq \delta\lambda] \leq P[\tilde{h}^* > (\beta - \delta - 1)\lambda],$$

so that

$$P[h^* > K\beta\lambda, \varphi^* \vee (4d^*) \leq \delta\lambda] \leq c \frac{3\delta\lambda}{(\beta - \delta - 1)\lambda} P[h^* > \lambda]$$

using (b) and (1.7). It follows that

$$(1.8) \quad \mathbf{E}\Phi(h^*) \leq c_3 \mathbf{E}\Phi(\varphi^* \vee 4d^*)$$

by Lemma 7.1 of [8]. We now return to the desired inequality for  $Ag$ . Since  $Ag_n = h_n + \sum_{k=1}^n v_k Ab_k$ , we see that

$$(1.9) \quad Ag^* \leq Kh^* + K\|A\| \sum_{k=1}^{\infty} \|v_k b_k\|_X \leq Kh^* + K \sum_{k=1}^{\infty} \|b_k\|_X.$$

Note that the  $\Delta_2$  condition and (1.6) give us

$$(1.10) \quad \mathbf{E}\Phi(\varphi^* \vee 4d^*) \leq \mathbf{E}\Phi(\varphi^* + 4d^*) \leq c_4[\mathbf{E}\Phi(\varphi^*) + \mathbf{E}\Phi(f^*)],$$

and that  $\varphi_n = f_n - \Psi_n$  together with (1.6) gives us

$$(1.11) \quad \mathbf{E}\Phi(\varphi^*) \leq c_5\mathbf{E}(f^*).$$

Putting together (1.9), (1.10) and (1.11) we finally arrive at the desired inequality

$$(1.12) \quad \mathbf{E}\Phi(Ag^*) \leq c_6\mathbf{E}\Phi(f^*).$$

If we let  $\Phi = \lambda^p$  with  $p > 1$ , Doob's inequality gives us  $\|f^*\|_p \leq q\|f\|_p$ , and  $\|Ag\|_p = \lim \|Ag_n\|_p \leq \|Ag^*\|_p$  gives, finally, condition (c).

(f)  $\Rightarrow$  (a): This implication is not explicit in [10]. Suppose that  $f, g$  and  $v$  are as above, and that  $(Ag)^* > 1$ , a.e. There is an  $N$  such that  $P[(Ag_N)^* > 1] > \frac{7}{8}$ , so we may restrict ourselves to the finite martingale,  $f_N$  with difference sequence  $d_1, \dots, d_N$ . Further, since  $A$  is bounded, hence continuous, and since  $\|\cdot\|_Y$  is a continuous quasinorm,

$$\tau(\omega) = \inf\{n \mid \|Ag_n\| > 1\}$$

is a stopping time. If we then replace  $f_n$  by  $f_{\tau \wedge n}$ , we have a martingale with the property that  $\|Ag_{\tau \wedge N}\|_Y > 1$  on a set of probability greater than  $\frac{7}{8}$ . To simplify matters, let us assume that we started with such a martingale, since also  $\|f_{\tau \wedge N}\|_{L^1} \leq \|f_N\|_{L^1}$ .

We start, then, with a finite martingale,  $f_1, f_2, \dots, f_N$  with difference sequence  $(d_k)$ , and a predictable sequence  $(u_k)$  such that the transform,  $g$ , of  $f$  by  $u$  satisfies  $P[\|Ag_N\|_Y > 1] > \frac{7}{8}$ , and call  $\|f_N\|_{L^1} = \alpha$ . Set  $\lambda = 8\alpha$ , so that Doob's inequality gives us  $P[f_N^* \leq \lambda] > \frac{7}{8}$  as well. Next, let  $A_k = [f_{k-1}^* \leq \lambda] \in \mathcal{F}_{k-1}$ , and  $B_k = [\|d_k\|_X > 2\lambda] \in \mathcal{F}_k$ . The set  $B_k$  has no reason to be in  $\mathcal{F}_{k-1}$ , a priori. If we set  $\Delta_k = d_k 1_{A_k}$ , then  $\sum_{k=1}^j \Delta_k$  is a stopped version of  $f$  which agrees with  $f$  on  $[f_N^* \leq \lambda]$ .

We need to produce from  $\sum \Delta_k$  an  $L^\infty$ -bounded martingale, and so we need to control the jumps of the  $\Delta_k$ 's. We will control the martingale difference sequence  $z_k - E_{k-1}z_k$ , where  $z_k = \Delta_k 1_{B_k} = d_k 1_{A_k \cap B_k}$ . The  $z_k$ 's are disjointly supported, and so

$$\sum_{k=1}^N \|E_{k-1}z_k\|_1 \leq \sum \|z_k\|_1 = \|\sum z_k\|_1 \leq 2\|f\|_1 = 2\alpha.$$

Thus, also  $\sum \|z_k - E_{k-1}z_k\|_1 \leq 4\alpha$ , and  $\|(E_{k-1}z_k)^*\|_1 \leq 2\alpha$ . Next, let

$$C_k = \left[ \left( \sum_{j=1}^k E_{j-1}(z_j) \right)^* < \lambda \right] \in \mathcal{F}_{k-1},$$

and consider the martingale difference sequence

$$D_k = (\Delta_k - z_k + E_{k-1}z_k)1_{C_k} = d_k 1_{(A_k \setminus B_k) \cap C_k} + E_{k-1}(z_k)1_{C_k}.$$

We have  $A_k \setminus B_k \subset A_k \subset A_{k-1} \setminus B_{k-1}$  and  $C_k \subset C_{k-1}$  so that for each  $\omega$ , there is an  $r = r(\omega)$ ,  $0 \leq r \leq j$  with  $\sum_{k=1}^j d_k 1_{(A_k \setminus B_k) \cap C_k} = \sum_{k=1}^r d_k(\omega)$ . Also, for each  $\omega$ , there is an  $s(\omega)$ ,  $0 \leq s \leq j$ , with  $\sum_{k=1}^j E_{k-1}(z_k)1_{C_k} = \sum_{k=1}^s E_{k-1}(z_k)1_{C_k}$ . These give

$$\left\| \sum_{k=1}^j d_k 1_{(A_k \setminus B_k) \cap C_k} \right\|_X \leq 3\lambda \quad \text{and} \quad \left\| \sum_{k=1}^j E_{k-1}(z_k)1_{C_k} \right\|_X \leq \lambda.$$

In other words,  $\|\sum_{k=1}^j D_k\|_{L_\infty(X)} \leq 4\lambda$ . From (1.3) we get

$$(1.13) \quad \left\| \sum u_k A(D_k) \right\|_{L_q(Y)} < 4C\lambda = 32C\alpha.$$

Notice, also, that the boundedness of  $A$  and the triangle inequality give

$$\left\| \sum u_k 1_{C_k} A(z_k - E_{k-1}(z_k)) \right\|_{L_1(Y)} \leq 4\alpha.$$

Let  $S$  be the event

$$S = \left[ \|Ag_N\|_Y > 1 \right] \cap A_N \cap \left[ (E_{N-1}z_N)^* \leq 2\lambda \right] \\ \cap \left[ \left\| \sum_{k=1}^N u_k 1_{C_k} A(z_k - E_{k-1}(z_k)) \right\|_Y \leq 32\alpha \|A\| \right].$$

From the estimates above and Chebyshev's inequality we see that  $P(S) \geq \frac{1}{2}$ . If  $\omega \in S$ , since  $S \supset A_N \cup C_N$ ,

$$\left\| \sum_{k=1}^N u_k(\omega) 1_{C_k}(\omega) A(\Delta_k(\omega)) \right\|_Y = \left\| \sum_{k=1}^N u_k(\omega) A d_k(\omega) \right\|_Y > 1.$$

Therefore, for these  $\omega$ 's, we have

$$\left\| \sum_{k=1}^N u_k(\omega) A(D_k(\omega)) \right\|_Y > \frac{1}{K} - 32\alpha \|A\|.$$

From this and (1.13), we get  $2^{-1/q}(1/K - 32\alpha \|A\|) \leq 32C\alpha$ , so that

$$(1.14) \quad \|f\|_1 = \alpha \geq (32K(2^{1/q}C + \|A\|))^{-1}$$

as desired for (1.4).

The purpose of Theorem 1.1 for us is to allow deduction of the boundedness of the vector-valued Hilbert transform as a map from  $L_2(L_1(T))$  to  $L_2(L_q(T))$  for  $0 < q < 1$ . Toward this, let  $X$  be an  $n$ -dimensional symmetric sequence space, and assume, without loss of generality, that  $\lambda_k = \|\sum_{i=1}^k e_i\|$  is concave as a function of  $k$ . Assume further that  $\lambda_n = 1$  [15, I, p. 119]. Let  $0 < \alpha < 1$ , and define  $X_\alpha$  to be  $X$  with the quasinorm

$$\|x\|_\alpha = \left\| |x|^\alpha \right\|^{1/\alpha}.$$

**THEOREM 1.2.** *For each  $\alpha \in (0, 1)$ , the natural injection of  $X$  to  $X_\alpha$  is an MT operator with constants (Theorem 1.1) independent of  $n$ .*

**PROOF.** We prove that there is a constant  $M$  such that for all  $n$  and all  $X$ -valued martingales  $G_n = \sum_{k=1}^n \Delta_k$ ,

$$(1.15) \quad \|A\tilde{G}_n^*\|_{L_\alpha} \leq M \|G_n\|_{L_\infty(X)}.$$

For this proof, it is convenient to use the result of Trautman [20] mentioned above that it is sufficient to consider only nonrandom martingale transforms, i.e.  $\tilde{G}_n = \sum_{k=1}^n \epsilon_k \Delta_k$  with  $\epsilon_k = \pm 1$  for all  $k$ .

The first part of the proof shows that it is enough to prove (1.15) for dyadic martingales defined on  $[0, 1]$ . The argument is a standard perturbation one. Since  $G_n$  and the  $\Delta_k$ 's are strongly measurable, we can, for any  $\eta > 0$ , choose finite subfields,  $\mathcal{F}_k \subset \mathcal{F}_{k+1}$ , such that

$$(1.16) \quad \|G_k - \mathbf{E}(G_n | \mathcal{F}_k)\|_{L_\infty(X)} < 4^{-k-1}\eta \|G_n\|_{L_\infty(X)}.$$

Set  $B_k = \mathbf{E}(G_n | \mathcal{F}_k) - \mathbf{E}(G_n | \mathcal{F}_{k-1})$  and  $C_k = \Delta_k - B_k$ , so that  $\|C_k\|_{L_\infty(X)} \leq 2^{-k}\eta \|G_n\|_{L_\infty(X)}$  by (1.16). One has

$$\left\| \sum \varepsilon_k A C_k \right\|_{L_\alpha(X_\alpha)} \leq \|A\| \left\| \sum \varepsilon_k C_k \right\|_{L_1(X)} \leq \eta \|A\| \|G_n\|_{L_\infty^X}.$$

Therefore

$$(1.17) \quad \|A \tilde{G}_n^*\|_{L_\alpha} \leq K \eta \|A\| \|G_n\|_{L_\infty(X)} + K \left\| \sum \varepsilon_k A B_k \right\|_{L_\alpha(X_\alpha)}$$

where  $K$  is the ‘‘triangle inequality’’ constant for  $X_\alpha$ .

Next, since each  $\mathcal{F}_k$  is a finite field, we may as well assume that  $\Omega$  is  $[0, 1]$ . Let  $\mathcal{D}_k$  denote the usual dyadic field, i.e.,  $\mathcal{D}_k = \langle [(l-1)/2^k, l/2^k) : 1 \leq l \leq 2^k \rangle$ . We can now choose  $m_1 < m_2 < \dots < m_n$  so that

$$(1.18) \quad \left\| \sum_{j=1}^k B_j - \mathbf{E}\left(\sum_{j=1}^n B_j | \mathcal{D}_{m_k}\right)\right\|_{L_1(X)} \leq 4^{-k-1}\eta \|G_n\|_{L_\infty(X)}.$$

Set  $D_k = \mathbf{E}(\sum_{j=1}^n B_j | \mathcal{D}_{m_k}) - \mathbf{E}(\sum_{j=1}^n B_j | \mathcal{D}_{m_{k-1}})$  and  $H_k = B_k - D_k$ . As before,  $\|\sum \varepsilon_k A H_k\|_{L_\alpha(X_\alpha)} \leq \eta \|A\| \|G_n\|_{L_\infty(X)}$ . We now have

$$(1.19) \quad \|A \tilde{G}_n^*\|_{L_\alpha} \leq \eta (K + K^2) \|A\| \|G_n\|_{L_\infty(X)} + K^2 \left\| \sum \varepsilon_k A D_k \right\|_{L_\alpha(X_\alpha)}.$$

Notice that  $\|\sum_{k=1}^n D_k\|_{L_\infty(X)} \leq \|\sum_{k=1}^n \Delta_k\|_{L_\infty(X)}$  by construction, and that  $\sum_{k=1}^l D_k$  is a blocking of the dyadic martingale  $\mathbf{E}(\sum_{j=1}^n B_j | \mathcal{D}_k)$ . Thus, if we prove (1.15) for dyadic martingales, we will have proved it in general.

Let  $F$ , then, be a dyadic martingale on  $[0, 1]$  with values in  $X$ . We write  $F_k = \sum f_k^{(j)} e_j$ , where each  $f_k^{(j)}$  is a real dyadic martingale. The reason for the reduction to the dyadic case is the fact that real dyadic martingales have  $|f_k - f_{k-1}| = |d_k|$  predictable, so that there is a distributional inequality of the form

$$(1.20) \quad P[\tilde{f}^* > 2\delta, f^* \leq \varepsilon\delta] \leq \gamma(\varepsilon) P[\tilde{f}^* > \delta],$$

with  $\gamma(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  (see, e.g., [8]). Use also the facts that  $|\tilde{f}|^\alpha = \int_0^\infty 1_{[|\tilde{f}|^\alpha > t]} dt$  and the concavity of  $\lambda(t)$ , the piecewise linear extension of  $\lambda(k) = \lambda_k$  for  $k = 1, \dots, m$  and  $\lambda(t) = 1$  for  $t \geq n$ . One obtains

$$(1.21) \quad \begin{aligned} \int_0^1 \left\| \sum \widetilde{f^{(j)}}^\alpha e_j \right\| dP &\leq \int_0^\infty \int_0^1 \left\| \sum 1_{[|\tilde{f}|^\alpha > t]} e_j \right\| \\ &\leq \int_0^\infty \lambda\left(\sum P\left[|\widetilde{f^{(j)}}| > t^{1/\alpha}\right]\right) dt. \end{aligned}$$

It follows from (1.20) that

$$(1.22) \quad \sum_j P\left[|\widetilde{f^{(j)}}^*| > t^{1/\alpha}\right] \leq \sum P[f_j^* > \varepsilon t^{1/\alpha}] + \gamma(\varepsilon) \sum P\left[|\widetilde{f^{(j)}}^*| > \frac{1}{2} t^{1/\alpha}\right].$$

Using the concavity of  $\lambda$  again, it follows that for some  $\varepsilon > 0, 0 < \delta < \frac{1}{2}$ ,

(1.23)

$$\lambda\left(\sum P\left[\left|\widetilde{f^{(j)}}^*\right| > t^{1/\alpha}\right]\right) \leq \lambda\left(\sum P\left[f_j^* > \varepsilon t^{1/\alpha}\right]\right) + \delta\lambda\left(\sum P\left[\left|\widetilde{f^{(j)}}^*\right| > \frac{1}{2}t^{1/\alpha}\right]\right).$$

Integrating, we find that

$$(1.24) \quad \int_0^\infty \lambda\left(\sum P\left[\left|\widetilde{f^{(j)}}^*\right| > t^{1/\alpha}\right]\right) dt \leq c \int_0^\infty \lambda\left(\sum P\left[f_j^* > t^{1/\alpha}\right]\right) dt.$$

By Doob’s inequality,  $P[f^* > s] \leq (1/s)\int_{\{|f|>s\}}|f|dP$  for  $s > 0$ . Further, a moment’s reflection shows that  $\lambda(\sum|x_j|) \leq \|\sum x_j e_j\|$  if  $|x_j| \geq 1$  or  $x_j = 0$  for all  $j$  (e.g. [4]).

We have, then,

$$(1.25) \quad \lambda\left(\sum P\left[f_j^* > t^{1/\alpha}\right]\right) \leq \lambda\left(t^{-1/\alpha}\sum_j \int_{\{|f_j|>t^{1/\alpha}\}}|f_j|dP\right) \\ \leq \left\|\lambda\left(t^{-1/\alpha}\sum|f_j|1_{\{|f_j|>t^{1/\alpha}\}}\right)\right\|_{L_\infty(\Omega)} \leq t^{-1/\alpha}\|F\|.$$

This finishes the proof since

$$\int_0^\infty (1 \wedge t^{-1/\alpha}\|F\|_\infty) dt = \left(\frac{1}{1-\alpha}\right)\|F\|_\infty^\alpha.$$

If one recognizes that the above estimates are local in nature and sets  $X = l^n$  in the theorem, one has  $X_\alpha = l^n$  and gets the following

**COROLLARY 1.3.** *Martingale transforms are bounded as operators from  $L_p(L_1)$  to  $L_p(L_q)$  when  $1 < p < \infty$  and  $0 < q < 1$ .*

In [9], Burkholder approximated the vector-valued Hilbert transform

$$(1.26) \quad \mathcal{H}f = \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon < |x-y| < \pi} \frac{f(y)}{x-y} \frac{dy}{\pi}$$

by (finite) martingale transforms to show that if  $f$  takes its values in a UMD space, then the Hilbert transform has (up to constants) the same boundedness properties as its scalar ancestor.

Since we use a somewhat more complicated version in the proof of Theorem 3.2, and for the sake of completeness, we include here a sketch of Burkholder’s approximation argument [9]. Let  $f \in L_1(X)$  and let  $u$  be its Poisson integral in the disc  $\{|z| < 1\}$ . Let  $\{r_k\}$  denote the Rademacher functions on  $[0, 1]$ ,  $\delta > 0$  and  $d_k = \delta r_{2k-1}$ ,  $e_k = \delta r_{2k}$ . With  $Z_k(t) = d_k(t) + ie_k(t)$ , the stopping time  $\tau = \inf\{n | |n_k| > 1\}$ , and  $w_k(t) = 1_{[\tau \geq k]}$ . Define martingales

$$(1.27) \quad U_n = \sum_{k=1}^n w_k \{u_x(Z_{k-1})d_k + u_y(Z_{k-1})e_k\}, \quad \text{and} \\ V_n = \sum_{k=1}^n w_k \{v_x(Z_{k-1})d_k + v_k(Z_{k-1})e_k\},$$

where  $v$  is the harmonic conjugate to  $u$ . Burkholder shows that  $U_n$  and  $V_n$  are, for  $\delta$  small and  $n$  large, good approximations in norm to the boundary values of  $u$  and  $v$ . We note that, in distribution,  $V_n$  is a martingale transform of  $U_n$ : it is clear that  $U_n$  has the same distribution as

$$W_n = \sum_{k=1}^n w_k \{ u_x(Z_{k-1})e_k + u_y(Z_{k-1})d_k \}.$$

The Cauchy-Riemann equations show that

$$V_n = \sum_{k=1}^n w_k \{ u_x(Z_{k-1})e_k - u_y(Z_{k-1})d_k \}.$$

$V_n$  is not a martingale transform of  $U_n$  as defined in §1. In order to formally deduce Corollary 1.4 from Corollary 1.3 above, we should have discussed a slightly larger class of transforms, which we did not do for the sake of simplicity. In fact, in the present case of the identity map  $L_p(D, L_1) \rightarrow L_p(D, L_q)$ , with  $D$  the Cantor group,  $U_n$  is  $L_1$  valued, and  $U_n, V_n$  have the same martingale square function with respect to the  $D$  variable. From Corollary 1.3 and the  $L_p(L_q)$  lattice structure, it follows that the  $L_p(L_q)$  norm of this square function can be estimated by the  $L_p(L_1)$  norm of  $U_n$ . Hence, to obtain Corollary 1.4, it suffices to use the following lemma.

**LEMMA.** *Let  $F(\epsilon, \omega)$  be a scalar function on the product space  $D \times \Omega$ , and  $S(F)$  the martingale square function with respect to the  $\epsilon$ -variable. Then, for  $p > q$*

$$\|F\|_{L_p(L_q)} \leq C(p, q) \|S(F)\|_{L_p(L_q)}.$$

**PROOF.** There is a function  $g \geq 0$  on  $D$ ,  $\|g\|_r = 1$ , where  $r/(r - 1) = p/q$  satisfying

$$\|F\|_{L_p(L_q)}^q = \iint |F(\epsilon, \omega)|^q g(\epsilon) d\epsilon d\omega.$$

It is possible, since  $r > 1$ , to replace  $g$  by a function  $g_1 \geq g$  with  $\|g_1\|_r \leq 2$ , which is an  $A_1$  weight in the Muckenhoupt sense. Thus,  $g_1^* \leq cg_1$ , where  $g_1^*$  is the maximal function. Since  $g_1$  is an  $A_\infty$  weight, it follows from [21] that for fixed  $\omega$

$$\int |F(\epsilon, \omega)|^q g_1(\epsilon) d\epsilon \leq C' \int |S(F)(\epsilon, \omega)|^q g_1(\epsilon) d\epsilon.$$

Integrating in  $\omega$ , the proof is easily completed from Hölder's inequality.

**COROLLARY 1.4.** *If  $1 < p < \infty$  and  $0 < q < 1$ , then*

$$\mathcal{H}: L_p(\mathbf{T}, L_1) \rightarrow L_p(\mathbf{T}, L_q).$$

The reader familiar with applications of Brownian motion to such questions will recognize that the above is an approximation of the formulas

$$(1.28) \quad u(Z_t) = \int_0^t u_x dX_s + \int_0^t u_y dY_s \quad \text{and} \quad v(Z_t) = \int_0^t u_x dX_s - \int_0^t u_y dY_s,$$

where  $Z_t$  is a Brownian motion starting at 0 in the plane and stopped at the boundary of the disc, and where (1.28) comes from the Ito calculus.

It is Corollary 1.4 which we shall want to use in later sections, in particular for the boundedness of the Riesz projection in this vector-valued setting.

**2. Cotype 2 of  $L_1/H_1$  revisited.** The results of the previous section can be used to give a shortened proof of the fact proved in [3, 6] that  $L_1/H_1$  has cotype 2. That is, if  $(\varepsilon_j)$  denotes an independent sequence of Bernoulli variables, and if  $(f_j)$  is a sequence in  $L_1/H_1^0$ , then

$$(2.1) \quad \int_{\Omega} \left\| \sum f_j \varepsilon_j \right\|_{L_1/H_1^0} d\omega \geq c \left( \sum \|f_j\|_{L_1/H_1^0}^2 \right)^{1/2}.$$

The major advantage of the proof given here is the explicit nature of the lifting of  $\sum f_j \varepsilon_j$  to  $L_1$ .

The fact that the cotype 2 inequality holds in  $L_1$  is classical, and scalar (Khintchine’s inequality). An obvious line of attack, then, is to lift the expression  $\sum f_j \varepsilon_j$  to  $L_1(L_1)$  and then to use that inequality. The difficulty is that good liftings are no longer necessarily of the form  $\sum F_j \varepsilon_j$ , and this is the point to be overcome by the previous theorem.

**THEOREM 2.1 [3, 6].**  $L_1/H_1^0$  has cotype 2.

**PROOF.** Let  $f_1, \dots, f_n \in L_1/H_1^0$  and let  $q: L_1 \rightarrow L_1/H_1^0$  be the quotient map. Let  $F(\varepsilon)$  be a lifting of  $\sum f_j \varepsilon_j$  to  $L_1(\varepsilon, L_1(\mathbb{T}))$  such that  $\|F(\varepsilon)\|_{L_1(\mathbb{T})} < (1 + \alpha) \|\sum f_j \varepsilon_j\|_{L_1/H_1^0}$  for each  $\varepsilon$ , where  $\alpha$  is small. Expand  $F(\varepsilon)$  in its Walsh series:

$$F(\varepsilon) = \varphi_0 + \sum \varepsilon_j \varphi_j + \sum_{k \geq 2} \sum_{|\beta|=k} w_{\beta}(\varepsilon) \varphi_{\beta},$$

so that  $q(\varphi_j) = f_j$ , and so that  $\varphi_0$  and  $\varphi_{\beta}$  are in  $H_1^0$  when  $|\beta| \leq 2$ . Define

$$h(\theta) = \int |F(\varepsilon)(\theta)| d\varepsilon.$$

Fix  $\chi > 0$  and let  $\Phi$  be an outer function on the disc  $D = \{|z| \leq 1\}$  such that  $|\Phi| = h + \chi$  on  $\mathbb{T}$ . Let  $\mathcal{R}_-$  denote the negative Riesz transform, i.e. the projection onto  $[e^{ik\theta} | k \leq 0]$  on the circle. It is clear that

$$\mathcal{R}_- [\Phi^{-1/2} F(\varepsilon)] = \sum \varepsilon_j \mathcal{R}_- [\Phi^{-1/2} \varphi_j].$$

Setting  $\alpha = \frac{1}{2}$  in Corollary 1.4, we get

$$\left\| \mathcal{R}_- (\Phi^{-1/2} F(\varepsilon)) \right\|_{L_2(L_{1/2})} \leq c \left\| \Phi^{-1/2} F(\varepsilon) \right\|_{L_2(L_1)}.$$

Since  $\mathcal{R}_- (\Phi^{-1/2} F(\varepsilon))$  is in the span of the Rademacher functions, and since, by Khintchine’s inequality the  $L_{1/2}$  and  $L_1$  norms agree up to a constant there, we have  $c > 0$  such that

$$(2.2) \quad c \left\| \left( \sum \left| \mathcal{R}_- [\Phi^{-1/2} \varphi_j] \right|^2 \right)^{1/2} \right\|_{L_2(\mathbb{T})} \leq \left\| \Phi^{-1/2} h \right\|_{L_2(\mathbb{T})} \leq \left\| h^{1/2} \right\|_{L_2(\mathbb{T})} = \left( \int h(\theta) \frac{d\theta}{2\pi} \right)^{1/2}.$$

Since  $\Phi^{1/2} \mathcal{R}_- [\Phi^{-1/2} \varphi_j] - \varphi_j \in H_1^0$ , we have  $q(\varphi_j) - q(\Phi^{1/2} \mathcal{R}_- [\Phi^{-1/2} \varphi_j]) = f_j$ .

Setting  $\Psi_j = \Phi^{1/2}\mathcal{R}_-[\Phi^{-1/2}\varphi_j]$ , we have

$$(2.3) \quad \int \left( \sum |\Psi_j|^2 \right)^{1/2} = \int \left( \sum |\mathcal{R}_-[\Phi^{-1/2}\varphi_j]|^2 \right)^{1/2} |\Phi|^{1/2} \\ \leq c \left( \int h \right)^{1/2} \left( \int |\Phi| \right)^{1/2}$$

by (2.2) and the Cauchy-Schwarz inequality. By the definitions of  $h$  and  $\Phi$ , the right-hand side of (2.3) is dominated by

$$c \left( \int \|\sum f_j \varepsilon_j\| d\varepsilon \right)^{1/2} \left( \int (\|\sum f_j \varepsilon_j\| + \chi) d\varepsilon \right)^{1/2}.$$

Also,  $2^{-1/2}(\sum \|f_j\|^2)^{1/2} \leq f(\sum |\Psi_j|^2)^{1/2}$ ; so, by letting  $\chi \rightarrow 0$ , we obtain the desired inequality (2.1).

The advantage of this proof is the “explicit” form,  $\Phi^{1/2}\mathcal{R}_-[\Phi^{-1/2}F(\varepsilon)]$ , of the lifting of  $\sum f_j \varepsilon_j$  from  $L_1/H_1^0$  to  $L_1$ .

**3. Complex uniform convexity.** In [12], notions of complex uniform convexity were introduced. In particular, a (complex) Banach space  $X$  is called uniformly PL-convex if for all  $\varepsilon > 0$  there is  $\delta > 0$  such that if  $x, y \in X$ ,  $\|x\| = 1$  and  $\|y\| \geq \varepsilon$  then

$$(3.1) \quad \int_0^{2\pi} \|x + e^{i\theta}y\| \frac{d\theta}{2\pi} \geq 1 + \delta.$$

In that paper, it was shown that spaces such as  $L_1$ ,  $C_1$  (the trace class), and duals of  $C^*$ -algebras are all uniformly PL-convex. It was also shown that uniformly PL-convex spaces have some cotype. In contrast to the result of the previous section here, it was also shown, using a result of Pisier, that the space  $L_1/H_1$  cannot be renormed to be uniformly PL-convex. The reason, roughly, that this occurs is that good complex martingales with values in  $L_1/H_1$  cannot generally be lifted to good martingales with values in  $L_1$ .

The renorming theorems of Enflo [14] and Pisier [18] have complex analogs in [12]. Here we look again at these renormings and use Corollary 1.4 to prove that the quotient spaces  $L_1/Y$  are uniformly PL-convex whenever  $Y$  is a reflexive subspace of  $L_1$ .

For this section, we do not need the full generality of uniform PL-convexity, but just the notion of 2-uniform PL-convexity:

$$(3.2) \quad \left( \int \|x + e^{i\theta}y\|^2 \frac{d\theta}{2\pi} \right)^{1/2} \geq (\|x\|^2 + \rho^2\|y\|^2)^{1/2}$$

for all  $x, y \in X$  where  $\rho > 0$  depends only on  $X$ . To avoid the cumbersome terminology, we shall call spaces satisfying (3.2) 2-cuc spaces (complex uniformly convex).

We first want an analog of Pisier’s renorming theorem [18] via martingales. This result appears in [12], but the martingales there are defined a bit differently.

Let  $\Omega$  denote the product space  $\mathbf{T} \times D \times \mathbf{T} \times D \times \cdots$  where  $D = \{-1, 1\}$ , and where the natural product Borel field,  $\mathcal{B}$ , is given. Further, let the filtration  $\mathcal{B}_n$  denote all the Borel sets in the product field depending only on the first  $n$

coordinates. Consider martingales with values in  $X$  of the form

$$(3.3) \quad F(\omega) = x_0 + x_1 e^{i\theta_1} + F_2(\theta_1) \varepsilon_2 + F_3(\theta_1, \varepsilon_2) e^{i\theta_2} + \dots$$

which are adapted to the filtration  $(\mathcal{B}_n)$ . We shall say that such a martingale is in class  $\mathcal{F}$  if there is  $\rho > 0$  such that

$$(3.4) \quad \|F\|_{L_2(X)} \geq \rho \left( \sum_{j=1}^{\infty} \|F_{2j-1}\|_{L_2(X)}^2 \right)^{1/2},$$

where  $F_1 = x_1$ . If there is a constant  $\rho > 0$  such that every martingale of the form (3.3) valued in  $X$  satisfies (3.4), then one can define a new norm on  $X$  by

$$(3.5) \quad \|\!|x|\!\|^2 = \inf \left\{ \|F\|_2^2 - \frac{\rho^2}{2} \sum \|F_{2j-1}\|_2^2 \mid \int F = x, F \in \mathcal{F} \right\}.$$

**THEOREM 3.1.** *Given  $\rho = \rho(X)$  as above,  $\|\!|\cdot|\!\|$  defines an equivalent norm on  $X$  under which  $X$  is 2-cuc.*

The proof is as in [12]. Some remarks are needed concerning the form of the martingales. The dependence on the two-point spaces,  $D$ , in the product is used to guarantee that (3.5) yields a norm. The dependence on the copies of the circle  $\mathbf{T}$  is used to give the 2-cuc property from inequality (3.4). The class  $\mathcal{F}$ , then, is the smallest class of martingales closed under pasting together of these two requirements.

If one insisted that the terms on the right-hand side in (3.4) included the even-numbered terms, then Pisier's theorem would force  $X$  to be uniformly convexifiable with modulus  $c\varepsilon^2$ . This is, of course, stronger than the 2-cuc property.

It is also clear that a 2-cuc space satisfies (3.4). One simply uses the inequality (3.2) repeatedly on martingales in  $\mathcal{F}$  dropping the even-numbered terms as they occur by expectation.

The main result of this section is

**THEOREM 3.2.** *Let  $Y$  be a reflexive subspace of  $L_1$  whose type is  $p - 1$  and whose type  $p$  constant in  $T_p$ . Then  $L_1/Y$  can be renormed to be 2-cuc, and the constant of norm equivalence depends only on  $T_p$ .*

**PROOF.** Fix  $0 < \alpha < 1$ , and, after a change of density if necessary, let  $c > 0$  be such that

$$(3.6) \quad \|y\|_{L_\alpha} \geq c \|y\|_{L_1} \quad \text{for } y \in Y$$

[19], where  $c = c(p, T_p)$ .

We need to verify (3.4) for finite martingales in  $\mathcal{F}$  taking values in  $L_1/Y$ . We once again use a lifting argument. Let  $q: L_1 \rightarrow L_1/Y$  be the quotient map, and let  $F \in \mathcal{F}(L_1/Y)$ . Again, for each  $\omega$  let  $\tilde{F}$  be a measurable function such that

$$(3.7) \quad \|\tilde{F}(\omega)\|_{L_1} \leq (1 + \varepsilon) \|F(\omega)\|_{L_1/Y}$$

where  $\varepsilon$  is small and  $q(\tilde{F}(\omega)) = F(\omega)$  for each  $\omega$ . Since the martingales are finite, that is  $F = F_0, F_1, \dots, F_m, F_m, \dots$ , we have identified, where appropriate,  $F$  with

its limit  $F_m$ . Set  $E_n = E(\mathcal{B}_n)$ , and write

$$(3.8) \quad \begin{aligned} \tilde{F}(\omega) &= \tilde{F}_0 + (E_1 - E_0)\tilde{F} + \dots \\ &= \tilde{F}_0 + \tilde{F}_1(\theta_1) + \tilde{F}_2(\theta_2, \varepsilon_1) + \tilde{F}_3(\theta_1, \varepsilon_1, \theta_2) + \dots \end{aligned}$$

We have

$$(3.9) \quad q(\tilde{F}_0) = x_0, \quad q(\tilde{F}_{2j}) = F_{2j}, \quad q\left(\int \tilde{F}_{2j-1} e^{-i\theta_j} \frac{d\theta_j}{2\pi}\right) = F_{2j-1},$$

and  $\int F_{2j-1} e^{-ik\theta_j} \frac{d\theta_j}{2\pi} \in Y$  for all  $j$ , and  $k \neq 1$ .

Consider the following transform on  $\mathcal{F}(L_1)$ : Let  $\mathcal{R}_{j-}$  be the negative Riesz transform with respect to the variable  $\theta_j$ . Let

$$(3.10) \quad T(\tilde{F}) = \sum \mathcal{R}_{j-} \tilde{F}_{2j-1}(\omega).$$

$T(\tilde{F})$  can be approximated as closely as desired by a martingale transform just as in [9] and the remarks following Corollary 1.3, since each  $\mathcal{R}_{j-} \tilde{F}_{2j-1}$  can be approximated by a finite martingale transform. We see from Corollary 1.3 that  $T$  is bounded from  $L_2(L_1)$  to  $L_2(L_\alpha)$ . That is, there is  $c_1$  such that

$$(3.11) \quad \|T\tilde{F}\|_{L_2(L_\alpha)} \leq c_1 \|\tilde{F}\|_{L_2(L_1)}.$$

From (3.7), (3.9) and (3.6), we see that there is  $c_2$  such that

$$(3.12) \quad \left\| \sum \mathcal{R}_{j-} \tilde{F}_{2j-1} \right\|_{L_2(Y)} \leq c_2 \|F\|_{L_2(X)}.$$

Subtracting from  $\tilde{F}$ , we get

$$(3.13) \quad \left\| \tilde{F}_0 + \sum \mathcal{R}_+ \tilde{F}_{2j-1} + \sum \tilde{F}_{2j} \varepsilon_j \right\|_{L_2(L_1)} \leq c_3 \|F\|_{L_2(X)}.$$

Apply the same procedure again with the Riesz projection onto  $[e^{2i\theta_j}, e^{3i\theta_j}, \dots]$  to get

$$(3.14) \quad \left\| \tilde{F}_0 + \sum \left( \int e^{-i\theta_j} \tilde{F}_{2j-1} \frac{d\theta_j}{2\pi} \right) + \sum \tilde{F}_{2j} \varepsilon_j \right\|_{L_2(L_1)} \leq c_4 \|F\|_{L_2(X)}.$$

The martingale appearing on the left in (3.14) is in  $\mathcal{F}(L_1)$ , and  $L_1$  is 2-cuc, so we get

$$(3.15) \quad \left( \sum \left\| \int e^{-i\theta_j} \tilde{F}_{2j-1} \frac{d\theta_j}{2\pi} \right\|_{L_2(L_1)}^2 \right)^{1/2} \leq c_5 \|F\|_{L_2(X)}.$$

Clearly

$$\left\| \int e^{-i\theta_j} \tilde{F}_{2j-1} \frac{d\theta_j}{2\pi} \right\| \geq \|F_{2j-1}\|$$

by (3.9), and so we have

$$\left( \sum \|F_{2j-1}\|_{L_2(X)}^2 \right)^{1/2} \leq c_5 \|F\|_{L_2(X)}$$

as desired.

**4. Convergence of analytic martingales.** A special subclass of the martingales considered in the previous section is of particular interest: Let  $\Omega = T^{\mathbb{N}}$  with the natural Borel field and the natural filtration. A martingale of the form

$$(4.1) \quad f(\Theta) = \sum f_j(\theta_1, \dots, \theta_{j-1}) e^{i\theta_j}$$

is called an analytic martingale. Bourgain has used these and their multi-indexed forms, for example in [5], to show that the polydisc algebras  $A(D^n)$  and  $A(D^m)$  are nonisomorphic when  $n \neq m$ .

In this section, we give a result based on transference principles which yields the convergence of these martingales in  $L_1(\Omega)$ . The roots of the argument seem to go back to the work of Bonami in [1], and yield

PROPOSITION 4.1. For  $1 \leq p < \infty$  one has

$$(4.2) \quad \|f(\Theta)\|_{L_p(\Omega)} \sim \|(Sf)(\Theta)\|_{L_p(\Omega)}, \quad \text{where } S(f) = \left(\sum |f_j(\Theta)|^2\right)^{1/2}.$$

PROOF. Assume that  $f_k = \sum \alpha_{k,\vec{j}} \exp(i(j_1\theta_1 + \dots + j_{k-1}\theta_{k-1}))$  with  $\vec{j} = (j_1, \dots, j_{k-1})$  and  $-M_k \leq j_l \leq M_k$  for each  $k, l$ . Introduce an auxiliary variable  $\zeta \in \mathbf{T}$ , and consider the new function

$$(4.3) \quad F(\Theta, \zeta) = \sum f_k(\theta_1 + N_1\zeta, \dots, \theta_{k-1} + N_{k-1}\zeta) e^{i\theta_k + iN_k\zeta}.$$

Due to the finiteness of the supports of the  $f_k$ 's, if one is given a sequence of disjoint intervals,  $I_j$ , in the positive integers with the length  $(I_j) \rightarrow \infty$ , one can recursively choose the  $N_j$ 's so that the support of the  $\zeta$ -variable Fourier transform of  $F$  lies in  $\cup I_j$ .

The Stein multiplier theorem (as in [16], for example) says that there are intervals  $(I_j)$ , as above, and a constant  $c$  such that with  $N_j$ 's chosen as indicated,

$$(4.4) \quad \left\| \sum \epsilon_k f_k(\Theta, \zeta) e^{i(\theta_k + N_k\zeta)} \right\|_{L_1(\zeta)} \leq c \|F(\Theta, \zeta)\|_{L_1(\zeta)},$$

where  $\epsilon_k = \pm 1$  for all  $k$ .

Averaging this inequality over all choices of  $(\epsilon_k)$ , one obtains

$$\left(\sum \|f_k(\Theta, \zeta)\|_{L_1(\zeta)}^2\right)^{1/2} \leq \tilde{c} \|F(\Theta, \zeta)\|_{L_1(\zeta)}.$$

Integrate the left-hand side with respect to  $\Theta$ , and notice that  $\int_{\zeta} \int_{\Theta} (\sum |f_k(\Theta, \zeta)|^2)^{1/2} d\Theta d\zeta = \int (\sum |f_k(\Theta)|^2)^{1/2} d\Theta$  using the change of variables  $\theta_j \rightarrow \theta_j + N_j\zeta$ . Similarly, on the right, we have

$$\int_{\Theta} \|F(\Theta, \zeta)\|_{L_1(\zeta)} d\Theta = \|F(\Theta)\|_{L_1(\Theta)}.$$

REMARK 4.2. The Burkholder-Gundy inequalities [8] give the equivalence of  $\|F\|_p$  and  $\|S(F)\|_p$  in the reflexive  $p$  range, and Doob's inequalities give the equivalence of  $\|F\|_p$  and  $\|F^*\|_p$  in that range. The B. Davis inequalities [11] give the equivalence of  $\|F^*\|_1$  and  $\|S(F)\|_1$  for all martingales, and so all of these functions are equivalent for analytic martingales.

REMARK 4.3. It is clear that the same argument gives the same result for martingales of the form  $\sum f_j(\theta_1, \dots, \theta_j)$  where each  $f_j$  is "analytic" in  $\theta_j$ , that is, as a function of  $\theta_j$ ,  $f_j \in H_1^0(\mathbf{T})$ .

Analytic martingales are just martingale transforms of the independent Steinhaus sequence  $(e^{i\theta_j})$ . One might be tempted to think that this independence is the cause of the validity of (4.2). That is not the case, though, since the martingale “double or nothing” is a martingale transform of the independent Bernoulli sequence, and this is the simplest example of a martingale for which  $\|F\|_{L_1}$  fails to be equivalent to  $\|S(F)\|_{L_1}$ .

The immediate consequence of the proposition is

**COROLLARY 4.4.** *If  $F$  is an  $L_1$ -bounded analytic martingale, then  $F$  converges a.e. and in  $L_1$ .*

One can prove these assertions in several ways, but it is probably best to simply note that such results were proved by Burkholder in [7]. The interest in this result comes from the fact that general martingales cannot be assumed to converge in  $L_1$ . The work of G. Edgar in [13] can also be applied to this special class of martingales to obtain Corollary 4.3.

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