ON $K_3$ OF TRUNCATED POLYNOMIAL RINGS

BY

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ABSTRACT. Group homology spectral sequences are used to investigate $K_3$ of truncated polynomial rings. If $F$ is a finite field of odd characteristic, we show that relative $K_2$ of the pair $(F[t]/(t^q), (t^k))$, which has been identified by van der Kallen and Stienstra, is isomorphic to $K_3(F[t]/(t^k), (t))$ when $q$ is sufficiently large. We also show that $H^3_{SL_2}(Z[t]/(t^k); Z) = Z^{k-1} \oplus Z/24$ and is isomorphic to the associated $K_3$ group modulo an elementary abelian 2-group.

1. Introduction. There are relatively few rings whose third algebraic $K$-groups are known fully. Group homology spectral sequence techniques, however, have proved a useful tool, yielding for example $K_3$ of the quotient rings of the rational integers [ALSS]. This paper again applies these methods. Our main results are for $k \geq 2$:

1.1 Theorem. There are exact sequences

$$Z \oplus Z \oplus Z/2 \overset{1 \oplus 1 \oplus 1}{\longrightarrow} K_3(Z[t]/(t^{k+1}))$$

$$= K_3(Z \oplus Z^k \oplus U_k + 1 \to K_3(Z[t]/(t^k)) \to Z,$$

where $U_{k+1}$ is an elementary abelian 2-group of rank at most $[(k+1)/2]$, and $l = 0$ if $k$ is even.

1.2 Theorem. If $F$ is a finite field of odd characteristic, there is an isomorphism

$$\partial^{-1} \circ \Delta_k: K_1(F[t]/(t^{2k}), (t))/(1 - \alpha t^k: \alpha \in F) \to K_3(F[t]/(t^k), (t)).$$

These theorems use, and sharpen, results of Van der Kallen and Stienstra [S, VKS]. Stienstra obtains 1.1 modulo information on the torsion group $U_k$; the theorem with $k = 2$ has essentially been proved by Kassel. Van der Kallen and Stienstra define the isomorphism $\Delta_k$ onto the relative $K$-group $K_2(F[t]/(t^k), (t^k))$, whenever $q \gg k$ and $F$ is a perfect field. Theorem 1.2 with $k = 2$ and 3 has been proved by Snaith, Lluis and Aisbett [LS, ALS].

The Hochschild-Serre spectral sequences studied are associated to the reduction $\Pi^k: SL_n R[t]/(t^k) \to SL_n R$, where $R$ is initially any commutative ring with identity for which $SK_1 R = 0$. More general results than those of Theorems 1.1 and 1.2 would need information on certain $E^2_{\ast \ast}$ terms.
The remainder of this paper is organized as follows. §2 lists notation, and introduces Hochschild-Serre spectral sequences from our perspective. §3 recursively estimates the second homology groups of the kernel of \( \Pi_k \), by means of spectral sequences associated to the reduction \( \ker \Pi_k \to \ker \Pi_{k-1} \); this also gives information on the \( \text{SL}_n R \)-coinvariance of the third homology groups. §4 deals with the integral case, to prove 1.1, and §5 deals with the finite fields to prove 1.2. The appendix contains constructive proofs of various module structural details needed elsewhere in the paper.

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2. Notation and conventions. This section introduces notation, describes differentials in Hochschild-Serre spectral sequences, and determines a formula for the \( d^2 \)-differential which will cover our applications. There are five subsections.

2.1 General notation and conventions.

2.1.1 \( R \) is a commutative associative ring with identity, such that for some \( N > 4 \) and all \( n \geq N \), \( SK_1(n, R) = 0 \) (i.e. the elementary matrices generate \( \text{SL}_n R \)) and \( K_2(n, R) = K_2 R \). Henceforth assume that \( n \geq N \).

2.1.2 \( R_k \) is the truncated polynomial ring \( R[t]/(t^k) \).

2.1.3 Any map \( \pi \) is a homomorphism induced by the reduction \( R_k \to R_r \) for some \( r < k \) which the context will indicate. \( \pi^k \) is induced by the reduction \( R_k \to R_{k-1} \).

2.1.4 All diagrams are commutative exact and all sequences are exact unless otherwise specified.

2.2 Notation for elements and \( \text{SL}_n R \)-submodules of \( \text{SL}_n R_k \).

2.2.1 \( \text{SL} R = \lim_n \text{SL}_n R \), where the special linear group \( \text{SL}_n R \) includes into \( \text{SL}_{n+1} R \) as the upper left corner matrices, say. An elementary matrix is denoted \( \alpha \), \( i \neq j \). \( \alpha / \eta = \eta (1, \ldots, 1, \alpha, \ldots, (1 + \alpha)^{-1}, \ldots, 1) \) when \( (1 + \alpha) \) is invertible in \( R \); here, the nontrivial entries are in the \( i \)th and \( j \)th positions.

2.2.2 \( \alpha_{ij} \) is the \( n \times n \) matrix over \( R \) with \( \alpha \in R \) in the \( (i, j) \)th position and all other entries zero; \( \alpha_{ij} = \alpha_{ii} - \alpha_{jj} \). \( M_n^0 \) is the \( \text{SL}_n R \)-module of zero-trace \( n \times n \) matrices over \( R \). (Here and elsewhere, subscript ranges are implied to be \( \{1, \ldots, n\} \).)

2.2.3 \( G_n^k = \ker(\pi_*: \text{SL}_n R_k \to \text{SL}_n R) \) so that \( \text{SL}_n R_k = \text{SL}_n R \ltimes G_n^k \). Let \( i: G_n^k \to \text{SL}_n R_k \) be the inclusion. The kernel of \( \pi_*: G_n^{k+1} \to G_n^k \) is central in \( G_n^{k+1} \) and is isomorphic to \( M_n^0 \) (identify \( \alpha_i (\alpha^k) \) with \( \alpha_{ij} \)). We denote it \( M_n^0 (t^k) \), or sometimes just \( M_n^0 \). Let \( j: M_n^0 (t^k) \to G_n^{k+1} \) be the inclusion.

2.3 Commutator relations in \( \text{SL}_n R_k \); the \( \text{SL}_n R \)-action on \( M_n^0 \). If \( x, y \in R_k \) and \( x^2 y^2 = 0 \),

\[
(2.1) \quad [e_{ij}(x), e_{ab}(y)] = \begin{cases} 
    e_{ib}(xy), & j = a, i \neq b, \\
    e_{aj}(-xy), & i = b, j \neq a, \\
    \hat{e}_{ij}(xy)e_{ij}(-x^2y)e_{ij}(xy^2), & i = b, j = a, \\
    0, & \text{else.}
\end{cases}
\]
If \( \alpha, \beta \in R \), the (left) \( \text{SL}_n R \)-action on \( M_n^0 \) is

\[
(e_{ij}(\alpha) - 1) \cdot \beta_{ab} = \begin{cases} 
\alpha \beta_{ib}, & j = a, i \neq b, \\
- \alpha \beta_{aj}, & a \neq j, i = b, \\
\alpha \beta_{ij} - \alpha^2 \beta_{ij}, & a = j, i = b, \\
0, & \text{otherwise}.
\end{cases}
\]

(The right action differs only by the sign reversal \( \alpha \to -\alpha \).)

### 2.4 Homology related definitions.

#### 2.4.1 \( H_\ast X \) denotes the integral homology group \( H_\ast(X; \mathbb{Z}) \) of a group \( X \).

#### 2.4.2 \( H_1(\text{SL}_n R; M_n^0) = HH_1(R, R) \), where \( HH \) denotes Hochschild homology \([K4, 2.16]\). Since \( R \) is commutative,

\[
HH_1(R, R) = R \otimes R / (\alpha \otimes \beta - \alpha \beta \otimes \gamma + \beta \otimes \gamma \alpha : \alpha, \beta, \gamma \in R)
\]

(e.g. \([I, p. 108]\)) and is isomorphic to the \( R \)-module of absolute Kähler differentials \( \Omega \equiv \Omega_{R/\mathbb{Z}} \) (e.g. \([K3]\)).

#### 2.4.3 \( B_\ast X \) is the standard normalized bar resolution with \( \mathbb{Z} \)-basis elements \( x_0[x_1] \cdots | x_n], \ x_i \in X \). The boundary map \( \partial_X : B_\ast X \to B_{\ast-1} X \) is

\[
\partial_X x_0[x_1] \cdots | x_n] = x_0 x_1[x_2] \cdots | x_n] - \left( \sum_{i=2}^{n} (-1)^{i-1} x_0[x_1] \cdots | x_{i-1} x_i] \cdots | x_n] \right) \\
+ (-1)^n x_0[x_1]x_2 \cdots | x_{n-1}].
\]

#### 2.4.4 If \( C \) is a right \( \mathbb{Z}[X] \) coefficient module, and \( x \in C \otimes_X B_\ast X \), we write \( \{ x \} \) for the class of \( x \) with respect to the equivalence relation induced by the boundary map \( 1 \otimes \partial_X \).

#### 2.4.5 \([x \cap y] \) denotes \([x]|y] - [y]|x] \in B_2 X \); similarly, \([x_1 \cap x_2 \cap \cdots \cap x_{r-1} \cap x_r] \) is \( \sum (-1)^{n} [\sigma x_1] [\sigma x_2] \cdots [\sigma x_n] \), where the sum is over the symmetric group \( \Sigma_n \). Generally, if \( z = z_0[z_1] \cdots | z_i] \in B_i X \) and \( z^1 = z_0[z_{i+1}] \cdots | z_{i+j}] \in B_j X \), then \( [z \cap z^1] \in B_{i+j} X \) is

\[
\sum \{ (-1)^{n} z_0 z_0^1 \cdots | z_{\sigma(i)}] : \sigma \in \Sigma_{i+j}; \ \sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(i) \\
\quad \text{and } \sigma^{-1}(i+1) < \sigma^{-1}(i+2) < \cdots < \sigma^{-1}(i+j) \}
\]

(That is, \( \sigma \) runs over all \( (i, j) \)—shuffles—see \([M, p. 243]\).) Extend \([- -] \) to a group homomorphism \( B_\ast X \otimes B_\ast X \to B_\ast X \).

#### 2.4.6 If \( X \) is abelian with operation \( +, \cap : H_i X \otimes H_j X \to H_{i+j} X \times X \to H_{i+j} X \) is the homology (shuffle) product. This operation defines an exterior algebra \( \wedge \ast X \) which injects into \( H_\ast X \) (see \([B, p. 123]\)).

#### 2.4.7 Suppose \( X_1 \) and \( X_2 \) are subgroups of \( X \) with \( [X_1, X_2] = 1 \). Identify \( x \in B_j X_1 \) and \( x' \in B_j X_2 \) with their images in \( B_\ast X \). Then

\[
\partial_X [x \cap x'] = [\partial_X (x) \cap x'] + (-1)^i [x \cap \partial_X (x')].
\]

Note that this is not true for general elements in \( B_\ast X \).
2.5 Hochschild-Serre spectral sequences. This subsection reviews aspects of Hochschild-Serre group homology spectral sequences and it introduces more notation [CE].

2.5.1 Take an extension $N \rightarrow Y \rightarrow X$, in a category of $S$-groups with $S$-equivariant maps, say. Identify $N$ with $i(N)$. Let $C$ be a right $Y$-module. Consider the spectral sequence

$$(2.4)\quad E_{i,n}^2(C) = H_*(X; H_*(N; C)) \Rightarrow H_*(Y; C)$$

induced from the bicomplex \(\{C \otimes_Y (B_*Y \otimes B_*X) \equiv (C \otimes_N B_*Y) \otimes_X B_*X; \quad \partial = d_1 + d_\Pi\}$. $B_*X$ is the standard bar resolution, with $d_\Pi = \partial_Y \otimes 1$ and $d_I(f \otimes g) = (-1)^{|f|}f \otimes \partial_X g$. $B_*Y \otimes B_*X$ is a left $Y$-module via diagonal action, and $C \otimes_N B_*Y$ is a right $X$-module with action $c \otimes y \cdot x = (c \cdot \hat{x}) \otimes (\hat{x}^{-1}y)$ ($\hat{x}$ is any lifting of $x$ to $Y$). This is well defined, since for all $n$ in $N$, $c \cdot \hat{x} \otimes \hat{x}^{-1}y = c \cdot \hat{x}n \otimes n^{-1}\hat{x}^{-1}y$.

2.5.2 If $x \in E_{a,b}^c(C)$ and $2 \leq c \leq a$, then $d_{a,b}^c x$ is calculated by choosing representatives $x_i \in C \otimes_Y (B_{b+i}Y \otimes B_{a-i}X)$ for $0 < i < c$, such that $x_0$ represents $x$ and $d_Ix_i = -d_\Pi x_{i+1}$; set $d_{a,b}^c x = \{d_Ix_{c-1}\}$. Let $d_{a,b}^c x_0$ denote $d_Ix_{c-1}$ (of course, this definition is not unique).

2.5.3 Suppose $C$ is a trivial $Y$-module. Suppose $Y_1$ and $Y_2$ are subgroups of $Y$ with $[Y_i, N] = 1$, $i = 1, 2$, and, if $f(Y) = X_i [X_1, X_2] = 1$. Let $u: X \rightarrow Y$ be any set map section to $f$ with $u(X_i) \subset Y_i$. Suppose

\[ x = \sum_{i \in I} (-1)^i [x_i^1 | x_i^2 | \cdots | x_i^a] \in B_a X_1 \]

and

\[ v = \sum_{j \in J} (-1)^j [v_j^1 | v_j^2 | \cdots | v_j^b] \in B_b X_2 \]

represent elements in $H_* X$ and $y \in C \otimes_N B_* N$ is a cycle. Then in the spectral sequence (2.4),

$$(2.5)\quad d_{a+b,s}^2 \{y \otimes [x \cap v]\} = \left\{ \left[ d_{a,s}^2 (y \otimes x) \cap v \right] \right\} + (-1)^{ab} \left\{ \left[ d_{b,s}^2 (y \otimes v) \cap x \right] \right\}$$

$$+ \left( \sum_{i \in I} \sum_{j \in J} (-1)^{a+i+j} \left[ \left[ u(x_i^j)^{-1}, u(v_j^i)^{-1} \right] \cap y \right] \right. \left. \otimes \left[ [x_i^1] \cdots [x_i^a] \cap [v_j^1] \cdots [v_j^b] \right] \right\}.$$ 

Note. (1) In the process of calculating the differential we prove that $y \otimes [x \cap v]$ represents an element in $H_{a+b}(X; H_*(N; C))$.

(2) The restricted definitions of $x, v$ and $y$ ensure that one can choose differential representatives on the right-hand side of (2.5) such that the shuffle products are cycles—again, this comes from the calculation. Indeed, as pointed out by the referee, if $N$ is central in $Y$, then the coefficients in (2.4) are trivial and $d_{a,b}^2([y \otimes x] \cap v)$ may be defined using a homology shuffle product

\[ \cap: H_{a-2}(X_1; H_{s+1}(N; C)) \otimes H_b(X_2) \rightarrow H_{a+b-2}(X; H_{s+1}(N; C)). \]
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PROOF. Use (2.3) to compute $d_1(y \otimes [x \cap v]) = -d_1 z$ for $z \in (C \otimes X B_{s+1}Y) \otimes X B_{a+h-1} X$, where

$$-z = \sum_{i \in I} (-1)^{i+1} \left[ \left[ \left( u(x_i^t) \right)^{-1} \right] \cap y \right] \otimes \left[ x_{i1} \cdots x_{in} \otimes v \right]$$

$$+ \sum_{j \in J} (-1)^{s+i+j} \left[ \left[ \left( u(v_j^t) \right)^{-1} \right] \cap y \right] \otimes x \cap [v_{j1} \cdots v_{jk}].$$

Then

$$d_1 z = \sum_{i} (-1)^i \left[ \left[ \left( u(x_i^t) \right)^{-1} \right] \cap y \right] \otimes \left[ \partial x [x_{i1} \cdots x_{ni}] \cap v \right]$$

$$+ \sum_{j} (-1)^j \left[ \left[ \left( u(v_j^t) \right)^{-1} \right] \cap y \right] \otimes \left[ x \cap \partial x [v_{j1} \cdots v_{jk}] \right]$$

$$+ \sum_{i} \sum_{j} (-1)^{a+i+j} \left[ \left[ \left( u(x_i^t)^{-1} - 1 \right) \left[ \left( u(v_j^t) \right)^{-1} \right] \right] \cap y \right]$$

$$
\otimes \left[ x_{i1} \cdots x_{ni} \cap [v_{j1} \cdots v_{jk}] \right].$$

Since $\langle f - 1 \rangle g - (g - 1)f = \partial y ([f \cap g] + [fg | f^{-1} g^{-1}]) + [[f, g]] \sim [[f, g]]$, the expression for $d_1 z$ is equivalent to the class in (2.5).

2.5.4 If $x$ and $y$ are as in 2.5.3 with $x = \{x_1 \cap x_2 \cap \cdots \cap x_a\}$, where $x_i, x_j = 1$ ($1 \leq i, j \leq a$), then repeated application of (2.5) yields

$$d^2_{a+i} \{ y \otimes x \} = \left\{ \sum_{\sigma} \left( \left[ \left[ \left( u(x_{\sigma(1)}) \right)^{-1}, (u(x_{\sigma(2)}) \right)^{-1} \right] \right] \cap y \right)$$

$$\otimes \left[ x_{\sigma(3)} \cdots | x_{\sigma(a)} \right] \right\},$$

where $\sigma$ runs over the alternating group $A_a$.

3. Low dimensional homology groups of $G_n^k$. This section inductively looks at low dimensional terms in the spectral sequences

$$H_s(G_n^{k-1}; H_s M_n^0(t^{k-1})) \Rightarrow H_s G_n^k, \quad k > 2.$$  

The first subsection is concerned with $E^\infty_{2,0}$ and $E^\infty_{0,0}$. Next, the $E^\infty_{2,1}$ term in (3.1) is computed, yielding enough information on $H^2 G_n^k$ to estimate $(E^\infty_{2,1})_{SL_n R}$ recursively. This gives us information on the $SL_n R$-coinvariance of the kernel of $\pi^k_s | H^2 G_n^k$. The final subsection looks at ker $(\pi^k_s H^2 G_n^k)_{SL_n R} \rightarrow (H^2 G_n^k)_{SL_n R}$.

3.1 LEMMA. (i) $\pi^k_s: H^2 G_n^k \rightarrow (H^2 G_n^k \cong M_n^0)$ is an isomorphism, $k > 2$.

(ii) If $k > r > 2$, $\text{im}(\pi^k_s): H^2 G_n^k \rightarrow H^2 G_n^r$ is an isomorphism.

(iii) $j^k_*(\Lambda^* M_n^0(t^k)) = 0$ in $H^0 G_n^{k+1}$; in particular, $j^k_*(H^2 M_n^0(t^k)) = 0$. For torsion-free $R$, in the spectral sequence (3.1), $E^{3,0}_{0,*} = 0$ whenever $* > 0$ and $k \geq 2$.

PROOF. Consider (3.1)(k + 1), $k \geq 2$. For any $\{i, j\} \subset \{1, \ldots, n\}$, fix $m \in \{i, j\}$. For $\alpha \in R$, define $b_{ij}(\alpha)$ to be $[e_{im}(t) \cap e_{mj}(at^{k-1})] \in Z_{G_n^2} B_{2} G_n{k}$. Check that this is a cycle. Let $u: G_n^k \rightarrow G_n^{k+1}$ be a set map section to the reduction; it can be assumed that the elementary matrix $e_{ab}(\sum \alpha m t^m) \in G_n^k$ is taken to the matrix of the
same form in \( G_n^{k+1} \). Apply formula (2.6) of 2.5.4 and the commutator relations 2.3 to compute

\[
-d_{2,0}^2 \{ b_{ij}(a) \} = \left\{ \left[ u(e_{im}(-t)), u(e_{mj}(-\alpha t^{k-1})) \right] \right\} = \left\{ \left[ e_{im}(-t), e_{mj}(-\alpha t^{k-1}) \right] \right\} = a_{ij} \quad \text{if } i \neq j, \quad \text{else } -a_{mm} + a_{ii}.
\]

Therefore \( E_{0,1}^3 = 0 \). Since \( k \) was arbitrary this proves (i).

(ii) \( \{ b_{ij}(a) \} : 1 \leq i, j \leq n; \alpha \in R \) is a set of generators for \( H_n^{2k}/\pi_*H_n^{2k+1} \). But also \( \{ b_{ij}(a) \} \in \ker(\pi_*: H_n^{2k} \to H_n^{2k-1}) \). Thus in \( H_n^{2k-1}, \pi_*H_n^{2k+1} = \pi_*H_n^{2k} \), which implies (ii).

(iii) Consider (3.1)(k + 1), \( k \geq 2 \). By the universal coefficient theorem [M, p. 171] there is an inclusion \( H_n^{2k} \otimes H_n^0(t^k) \to E_{2,0}^2 \). The proof of (i) shows that \( d_{2,0}^2(H_n^{2k}) = M_n^0(t^k) \). So, by (2.6), \( d_{2,0}^2(E_{2,0}^2) \) contains the homology product \( H_n^0(t^k) \cap \Lambda^a M_n^0(t^k) \), which is just \( \Lambda^{a+1}M_n^0(t^k) = H_n^2M_n^0(t^k) \) and, if \( R \) is torsion-free, \( \Lambda^a M_n^0(t^k) = H_n^2M_n^0(t^k) \) [B, p. 123] .

3.2 REMARKS. Let \( J_n = \text{im}(\pi_*: H_n^{2k} \to H_n^{2k+1}) \), \( k > 2 \). (This is independent of \( k \), by 3.1(ii).) Then (3.1)(3) contains an exact sequence

\[
J_n \to \Lambda^2 M_n^0 \to M_n^0.
\]

Let \( L_n \) be the associated module defined as \( \ker([ \cdot , \cdot ]: M_n^0 \otimes M_n^0 \to M_n^0) \), where \([ \cdot , \cdot ] \) is the composite of the homology product with \( d^2 \). The identification (2.2.3) of \( M_n^0 \) takes the matrix \( a \) to \( 1 + at \in \text{SL}_n R_2 \). Thus if \( a, b \in M_n^0 \), using definition (2.5) \([a, b]\) is the commutator of \( 1 + at \) and \( 1 + bt \) evaluated in \( \text{SL}_n R_3 \). Since \( 1 + at^{-1} \equiv 1 - at + a^2t^2 \) (mod \( t^3 \)), this commutator is readily seen to equal \( 1 + abt^2 - bat^2 \). As an element of \( M_n^0 \equiv \ker(\pi^3: \text{SL}_n R_3 \to \text{SL}_n R_2) \), this is \( ab - ba \); thus \( [a, b] \) is the usual Lie bracket.

Let \( \Omega \to \text{St}(R, R) \to M_n^0 \) be the universal \( \text{SL}_n R \)-central extension of \( M_n^0 \) [K4, 2.15]. In the Appendix (Lemma A.1(iii)) we show that \( (M_n^0 \otimes M_n^0)/I \equiv \text{St}(R, R) \otimes R \), where \( I \) is the \( \text{SL}_n R \)-submodule generated by \( 1_{12} \otimes 1_{23} + 1_{43} \otimes 1_{14} \).

We use this to estimate \( \ker(\pi_*^k + 1|H_n^{2k+1}) \), which is the \( E_{1,1}^{\infty} \) term in (3.1) \((k + 1)\).

3.3 PROPOSITION. (i) There is a commutative diagram

\[
\begin{array}{ccc}
\Omega \otimes R & \rightarrow & \text{St}(R, R) \otimes R \\
\downarrow \phi \otimes 0 & & \downarrow \psi \otimes 0 \\
\Omega \otimes R/\ker d_{2,0}^2 & \rightarrow & M_n^0 \\
\end{array}
\]

where \( d_{2,0}^2 \) is the differential in the spectral sequence (3.1)(k + 1) and \( D \) is the restriction of the \( d_{2,0}^2 \)-differential in (3.1)(k + 2).

(ii) \( \ker(\pi_*^k + 1|H_n^{2k+1})_\text{SL}_n R = \text{im} \psi \).

PROOF. Let \( u \) be a section to \( \pi: G_n^{2k+1} \to G_n^k \) with \( u(e_{ij}(t')) = e_{ij}(t') \). Use 3.1 to identify \( E_{1,1}^2 \) with \( M_n^0 \otimes M_n^0 \). Use the commutator relations 2.3 to check that

\[
x = - \left[ e_{12}(t) \cap e_{24}(t^{k-1}) \cap e_{43}(t) \right]
\]
is a cycle in \( \mathbb{Z} \otimes G_n^k \). Use (2.6) to compute

\[
(3.4) \quad d_{3,0}^2 \{x\} = \{ \{ [e_{43}(-t), e_{24}(-t^{k-1})] \} \otimes [e_{12}(t)] \}
\]

\[
= \{ l_{12} \otimes l_{13} + l_{14} \otimes l_{14} \in H_1 G_n^k \otimes M_n^0(t^k) = M_n^0 \otimes M_n^0. \}
\]

So \( I \subseteq d_{3,0}^2 \).

As in the proof of 3.1(ii), \( H_2 G_n^{k+1} / \pi^* H_2 G_n^{k+2} \cong M_n^0 \) and is represented in \( \ker \pi_0^* = E_{1,1}^\infty \). Moreover by the proof of 3.1(i) and Remarks 3.2, in the spectral sequence (3.1)(k + 2), \( d_{2,0}^2 \ker \pi_0^* \) is induced by the Lie bracket: \( (E_{1,1}^2 \cong M_n^0 \otimes M_n^0) \rightarrow M_n^0 \). According to A.1(ii) if \( \text{St}(R, R) \) is identified with \( (M_n^0 \otimes M_n^0) / \langle I \rangle \), then also \( \phi \) is induced by the Lie bracket. This gives the right square of the diagram in the proposition statement. Moreover, because \( \ker \pi^*_{k+1} = E_{1,1}^\infty = (M_n^0 \otimes M_n^0) / \text{im} d_{3,0}^2 \), \( \rho \oplus \psi \) has kernel isomorphic to \( \text{im} d_{3,0}^2 / I \); as a submodule of \( (\Omega \oplus R = \ker(\phi \oplus 0)) \), \( \text{im} d_{3,0}^2 / I \) has trivial \( \text{SL}_n R \)-action. By Lemma A.1(i) \( (\text{I})_{\text{SL}_n R} = 0 \) so \( \text{im} d_{3,0}^2 / I \equiv \text{im} d_{3,0}^2 / \text{SL}_n R \). This implies the left square of the diagram.

Finally, \( \text{(St}(R, R))_{\text{SL}_n R} = 0 \) \([K4, 1.7 and 1.4]\); hence part (ii) is implied by the isomorphism \( \ker \pi_{k+1}^* = \text{St}(R, R) \oplus R / \text{im} d_{3,0}^2 / \text{SL}_n R \). \( \Box \)

We next want to estimate \( (H_3 G_n^k)_{\text{SL}_n R} \). The following lemma will be used in 3.5 to investigate \( (E_{2,1}^\infty)_{\text{SL}_n R} \).

3.4 LEMMA. For \( k > 2 \), \( \text{SL}_n R \) acts trivially on \( \ker(\pi_0^* \circ \cdots \circ \pi_{k-1}^* : \pi_{k}^* H_2 G_n^k \rightarrow J_n) \).

PROOF. Denote the kernel of \( \pi_0^* \circ \cdots \circ \pi_{k}^* \) by \( U_k \) when it has domain \( H_2 G_n^k \) and by \( \bar{U}_k \) when it has domain \( \pi_{k+1}^* H_2 G_n^{k+1} \). Use (3.3)(k) to fit the sequence \( \ker \pi_{k}^* : H_2 G_n^k \rightarrow U_k \rightarrow \bar{U}_k \) into the following diagram, the top row of which is therefore exact. (In (3.5), \( D \) is the restriction of the \( d_{2,0}^2 \) differential in the spectral sequence (3.1)(k + 1).)

\[
\begin{array}{cccc}
(\Omega \oplus R) / (\text{im} d_{3,0}^2)_{\text{SL}_n R} & \rightarrow & ? & \rightarrow \bar{U}_k^{-1} \\
\downarrow & & \downarrow & \downarrow \\
\ker \pi_k^* & \rightarrow & U_k & \rightarrow \bar{U}_k^{-1} \\
D & \downarrow & D & \downarrow \\
M_n^0 & = & M_n^0
\end{array}
\]

Inductively assume that both the base and fibre modules in the top row of (3.5) have trivial action \( \bar{U}^2 = 0 \). Therefore, because \( H_1 \text{SL}_n R = 0 \), \( ? \) is also a trivial \( \text{SL}_n R \)-module. Moreover, the middle column of (3.5) fits into the following diagram, i.e., \( ? = \bar{U}_k \). This gives the inductive step.

\[
\begin{array}{cccc}
? & \rightarrow & \pi_{k+1}^* H_2 G_n^{k+1} & \rightarrow J_n \\
\downarrow & & \downarrow & \downarrow \\
U_k & \rightarrow & H_2 G_n^k & \rightarrow J_n \\
D & \downarrow & d_{2,0}^2 & \downarrow \\
M_n^0 & = & M_n^0 & \Box
\end{array}
\]
3.5 Proposition. In the spectral sequence (3.1)(k + 1) for \( k \geq 3 \),
(i) there is an epimorphism: \( R \rightarrow (E^\infty_{1,2})_{SL_n R} \);
(ii) there is an exact sequence

\[
R \rightarrow (\ker d^2_{2,1})_{SL_n R} \rightarrow \left( \text{Tor}(M^0_n, M^0_n) \right)_{SL_n R},
\]
hence for some quotient \( T \) of \( (\text{Tor}(M^0_n, M^0_n))_{SL_n R} \), there is an exact sequence

\[
R \rightarrow \left( \frac{\left( \ker d^2_{2,1} \right)}{\text{im} d^2_{0,0}} \right) = E^\infty_{2,1} \rightarrow T.
\]

Proof. (i) \( E^\infty_{1,2} \) is a quotient of \( E^2_{1,2} = H_1 G_n^k \otimes H_2 M^0_n(t^k) \). By 3.1, \( H_1 G_n^k \cong M^0_n \), so \( E^2_{1,2} \cong M^0_n \otimes \Lambda^2 M^0_n \). Use Lemma A.2(iv) to see that \( (E^2_{1,2})_{SL_n R} \cong R \), where \( \alpha \) corresponds to the class of \( g_{12} \otimes 1_{23431} \); \( (E^\infty_{1,2})_{SL_n R} \) is a quotient of this.

(ii) Using the notation and proof of Lemma 3.4, filter \( U^k \otimes M^0_n \) as

\[
\hat{U}^k \otimes M^0_n \rightarrow U^k \otimes M^0_n \rightarrow M^0_n \otimes M^0_n,
\]
where in the base group \( \alpha_{rs} \otimes \beta_{uv} \) is the image of \( \{ b_{rs}(\alpha) \} \otimes \beta_{uv} \) for \( \{ b_{rs} \} \) as in 3.1(i)(proof). Hence if \( \cap \) is the product \( M^0_n \otimes M^0_n \rightarrow M^0_n \), the composite \( \cap \circ (D \otimes 1) \) induces an epimorphism \( g^0 : M^0_n \otimes M^0_n \rightarrow \Lambda^2 M^0_n = E^0_{2,2} \), equivalent to the product map. This has kernel \( \Gamma(M^0_n) \), the Whitehead gamma group which projects onto \( M^0_n \otimes \mathbb{Z}/2 \) with kernel \( S^2 M^0_n \), the 2-fold symmetric product of \( M^0_n \) (see, for example, [A]). So \( (\Gamma(M^0_n))_{SL_n R} \) is a quotient of \( (S^2 M^0_n)_{SL_n R} = R \) (by Lemma A.2(ii)). Moreover, (3.8) restricts to

\[
\hat{U}^k \otimes M^0_n \rightarrow \ker \cap \circ (D \otimes 1) \rightarrow \Gamma(M^0_n) = \ker g.
\]
Since \( \hat{U}^k \) has trivial \( SL_n R \)-action, \( (\hat{U}^k \otimes M^0_n)_{SL_n R} = 0 \); so \( \ker \cap \circ (D \otimes 1) \) also has \( SL_n R \)-coinvariance a quotient of \( R \). \( U^k \) is, by definition, \( \ker(\pi^*: H^3 G_n^k \rightarrow J_n) \), and \( D \) is the restriction of \( d^2_{2,0} \). The above shows that the restriction of \( d^2_{2,1} \) to the image of \( U^k \otimes M^0_n \) in \( H^3 G_n^k \otimes M^0_n \) is onto \( E^0_{2,2} \). Hence we have an exact sequence

\[
\ker \circ (D \otimes 1) \rightarrow \ker(\left. d^2_{2,1} \right| H^3 G_n^k \otimes M^0_n) \rightarrow J_n \otimes M^0_n.
\]
Apply \( H_0(\text{SL}_n R; -) \) to this and insert result \( (J_n \otimes M^0_n)_{SL_n R} = 0 \) from Lemma A.2(vi) to see that there is an epimorphism: \( R \rightarrow \left( \ker d^2_{2,1} \right| H^3 G_n^k \otimes M^0_n)_{SL_n R} \).

Finally, the universal coefficient theorem provides a sequence \( H^3 G_n^k \otimes M^0_n \rightarrow E^2_{2,1} \rightarrow \text{Tor}(H^3 G_n^k, M^0_n) \); because the restriction of \( d^2_{2,1} \) to \( H^3 G_n^k \otimes M^0_n \) is onto \( E^0_{2,2} \), there is a sequence

\[
\ker d^2_{2,1} \rightarrow H^3 G_n^k \otimes M^0_n \rightarrow \ker d^2_{2,1} \rightarrow \text{Tor}(H^3 G_n^k, M^0_n).
\]
Apply \( H_0(\text{SL}_n R; -) \) to this and identify \( H^3 G_n^k \) with \( M^0_n \) to get (3.6). \( \square \)

3.6 Proposition. Whenever \( k > 2 \), there is an exact sequence

\[
(3.9) \quad H_1(\text{SL}_n R; I) \xrightarrow{\partial} \left( \pi^* \left( H^3 G_n^k \right) \right)_{\text{SL}_n R} \xrightarrow{\iota^*} \left( H^3 G_n^{k-1} \right)_{\text{SL}_n R} \rightarrow \left( \text{im} d^2_{3,0} \right)_{\text{SL}_n R}.
\]
where \( I \) is the \( \text{SL}_n R \)-submodule of \( M^0_n \otimes M^0_n \) generated by \( 1_{12} \otimes 1_{23} + 1_{43} \otimes 1_{14} \).

Proof. Consider the spectral sequence (3.1)(k). There can be no transgressions from \( E^3_{3,0} \), as \( E^3_{3,0} = 0 \) (proof of Proposition 3.1(iii)). Thus there is a sequence \( \pi^* H^3 G_n^k \rightarrow H^3 G_n^{k-1} \rightarrow \text{im} d^2_{3,0} \). Apply \( H_*(\text{SL}_n R; -) \) to this to get (3.9) except that \( \iota_\ast \) has kernel \( \partial(\left. H_1(\text{SL}_n R; -) \right| d^2_{3,0}) \).
Identify $H_1G_n^{k-1} \otimes M_0^0(\ell^{k-1})$ with $M_0^0 \otimes M_n^0$ via 3.1(i) and identify $I$ with $(d_{3,0}^2(x))$ as in 3.3(proof). As in the last part of the proof of 3.3, $\text{im} \ d_{3,0}^2/I$ is the trivial $\text{SL}_n R$-module (im $d_{3,0}^2$)$_{\text{SL}_n R}$. So $H_1(\text{SL}_n R; \text{im} \ d_{3,0}^2/I) = 0$, implying an epimorphism: $H_1(\text{SL}_n R; \text{im} \ d_{3,0}^2/I) \rightarrow H_1(\text{SL}_n R; \text{im} \ d_{3,0}^2)$. $\partial k$ is the composite of this epimorphism and $\partial$.

4. $K_3$ of truncated polynomial rings over the integers. The first three subsections refer to general rings $R$ and concern the $E_{1,2}^*$ terms in the spectral sequences

$$(4.1)(k) \quad kE_{1,2}^* = H_*(\text{SL}_n R; \quad H_0G_n^k) \Rightarrow H_*(\text{SL}_n R), \quad k \geq 2.$$ 

Lemmas 4.4 and 4.5 determine $H_1(\text{SL}_n \mathbb{Z}; \wedge^2 M_0^0)$ and $H_1(\text{SL}_n \mathbb{Z}; I)$. This and earlier work yield the main theorem which computes $H_3(\text{SL}_n \mathbb{Z}[t]/(t^k), k \geq 2$.

4.1 Proposition. For $k > 2$ there is an exact sequence

$$(4.2) \quad (R + N_k)/R \rightarrow H_1(\text{SL}_n R; \quad H_2G_n^k) \rightarrow H_1(\text{SL}_n R; J_n = \pi*H_2G_n^k),$$

where $N_k$ is the $\text{SL}_n R$-coinvariance of the image of the $d_{3,0}^2$ differential in the spectral sequence (3.1)(k), and $R + N_k = R$ if $H_1(\text{SL}_n R; \quad M_0^0) = 0$ (see Proposition 3.3).

Proof. Proposition 3.3 provides a sequence $N_k \rightarrow \Omega \otimes R \rightarrow \text{ker} \pi^k \rightarrow M_n^0$, in which $(\text{ker} \pi^k)_{\text{SL}_n R} \equiv \text{im} R = (R + N_k)/N_k$ (implying the splitting $(\Omega \otimes R)/N_k \equiv ((R + N_k)/N_k) \oplus (\Omega \otimes R)/(R + N_k)$). Application of $H_*(\text{SL}_n R; \cdot)$ to the sequence (3.3)(k) yields

$$(4.3) \quad H_1(\text{SL}_n R; \quad \text{ker} \pi^k) \rightarrow H_1(\text{SL}_n R; \quad M_0^0)$$

$$\Rightarrow \quad \Omega \rightarrow (\Omega \otimes R)/N_k \rightarrow \text{ker} \pi^k)_{\text{SL}_n R} = (R + N_k)/N_k.$$

Thus $H_1(\text{SL}_n R; \quad \text{ker} \pi^k) \equiv (R + N_k)/R$; its image under the map induced by inclusion is the kernel of

$$\pi* : \left( H_1(\text{SL}_n R; \quad H_2G_n^k) \rightarrow H_1(\text{SL}_n R; \quad \pi*H_2G_n^k) \rightarrow H_1(\text{SL}_n R; \quad J_n) \right),$$

where the injection $\sigma$ is implied by Lemma 3.4.

4.2 Proposition. If $H_2(\text{SL}_n R; \quad M_0^0) = 0$ and $s : J_n \rightarrow \wedge^2 M_n^0$ is as in (3.2), then there is an injection

$$s* : H_1(\text{SL}_n R; \quad J_n) \rightarrow H_1(\text{SL}_n R; \quad H_2G_n^2 = \wedge^2 M_n^0) .$$

If $H_1(\text{SL}_n R; \quad M_n^0) = 0$, $s*$ is an isomorphism.

Proof. Apply $H_*(\text{SL}_n R; \cdot)$ to the exact sequence $J_n \rightarrow \wedge^2 M_n^0 \rightarrow M_n^0$ of (3.2) and use the assumption.

4.3 Proposition. Suppose $H_i(\text{SL}_n R; \quad M_n^0) = 0$ for $i = 1, 2$. Denote $E_{1,2}^*$ terms in the spectral sequence (4.1)(k) by $kE_{1,2}^*$, $k \geq 2$. Then $\pi* : kE_{1,2}^* \rightarrow 2E_{1,2}^*$ is an injection.

Proof. By 4.1 and the assumption, $kE_{1,2}^*$ injects into $H_1(\text{SL}_n R; \quad J_n)$ which by 4.2 is isomorphic to $2E_{1,2}^*$. If $j < 2$, $\pi* : H_jG_n^k \rightarrow H_jG_n^2$ is an isomorphism (Lemma 3.1) so that $\pi* : kE_{1,2}^* \rightarrow 2E_{1,2}^*$ is an injection for $r \geq 2$. \□
The remainder of this section is devoted to computing $K_3 \mathbb{Z}[t]/(t^k)$; two preliminary lemmas are required. Let $M_n^0R$ denote the submodule of zero-trace matrices in the $SL_n R$-module $M_n R$ of $n \times n$ matrices over $R$.

4.4 LEMMA. If $n$ is large and $p$ is any prime,

(i) $H_1(SL_n \mathbb{Z}/p^2; M_n^0 \mathbb{Z}/p \otimes M_n^0 \mathbb{Z}/p) = 0$; $H_1(SL_n \mathbb{Z}; M_n^0 \mathbb{Z}/p \otimes M_n^0 \mathbb{Z}/p) = 0$;

(ii) $H_1(SL_n \mathbb{Z}; M_n^0 \mathbb{Z} \otimes M_n^0 \mathbb{Z}) = 0$;

Proof. (i) Take $n$ large and prime to $p$ and to $p-1$. Consider the spectral sequence

\[ E_1^2 = H_*(SL_n \mathbb{Z}/p; H_*(M_n^0 \mathbb{Z}/p; C)) \Rightarrow H_*(SL_n \mathbb{Z}/p^2; C) \]

firstly with $C = M_n^0 \mathbb{Z}/p$. The proof of [ALSS, part 1, VI, 1.1] demonstrates an isomorphism which is dual to $1.0$. Moreover, $M_n^0 \mathbb{Z}/p$ may be viewed as a direct summand of $M_n^0 \mathbb{Z}/p \otimes M_n^0 \mathbb{Z}/p$ using either of the inclusions

\[ \Phi: \alpha_{ij} \mapsto \left[ \sum_{k=1}^{n} \alpha_{ik} \otimes 1_{k,j} \right] \quad \text{or} \quad \Phi^T: \alpha_{ij} \mapsto \left[ \sum_{k=1}^{n} -1_{k,j} \otimes \alpha_{ik} \right], \]

where if $a \otimes b \in M_n^0 \mathbb{Z}/p \otimes M_n^0 \mathbb{Z}/p$, $[a \otimes b]$ is its image in $M_n^0 \mathbb{Z}/p \otimes M_n^0 \mathbb{Z}/p$ under the canonical projection $(M_n^0 \mathbb{Z}/p)^{\otimes 2} = (M_n^0 \mathbb{Z}/p \otimes \mathbb{Z}/p)^{\otimes 2} \rightarrow (M_n^0 \mathbb{Z}/p)^{\otimes 2}$. (Since $p + n$, $\Phi$ and $\Phi^T$ are split by the Lie bracket.)

Now

\[ E_1^2 = (M_n^0 \mathbb{Z}/p \otimes M_n^0 \mathbb{Z}/p) \cong \left( (M_n^0 \mathbb{Z}/p)^{\otimes 2} \right)_{SL_n \mathbb{Z}/p} = \Phi^T(E_1^2), \]

from A.2(i) and (iii). Therefore by naturality,

\[ E_1^2 = d_2(\Phi^T(E_1^2) + \Phi^T(E_2^2)). \]

The proof of the first equality in part (i) is completed by recalling that

\[ (E_2^2 = 1.0) \Rightarrow H_1(SL_n \mathbb{Z}/p; M_n^0 \mathbb{Z}/p \otimes M_n^0 \mathbb{Z}/p) = 0, \]

dually to [ALSS, part 1, II, 1.4]. Given this, Kassel [K2, 3.4] asserts that the second equality holds. This then implies that multiplication by $p$ is an epimorphism on $H_1(SL_n \mathbb{Z}; M_n^0 \mathbb{Z} \otimes M_n^0 \mathbb{Z})$ which, since we are dealing with a finitely generated abelian group (e.g. [B, p. 217]) means the group is torsion with trivial $p$-component. However, for large enough $n$ the inclusion: $M_n^0 \mathbb{Z} \rightarrow M_{n+m}^0 \mathbb{Z}$ induces isomorphisms

\[ H_1(SL_n \mathbb{Z}; M_n^0 \mathbb{Z} \otimes M_n^0 \mathbb{Z}) \rightarrow H_1(SL_{n+m} \mathbb{Z}; M_{n+m}^0 \mathbb{Z} \otimes M_{n+m}^0 \mathbb{Z}), \]

for $m \geq 1$ (e.g. [VK, §5]). Thus for $n$ sufficiently large, $H_1(SL_n \mathbb{Z}; M_n^0 \mathbb{Z} \otimes M_n^0 \mathbb{Z})$ has no $p$-component for any prime $p$, giving (i).

(ii) If $I \subset M_n^0 \mathbb{Z} \otimes M_n^0 \mathbb{Z}$ is as in 3.2, $I \otimes \mathbb{Z}/p \subset M_n^0 \mathbb{Z}/p \otimes M_n^0 \mathbb{Z}/p$ is the $SL_n \mathbb{Z}/p$-module generated by $1_{12} \otimes 1_{33} + 1_{43} \otimes 1_{14}$, and equation (A.7) (see A.2(iii) proof) shows that $(I \otimes M_n^0 \mathbb{Z}/p)_{SL_n \mathbb{Z}/p} = \mathbb{Z}/p$, generated by the class of $1_{12} \otimes 1_{33} \otimes 1_{31} + 1_{43} \otimes 1_{14} \otimes 1_{31}$. Embed $M_n^0 \mathbb{Z}/p$ into $I \otimes \mathbb{Z}/p$ with the map $\Phi - \Phi^T$, then argue as for part (i). \[\square\]
4.5 Lemma. If $n$ is large and $p$ is any odd prime, 
$H_1(SL_n \mathbb{Z}; \wedge^2 M_n^0 \mathbb{Z}/p) = 0$; 
$H_1(SL_n \mathbb{Z}; \wedge^2 M_n^0 \mathbb{Z}/2) = \mathbb{Z}/2$; 
$H_1(SL_n \mathbb{Z}; \wedge^2 M_n^0 \mathbb{Z}) = 0$.

Proof. If $p$ is odd, $\wedge^2 M_n^0 \mathbb{Z}/p$ is a direct summand of $M_n^0 \mathbb{Z}/p \otimes M_n^0 \mathbb{Z}/p$, so the first equality follows from 4.4(i).

Define $\Phi': M_n^0 \mathbb{Z}/2 \to \wedge^2 M_n^0 \mathbb{Z}/2$ to be the composite of $\Phi$ (defined in (4.5)) with the homology product on $H_*(M_n^0 \mathbb{Z}/2)$. The map $\Phi': E^2_{0,1}(M_n^0 \mathbb{Z}/2) \to E^2_{0,1}(\wedge^2 M_n^0 \mathbb{Z}/2)$ between terms in the spectral sequences (4.4) is an isomorphism, by A.2. Arguing as in the proof of 4.4, we conclude that $E^3_{0,1}(\wedge^2 M_n^0 \mathbb{Z}/2) = 0$. Thus the reduction epimorphism: $H_1(SL_n \mathbb{Z}; \wedge^2 M_n^0 \mathbb{Z}/2) \to H_1(SL_n \mathbb{Z}/2; \wedge^2 M_n^0 \mathbb{Z}/2)$ is injective; its image is identified as $\mathbb{Z}/2$ in [ALSS, part 3, 9.16]. Again apply the Kassel result [K2, 3.4] to conclude that $H_1(SL_n \mathbb{Z}; \wedge^2 M_n^0 \mathbb{Z}/2) = \mathbb{Z}/2$.

Now $SL_n \mathbb{Z}$ acts on $\wedge^2 M_n^0 \mathbb{Z}/2$ via reduction to $SL_n \mathbb{Z}/2$. Thus we can use Lemma A.2 to see that each of the terms in the exact sequence $\wedge^2 M_n^0 \mathbb{Z} \to \wedge^2 M_n^0 \mathbb{Z} \to \wedge^2 M_n^0 \mathbb{Z}/2$ has $SL_n \mathbb{Z}$-coinvariance $\mathbb{Z}/2$. In the long exact sequence obtained on application of $H_*(SL_n \mathbb{Z}; -)$ to this coefficient sequence, the connecting homomorphism $H_1(SL_n \mathbb{Z}; \wedge^2 M_n^0 \mathbb{Z}/2) \to \mathbb{Z}/2$ must therefore be onto, hence injective.

So multiplying the coefficients in $H_1(SL_n \mathbb{Z}; \wedge^2 M_n^0 \mathbb{Z})$ by two results in an onto map. Since we are dealing with a finitely generated group, it must be an odd torsion group. The first equality in the statement of the lemma then implies it is trivial. \(\square\)

4.6 Theorem. If $k \geq 2$ and $n$ is large there are exact sequences $\mathbb{Z} \oplus \mathbb{Z} \to H_3 SL_n \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2 \to K_3 \mathbb{Z}/k \to \mathbb{Z}/24 \to \mathbb{Z}$, and $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2 \to K_3 \mathbb{Z}/k \to \mathbb{Z}/24 \to \mathbb{Z}$, where $U_k$ is an elementary 2-group of rank at most $[k/2]$ and $\iota = 0$ if $k$ is even. (Here $\mathbb{Z}_k$ denotes $\mathbb{Z}[t]/(t^k).$)

Proof. Take $R = \mathbb{Z}$. If $n > 10$, $H_2(SL_n \mathbb{Z}; M_n^0) = 0$ [K1], and of course, $\Omega = \Omega^1_{2/\mathbb{Z}} = 0$. $H_1(SL_n \mathbb{Z}; \wedge^2 M_n^0) = 0$ by Lemma 4.5, so that by Propositions 3.1 and 4.3, in the spectral sequence (4.1)(k), $k \geq 2$, $\delta^1_{1,2} = 0$ and $\delta^2_{2,1} = 0$. Thus there is an epimorphism, induced by the inclusion of $G^k_n$ into $SL_n R_k$,

\begin{equation}
(4.6) \quad i_*^k: (H_3 G^k_n)_{SL_n R} \to H_3 SL_n R_k / H_3 SL_n R.
\end{equation}

Take $k \geq 3$. By (4.6), $\ker \pi_*^k | H_3 SL_n R_k$ is the quotient of $V_k = \ker \left( i_*^k \circ \pi_*^k: (H_3 G^k_n)_{SL_n R} \to H_3 SL_n R_k \right)$ by $\ker i_*^k$. Use (3.9) and Lemma 4.4 to identify $\ker \left( (\pi_*^k H_3 G^k_n)_{SL_n R} \to (H_3 G^k_{n-1})_{SL_n R} \right) = 0$, then use this fact in constructing the exact sequence

\begin{equation}
(4.7) \quad (\ker \pi_*^k | H_3 G^k_n)_{SL_n R} \to V_k \to \ker \left( i_*^{k-1} \circ \pi_*^{k-1} | (H_3 G^k_{n-1})_{SL_n R} \right).
\end{equation}
By Propositions 3.5 and 3.1, imσ has at most two generators. For the case \( k = 2 \), we have \( H_3SL_n \mathbb{Z} = \mathbb{Z}/24 \) (e.g. [ALSS, part 1, VI]) and by A.2(v), \( (H_3^n)^2_{SL_nR} = \mathbb{Z} \). So \( H_3SL_n R_2 \cong \mathbb{Z} \oplus \mathbb{Z}/24 \).

Now suppose \( n = \infty \). Theorem 2.1 of [A] proposes an exact sequence

\[ (4.8) \quad K_2 R_r \otimes \mathbb{Z}/2 \rightarrow K_3 R_r \rightarrow H_2 SL_R, \quad (r \geq 1). \]

There is a canonical epimorphism: \( K_3 Z_k \rightarrow Z^{k-1} \), with \( K_3 Z_{k-1}/\pi k K_3 Z_k = \mathbb{Z} \); hence \( H_3 SL_n R_2 = \mathbb{Z} \oplus \mathbb{Z}/24 \) and if \( k \geq 3 \) there is an induced epimorphism: \( \ker (K_3 Z_k \rightarrow K_3 Z_{k-1}) \rightarrow \mathbb{Z} \oplus \mathbb{Z} \) (Stienstra [S, Theorem 1.13]). As \( K_3 R_k = H_3 SL R_k \) off torsion, an inductive argument from the initial case \( k = 3 \) shows that in (4.7), \( \ker \pi_k^* = 0 \) and \( \im \sigma = \mathbb{Z} \oplus \mathbb{Z} = \ker \pi_k^*|H_3 SL_n R_k \). The first of the exact sequences in the theorem statement follows.

The theorem for general \( n \) is a consequence of the stability of the generators. The \( K \)-theory result is implied by the \( n = \infty \) part of the earlier statement, and (4.8), together with the identifications [Ro, Theorem 4] \( K_2 Z_k = Z/2 \oplus (\otimes_{i=2}^k Z/i) \) and \( K_3 Z = Z/48 \). 

5. \( K_3 \) of truncated polynomial rings over finite fields. Throughout this section, \( R \) is a finite field of characteristic \( p \) greater than 2, and \( n \) is a large integer. We obtain \( K_3 R_k \) by using a van der Kallen-Stienstra result in conjunction with an estimate of the quotient of group orders \#\( K_3 R_{k+1} \)/\#\( K_3 R_k \). The latter is obtained from the spectral sequences and associated maps:

\[ (5.1)(k+1) \quad H_*^{\mathbb{A}}(SL_n R_k; H_* M^n_0 (t^k)) \Rightarrow H_*^{\mathbb{A}}SL_n R_{k+1} \]

\[ \uparrow (i_{*+1}) \]

\[ (5.2)(k+1) \quad H_*^{\mathbb{A}}(G^n_0^*; H_* M^n_0 (t^k)) \Rightarrow H_*^{\mathbb{A}}G^n_{k+1} \]

The first subsection reviews various groups \( H_j(SL_n R; H^j M^n_0) \) for \( j \leq 3 \). The next four subsections look at \( E_*^{\mathbb{A}} \) terms of total degree 3 in the spectral sequence \((5.1)(k+1)\), from which \( \ker \pi_k^{*+1} H_3 SL_n R_{k+1} \) is estimated in 5.6. Proposition 5.7 determines \( H_3 SL_n R_k / \pi_k H_3 SL_n R_{k+1} \) then obtains the desired quotient of group orders. Finally, the computation of \( K_2 (R_q^* (t^k)) \) in [VKS] is invoked to give the main theorem, 5.9.

Our constraints on the dimension \( n \) are introduced by Proposition 5.1. The results of Lluis [ALSS], phrased in terms of the general linear group \( GL_n R \) and the full matrix group \( M_n R \), hold for large \( n \). By restricting to \( n \) relatively prime to \( p \) and \( p - 1 \), his results are simply expressed in terms of the respective direct summands \( SL_n R \) and \( M^n_0 \), however, applying stability theorems such as those in [VK], we need only assume \( n \) to be “sufficiently large”.

5.1 PROPOSITION. (i) \( H_i(SL_n R; M^n_0) = 0 = H_i(SL_n R; \wedge^2 M^n_0) \) for \( i = 0 \) or 1.
(ii) \( H_0(SL_n R; H_1 M^n_0) = (S^2 M^n_0)_{SL_n R} \oplus (\wedge^3 M^n_0)_{SL_n R} = R \oplus R. \)
(iii) \( H_1(SL_n R; M^n_0 \otimes M^n_0) = 0. \)
(iv) \( H_2(SL_n R; M^n_0) = R. \)

PROOF. (i), (ii) and (iv) are derived from Lluis [ALSS]. The proof of (iii) is analogous to that used in [ALSS] to prove \( H_i(SL_n R; \wedge^2 M^n_0) = 0. \)
5.2 Lemma. For \( k \geq 2 \), \( R \equiv H_1(\text{SL}_n R_k; \Lambda^n M_n^0(t^k)) \).

Proof. \( \text{SL}_n R_k = \text{SL}_n R \ltimes G_n^k \), so that for any coefficient module \( C \),

\[
H_1(\text{SL}_n R_k; C) = H_1(\text{SL}_n R; C) \oplus \left( H_1(G_k; C) \right)_{\text{SL}_n R}.
\]

Set \( C = \Lambda^n M_n^0 \) and use 5.1(i), the isomorphism \( \pi_* : H_k G_k \to M_n^0 \) of 3.1(i), and A.2(iv). \( \square \)

5.3 Lemma. For \( k \geq 2 \), \( j_* H_3 M_n^0(t^k) = 0 \) in \( H_3 G_k \) and hence \( i_* j_* H_3 M_n^0(t^k) = 0 \) in \( H_3 \text{SL}_n R_{k+1} \).

Proof. The case \( k = 2 \) is [ALS, 2.7 and 2.8(iii)]. Take \( k \geq 3 \). Then according to Proposition 3.1(iii), \( E_3^{0,3} = 0 \) in the spectral sequence (5.2)(\( k+1 \)). Further, it implies \( H_2 G_k^k/\pi_* H_2 G_k^{k+1} \) is represented in \( E_\infty^{0,1} \).

If \( \beta \) is the homology Bockstein associated to the coefficient sequence \( \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p \), and \( s \geq 1 \), \( \beta H_{k+1}(M_n^0; \mathbb{Z}/p) = H_k M_n^0 \) so the lemma is proved by showing that \( j_* H_s(M_n^0; \mathbb{Z}/p) = 0 \) in \( H_s G_k^k; \mathbb{Z}/p \), or, dually, that \( j^* H_s(G_k^k; \mathbb{Z}/p) = 0 \). Consider the cohomology spectral sequence

\[
(5.3)(k+1) H^*(G_k^k; \mathbb{Z}/p) \otimes H^*(M_n^0; \mathbb{Z}/p) \Rightarrow H^*(G_{k+1}; \mathbb{Z}/p), \quad k > 2.
\]

Denote terms in this sequence by \( k^{+1} E_\infty^{*,*} \).

Dually to the homology results, \( k^{+1} E_3^{0,1} = 0 \) and \( \text{im} d_2^{0,1} \) lies in a direct summand of \( H^2(G_k^k; \mathbb{Z}/p) \) which is represented in \( E_\infty^{1,1} \). (In (3.3)(k), the map \( D \) induced by the homology differential is split since \( H_1(\text{SL}_n R; M_n^0) = 0 \).)

Because \( d_2^{0,*} | \Lambda^* M_n^0 \) is a derivation with image in the \( \mathbb{Z}/p \)-vector space \( H^2 G_k^k \otimes \Lambda^* M_n^0 \), the injectivity of \( d_2^{0,*} \) implies that of \( d_2^{0,*} | \Lambda^* M_n^0 \). Now suppose \( \beta \) is the cohomology Bockstein associated to the coefficient sequence \( \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \). Since \( M_n^0 \) is a \( \mathbb{Z}/p \)-vector space, \( H^*(M_n^0; \mathbb{Z}/p) = \Lambda^* M_n^0 \otimes S^*(\beta \otimes M_n^0) \). Thus \( (E_3^{0,*} = \text{ker} d_3^{0,*} = S^*(\beta \otimes M_n^0) \). We want to show that \( d_3^{0,1} \beta \otimes M_n^0 = \text{ker} d_3^{0,1} \).

Observe first that the connecting homomorphism \( \beta \) commutes with spectral sequence differentials which are transgressive. (This can be seen from the geometric description of the transgression of e.g. [M, p. 335] so \( d_3^{0,1} \beta \otimes M_n^0 = \beta d_3^{0,1} M_n^0 \), modulo \( \text{im} d_2^{1,1} \).

We claim next that \( \beta \otimes d_2^{0,1} M_n^0 \) is represented in \( k E_\infty^{1,2} \) by a submodule isomorphic to \( M_n^0 \). To see this, at the cochain level apply the Cartan formula, \( \beta \otimes (a \otimes b) = (\beta \otimes a) \otimes b \pm a \otimes (\beta \otimes b) \) which is a consequence of the definition of the connecting homomorphism in terms of the cochain boundary maps, which are derivations (see e.g. [M, pp. 190, 52]). Note that here we are dealing with the cochain bicomplex underlying the spectral sequence (5.3)(k), not (5.3)(\( k+1 \)). If \( \tilde{x} \neq 0 \) in \( k^{+1} E_2^{0,1} \) and \( x \neq 0 \) represents \( d_2^{0,1} \tilde{x} \) in \( k E_2^{1,1} \), then \( (1 \otimes \beta) x \) represents \( \beta \otimes d_2^{0,1} \tilde{x} \) in \( k E_2^{1,2} \), and is nonzero because \( 1 \otimes \beta \) is injective. Since there are no higher differentials hitting \( k E_2^{1,*} \) terms, \((1 \otimes \beta)x \) is nonzero in \( k E_\infty^{1,2} \). So \( \beta \otimes d_2^{0,1} M_n^0 = M_n^0 \) and is represented in \( k E_\infty^{1,2} \).
On the other hand, because the differential is a derivation, im\( d_2^1 \) is represented in
\[
\left( E_{\infty}^{1,0} \cup E_{\infty}^{1,1} \right) \cup H^*(M^0_n, \mathbb{Z}/p) = E_{\infty}^{2,1} \cup H^*(M^0_n, \mathbb{Z}/p)
\]
or in a lower filtration. Therefore \( \text{im}(\beta^*d_2^{0,1}) \cap \text{im}(d_1^1) = 0 \).

We conclude that \( d_2^{0,2} | \beta^*M^{0\#}_n \) is injective. Hence so is \( d_3^3 | S^*(\beta^*M^{0\#}_n) \).

5.4 LEMMA. \( H_2(\text{SL}_n R_k; M^0_n) \leq R \oplus R \) whenever \( k \geq 2 \).

PROOF. In the spectral sequence
\[
k^D_{\ast,\ast} = H_\ast(\text{SL}_n R; H_\ast(G^k_n; M^0_n)) \Rightarrow H_\ast(\text{SL}_n R_k; M^0_n), \quad kD^2_{1,1} = 0
\]
(by 5.1(iii) and 3.1) and \( kD^2_{2,0} = R \) (by 5.1(iv)). Further, there is a sequence
\[5.4(k)\]
\[
H_2G^k_n \otimes M^0_n)_{\text{SL}_n R} \rightarrow kD^3_{0,2} = (H_2(G^k_n; \mathbb{Z}/p) \otimes M^0_n)_{\text{SL}_n R} \rightarrow (\text{Tor}(M^0_n, M^0_n))_{\text{SL}_n R},
\]
where the final term is isomorphic to \( (M^0_n \otimes M^0_n)_{\text{SL}_n R} \).

Take \( k > 2 \). Using the notation and proof of 3.4, there are exact sequences
\[
U^k \otimes M^0_n \rightarrow H_2G^k_n \otimes M^0_n \rightarrow J_n \otimes M^0_n, \quad \tilde{U}^k \otimes M^0_n \rightarrow U^k \otimes M^0_n \rightarrow M^0_n \otimes M^0_n,
\]
where \( \text{SL}_n R \) acts trivially on \( \tilde{U}_k \). Take \( \text{SL}_n R \)-coinvariance of these and apply the A.2 results \( (J_n \otimes M^0_n)_{\text{SL}_n R} = 0 \) and \( (M^0_n \otimes M^0_n)_{\text{SL}_n R} = R \) to conclude that \( (H_2G^k_n \otimes M^0_n)_{\text{SL}_n R} \) is dominated by \( R \). Thus \( kD^3_{0,2} \leq R \otimes R \). When \( k = 2 \), the same result holds because \( H_2G^k_n \otimes M^0_n \equiv \wedge^2 M^0_n \otimes M^0_n \) and so by A.2(iv) has \( \text{SL}_n R \)-coinvariance \( R \).

The next lemma shows that there is a surjection \( \text{im} d^2_{2,1} \rightarrow R \). Since
\[
H_2(\text{SL}_n R_k; M^0_n) = kD^2_{2,0} \oplus kD^2_{1,1} \oplus kD^3_{0,2},
\]
it will prove this lemma.

5.5 LEMMA. In the spectral sequence \( kD_{\ast,\ast} \Rightarrow H_\ast(\text{SL}_n R_k; M^0_n) \) of 5.4 (proof), \( \text{im} d^2_{2,1} \) maps onto \( R \).

PROOF. The case \( k = 2 \) is covered by [ALS, Theorem 2.2], which shows that \( \text{im} d^2_{2,1} \) maps onto \( (\text{Tor}(M^0_n, M^0_n))_{\text{SL}_n R} \).

If \( k \geq 3 \), let \( \tau: M^0_n(t^k) \rightarrow M^0_n(t^2) \) be the coefficient isomorphism. The spectral sequence map \( \pi_{\ast \tau_{\ast}}: E_{0,2}^2 \rightarrow E_{0,2}^2 \) induces an isomorphism between the terms \( (\text{Tor}(M^0_n, M^0_n))_{\text{SL}_n R} \) and \( \pi_{\ast \tau_{\ast}} \) also induces an isomorphism
\[
H_2(\text{SL}_n R; H_1(G^k_n; M^0_n(t^k))) \rightarrow H_2(\text{SL}_n R; H_1(M^0_n; M^0_n(t^2))),
\]
because \( \pi_{\ast}: H_1(G^k_n; \mathbb{Z}/p) \equiv H_1(G^k_n; \mathbb{Z}/p) \) by 3.1(i). The lemma, and hence Lemma 5.4, follows. □ □

5.6 PROPOSITION. If \( k > 2 \), \#ker(\pi_{\ast}) = H_3SL_n R_k \rightarrow H_3SL_n R_{k-1} \leq \#(R)^3 \).

PROOF. The spectral sequence \( (5.1)(k) \) has \( E_{0,3}^\infty = 0 \) (by 5.3), \( E_{1,2}^\infty \leq R \) (by 5.2) and \( E_{2,1}^\infty \leq R \oplus R \) (by 5.4). □
We next investigate
\[ H_3 \text{SL}_n R_{k-1}/\pi^k_* H_3 \text{SL}_n R_k \]
to estimate \( \#H_3 \text{SL}_n R_k/\#H_3 \text{SL}_n R_{k-1} \).

5.7 Proposition. If \( k > 2 \), there is an exact sequence
\[ \ker \pi^k_* \rightarrow H_3 \text{SL}_n R_k \xrightarrow{\pi^k_*} H_3 \text{SL}_n R_{k-1} \rightarrow R. \]

Hence with Proposition 5.6 we have
\[ (5.6) \quad \#H_3 \text{SL}_n R_k/\#H_3 \text{SL}_n R_{k-1} \leq (\#R)^2. \]

Proof. In the spectral sequence (5.1)(k), \( E^{2}_{0,2} = ((H_2 M^0_n(t^k))_{\text{SL}_n R}) G^2_n = 0 \) by 5.1(i). Therefore \( \text{coker } \pi^k_* = \text{id}_{R^2} \), and this is \( E^{2}_{1,1} \) because \( H_2 \text{SL}_n R_k = K_2(n, R_k) = 0 \) [DS, 4.4]. Finally, \( E^{1}_{1,1} = H_1(\text{SL}_n R; M^0_n(t^k)) \oplus (H_2 G^2_n \otimes M^0_n)_{\text{SL}_n R} \cong R \), following the proof of 5.2 and substituting the isomorphism of 3.1(i) and the results 5.1(i) and A.2(i).

The main theorem is an easy corollary to 5.7 and the following theorem.

5.8 Theorem (van der Kallen and Stienstra). If \( q \gg k \), there is an isomorphism
\[ (5.7) \quad \Delta_k : K_1(R_{2k}, (t))/\{1 - at^k : \alpha \in R\} \rightarrow K_2(R_q, (t^k)). \]

Proof. This is a special case of [VKS, 4.3], given that \( K_2 R_q = 0 \) by the previously quoted result of Dennis and Stein.

Note that because \( K_2 R_k = 0 \), \( K_3(R_k, (t)) = H_3 \text{SL}_n R_k/\text{H}_3 \text{SL}_n R \).

5.9 Theorem. If \( R \) is a finite field of odd characteristic \( p \), there is an isomorphism
\[ \partial^{-1} \circ \Delta_k : K_1(R_{2k}, (t))/\{1 - at^k : \alpha \in R\} \rightarrow K_3(R_k, (t)), \]
where \( \Delta_k \) is as in 5.8 and \( \partial : K_3(R_k, (t)) \rightarrow K_2(R_q, (t^k)) \) is the connecting homomorphism in the long exact K-sequence \( (q \gg k) \).

If \( n \) is large then also \( H_2 \text{SL}_n R_k/\text{H}_3 \text{SL}_n R \cong K_3(R_k, (t)) \) under the map induced by the inclusion \( \text{SL}_n R_k \rightarrow \text{SL} R_k \).

Proof. If \( k = 2 \) or 3 the theorem comes from [ALS, Theorem 1.1], given the stability of the generators used in that proof. Inductive application of (5.6) yields the group order estimate
\[ \#H_3 \text{SL}_n R_k/\#H_3 \text{SL}_n R \leq (\#R)^{2k-2}. \]
However, the order of the domain of \( \partial^{-1} \circ \Delta_k \) is \( (\#R)^{2k-2} \); since \( \partial \) is onto, \( \#H_3 \text{SL}_n R_k/\#H_3 \text{SL}_n R \) must be at least as great. This implies the theorem when \( n = \infty \). The general case comes from the stability theorems of, for example, [VK].

Appendix. This Appendix investigates the \( \text{SL}_n R \)-structure of various modules related to \( M^0_n \otimes i \), \( i = 2 \) or 3. There are 2 subsections.

Notation is as in §2.
A.1 Lemma. Let I be the $\text{SL}_n R$-submodule of $M_n^0 \otimes M_n^0$ generated by $1_{12} \otimes 1_{23} + 1_{43} \otimes 1_{14}$. Let $(M_n^0 \otimes M_n^0)^0$ be the kernel of the quotient map of $M_n^0 \otimes M_n^0$ onto its $\text{SL}_n R$-coinvariance $R$ [K3, 3.7]. Let $\text{St}(R, R)$ be the additive Steinberg group (defined in [K4, 1.4]) where by [K4, 2.15] there is an epimorphism $\Phi: \text{St}(R, R) \twoheadrightarrow M_n^0$ such that $(\text{St}(R, R), \Phi)$ is the universal $\text{SL}_n R$-central extension of $M_n^0$, with kernel $\Omega$.

Then

(i) $(I)_{\text{SL}_n R} = 0$.

(ii) $(M_n^0 \otimes M_n^0)^0/I \cong \text{St}(R, R)$, and if $D^0: (M_n^0 \otimes M_n^0)^0/I \to M_n^0$ is the map induced by the Lie bracket, $((M_n^0 \otimes M_n^0)^0/I, D^0)$ is the universal $\text{SL}_n R$-central extension of $M_n^0$.

(iii) $(M_n^0 \otimes M_n^0)/I \cong \text{St}(R, R) \otimes R$.

Proof: (i) $I$ is $\text{SL}_n R$-generated by $(e_{15} - 1) \cdot (1_{52} \otimes 1_{23} + 1_{43} \otimes 1_{54})$ (use the actions given in 2.3).

(ii) We will use the following definitions:

\[ D: \left( M_n^0 \otimes M_n^0 \right)^0 / I \to M_n^0 \]

is the map induced by $[,]$;

\[ F_{ij}(\alpha, \beta) = \alpha_{ij} \otimes \beta_{ji} \in M_n^0 \otimes M_n^0; \]

\[ \Psi: R \otimes R \to \left( M_n^0 \otimes M_n^0 \right)^0 / I \] is: $\alpha \otimes \beta \to \{ F_{12}(\alpha, \beta) + F_{23}(\alpha, \beta) - F_{13}(\alpha, \beta) \}$.

The proof is divided into two parts. The first shows that $\Psi$ is equivariant and that $\text{im} \Psi = \ker D$, so that $((M_n^0 \otimes M_n^0)^0/I, D^0)$ is $\text{SL}_n R$-central; the second shows that it is universal by exhibiting a map $\sigma: (M_n^0 \otimes M_n^0)^0/I \to \text{St}(R, R)$ with $\Phi \circ \sigma = D^0$.

A. Define a group homomorphism $g: M_n^0 \to ((M_n^0 \otimes M_n^0)^0/I)/\text{im} \Psi$ by choosing for each pair $i \neq j$, an $m \notin \{ i, j \}$ and setting $g(\alpha_{ij}) = \{ 1_{im} \otimes \alpha_{mj} \}$, and, if $j > 1$, $g(\bar{a}_{ij}) = \{ 1_{j1} \otimes \alpha_{ij} \}$. We are going to show that $g$ is an epimorphism. Use 2.3 to check that $\text{im} \Psi$ lies in ker $D$; hence $[,]$ induces an epimorphism

\[ \left( \left( M_n^0 \otimes M_n^0 \right)^0 / I \right) / \text{im} \Psi \to M_n^0 \]

which is a left inverse to $g$, so that $\text{im} \Psi = \ker D$ as claimed.

Consider each of the ten elements listed below as representative of the subset of the canonical $\mathbb{Z}$-basis of $M_n^0 \otimes M_n^0$ which can be obtained from (a) the action of the permutation matrices and (b) switching of the modules $M_n^0$ which form the tensor product. (Recall that $\bar{a}_{ij} = \bar{a}_{im} + \bar{a}_{mj} = -\bar{a}_{ji}$, $m \notin \{ i, j \}$.) The subsets are based on subscript configurations, and partition the basis. It thus suffices to show that the projections of these subsets to $((M_n^0 \otimes M_n^0)^0/I)/\text{im} \Psi$ lie in the image of $g$.

1. $\alpha_{12} \otimes \beta_{34}$
2. $\bar{a}_{12} \otimes \beta_{34}$
3. $\alpha_{12} \otimes \beta_{32}$
4. $\bar{a}_{12} \otimes \beta_{13}$
5. $\alpha_{12} \otimes \beta_{12}$
6. $\alpha_{12} \otimes \beta_{23}$
7. $\alpha_{12} \otimes \beta_{13}$
8. $\bar{a}_{12} \otimes \beta_{23}$
9. $\bar{a} \otimes \beta_{13}$
10. $\bar{a}_{12} \otimes \beta_{21}$

All elements of type 1–5 are zero modulo $I$. Look at $(e_{m1}(\alpha) - 1) \cdot 1_{12} \otimes 1_{34}$, $(e_{m3}(\beta) - 1) \cdot 1_{12} \otimes 1_{34}$, $(e_{12} - 1) \cdot 1_{21} \otimes \beta_{34}$, etc., and use the symmetry of the $\text{SL}_n R$-action on the modules $M_n^0$ forming the tensor product, plus the action of the
permutation matrices. I also contains elements of the types \( \gamma \alpha_{12} \otimes \beta_{23} - \gamma_{14} \otimes \alpha_{24} \)
and \( \frac{1}{2} \alpha_{32} \otimes \beta_{12} - \alpha_{43} \otimes \gamma_{24} \)—consider
\((e_{42}(-\gamma) - 1) \cdot \alpha_{14} \otimes \beta_{23} + \gamma_{12} \otimes \gamma_{24} \). Similarly, it
contains elements of type
\[
(A.1) \quad \frac{1}{2} \alpha_{12} \otimes \beta_{23} - \alpha_{14} \otimes \gamma_{24} \left( (e_{42}(-\gamma) - 1) \cdot \alpha_{14} \otimes \beta_{23} + \gamma_{12} \otimes \gamma_{24} \right).
\]

The class of each element of type \( 6-8 \) is in \( \text{im } g \). As above, \( \xi_{lm} \otimes \mu_{mj} \equiv \frac{1}{2} \xi_{lm} \otimes \beta_{mj} \equiv \alpha_{ij} \otimes \beta_{im} \) modulo \( I \), whenever \( \alpha \beta = e \mu \) and \( i \), \( j \) and \( m \) are distinct. The same sort of reasoning used above, applied to the generator \( \frac{1}{2} \beta_{13} \otimes \frac{1}{2} \beta_{13} + \frac{1}{2} \beta_{14} \otimes \frac{1}{2} \beta_{14} \), shows that
modulo \( I \), \(-\frac{1}{2} \xi_{mj} \otimes \beta_{mj} \equiv \alpha_{sj} \otimes \beta_{is} \equiv \alpha_{ij} \otimes \beta_{is} \) whenever \( s \), \( m \), and \( i \) are mutually distinct.

We can now check that \( \Psi \) is an equivariant map; e.g. if \( m \notin \{1, 2, 3\} \),
\[
(e_{im}(-\gamma) - 1) \cdot \Psi(\alpha, \beta) = \alpha_{12} \otimes \gamma_{2m} \alpha_{13} \otimes \beta_{y_{3m}} = 0 \mod I
\]
and
\[
\left( e_{12}(-\gamma) - 1 \right) \cdot \Psi(\alpha, \beta) = \alpha_{12} \otimes \left( \beta_{y_{21}} - \gamma_{12} \right)
- \alpha_{13} \otimes \beta_{y_{32}} = 0 \mod I.
\]

The class of each type \( 10 \) element is in \( \text{im } g \). Take \( i = \neq j \), \( \{i, j\} \cap \{a, b\} = \emptyset \) and
\( \alpha \beta = e \mu \). Look at \( (e_{21} - 1) \alpha_{y_{13}} \otimes \beta_{y_{32}} - \alpha_{14} \otimes \gamma_{42} \) and permutations to see that
\[
(A.2) \quad -F_{l_{a}}(\alpha \gamma, \beta) + F_{j_{a}}(\alpha \gamma, \beta) + F_{l_{b}}(\alpha, \gamma \beta) - F_{j_{b}}(\alpha, \gamma \beta) \in I.
\]

Apply \( (e_{ji} - 1) \) to \( \xi_{ia} \otimes \mu_{ja} + \alpha_{bj} \otimes \beta_{ib} \) to see that
\[
(A.3) \quad -F_{j_{a}}(\epsilon, \mu) + F_{i_{a}}(\epsilon, \mu) + F_{b_{i}}(\alpha, \beta) - F_{b_{j}}(\alpha, \beta) \in I.
\]

Hence modulo \( I + \text{im } \Psi \),
\[
F_{12}(\epsilon, \mu) + F_{b_{i}}(\alpha, \beta) - F_{b_{j}}(\alpha, \beta) = 0;
\]
i.e., \( F_{b_{i}}(\alpha, \beta) \equiv F_{12}(\epsilon, \mu) + F_{b_{i}}(\alpha, \beta) \equiv F_{j_{2}}(\epsilon, \mu) + F_{b_{j}}(\alpha, \beta) \), by (A.3) so \( F_{b_{j}}(\alpha, \beta) \equiv F_{b_{i}}(\alpha, \beta) - F_{j_{2}}(\alpha, \beta) = F_{b_{j}}(1, \alpha \beta) - F_{j_{2}}(1, \alpha \beta) \), by (A.2). Set \( a = 1 \).

The class of each type \( 9 \) element is in \( \text{im } g \). As shown above \( \alpha_{12} \otimes \beta_{13} - \alpha_{43} \otimes \beta_{43} \)
\( \in I \). Apply \( (e_{31} - 1) \) to it and use the facts that \( F_{31}(\alpha, \beta) + F_{4i}(\alpha, \beta) - F_{4j}(\alpha, \beta) \)
\( \in (I + \text{im } \Psi) \), and, modulo \( I \), \(-\alpha_{12} \otimes \beta_{13} - \alpha_{43} \otimes \beta_{43} \). This gives \( \alpha_{12} \otimes \beta_{13} = 0 \) in \( ((M_{n}^{0} \otimes M_{n}^{0})/I)/\text{im } \Psi \).

This completes part A.

B. Before going into the main part of the proof, we recall some facts about
Hochschild homology \([I]\).

If \( S \) is any associative ring and \( N \) is an \( S \)-bimodule, the Hochschild homology
groups \( HH_{*}(S, N) \) are defined as the homology of the complex \( C_{H}(S, N) = ((S^{**}) \otimes N, \delta_{H}) \), where
\[
\delta_{H}(s_{1} \otimes \cdots \otimes s_{m} \otimes x) = s_{1} \otimes \cdots \otimes s_{m-1} \otimes s_{m} x
+ \sum_{i=1}^{m-1} (-1)^{i} s_{1} \otimes \cdots \otimes s_{i} s_{i+1} \otimes \cdots \otimes x
+ (-1)^{m} s_{2} \otimes \cdots \otimes s_{n} \otimes x s_{1}.
\]
The Morita equivalence $\text{HH}_*(M^0_n, M^0_n) \to \text{HH}_*(R, R)$ is induced by the chain equivalence

$$\text{Tr}(X_1 \otimes \cdots \otimes X_m) = \sum_{a, b, \ldots, z} x_{ab}^1 \otimes x_{bc}^2 \otimes \cdots \otimes x_{yz}^{m-1} \otimes x_{za}^m$$

where $X_i = \sum_{a, b = 1}^n x_{ab}^i$. Define $f: M^0_n \otimes M^0_n \to \text{HH}_1(R, R) = \Omega$ by $f(a \otimes b) = \{\text{Tr}(a \otimes b)\}$. In $C_\mu(M^0_n, M^0_n)$, for any $r \in \text{SL}_n R$ and $a, b \in M^0_n$, compute

(A.4)

$$rar^{-1} \otimes rbr^{-1} = r \otimes [a, b] r^{-1} + a \otimes b + \delta_H (rar^{-1} \otimes r \otimes br^{-1} - r \otimes a \otimes br^{-1}).$$

Thus, since $\text{Tr}$ is a chain equivalence,

$$f(rar^{-1} \otimes rbr^{-1}) = f(r \otimes [a, b] r^{-1}) + f(a \otimes b).$$

As a group, $\text{St}(R, R) \equiv \Omega \oplus M^0_n$. Define $\sigma: (M^0_n \otimes M^0_n)^0 \to \text{St}(R, R)$ by

$$\sigma(a \otimes b) = (f(a \otimes b), [a, b]).$$

Check that $\sigma(1_{12} \otimes 1_{23} + 1_{43} \otimes 1_{14}) = 0$. Hence if we can show that $\sigma$ is an $\text{SL}_n R$-module map, we will have an induced map

$$\left( M^0_n \otimes M^0_n \right)^0 \to \text{St}(R, R)$$

which is inverse to the canonical map.

With the identification $\text{St}(R, R) = \Omega \oplus M^0_n$, the $\text{SL}_n R$-action is given by

$$r \cdot (h, u) = (f(r \otimes ur^{-1}) + h, ru^{-1}); \quad r \in \text{SL}_n R, \ h \in \Omega, \ u \in M^0_n$$

(to check this, one first checks that the action is well defined, then that it is compatible with the usual description of the action on $\text{St}(R, R)$ defined in terms of the generators $y_{ij}(a)$ [K4, 1.4]). Then because $\text{SL}_n R$ acts on $M^0_n$ by conjugation,

$$\sigma(r \cdot (a \otimes b)) = \sigma(rar^{-1} \otimes rbr^{-1}) = (f(rar^{-1} \otimes rbr^{-1}), r[a, b] r^{-1})$$

$$= (f \otimes [a, b] r^{-1} + f(a \otimes b), r[a, b] r^{-1})$$

$$= r \cdot (f(a \otimes b), [a, b]) = r \cdot \sigma(a \otimes b).$$

Thus $\sigma$ is an $\text{SL}_n R$-module map.

(iii) This is read from the commutative diagram:

\[
\begin{array}{ccc}
\Omega & \xrightarrow{\sim} & \text{im}\Psi \\
\downarrow & & \downarrow \| \\
(M^0_n \otimes M^0_n)^0/I & \xrightarrow{\sim} & (M^0_n \otimes M^0_n)/I \rightarrow R \\
\downarrow D^0 & & \downarrow D \\
M^0_n & = & M^0_n
\end{array}
\]

A,2 Lemma. (i) $(M^0_n \otimes M^0_n)_{\text{SL}_n R} = R$, with $\alpha \leftrightarrow \{1_{12} \otimes 2_{21}\}$,

(ii) $(\Lambda^2 M^0_n)_{\text{SL}_n R} = R/2 R$, and $(S^2 M^0_n)_{\text{SL}_n R} = R$.
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(iii) \((M_n^0 \otimes^3)_{SL_n R} = R \oplus R\) with \((\alpha, \beta) \leftrightarrow \{1_{12} \otimes 1_{23} \otimes \alpha_{31} \} + \{1_{23} \otimes 1_{12} \otimes \beta_{31}\}\)

(iv) \((\Lambda^2 M_n^0 \otimes M_n^0)_{SL_n R} = R\), with \(\alpha \leftrightarrow \{1_{12} \otimes 1_{23} \otimes \alpha_{31}\}\)

(v) \((\Lambda^3 M_n^0)_{SL_n R} = R\), with \(\alpha \leftrightarrow \{1_{12} \otimes 1_{23} \otimes \alpha_{31}\}\)

(vi) \((J_n \otimes M_n^0)_{SL_n R} = 0\).

**Proof.** (i) ([Kassel K3, 3.7]) The coinvariance is detected by the equivariant map

\[ T: M_{no} \otimes M_{no} \to R, \quad T(\alpha \otimes b) = \text{Tr}(ab). \]

(ii) \(A^2 M_n^0\) is \(M_{no} \otimes M_{no}\) quotiented by the group generated by \( \{\alpha \otimes b + b \otimes \alpha; a, b \in M_n^0\} \), so the map induced by \(T\) detects the coinvariance [K3, 3.7]. Similarly \(S^2 M_n^0\) is the quotient of \(M_n^0 \otimes M_n^0\) by the group generated by \( \{\alpha \otimes b - b \otimes \alpha; a, b \in M_n^0\} \); again, the map induced by \(T\) detects.

(iii) Let \(I\) be the \(SL_n R\)-submodule of \(M_{no} \otimes M_{no}\) generated by \(1_{12} \otimes 1_{23} + 1_{42} \otimes 1_{14}\). Lemma A.1(ii) exhibits a sequence \(\text{im} \Psi \to (M_{n0}^0 \otimes M_{n0}^0)/I \overset{D}{\to} M_{n0}^0\), where \(D\) is induced by the commutator map \([\ , \ ]\) of 3.2, and \(SL_n R\) acts trivially on \(\text{im} \Psi\). Since \((M_{n0}^0)_{SL_n R} = 0\), \(D_{\ast}(\big((M_{n0}^0 \otimes M_{n0}^0)/I\big) \otimes M_{n0}^0)_{SL_n R} \to \big(M_{n0}^0 \otimes M_{n0}^0\big)_{SL_n R} = R\) is an isomorphism. We will compute \((I \otimes M_{n0}^0)_{SL_n R}\) by determining the \(SL_n R\)-covariance classes of its generators, \(\{y_{ab}(\alpha) = (1_{12} \otimes 1_{23} + 1_{43} \otimes 1_{14}) \otimes \alpha_{ab}; a \neq b\}\), where \(\alpha_{ij}\) denotes "either \(\alpha_{ij}\) or \(\alpha_{ji}\)" and \(y_{ij}\) denotes "either \(y_{ij}\) or \(y_{ji}\)". Congruence in what follows is with respect to coinvariance class.

If \(a \not\in \{2, 3, 4\}\), choose \(c \not\in \{1, 2, 4, a\}\). Then \(y_{ab}(\alpha) = (e^{-1} \cdot y_{cb}(\alpha) (or, if \(c = b, (e \cdot y_{ab})\), say). Apply this sort of argument again, to conclude

\[ (A.5) \quad y_{ab}(\alpha) \equiv 0 \quad \text{if} \quad a \not\in \{2, 3, 4\} \text{ or } b \not\in \{1, 2, 4\}. \]

If \(a \not\in \{2, 3, 4\}\) and \(b \not\in \{1, 2, 4\}\), \(y_{ab}(\alpha) = (e^{-1} \cdot y_{ba}(\alpha) + y_{ab}(\alpha)\). This together with the identities \(y_{ab}(\alpha) = y_{ab}(\alpha) + y_{ab}(\alpha)\) and \(y_{ab}(\alpha) = y_{ab}(\alpha)\), implies

\[ (A.6) \quad y_{ab}(\alpha) \equiv 0 \quad \text{unless} \quad \{a, b\} \cap \{2, 4\} \neq \emptyset. \]

Hence we need only consider \(\{a, b\} \subset \{1, 2, 3, 4\}\).

\(P(i, j)\) is the permutation matrix \(e_{ij} e_{ji} (-1) e_{ij}\). Check that if \(1 \not\in \{a, b\}\), \((e \cdot y_{ab})\), and if \(3 \not\in \{a, b\}\), \(-y_{ab} = (e^{-1} \cdot (P(5, 3) \cdot y_{ab})\).

This eliminates classes of all the generating set of \(I \otimes M_n^0\) other than \(\{y_{31}(\alpha)\}\). However, the equivariant function on \(M_n^0 \otimes M_n^0\), \(A \otimes B \otimes C \to \text{Tr}(ABC)\), takes \(y_{31}(\alpha)\) to \(\alpha\), so there is a copy of \(R\) in \((I \otimes M_n^0)_{SL_n R}\). Thus there is a split sequence

\[ (A.7) \quad R = (I \otimes M_n^0)_{SL_n R} \overset{i}{\to} (M_n^0 \otimes^3)_{SL_n R} \overset{D_{\ast}}{\to} R, \]

where \(i(\alpha)\) is the class of \(1_{12} \otimes 1_{23} \otimes \alpha_{31} + 1_{43} \otimes 1_{14} \otimes \alpha_{31}\), and \(D_{\ast}^{-1}(\alpha)\) is the class of \(1_{12} \otimes 1_{23} \otimes \alpha_{31}\).

(iv) The homology product \(M_n^0 \otimes^3 \to \Lambda^2 M_n^0 \otimes M_n^0\) takes \(y_{31}(\alpha)\) to

\[ \big(\otimes 1\big) y_{31}(\alpha) = 1_{12123} \otimes \alpha_{31} - 1_{4343} \otimes \alpha_{31} = -(e_{42} \cdot 1_{1413} \otimes \alpha_{31}). \]

\(D_{\ast}^{-1}(\alpha)\) maps through \(1_{12123} \otimes \alpha_{31} \in \Lambda^2 M_n^0 \otimes M_n^0\) to \(1_{13} \otimes \alpha_{31}\) in \(M_n^0 \otimes M_n^0\). Hence \((\Lambda^2 M_n^0 \otimes M_n^0)_{SL_n R}\) is as described.
(v) By (iv), the homology product induces an epimorphism $R \to (\Lambda^3 M_n^0)^{SL_n R}$. Observe this is an isomorphism, since there is an $SL_n R$-invariant function in $(\Lambda^3 M_n^0)^\ast$, $A \cap B \cap C \to \text{Tr}(ABC - BAC)$, $A, B, C \in M_n^0$, which takes $1_{12} \cap 1_{23} \cap \alpha_3 \to \alpha$.

(vi) $J_n$ is the image under the homology product of $L_n$ (see 3.2) whereas A.1(ii) proves $(L_n/ I \equiv \ker D)$ is $SL_n R$-invariant. Thus, if $I'$ is the image of $I$ under the product, it suffices to show that $(I' \otimes M_n^0)^{SL_n R} = 0$. But $(I' \otimes M_n^0)^{SL_n R}$ is generated by the class of $(\cap \otimes 1)y_3(\alpha)$, and this is the zero class in $(I' \otimes M_n^0)^{SL_n R}$ as well as in $(\Lambda^2 M_n^0 \otimes M_n^0)^{SL_n R}$.

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