ON $K_3$ OF TRUNCATED POLYNOMIAL RINGS

BY

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ABSTRACT. Group homology spectral sequences are used to investigate $K_3$ of truncated polynomial rings. If $F$ is a finite field of odd characteristic, we show that relative $K_2$ of the pair $(F[t]/(t^q), (t^k))$, which has been identified by van der Kallen and Stienstra, is isomorphic to $K_3(F[t]/(t^k), (t))$ when $q$ is sufficiently large. We also show that $H_3(SL\mathbb{Z}[t]/(t^k)); \mathbb{Z}) = \mathbb{Z}^{k-1} \oplus \mathbb{Z}/24$ and is isomorphic to the associated $K_3$ group modulo an elementary abelian 2-group.

1. Introduction. There are relatively few rings whose third algebraic $K$-groups are known fully. Group homology spectral sequence techniques, however, have proved a useful tool, yielding for example $K_3$ of the quotient rings of the rational integers [ALSS]. This paper again applies these methods. Our main results are for $k \geq 2$:

1.1 Theorem. There are exact sequences

$$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2 \rightarrow K_3\mathbb{Z}[t]/(t^{k+1}) \rightarrow K_3\mathbb{Z}[t]/(t^k) \rightarrow \mathbb{Z},$$

where $U_{k+1}$ is an elementary abelian 2-group of rank at most $[(k+1)/2]$, and $i=0$ if $k$ is even.

1.2 Theorem. If $F$ is a finite field of odd characteristic, there is an isomorphism

$$\partial^{-1} \circ \Delta_k: K_1(F[t]/(t^{2k}), (t))/\{1 - \alpha t^k: \alpha \in F\} \rightarrow K_3(F[t]/(t^k), (t)).$$

These theorems use, and sharpen, results of Van der Kallen and Stienstra [S, VKS]. Stienstra obtains 1.1 modulo information on the torsion group $U_k$; the theorem with $k=2$ has essentially been proved by Kassel. Van der Kallen and Stienstra define the isomorphism $\Delta_k$ onto the relative $K$-group $K_2(F[t]/(t^k), (t^k))$, whenever $q \gg k$ and $F$ is a perfect field. Theorem 1.2 with $k=2$ and 3 has been proved by Snaith, Lluis and Aisbett [LS, ALS].

The Hochschild-Serre spectral sequences studied are associated to the reduction $\Pi^k: SL_n R[t]/(t^k) \rightarrow SL_n R$, where $R$ is initially any commutative ring with identity for which $SK_1 R = 0$. More general results than those of Theorems 1.1 and 1.2 would need information on certain $E^{2}_{\ast \ast}$ terms.
The remainder of this paper is organized as follows. \S 2 lists notation, and
introduces Hochschild-Serre spectral sequences from our perspective. \S 3 recursively
estimates the second homology groups of the kernel of $\Pi^k$, by means of spectral
sequences associated to the reduction $\ker\Pi^k \to \ker\Pi^{k-1}$; this also gives informa-
tion on the SL$_n$ R-coinvariance of the third homology groups. \S 4 deals with the
integral case, to prove 1.1, and \S 5 deals with the finite fields to prove 1.2. The
appendix contains constructive proofs of various module structural details needed
elsewhere in the paper.

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2. Notation and conventions. This section introduces notation, describes differen-
tials in Hochschild-Serre spectral sequences, and determines a formula for the
d$^2$-differential which will cover our applications. There are five subsections.

2.1 General notation and conventions.
2.1.1 $R$ is a commutative associative ring with identity, such that for some $N > 4$
and all $n \geq N$, $SK_1(n, R) = 0$ (i.e. the elementary matrices generate SL$_n$R) and
$K_2(n, R) = K_2R$. Henceforth assume that $n \geq N$.
2.1.2 $R_k$ is the truncated polynomial ring $R[t]/(t^k)$.
2.1.3 Any map $\pi$ is a homomorphism induced by the reduction $R_k \to R_r$ for
some $r < k$ which the context will indicate. $\pi^k$ is induced by the reduction
$R_k \to R_{k-1}$.
2.1.4 All diagrams are commutative exact and all sequences are exact unless
otherwise specified.

2.2 Notation for elements and SL$_n$ R-submodules of SL$_n$R.
2.2.1 $\text{SL}_n R = \text{SL}_\infty R = \lim_n \text{SL}_n R$, where the special linear group SL$_n$ R includes
into SL$_{n+1}$ R as the upper left corner matrices, say. An elementary matrix is denoted
$e_{ij}(\alpha)$, $i \neq j$, $\alpha \in R$. Set $\tilde{e}_{ij}(\alpha) = \text{diag}(1, 1, \ldots, 1 + \alpha, \ldots, (1 + \alpha)^{-1}, \ldots 1)$ when
$(1 + \alpha)$ is invertible in $R$; here, the nontrivial entries are in the $i$th and $j$th
positions.
2.2.2 $e_{ij}$ is the $n \times n$ matrix over $R$ with $\alpha \in R$ in the $(i, j)$th position and all
other entries zero; $\bar{a}_{ij} = a_{ii} - a_{jj}$. $M^0_n$ is the SL$_n$ R-module of zero-trace
$n \times n$ matrices over $R$. (Here and elsewhere, subscript ranges are implied to be
$\{1, \ldots, n\}$.)
2.2.3 $G^k_n = \ker(\pi_\bullet : \text{SL}_n R_k \to \text{SL}_n R)$ so that $\text{SL}_n R_k = \text{SL}_n R \ltimes G^k_n$. Let $i : G^k_n \to $ SL$_n$ R$_k$ be the inclusion. The kernel of $\pi_\bullet$: $G^{k+1}_n \to G^k_n$ is central in $G^{k+1}_n$ and is
isomorphic to $M^0_n$ (identify $e_{ij}(\alpha t^k)$ with $\bar{a}_{ij}$). We denote it $M^0_n(t^k)$, or sometimes
just $M^0_n$. Let $j : M^0_n(t^k) \to G^{k+1}_n$ be the inclusion.
2.3 Commutator relations in SL$_n$ R$_k$: the SL$_n$ R-action on $M^0_n$. If $x$, $y \in R_k$ and
$x^2 y^2 = 0$, 

\begin{equation}
\begin{aligned}
\left[ e_{ij}(x), e_{ab}(y) \right] &= \\
&= \begin{cases}
  e_{ib}(xy), & j = a, i \neq b, \\
  e_{aj}(-xy), & i = b, j \neq a, \\
  \tilde{e}_{ij}(xy) e_{ij}(-x^2 y) e_{ij}(xy^2), & i = b, j = a, \\
  0, & \text{else.}
\end{cases}
\end{aligned}
\end{equation}
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If $\alpha, \beta \in R$, the (left) $\text{SL}_n R$-action on $M_n^0$ is

$$
(e_{ij}(\alpha) - 1) \cdot \beta_{ab} = \begin{cases} 
\alpha \beta_{i,b}^a, & j = a, i \neq b, \\
-\alpha \beta_{a,j}^b, & a \neq j, i = b, \\
\alpha \beta_{i,j}^b - \alpha^2 \beta_{a,b}^a, & a = j, i = b, \\
0, & \text{otherwise.}
\end{cases}
$$

(The right action differs only by the sign reversal $\alpha \to -\alpha$.)

2.4 Homology related definitions.

2.4.1 $H_* X$ denotes the integral homology group $H_*(X; \mathbb{Z})$ of a group $X$.

2.4.2 $H_1(\text{SL}_n R; M_n^0) = HH_1(R, R)$, where $HH$ denotes Hochschild homology [K4, 2.16]. Since $R$ is commutative,

$$HH_1(R, R) = R \otimes R/\langle \alpha \otimes \beta \gamma - \alpha \beta \otimes \gamma + \beta \otimes \gamma \alpha : \alpha, \beta, \gamma \in R \rangle$$

(e.g. [I, p. 108]) and is isomorphic to the $R$-module of absolute Kähler differentials $\Omega \equiv \Omega_{R/\mathbb{Z}}$ (e.g. [K3]).

2.4.3 $B_* X$ is the standard normalized bar resolution with $\mathbb{Z}$-basis elements $x_0[x_1 \cdots | x_n], x_i \in X$. The boundary map $\partial_X: B_* X \to B_{*+1} X$ is

$$\partial_X x_0[x_1 \cdots | x_n] = x_0 x_1[x_2 \cdots | x_n] - \left( \sum_{i=2}^n (-1)^i x_0[x_1 \cdots | x_{i-1} x_i \cdots | x_n] \right) + (-1)^n x_0[x_1 \cdots | x_{n-1}].$$

2.4.4 If $C$ is a right $\mathbb{Z}[X]$ coefficient module, and $x \in C \otimes_X B_* X$, we write $\{x\}$ for the class of $x$ with respect to the equivalence relation induced by the boundary map $1 \otimes \partial_X$.

2.4.5 $[x \cap y]$ denotes $[x|y] - [y|x] \in B_2 X$; similarly, $[x_1 \cap x_2 \cap \cdots \cap x_{r-1} \cap x_r]$ is $\sum (-1)^r \sigma[x_1 \sigma|x_2 \cdots \sigma|x_r]$, where the sum if over the symmetric group $\Sigma_r$. Generally, if $z = z_0[z_1 \cdots z_l] \in B_l X$ and $z^1 = z_0[z_{i+1} \cdots z_{i+j}] \in B_j X$, then $[z \cap z^1] \in B_{i+j+1} X$ is

$$\sum \left\{ (-1)^r z_0 z_1^r \bigg| z_{\sigma(i)} \bigg| \cdots \bigg| z_{\sigma(i+j)} \right\} : \sigma \in \Sigma_{i+j}; \sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(i) \text{ and } \sigma^{-1}(i+1) < \sigma^{-1}(i+2) < \cdots < \sigma^{-1}(i+j).$$

(That is, $\sigma$ runs over all $(i, j)$—shuffles—see [M, p. 243].) Extend $[\cdot \cap \cdot]$ to a group homomorphism $B_* X \otimes B_* X \to B_* X$.

2.4.6 If $X$ is abelian with operation $+$, $\cap: H_i X \otimes H_j X \to H_{i+j} X \times X \to H_{i+j} X$ is the homology (shuffle) product. This operation defines an exterior algebra $\Lambda^* X$ which injects into $H_0 X$ (see [B, p. 123]).

2.4.7 Suppose $X_1$ and $X_2$ are subgroups of $X$ with $[X_1, X_2] = 1$. Identify $x \in B_j X_1$ and $x' \in B_j X_2$ with their images in $B_* X$. Then

$$\partial_X [x \cap x'] = [\partial_X(x) \cap x'] + (-1)^j [x \cap \partial_X(x')].$$

Note that this is not true for general elements in $B_* X$. 

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2.5 Hochschild-Serre spectral sequences. This subsection reviews aspects of Hochschild-Serre group homology spectral sequences and it introduces more notation [CE].

2.5.1 Take an extension \( N \to Y \to X \), in a category of \( S \)-groups with \( S \)-equivariant maps, say. Identify \( N \) with \( i(N) \).

Let \( C \) be a right \( Y \)-module. Consider the spectral sequence

\[
E_{\ast, \ast}(C) = H_{\ast}(X; H_{\ast}(N; C)) \Rightarrow H_{\ast}(Y; C)
\]

induced from the bicomplex \( \{ C \otimes_Y (B_s Y \otimes B_s X) \} = (C \otimes_Y B_s Y) \otimes_X B_s X ; \; \partial = d_1 + d_\Pi \}. \) \( B_s X \) is the standard bar resolution, with \( d_\Pi = \partial_Y \otimes 1 \) and \( d_1(f \otimes g) = (-1)^{ij} \partial \otimes \partial_Y g \). \( B_s Y \otimes B_s X \) is a left \( Y \)-module via diagonal action, and \( C \otimes_Y B_s Y \) is a right \( X \)-module with action \( c \otimes y \cdot x = (\hat{x} \cdot x) \otimes (\hat{x}^{-1} y) \) (\( \hat{x} \) is any lifting of \( x \) to \( Y \)). This is well defined, since for all \( n \) in \( N \), \( c \cdot \hat{x} \otimes \hat{x}^{-1} y = c \cdot \hat{x} n \otimes n^{-1} \hat{x}^{-1} y \).

2.5.2 If \( x \in E^2_{c, b}(C) \) and \( 2 \leq c \leq a \), then \( d^c_{a, b}x \) is calculated by choosing representatives \( x_i \in C \otimes_Y (B_{b+i} Y \otimes B_{a-i} X) \) for \( 0 \leq i < c \), such that \( x_0 \) represents \( x \) and \( d_1 x_i = -d_{\Pi} x_{i+1} \); set \( d^c_{a, b}x_0 = \{ d_1 x_{c-1} \} \). Let \( d^c_{a, b}x_{0} \) denote \( d_1 x_{c-1} \) (of course, this definition is not unique).

2.5.3 Suppose \( C \) is a trivial \( Y \)-module. Suppose \( Y_1 \) and \( Y_2 \) are subgroups of \( Y \) with \( [Y_i, N] = 1 \), \( i = 1, 2 \), and, if \( f(Y) = X_i \), \( [X_1, X_2] = 1 \). Let \( u: X \to Y \) be any set map section to \( f \) with \( u(X_i) \subset Y_i \). Suppose

\[
x = \sum_{i \in I} (-1)^i [x_1^i | x_2^i | \cdots | x_a^i] \in B_a X_1
\]

and

\[
v = \sum_{j \in J} (-1)^j [v_1^j | v_2^j | \cdots | v_b^j] \in B_b X_2
\]

represent elements in \( H_{\ast} X \) and \( y \in C \otimes_Y B_s N \) is a cycle. Then in the spectral sequence (2.4),

\[
d^2_{a+b, s}(y \otimes [x \cap v]) = \{ [d^2_{a, s}(y \otimes x) \cap v] + (-1)^{ab} \{ [d^2_{b, s}(y \otimes v) \cap x] \}
\]

\[
+ \{ \sum_{i \in I} \sum_{j \in J} (-1)^{a+i+j} \{ [u(x_i^{-1}) \cap u(v_j^{-1})] \cap y
\]

\[
\otimes [x_1^i | \cdots | x_a^i] \cap [v_1^j | \cdots | v_b^j] \}
\]

Note. (1) In the process of calculating the differential we prove that \( y \otimes [x \cap v] \) represents an element in \( H_{a+b}(X; H_s(N; C)) \).

(2) The restricted definitions of \( x, v \) and \( y \) ensure that one can choose differential representatives on the right-hand side of (2.5) such that the shuffle products are cycles—again, this comes from the calculation. Indeed, as pointed out by the referee, if \( N \) is central in \( Y \), then the coefficients in (2.4) are trivial and \( d^c_{a, b} \{ (y \otimes x) \cap v \} \) may be defined using a homology shuffle product

\[
\cap: H_{a-2}(X_1; H_{s+1}(N; C)) \otimes H_{b}(X_2) \to H_{a+b-2}(X; H_{s+1}(N; C)).
\]
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PROOF. Use (2.3) to compute \( d_\ell(y \otimes [x \cap v]) = -d_{11}z \) for \( z \in (C \otimes X B_{s+1} Y) \otimes X B_{a+h-1} X \), where
\[
- z = \sum_{i \in I} (-1)^{i+1} \left[ \left[ (u(x_i))^{-1} \right] \cap y \right] \otimes \left[ x_i^j \cdots x_i^l \cap v \right]
+ \sum_{j \in J} (-1)^{a+j+1} \left[ \left[ (u(v_i))^{-1} \right] \cap y \right] \otimes \left[ x \cap [v_i^j \cdots [v_i^k] \right].
\]

Then
\[
d_{12}z = \sum_i (-1)^i \left[ \left[ (u(x_i))^{-1} \right] \cap y \right] \otimes \partial_X \left[ x_i^j \cdots x_i^l \cap v \right]
+ \sum_j (-1)^j \left[ \left[ (u(v_i))^{-1} \right] \cap y \right] \otimes \left[ x \cap \partial_X [v_i^j \cdots [v_i^k] \right]
+ \sum_i \sum_j (-1)^{a+i+j} \left[ \left[ (x_i^j - 1) \right] \left[ (u(v_i)^{-1} \right] \right]
- \left[ (u(v_i)^{-1} \right] \left[ (u(x_i))^{-1} \right] \right] \cap y
\otimes \left[ [x_i^j \cdots x_i^l] \cap [v_i^j \cdots [v_i^k] \right].
\]

Since \( (f - 1)g - (g - 1)f = \partial_y ([f \cap g] + [fg \cap f^{-1}g^{-1}] - [gf \cap f^{-1}g^{-1}] + [ff, g]) \approx ([f, g]) \), the expression for \( d_{12}z \) is equivalent to the class in (2.5).

2.5.4 If \( x \) and \( y \) are as in 2.5.3 with \( x = [x_1 \cap x_2 \cap \cdots \cap x_a] \), where \( [x_i, x_j] = 1 \) \((1 \leq i, j \leq a)\), then repeated application of (2.5) yields
\[
d_{a+1} \{ y \otimes x \} = \left\{ \sum_{\sigma} - \left[ \left[ (u(x_{\sigma(1)}))^{-1}, (u(x_{\sigma(2)}))^{-1} \right] \right] \cap y \right\}
\otimes \left[ x_{\sigma(3)} \cdots [x_{\sigma(a)} \right],
\]
where \( \sigma \) runs over the alternating group \( A_a \).

3. Low dimensional homology groups of \( G_n^k \). This section inductively looks at low dimensional terms in the spectral sequences
\[
(3.1)(k) \quad H_*(G_n^{k-1}; H_* M_n^0(t^{k-1})) \Rightarrow H_* G_n^k, \quad k > 2.
\]
The first subsection is concerned with \( E_2^{\infty, 0} \) and \( E_2^{\infty, \ast} \). Next, the \( E_4^{\infty, \ast} \) term in (3.1) is computed, yielding enough information on \( H_* G_n^k \) to estimate \( E_{6, \ast} \) recursively. This gives us information on the \( SL_n \)-coinvariance of the kernel of \( \pi_*^k \mid H_* G_n^k \). The final subsection looks at \( ker \left( \pi_*^k \right) \mid H_* G_n^k \). For torsion-free \( R \), in the spectral sequence (3.1)(k + 1), \( E_3^{0, \ast} = 0 \) whenever \( \ast > 0 \) and \( k \geq 2 \).

PROOF. Consider (3.1)(k + 1), \( k \geq 2 \). For any \( \{ i, j \} \subset \{ 1, \ldots, n \} \), fix \( m \in \{ i, j \} \). For \( \alpha \in R \), define \( b_{ij}(\alpha) \) to be \( [e_{im}(t) \cap e_{mj}(at^{-1})] \in Z \otimes G_\alpha^R B_2 G_n^k \). Check that this is a cycle. Let \( u: G_n^k \rightarrow G_n^{k+1} \) be a set map section to the reduction; it can be assumed that the elementary matrix \( e_{a0}(\sum \alpha_m t^m) \in G_n^k \) is taken to the matrix of the
same form in $G^k_{n+1}$. Apply formula (2.6) of 2.5.4 and the commutator relations 2.3 to compute

$$-d_{2,0}^2(b_{ij}(a)) = \left\{ \left[u\left(e_{im}(-t)\right), u\left(e_{mj}(-\alpha t^{k-1})\right)\right]\right.$$  

$$= \left\{e_{im}(-t), e_{mj}(-\alpha t^{k-1})\right\}$$  

$$= a_{ij} \text{ if } i \neq j, \text{ else } -a_{mm} + a_{ii}.$$  

Therefore $E^3_{0,1} = 0$. Since $k$ was arbitrary this proves (i).

(ii) $\{b_{ij}(a)\}: 1 \leq i, j \leq n; \alpha \in R$ is a set of generators for $H^k_{2G_n}/\pi_*H^k_{2G_{n+1}}$. But also $\{b_{ij}(a)\} \in \ker(\pi_*: H^k_{2G_n} \to H^k_{2G_{n+1}})$. Thus in $H^k_{2G_{n-1}}, \pi_*H^k_{2G_{n+1}} = \pi_*H^k_{2G_n}$, which implies (ii).

(iii) Consider $(3.1)(k + 1), k \geq 2$. By the universal coefficient theorem [M, p. 171] there is an inclusion $H^k_{2G_n} \otimes H_*M^0_n(t^k) \to E_{2,0}^2$. The proof of (i) shows that $d_{2,0}^2(H^k_{2G_n}) = M^0_n(t^k)$. So, by (2.6), $d_{2,0}^2(E_{2,0}^2)$ contains the homology product $H^k_{2G_n} \otimes \Lambda^a M^0_n(t^k)$, which is just $\Lambda^{a+1}M^0_n(t^k)$. $\Lambda^2 M^0_n(t^k) = H^2_nM^0_n(t^k)$ and, if $R$ is torsion-free, $\Lambda^1 M^0_n(t^k) = H^1_nM^0_n(t^k)$ [B, p. 123].

3.2 REMARKS. Let $J_n = \text{im}(\pi_*: H^k_{2G_n} \to H^k_{2G_n}), k > 2$. (This is independent of $k$, by 3.1(ii).) Then (3.1)(3) contains an exact sequence

$$J_n \to \Lambda^2 M^0_n \to M^0_n.$$  

Let $L_n$ be the associated module defined as $\ker(\{ , \}: M^0_n \otimes M^0_n \to M^0_n)$, where $\{ , \}$ is the composite of the homology product with $d^2$. The identification (2.2.3) of $M^0_n$ takes the matrix $a$ to $1 + at \in \text{SL}_n R_2$. Thus if $a, b \in M^0_n$, using definition (2.5) $[a, b]$ is the commutator of $1 + at$ and $1 + bt$ evaluated in $\text{SL}_n R_3$. Since $(1 + at)^{-1} \equiv 1 - at + a^2t^2 \pmod{t^3}$, this commutator is readily seen to equal $1 + abt^2 - bat^2$. As an element of $M^0_n \equiv \ker(\pi^3: \text{SL}_n R_3 \to \text{SL}_n R_2)$, this is $ab - ba$; thus $[a, b]$ is the usual Lie bracket.

Let $\Omega \to \text{St}(R, R) \to M^0_n$ be the universal $\text{SL}_n R$-central extension of $M^0_n$ [K4, 2.15]. In the Appendix (Lemma A.1(iii)) we show that $(M^0_n \otimes M^0_n)/I \equiv \text{St}(R, R) \otimes R$, where $I$ is the $\text{SL}_n R$-submodule generated by $1_{12} \otimes 1_{23} + 1_{43} \otimes 1_{14}$.

We use this to estimate $\ker(\pi_*^{k+1}|H^k_{2G_{n+1}})$, which is the $E_{1,1}^\infty$ term in (3.1) $(k + 1)$.  

3.3 PROPOSITION. (i) There is a commutative diagram

$$\begin{array}{ccc}
\Omega \otimes R & \longrightarrow & \text{St}(R, R) \otimes R \\
\downarrow \phi \otimes 0 & & \downarrow \rho \otimes \psi \\
(3.3)(k + 1) & \Omega \otimes R/\text{ker}(d^2_{3,0})_{\text{SL}_n R} & \to & \ker(\pi_*^{k+1}|H^k_{2G_{n+1}}) \to M^0_n \ (k \geq 2),
\end{array}$$

where $d^2_{3,0}$ is the differential in the spectral sequence (3.1)$(k + 1)$ and $D$ is the restriction of the $d^2_{3,0}$-differential in (3.1)$(k + 2)$.

(ii) $(\ker(\pi_*^{k+1}|H^k_{2G_{n+1}}))_{\text{SL}_n R} = \text{im} \psi$.

PROOF. Let $u$ be a section to $\pi: G^k_{n+1} \to G^k_n$ with $u(e_{i,j}(t')) = e_{i,j}(t')$. Use 3.1 to identify $E^2_{1,1}$ with $M^0_n \otimes M^0_n$. Use the commutator relations 2.3 to check that

$$x = -\left[e_{12}(t) \cap e_{24}(t^{k-1}) \cap e_{43}(t)\right].$$
is a cycle in $\mathbb{Z} \otimes G_n^* B_3 G_n^k$. Use (2.6) to compute

\[(3.4) \quad d^2_{3,0} \{ x \} = \{(e_{43}(-t), e_{24}(-t^{-k-1})) \otimes \{ e_{12}(t) \} \}
\]

\[= 1_{12} \otimes 1_{23} + 1_{43} \otimes 1_{14} \in \ker \pi^k_{k+1} = M_n^0 \otimes M_n^0.
\]

So $I \subset \im d^2_{3,0}$.

As in the proof of 3.1(ii), $H_2 G_n^{k+1}/\pi^*_n H_2 G_n^{k+2} \cong M_n^0$ and is represented in $\ker \pi^k_{k+1} = E^\infty_{1,1}$. Moreover by the proof of 3.1(i) and Remarks 3.2, in the spectral sequence (3.1)(k + 2), $d^2_{2,0} \ker \pi^k_{k+1}$ is induced by the Lie bracket: $(E^2_{1,1} \cong M_n^0 \otimes M_n^0) \to M_n^0$. According to A.1(ii) if $\St(R, R)$ is identified with $(M_n^0 \otimes M_n^0)^0/I$, then also $\phi$ is induced by the Lie bracket. This gives the right square of the diagram in the proposition statement. Moreover, because $\ker \pi^k_{k+1} = E^\infty_{1,1} = (M_n^0 \otimes M_n^0)/\im d^2_{3,0}$, $\rho \otimes \psi$ has kernel isomorphic to $\im d^2_{2,0}/I$: as a submodule of $(\Omega \otimes R = \ker(\phi \otimes 0))$, $\im d^2_{2,0}/I$ has trivial $\SL_n(R)$-action. By Lemma A.1(i) $(I)_{\SL_n R} = 0$ so $\im d^2_{3,0}/I \cong (\im d^2_{3,0})_{\SL_n R}$. This implies the left square of the diagram.

Finally, $(\St(R, R))_{\SL_n R} = 0$ [K4, 1.7 and 1.4]; hence part (ii) is implied by the isomorphism $\ker \pi^k_{k+1} = (\St(R, R) \otimes R)/(\im d^2_{3,0})_{\SL_n R}$. □

We next want to estimate $(H_3 G_n^k)_{\SL_n R}$. The following lemma will be used in 3.5 to investigate $(E^\infty_{2,1})_{\SL_n R}$.

**3.4 Lemma.** For $k > 2$, $\SL_n R$ acts trivially on $\ker(\pi^*_n \circ \cdots \circ \pi^k_{k+1}: \pi^k_n H_2 G_n^k \to J_n)$.

**Proof.** Denote the kernel of $\pi^*_n \circ \cdots \circ \pi^k_{k+1}$ by $U^k$ when it has domain $H_2 G_n^k$ and by $\bar{U}^k$ when it has domain $\pi^k_n H_2 G_n^k + 1$. Use (3.3)(k) to fit the sequence $\ker \pi^k_n H_2 G_n^k \to U^k \to \bar{U}^k - 1$ into the following diagram, the top row of which is therefore exact. (In (3.5), $D$ is the restriction of the $d^2_{3,0}$ differential in the spectral sequence (3.1)(k + 1).)

\[(\Omega \otimes R)/(\im d^2_{3,0})_{\SL_n R} \quad \sim \quad \bar{U}^k - 1
\]

\[\begin{array}{ccc}
\downarrow & \downarrow & \downarrow \\
\ker \pi^k_n & \to & \bar{U}^k - 1 \\
D & \downarrow & \Downarrow \\
M_n^0 & = & M_n^0
\end{array}
\]

Inductively assume that both the base and fibre modules in the top row of (3.5) have trivial action ($\bar{U}^2 = 0$). Therefore, because $H_1 \SL_n R = 0$, $? = \bar{U}^k$. This gives the inductive step.

\[
\begin{array}{ccc}
? & \to & \pi^k_n H_2 G_n^k + 1 \\
\downarrow & \downarrow & \downarrow \\
\gamma & \to & \gamma \\
U^k & \to & H_2 G_n^k \\
\downarrow & \downarrow & \downarrow \\
D & \downarrow d^2_{3,0} & \Downarrow \\
M_n^0 & = & M_n^0
\end{array}
\]
3.5 Proposition. In the spectral sequence (3.1)(k + 1) for \( k \geq 3 \),
(i) there is an epimorphism: \( R \to (E^\infty_{1,2})_{SL_n R} \);
(ii) there is an exact sequence

\[
R \to (\ker d^2_{2,1})_{SL_n R} \to (\Tor(M^0, M^0)_{SL_n R};
\]
hence for some quotient \( T \) of \( (\Tor(M^0, M^0))_{SL_n R} \), there is an exact sequence

\[
R \to ((\ker d^2_{2,1}/\im d^2_{1,0}) = E^\infty_{2,1})_{SL_n R} \to T.
\]

Proof. (i) \( E^\infty_{1,2} \) is a quotient of \( E^{2}_{1,2} = H_2G^k_n \otimes H_2M^0_n(t^k) \). By 3.1, \( H_1G^k_n \equiv M^0_n \), so \( E^{2}_{1,2} \equiv M^0_n \otimes \Lambda^2 M^0_n \). Use Lemma A.2(iv) to see that \( (E^{2}_{1,2})_{SL_n R} \equiv R \), where \( \alpha \) corresponds to the class of \( g_{12} \otimes 1_{233131}; (E^{\infty}_{1,2})_{SL_n R} \) is a quotient of this.

(ii) Using the notation and proof of Lemma 3.4, filter \( U_k \otimes M^0_n \) as

\[
\begin{align*}
\hat{U}^k \otimes M^0_n &\to U^k \otimes M^0_n \to M^0_n \otimes M^0_n, \\
\end{align*}
\]
where in the base group \( a_{rs} \otimes b_{uv} \) is the image of \( \{ b_{rs}(\alpha) \} \otimes \beta_{uv} \) for \( \{ b_{rs} \} \) as in 3.1(i)(proof). Hence if \( \cap \) is the product \( M^0_n \otimes M^0_n \to M^0_n \), the composite \( \cap \circ (D \otimes 1) \) induces an epimorphism \( g: M^0_n \otimes M^0_n \to \Lambda^2 M^0_n = E^{3,2}_{2,1} \), equivalent to the product map. This has kernel \( \Gamma (M^0_n) \), the Whitehead gamma group which projects onto \( M^0_n \otimes \mathbb{Z}/2 \) with kernel \( S^2M^0_n \), the 2-fold symmetric product of \( M^0_n \) (see, for example, [AD, SO]). So \( (\Gamma (M^0_n))_{SL_n R} \) is a quotient of \( (S^2M^0_n)_{SL_n R} = R \) (by Lemma A.2(ii)). Moreover, (3.8) restricts to

\[
\hat{U}^k \otimes M^0_n \to \ker \cap \circ (D \otimes 1) \to \Gamma (M^0_n) = \ker g.
\]
Since \( \hat{U}^k \) has trivial \( SL_n R \)-action, \( (\hat{U}^k \otimes M^0_n)_{SL_n R} = 0 \); so \( \ker \cap \circ (D \otimes 1) \) also has \( SL_n R \)-coinvariance a quotient of \( R \). \( U_k \) is, by definition, \( \ker (\tau_1: \pi_1G_k \to J_{\lambda}) \), and \( D \) is the restriction of \( d^2_{2,1} \) to the image of \( U_k \otimes M^0_n \) in \( H_2G^k_n \otimes M^0_n \); hence we have an exact sequence

\[
\ker \cap \circ (D \otimes 1) \to \ker (d^2_{2,1} | H_2G^k_n \otimes M^0_n) \to J_{\lambda} \otimes M^0_n.
\]
Apply \( H_0(SL_n R; -) \) to this and insert result (\( J_{\lambda} \otimes M^0_n)_{SL_n R} = 0 \) from Lemma A.2(vi) to see that there is an epimorphism: \( R \to (\ker d^2_{2,1} | H_2G^k_n \otimes M^0_n)_{SL_n R} \).

Finally, the universal coefficient theorem provides a sequence \( H_2G^k_n \otimes M^0_n \to E^{3,2}_{2,1} \to \Tor(H_2G^k_n, M^0_n) \); because the restriction of \( d^2_{2,1} \) to \( H_2G^k_n \otimes M^0_n \) is onto \( E^{3,2}_{2,1} \), there is a sequence

\[
\ker d^2_{2,1} | H_2G^k_n \otimes M^0_n \to \ker d^3_{2,1} \to \Tor(H_2G^k_n, M^0_n).
\]
Apply \( H_0(SL_n R; -) \) to this and identify \( H_2G^k_n \) with \( M^0_n \) to get (3.6). \( \square \)

3.6 Proposition. Whenever \( k > 2 \), there is an exact sequence

\[
H_1(SL_n R; I) \otimes \pi_2H_2G^k_n \to (H_2G^k_n)^{-1} \to \Tor(H_2G^k_n, M^0_n) \to (\im d^2_{3,0})_{SL_n R},
\]
where \( I \) is the \( SL_n R \)-submodule of \( M^0_n \otimes M^0_n \) generated by \( 1_{12} \otimes 1_{23} + 1_{43} \otimes 1_{34} \).

Proof. Consider the spectral sequence (3.1)(k). There can be no transgressions from \( E^3_{3,0} \), as \( E^3_{3,0} = 0 \) (proof of Proposition 3.1(iii)). Thus there is a sequence \( \pi_2H_2G^k_n \to H_2G^k_n^{-1} \to \im d^2_{3,0} \). Apply \( H_1(SL_n R; -) \) to this to get (3.9) except that \( \iota_* \) has kernel \( \partial (H_1(SL_n R; \im d^3_{3,0})) \).
Identify $H_1G_n^{k-1} \otimes M_0^0(t^{k-1})$ with $M_0^0 \otimes M_0^0$ via 3.1(i) and identify $I$ with $(d_{0,0}^2(x))$ as in 3.3(proof). As in the last part of the proof of 3.3, $\im d_{0,0}^2/I$ is the trivial $\SL_n R$-module $(\im d_{0,0}^2)_{\SL_n R}$. So $H_1(\SL_n R; \im d_{0,0}^2/I) = 0$, implying an epimorphism: $H_1(\SL_n R; I) \to H_1(\SL_n R; \im d_{0,0}^2)$. $\partial_k$ is the composite of this epimorphism and $\partial$. □

4. $K_3$ of truncated polynomial rings over the integers. The first three subsections refer to general rings $R$ and concern the $E_{i,2}^\ast$ terms in the spectral sequences

\[(4.1)(k) \quad kE_{2}^{\ast} = H_\ast(\SL_n R; H_2G_n^k) \Rightarrow H_\ast(\SL_n R, k) \quad k \geq 2.\]

Lemmas 4.4 and 4.5 determine $H_1(\SL_n \Z; \Lambda^2 M_0^0)$ and $H_1(\SL_n \Z; I)$. This and earlier work yield the main theorem which computes $H_3(\SL_n \Z[t]/(t^k), k \geq 2)$.

4.1 PROPOSITION. For $k > 2$ there is an exact sequence

\[(4.2) \quad (R + N_k)/R \to H_1(\SL_n R; H_2G_n^k) \xrightarrow{\pi} H_1(\SL_n R; J_n = \pi_\ast H_2G_n^k),\]

where $N_k$ is the $\SL_n R$-coinvariance of the image of the $d_{0,0}^2$ differential in the spectral sequence $(3.1)(k)$, and $R + N_k = R$ if $H_1(\SL_n R; M_0^0) = 0$ (see Proposition 3.3).

PROOF. Proposition 3.3 provides a sequence $N_k \Rightarrow \Omega \oplus R \to \ker \pi_k^\ast \Rightarrow M_0^0$, in which $(\ker \pi_k^\ast)_{\SL_n R} \equiv \im R = (R + N_k)/N_k$ (implying the splitting $(\Omega \oplus R)/N_k \equiv ((R + N_k)/N_k) \oplus ((\Omega \oplus R)/(R + N_k))$). Application of $H_\ast(\SL_n R; -)$ to the sequence $(3.3)(k)$ yields

\[(4.3) \quad H_1(\SL_n R; \ker \pi_k^\ast) \Rightarrow H_1(\SL_n R; M_0^0) = \Omega \to (\Omega \oplus R)/N_k \to (\ker \pi_k^\ast)_{\SL_n R} = (R + N_k)/N_k.\]

Thus $H_1(\SL_n R; \ker \pi_k^\ast) \equiv (R + N_k)/R$; its image under the map induced by inclusion is the kernel of

$\pi_\ast: \left( H_1(\SL_n R; H_2G_n^k) \to H_1(\SL_n R; \pi_\ast H_2G_n^k) \Rightarrow H_1(\SL_n R; J_n) \right)$,

where the injection $\sigma$ is implied by Lemma 3.4. □

4.2 PROPOSITION. If $H_3(\SL_n R; M_0^0) = 0$ and $s: J_n \Rightarrow \Lambda^2 M_0^0$ is as in (3.2), then there is an injection

$s_\ast: H_1(\SL_n R; J_n) \to H_1(\SL_n R; H_2G_n^2 = \Lambda^2 M_0^0).$

If $H_1(\SL_n R; M_0^0) = 0$, $s_\ast$ is an isomorphism.

PROOF. Apply $H_\ast(\SL_n R; -)$ to the exact sequence $J_n \Rightarrow \Lambda^2 M_0^0 \Rightarrow M_0^0$ of (3.2) and use the assumption. □

4.3 PROPOSITION. Suppose $H_i(\SL_n R; M_0^0) = 0$ for $i = 1, 2$. Denote $E_{i,2}^\ast$ terms in the spectral sequence $(4.1)(k)$ by $kE_{i,2}^\ast$, $k \geq 2$. Then $\pi_\ast: \left. kE_{i,2}^\ast \Rightarrow 2E_{i,2}^\ast \right.$ is an injection.

PROOF. By 4.1 and the assumption, $kE_{1,2}^2$ injects into $H_1(\SL_n R; J_n)$ which by 4.2 is isomorphic to $2E_{1,2}^2$. If $j < 2$ and $\pi_\ast: H_jG_n^k \Rightarrow H_jG_n^2$ is an isomorphism (Lemma 3.1) so that $\pi_\ast: \left. kE_{1,2}^r \Rightarrow 2E_{1,2}^r \right.$ is an injection for $r \geq 2$. □
The remainder of this section is devoted to computing $K_3 \mathbb{Z}[t]/(t^k)$; two preliminary lemmas are required. Let $M_0^0R$ denote the submodule of zero-trace matrices in the $SL_n R$-module $M_n R$ of $n \times n$ matrices over $R$.

4.4 LEMMA. If $n$ is large and $p$ is any prime,

(i) $H_1(SL_n \mathbb{Z}/p^2; M_0^0 \mathbb{Z}/p \otimes M_0^0 \mathbb{Z}/p) = 0$; $H_1(SL_n \mathbb{Z}; M_0^0 \mathbb{Z}/p \otimes M_0^0 \mathbb{Z}/p) = 0$;

(ii) $H_1(SL_n \mathbb{Z}; M_0^0 \mathbb{Z} \otimes M_0^0 \mathbb{Z}) = 0$;

(iii) $H_1(SL_n \mathbb{Z}; I) = 0$, where $I$ is as in 3.2.

PROOF. (i) Take $n$ large and prime to $p$ and to $p - 1$. Consider the spectral sequence

$$E^2_{ij}(C) = H_*(SL_n \mathbb{Z}/p; H_*(M_0^0 \mathbb{Z}/p; C)) \Rightarrow H_*(SL_n \mathbb{Z}/p^2; C),$$

firstly with $C = M_0^0 \mathbb{Z}/p$. The proof of [ALSS, part 1, VI, 1.1] demonstrates an isomorphism which is dual to $d_{i,o}$. Moreover, $M_0^0 \mathbb{Z}/p$ may be viewed as a direct summand of $M_0^0 \mathbb{Z}/p \otimes M_0^0 \mathbb{Z}/p$ using either of the inclusions

$$\Phi: \alpha_{ij} \rightarrow \left[ \sum_{k=1}^{n} \alpha_{ik} \otimes 1_{kj} \right] \quad \text{or} \quad \Phi^T: \alpha_{ij} \rightarrow \left[ \sum_{k=1}^{n} -1_{kj} \otimes \alpha_{ik} \right],$$

where if $a \otimes b \in M_0^0 \mathbb{Z}/p \otimes M_0^0 \mathbb{Z}/p$, $[a \otimes b]$ is its image in $M_0^0 \mathbb{Z}/p \otimes M_0^0 \mathbb{Z}/p$ under the canonical projection $(M_0^0 \mathbb{Z}/p)^{\otimes 2} \rightarrow (M_0^0 \mathbb{Z}/p)^{\otimes 2}$. (Since $p + n$, $\Phi$ and $\Phi^T$ are split by the Lie bracket.)

Now

$$E^2_{0,1}(M_0^0 \mathbb{Z}/p \otimes M_0^0 \mathbb{Z}/p) \cong \left[ (M_0^0 \mathbb{Z}/p)^{\otimes 3} \right]_{SL_n \mathbb{Z}/p} = \Phi_*(E^2_{0,1}(M_0^0 \mathbb{Z}/p)) \oplus \Phi^T_*(E^2_{0,1}(M_0^0 \mathbb{Z}/p)),$$

from A.2(i) and (iii). Therefore by naturality,

$$E^2_{0,1}(M_0^0 \mathbb{Z}/p \otimes M_0^0 \mathbb{Z}/p) = d^2_{0,0}(\Phi_*(E^2_{0,0}(M_0^0 \mathbb{Z}/p)) + \Phi^T_*(E^2_{0,0}(M_0^0 \mathbb{Z}/p))).$$

The proof of the first equality in part (i) is completed by recalling that

$$E^2_{0,0}(M_0^0 \mathbb{Z}/p \otimes M_0^0 \mathbb{Z}/p) = H_1(SL_n \mathbb{Z}/p; M_0^0 \mathbb{Z}/p \otimes M_0^0 \mathbb{Z}/p) = 0,$$

dually to [ALSS, part 1, II, 1.4]. Given this, Kassel [K2, 3.4] asserts that the second equality holds. This then implies that multiplication by $p$ is an epimorphism on $H_1(SL_n \mathbb{Z}; M_0^0 \mathbb{Z} \otimes M_0^0 \mathbb{Z})$ which, since we are dealing with a finitely generated abelian group (e.g. [B, p. 217]) means the group is torsion with trivial $p$-component.

However, for large enough $n$ the inclusion: $M_0^0 \mathbb{Z} \to M_0^0 + m \mathbb{Z}$ induces isomorphisms

$$H_1(SL_n \mathbb{Z}; M_0^0 \mathbb{Z} \otimes M_0^0 \mathbb{Z}) \cong H_1(SL_n + m \mathbb{Z}; M_0^0 \mathbb{Z} \otimes M_0^0 \mathbb{Z}), \quad m \geq 1$$

(e.g. [VK, §5]). Thus for $n$ sufficiently large, $H_1(SL_n \mathbb{Z}; M_0^0 \mathbb{Z} \otimes M_0^0 \mathbb{Z})$ has no $p$-component for any prime $p$, giving (i).

(ii) If $I \subset M_0^0 \mathbb{Z} \otimes M_0^0 \mathbb{Z}$ is as in 3.2, $I \otimes \mathbb{Z}/p \subset M_0^0 \mathbb{Z}/p \otimes M_0^0 \mathbb{Z}/p$ is the $SL_n \mathbb{Z}/p$-module generated by $1_{12} \otimes 1_{23} + 1_{43} \otimes 1_{14}$, and equation (4.7) (see A.2(iii) proof) shows that $(I \otimes M_0^0 \mathbb{Z}/p)_{SL_n \mathbb{Z}/p} = \mathbb{Z}/p$, generated by the class of $1_{22} \otimes 1_{23} \otimes 1_{31} + 1_{43} \otimes 1_{14} \otimes 1_{31}$. Embed $M_0^0 \mathbb{Z}/p$ into $I \otimes \mathbb{Z}/p$ with the map $\Phi - \Phi^T$, then argue as for part (i). □
4.5 Lemma. If $n$ is large and $p$ is any odd prime, $H_1(\text{SL}_n\mathbb{Z}; \wedge^2 M_n^0 \mathbb{Z}/p) = 0$; $H_1(\text{SL}_n\mathbb{Z}; \Lambda^2 M_n^0 \mathbb{Z}/2) = \mathbb{Z}/2$; $H_1(\text{SL}_n\mathbb{Z}; \Lambda^2 M_n^0 \mathbb{Z}) = 0$.

Proof. If $p$ is odd, $\Lambda^2 M_n^0 \mathbb{Z}/p$ is a direct summand of $M_n^0 \mathbb{Z}/p \otimes M_n^0 \mathbb{Z}/p$, so the first equality follows from 4.4(i).

Define $\Phi'$: $M_n^0 \mathbb{Z}/2 \rightarrow \Lambda^2 M_n^0 \mathbb{Z}/2$ to be the composite of $\Phi$ (defined in (4.5)) with the homology product on $H_\ast M_n^0 \mathbb{Z}/2$. The map $\Phi'': E_{0,1}^2(\Lambda^2 M_n^0 \mathbb{Z}/2) \rightarrow E_{0,1}^2(\Lambda^2 M_n^0 \mathbb{Z}/2)$ between terms in the spectral sequences (4.4) is an isomorphism, by A.2. Arguing as in the proof of 4.4, we conclude that $E_{0,1}^2(\Lambda^2 M_n^0 \mathbb{Z}/2) = 0$. Thus the reduction epimorphism: $H_1(\text{SL}_n\mathbb{Z}; \Lambda^2 M_n^0 \mathbb{Z}/2) \rightarrow H_1(\text{SL}_n\mathbb{Z}/2; \Lambda^2 M_n^0 \mathbb{Z}/2)$ is injective; its image is identified as $\mathbb{Z}/2$ in [ALSS, part 3, 9.16]. Again apply the Kassel result [K2, 3.4] to conclude that $H_1(\text{SL}_n\mathbb{Z}; \Lambda^2 M_n^0 \mathbb{Z}/2) = \mathbb{Z}/2$.

Now $\text{SL}_n\mathbb{Z}$ acts on $\Lambda^2 M_n^0 \mathbb{Z}/2$ via reduction to $\text{SL}_n\mathbb{Z}/2$. Thus we can use Lemma A.2 to see that each of the terms in the exact sequence $\Lambda^2 M_n^0 \mathbb{Z} \rightarrow \Lambda^2 M_n^0 \mathbb{Z} \rightarrow \Lambda^2 M_n^0 \mathbb{Z}/2$ has $\text{SL}_n\mathbb{Z}$-coinvariance $\mathbb{Z}/2$. In the long exact sequence obtained on application of $H_\ast(\text{SL}_n\mathbb{Z}; \ast)$ to this coefficient sequence, the connecting homomorphism $H_1(\text{SL}_n\mathbb{Z}; \Lambda^2 M_n^0 \mathbb{Z}/2) \rightarrow \mathbb{Z}/2$ must therefore be onto, hence injective.

So multiplying the coefficients in $H_1(\text{SL}_n\mathbb{Z}; \Lambda^2 M_n^0 \mathbb{Z})$ by two results in an onto map. Since we are dealing with a finitely generated group, it must be an odd torsion group. The first equality in the statement of the lemma then implies it is trivial. $\square$

4.6 Theorem. If $k \geq 2$ and $n$ is large there are exact sequences

$$\mathbb{Z} \oplus \mathbb{Z} \rightarrow H_3 \text{SL}_n\mathbb{Z} \oplus \mathbb{Z} \rightarrow H_3 \text{SL}_n\mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z},$$

and

$$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2 \rightarrow K_3 \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2 \rightarrow K_3 \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2 \rightarrow \mathbb{Z},$$

where $U_k$ is an elementary 2-group of rank at most $[k/2]$ and $\iota = 0$ if $k$ is even. (Here $\mathbb{Z}_k$ denotes $\mathbb{Z}[(t)/(t^k)].$

Proof. Take $R = \mathbb{Z}$. If $n > 10$, $H_2(\text{SL}_n\mathbb{Z}; M_n^0) = 0$ [K1], and of course, $\mathbb{Z} = \mathbb{Z}_2/\mathbb{Z} = 0$. $H_1(\text{SL}_n\mathbb{Z}; \Lambda^2 M_n^0) = 0$ by Lemma 4.5, so that by Propositions 3.1 and 4.3. in the spectral sequence (4.1), $k \geq 2, E_{1,\infty}^2 = 0$ and $E_{2,1}^2 = 0$. Thus there is an epimorphism, induced by the inclusion of $G_k^{\ast}$ into $\text{SL}_n R_k$,

$$i_k^{\ast}: (H_3 G_k^{\ast})_{\ast \ast} R \rightarrow H_3 \text{SL}_n R_k/H_3 \text{SL}_n R.$$

Take $k \geq 3$. By (4.6), $\ker\pi_k^{\ast}| H_3 \text{SL}_n R_k$ is the quotient of

$$V_k = \ker(i_k^{k-1} \circ \pi_k^{\ast}: (H_3 G_k^{\ast})_{\ast \ast} R \rightarrow H_3 \text{SL}_n R_{k-1})$$

by $\ker i_k^{\ast}$. Use (3.9) and Lemma 4.4 to identify

$$\ker((\mathbb{Z} (H_3 G_k^{\ast})_{\ast \ast} R) \rightarrow (H_3 G_k^{k-1})_{\ast \ast} R) = 0,$$

then use this fact in constructing the exact sequence

$$\ker(i_k^{k-1}: (H_3 G_k^{k-1})_{\ast \ast} R \rightarrow V_k \rightarrow \ker(i_k^{k-1}| (H_3 G_k^{k-1})_{\ast \ast} R).$$
By Propositions 3.5 and 3.1, imσ has at most two generators. For the case \( k = 2 \), we have \( H_3\text{SL}_n \mathbb{Z} = \mathbb{Z}/24 \) (e.g. [ALSS, part 1, VI]) and by A.2(v), \( (H_3G_n^2)_{\text{SL}_nR} = \mathbb{Z} \). So \( H_3\text{SL}_n R_2 \leq \mathbb{Z} \oplus \mathbb{Z}/24 \).

Now suppose \( n = \infty \). Theorem 2.1 of [A] proposes an exact sequence

\[
K_2R_r \otimes \mathbb{Z}/2 \to K_3R_r \to H_3\text{SL}_n R_r \quad (r \geq 1).
\]

There is a canonical epimorphism: \( K_3Z_k \to Z^{k-1} \), with \( K_3Z_{k-1}/\pi^k K_3Z_k = Z \); hence \( H_3\text{SL}_n R_2 = \mathbb{Z} \oplus \mathbb{Z}/24 \) and if \( k \geq 3 \) there is an induced epimorphism: \( \ker(K_3Z_k \to K_3Z_{k-1}) \to \mathbb{Z} \oplus \mathbb{Z} \) (Stienstra [S, Theorem 1.13]). As \( K_3R_k = H_3\text{SL}_n R_k \) off torsion, an inductive argument from the initial case \( k = 3 \) shows that in (4.7), \( \ker\pi^{k-1}_{i+1} = 0 \) and \( \text{im} \sigma = \mathbb{Z} \oplus \mathbb{Z} = \ker\pi^k | H_3\text{SL}_n R_k \). The first of the exact sequences in the theorem statement follows.

The theorem for general \( n \) is a consequence of the stability of the generators. The K-theory result is implied by the \( n = \infty \) part of the earlier statement, and (4.8), together with the identifications [Ro, Theorem 4] \( K_3Z_k = \mathbb{Z}/2 \oplus (\bigoplus_{i=2}^k \mathbb{Z}/i) \) and \( K_3Z = \mathbb{Z}/48 \). □

5. \( K_3 \) of truncated polynomial rings over finite fields. Throughout this section, \( R \) is a finite field of characteristic \( p \) greater than 2, and \( n \) is a large integer. We obtain \( K_3R_k \) by using a van der Kallen-Stienstra result in conjunction with an estimate of the quotient of group orders \( \#K_3R_{k+1}/\#K_3R_k \). The latter is obtained from the spectral sequences and associated maps:

\[
(5.1) (k + 1) \quad H_k^*(\text{SL}_n R_k; H_*^0 M_n^0 (t^k)) \Rightarrow H_*^0 \text{SL}_n R_{k+1}
\]

\[
(5.2) (k + 1) \quad H_k^* (G_n^k; H_*^0 M_n^0 (t^k)) \Rightarrow H_*^0 G_{n+k+1}^k
\]

The first subsection reviews various groups \( H_1(\text{SL}_n R; H_*^0 M_n^0) \) for \( j \leq 3 \). The next four subsections look at \( E^* \) terms of total degree 3 in the spectral sequence (5.1)(k + 1), from which \( \ker\pi^{k+1}_* | H_3\text{SL}_n R_{k+1} \) is estimated in 5.6. Proposition 5.7 determines \( H_3\text{SL}_n R_k/\pi^k H_3\text{SL}_n R_{k+1} \) then obtains the desired quotient of group orders. Finally, the computation of \( K_2(R_q^1(t^k)) \) in [VKS] is invoked to give the main theorem, 5.9.

Our constraints on the dimension \( n \) are introduced by Proposition 5.1. The results of Lluis [ALSS], phrased in terms of the general linear group \( \text{GL}_n R \) and the full matrix group \( M_n R \), hold for large \( n \). By restricting to \( n \) relatively prime to \( p \) and \( p - 1 \), his results are simply expressed in terms of the respective direct summands \( \text{SL}_n R \) and \( M_n^0 \); however, applying stability theorems such as those in [VK], we need only assume \( n \) to be “sufficiently large”.

5.1 PROPOSITION. (i) \( H_1(\text{SL}_n R; M_n^0) = 0 = H_1(\text{SL}_n R; \wedge^2 M_n^0) \) for \( i = 0 \) or 1.

(ii) \( H_0(\text{SL}_n R; H_*^0 M_n^0) = (S^2 M_n^0)_{\text{SL}_n R} \oplus (\Lambda^3 M_n^0)_{\text{SL}_n R} = R \oplus R \).

(iii) \( H_1(\text{SL}_n R; M_n^0 \otimes M_n^0) = 0 \).

(iv) \( H_2(\text{SL}_n R; M_n^0) = R \).

PROOF. (i), (ii) and (iv) are derived from Lluis [ALSS]. The proof of (iii) is analogous to that used in [ALSS] to prove \( H_1(\text{SL}_n R; \wedge^2 M_n^0) = 0 \). □
5.2 Lemma. For \( k \geq 2 \), \( R \equiv H_1(SL_n R_k; \Lambda^2 M_n^0(t^k)) \).

Proof. \( SL_n R_k = SL_n R \otimes G_n^k \), so that for any coefficient module \( C \),

\[
H_1(SL_n R_k; C) = H_1(SL_n R; C) \otimes (H_1(G^k; C))_{SL_n R}.
\]

Set \( C = \Lambda^2 M_n^0 \) and use 5.1(i), the isomorphism \( \pi_*: H_* G^k \to M_0^k \) of 3.1(i), and A.2(iv). \( \square \)

5.3 Lemma. For \( k \geq 2 \), \( j_* H_3 M_n^0(t^k) = 0 \) in \( H_* G^k_{k+1} \) and hence \( i_* \circ j_* H_3 M_n^0(t^k) = 0 \) in \( H_* SL_n R_{k+1} \).

Proof. The case \( k = 2 \) is [ALS, 2.7 and 2.8(iii)]. Take \( k \geq 3 \). Then according to Proposition 3.1(iii), \( E_3^{0,1} = 0 \) in the spectral sequence (5.2)(\( k+1 \)). Further, it implies \( H_2 G^k_{k}/\pi_* H_2 G^k_{k+1} \) is represented in \( E_1^{1,1} \).

If \( \beta \) is the homology Bockstein associated to the coefficient sequence \( \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p \) and \( s \geq 1 \), \( \beta H_{s+1}(M^0_n; \mathbb{Z}/p) = H_s M^0_n \) so the lemma is proved by showing that \( j_* H_s(M^0_n; \mathbb{Z}/p) = 0 \) in \( H_*(G^k_n; \mathbb{Z}/p) \), or, dually, that \( j_* H_*(G^k_n; \mathbb{Z}/p) = 0 \). Consider the cohomology spectral sequence

\[
(5.3)(k+1) H^*(G^k_n; \mathbb{Z}/p) \otimes H^*(M^0_n; \mathbb{Z}/p) \Rightarrow H^*(G^k_{k+1}; \mathbb{Z}/p), \quad k > 2.
\]

Denote terms in this sequence by \( k+1 E_*^{**} \).

Dually to the homology results, \( k+1 E_3^{0,1} = 0 \) and \( im d_2^{0,1} \) lies in a direct summand of \( H^2(G^k_n; \mathbb{Z}/p) \) which is represented in \( k E_3^{1,1} \). (In (3.3)(k), the map \( D \) induced by the homology differential is split since \( H_1(SL_n R; M^0_n) = \Omega = 0 \).)

Because \( d_2^{0,*}|\Lambda^* M_n^{0,*} \) is a derivation with image in the \( \mathbb{Z}/p \)-vector space \( H^2 G^k_n \otimes \Lambda^* M_n^{0,*} \), the injectivity of \( d_2^{0,*}|\Lambda^* M_n^{0,*} \) implies that of \( d_2^{0,*}|\Lambda^* M_n^{0,*} \). Now suppose \( \beta^* \) is the cohomology Bockstein associated to the coefficient sequence \( \mathbb{Z}/p \to \mathbb{Z}/p \to \mathbb{Z}/p \). Since \( M^0_n \) is a \( \mathbb{Z}/p \)-vector space, \( H^*(M^0_n; \mathbb{Z}/p) = \Lambda^* M_n^{0,*} \otimes S^*(\beta^* M_n^{0,*}) \). Thus \( (E_3^{0,*} = ker d_3^{0,*}) = S^*(\beta^* M_n^{0,*}) \). We want to show that \( d_3^{0,*} \beta^* M_n^{0,*} = M_n^{0,*} \).

Observe first that the connecting homomorphism \( \beta^* \) commutes with spectral sequence differentials which are transgressive. (This can be seen from the geometric description of the transgression of e.g. [M, p. 335].) So \( d_3^{0,*} \beta^* M_n^{0,*} = \beta^* d_2^{0,1} M_n^{0,*} \) modulo \( im d_2^{1,1} \).

We claim next that \( \beta^* d_2^{0,1} M_n^{0,*} \) is represented in \( k E_3^{1,2} \) by a submodule isomorphic to \( M_n^{0,*} \). To see this, at the cochain level apply the Cartan formula, \( \beta^*(a \otimes b) = (\beta^* a) \otimes b \pm a \otimes \beta^* b \), which is a consequence of the definition of the connecting homomorphism in terms of the cochain boundary maps, which are derivations (see e.g. [M, pp. 190, 52]). Note that here we are dealing with the cochain bicomplex underlying the spectral sequence (5.3)(k), not (5.3)(k+1). If \( x \neq 0 \) in \( k+1 E_2^{0,1} \) and \( x \neq 0 \) represents \( d_2^{0,1} x \) in \( k E_2^{1,1} \), then \( (1 \otimes \beta^*) x \) represents \( \beta^* d_2^{0,1} x \) in \( k E_2^{1,2} \), and is nonzero because \( 1 \otimes \beta^* |k E_2^{1,1} \) is injective. Since there are no higher differentials hitting \( k E_1^{r,1} \) terms, \((1 \otimes \beta^*) x \) is nonzero in \( k E_2^{1,2} \). So \( \beta^* d_2^{0,1} M_n^{0,*} \equiv M_n^{0,*} \) and is represented in \( k E_3^{1,2} \).
On the other hand, because the differential is a derivation, \( \text{im} d_{1,1}^1 \) is represented in

\[
(kE_{\infty}^{1,0} \cup kE_{\infty}^{1,1}) \cup H^*(M_n^0; \mathbb{Z}/p) = kE_{\infty}^{2,1} \cup H^*(M_n^0; \mathbb{Z}/p)
\]
or in a lower filtration. Therefore \( \text{im}(\beta^*d_{2,1}^0) \cap \text{im}(d_{2,1}^1) = 0 \).

We conclude that \( d_{2,1}^0 \beta^*M_n^0 \) is injective. Hence so is \( d_{3,0}^3 \beta^*M_n^0 \). \( \square \)

5.4 LEMMA. \( H_2(\text{SL}_n R_k; M_n^0) \leq R \oplus R \) whenever \( k \geq 2 \).

PROOF. In the spectral sequence

\[
kD_{**}^2 = H_* (\text{SL}_n R; H_* (G_n^k; M_n^0)) \Rightarrow H_* (\text{SL}_n R_k; M_n^0),
\]

(by 5.1(iii) and 3.1) and \( kD_{2,0}^2 = R \) (by 5.1(iv)). Further, there is a sequence (5.4) \((k)\)

\[
H_2 (G_n^k \otimes M_n^0) \rightarrow kD_{2,0}^2 = (H_2 (G_n^k; \mathbb{Z}/p) \otimes M_n^0)_{\text{SL}_n R} \rightarrow (\text{Tor}(M_n^0, M_n^0))_{\text{SL}_n R},
\]

where the final term is isomorphic to \((M_n^0 \otimes M_n^0)_{\text{SL}_n R}\).

The next lemma shows that there is a surjection \( \text{im} d_{2,1}^1 \rightarrow R \). Since

\[
H_2 (\text{SL}_n R_k; M_n^0) = kD_{2,0}^2 \oplus kD_{1,1}^2 \oplus kD_{0,2}^3,
\]

it will prove this lemma.

5.5 LEMMA. In the spectral sequence \( kD_{**}^2 \Rightarrow H_* (\text{SL}_n R_k; M_n^0) \) of 5.4 (proof), \( \text{im} d_{2,1}^2 \) maps onto \( R \).

PROOF. The case \( k = 2 \) is covered by [ALS, Theorem 2.2], which shows that \( \text{im} d_{2,1}^2 \) maps onto \((\text{Tor}(M_n^0, M_n^0))_{\text{SL}_n R}\).

If \( k \geq 3 \), let \( \tau: M_n^0 (t^k) \rightarrow M_n^0 (t^2) \) be the coefficient isomorphism. The spectral sequence map \( \pi_* \tau_*: kE_{0,2}^2 \rightarrow kE_{0,2}^2 \) induces an isomorphism between the terms \((\text{Tor}(M_n^0, M_n^0))_{\text{SL}_n R}\); and \( \pi_* \tau_* \) also induces an isomorphism

\[
H_2 (\text{SL}_n R; H_1 (G_n^k; M_n^0 (t^k))) \rightarrow H_2 (\text{SL}_n R; H_1 (M_n^0; M_n^0 (t^2))),
\]

because \( \pi_*: H_1 (G_n^k; \mathbb{Z}/p) = H_1 (G_n^2; \mathbb{Z}/p) \) by 3.1(i). The lemma, and hence Lemma 5.4, follows. \( \square \)

5.6 PROPOSITION. If \( k > 2 \), \( \# \ker (\pi_*): H_3 \text{SL}_n R_k \rightarrow H_3 \text{SL}_n R_{k-1} \leq \# (R)^3 \).

PROOF. The spectral sequence (5.1) \((k)\) has \( E_{0,3}^\infty = 0 \) (by 5.3), \( E_{1,2}^\infty \leq R \) (by 5.2) and \( E_{2,1}^\infty \leq R \oplus R \) (by 5.4). \( \square \)
We next investigate

$$H_3 SL_n R_{k-1}/\pi_* H_3 SL_n R_k$$

to estimate $\# H_3 SL_n R_k/\# H_3 SL_n R_{k-1}$.

5.7 PROPOSITION. If $k > 2$, there is an exact sequence

$$\ker \pi_*^k \to H_3 SL_n R_k \to H_3 SL_n R_{k-1} \to R.$$ 

Hence with Proposition 5.6 we have

(5.6) $\# H_3 SL_n R_k/\# H_3 SL_n R_{k-1} \leq (\# R)^2$.

PROOF. In the spectral sequence (5.1)(k), $E^2_{0,2} = ((H_2 M_0^0(t^k))_{SL_n R}) G_n^k = 0$ by 5.1(i). Therefore $\text{coker} \pi_*^k = \text{im} d_{3,0}^2$, and this is $E^2_{1,1}$ because $H_2 SL_n R_k \equiv K_2(n, R_k) = 0$ [DS, 4.4]. Finally, $E^1_{1,1} = H_1(SL_n R; M_0^0(t^k)) \oplus (H_1 G_n^k \otimes M_0^0)_{SL_n R} \cong R$, following the proof of 5.2 and substituting the isomorphism of 3.1(i) and the results 5.1(i) and A.2(i).

The main theorem is an easy corollary to 5.7 and the following theorem.

5.8 THEOREM (VAN DER KALLEN AND STIENSTRA). If $q \gg k$, there is an isomorphism

(5.7) $\Delta_k: K_1(R_{2k}, (t))/\{1 - at^k: \alpha \in R\} \to K_2(R_q, (t^k)).$

PROOF. This is a special case of [VKS, 4.3], given that $K_2 R_q = 0$ by the previously quoted result of Dennis and Stein.

Note that because $K_2 R_k = 0$, $K_3(R_k, (t)) = H_3 SL R_k/H_3 SL R$.

5.9 THEOREM. If $R$ is a finite field of odd characteristic $p$, there is an isomorphism

$$\partial^{-1} \circ \Delta_k: K_1(R_{2k}, (t))/\{1 - at^k: \alpha \in R\} \to K_3(R_k, (t)),$$

where $\Delta_k$ is as in 5.8 and $\partial: K_3(R_k, (t)) \to K_2(R_q, (t^k))$ is the connecting homomorphism in the long exact $K$-sequence ($q \gg k$).

If $n$ is large then also $H_3 SL_n R_k/H_3 SL_n R \cong K_3(R_k, (t))$ under the map induced by the inclusion $SL_n R_k \to SL R_k$.

PROOF. If $k = 2$ or 3 the theorem comes from [ALS, Theorem 1.1], given the stability of the generators used in that proof. Inductive application of (5.6) yields the group order estimate

$$\# H_3 SL_n R_k/\# H_3 SL_n R \leq (\# R)^{2k-2}.$$

However, the order of the domain of $\partial^{-1} \circ \Delta_k$ is $(\# R)^{2k-2}$; since $\partial$ is onto, $\# H_3 SL R_k/\# H_3 SL R$ must be at least as great. This implies the theorem when $n = \infty$. The general case comes from the stability theorems of, for example, [VK].

Appendix. This Appendix investigates the $SL_n R$-structure of various modules related to $M_n^{0 \otimes i}$, $i = 2$ or 3. There are 2 subsections.

Notation is as in §2.
A.1 Lemma. Let \( I \) be the \( SL_n R \)-submodule of \( M_n^0 \otimes M_n^0 \) generated by \( 1_{12} \otimes 1_{23} + 1_{43} \otimes 1_{14} \). Let \( (M_n^0 \otimes M_n^0)^0 \) be the kernel of the quotient map of \( M_n^0 \otimes M_n^0 \) onto its \( SL_n R \)-coinvariance \( R \) [K3, 3.7]. Let \( St(R, R) \) be the additive Steinberg group (defined in [K4, 1.4]) where by [K4, 2.15] there is an epimorphism \( \phi: St(R, R) \to M_n^0 \) such that

\( (St(R, R), \phi) \) is the universal \( SL_n R \)-central extension of \( M_n^0 \), with kernel \( \Omega \).

Then

(i) \( (I)_{SL_n R} = 0 \).

(ii) \( (M_n^0 \otimes M_n^0)^0/I \equiv St(R, R) \), and if \( D^0: (M_n^0 \otimes M_n^0)^0/I \to M_n^0 \) is the map induced by the Lie bracket, \( ((M_n^0 \otimes M_n^0)^0/I, D^0) \) is the universal \( SL_n R \)-central extension of \( M_n^0 \).

(iii) \( (M_n^0 \otimes M_n^0)^0/I \equiv \ker D \).

Proof. (i) \( I \) is \( SL_n R \)-generated by \( (e_{15} - 1) \cdot 1_{52} \otimes 1_{23} + 1_{43} \otimes 1_{54} \) (use the actions given in 2.3).

(ii) We will use the following definitions:

\[ D: (M_n^0 \otimes M_n^0)^0/I \to M_n^0 \] is the map induced by the Lie bracket, \( (M_n^0 \otimes M_n^0)^0/I, D^0) \) is the universal \( SL_n R \)-central extension of \( M_n^0 \).

The proof is divided into two parts. The first shows that \( \Psi \) is equivariant and that \( \ker \Psi = \ker D \); hence \( (M_n^0 \otimes M_n^0)^0/I, D^0) \) is the universal \( SL_n R \)-central; the second shows that it is universal by exhibiting a map \( \sigma: (M_n^0 \otimes M_n^0)^0/I \to St(R, R) \) with \( \phi \circ \sigma = D^0 \).

A. Define a group homomorphism \( g: M_n^0 \to ((M_n^0 \otimes M_n^0)^0/I)/\ker D \) by choosing for each pair \( i \neq j \), an \( m \notin \{i, j\} \) and setting \( g(\alpha_{ij}) = 1_{im} \otimes \alpha_{mj} \); and, if \( j > 1 \),

\[ g(\alpha_{ij}) = \{1_{i1} \otimes \alpha_{1j}\} \]. We are going to show that \( g \) is an epimorphism. Use 2.3 to check that \( \ker D \) lies in \( \ker \Psi \); hence \( \ker \Psi \) induces an epimorphism

\[ ((M_n^0 \otimes M_n^0)^0/I)/\ker \Psi \to M_n^0 \]

which is a left inverse to \( g \), so that \( \ker \Psi = \ker D \) as claimed.

Consider each of the ten elements listed below as representative of the subset of the canonical \( Z \)-basis of \( M_n^0 \otimes M_n^0 \) which can be obtained from (a) the action of the permutation matrices and (b) switching of the modules \( M_n^0 \), forming the tensor product. (Recall that \( \tilde{\alpha}_{ii} = \tilde{\alpha}_{im} + \tilde{\alpha}_{mj} = -\tilde{\alpha}_{ji} \).) The subsets are based on subscript configurations, and partition the basis. It thus suffices to show that the projections of these subsets to \( (M_n^0 \otimes M_n^0)^0/I)/\ker \Psi \) lie in the image of \( g \).

1. \( \alpha_{12} \otimes \beta_{34} \)
2. \( \tilde{\alpha}_{12} \otimes \beta_{34} \)
3. \( \alpha_{12} \otimes \beta_{32} \)
4. \( \alpha_{12} \otimes \beta_{13} \)
5. \( \alpha_{12} \otimes \tilde{\beta}_{12} \)
6. \( \alpha_{12} \otimes \beta_{23} \)
7. \( \alpha_{12} \otimes \tilde{\beta}_{13} \)
8. \( \alpha_{12} \otimes \tilde{\beta}_{23} \)
9. \( \tilde{\alpha} \otimes \tilde{\beta}_{13} \)
10. \( \alpha_{12} \otimes \tilde{\beta}_{21} \).

All elements of type 1–5 are zero modulo \( I \). Look at \( (e_{m1}(\alpha) - 1) \cdot 1_{12} \otimes 1_{34}, \)

\( (e_{m3}(\beta) - 1) \cdot \alpha_{12} \otimes 1_{34}, (e_{12} - 1) \cdot \alpha_{21} \otimes \beta_{34}, \) etc., and use the symmetry of the \( SL_n R \)-action on the modules \( M_n^0 \), forming the tensor product, plus the action of the
permutation matrices. I also contains elements of the types $\gamma a_{12} \otimes \beta_{23} - \gamma_{14} \otimes \alpha \beta_{43}$ and $\gamma a_{14} \otimes \beta_{12} - \alpha_{43} \otimes \gamma_{24}$—consider $(e_{42}(-\alpha) - 1) \cdot \gamma_{14} \otimes \beta_{23}$, etc. Similarly, it contains elements of type

\[(A.1) \quad \gamma a_{12} \otimes \beta_{23} - \alpha_{14} \otimes \gamma_{24} \cdot (e_{42}(-\gamma) - 1) \cdot \alpha_{14} \otimes \beta_{23} + \gamma a_{12} \otimes \gamma_{24}.\]

The class of each element of type 6–8 is in $\text{im } g$. As above, $e_{1m} \otimes \mu_{mj} = \tilde{a}_{1m} \otimes \beta_{1j}$ modulo $I$, whenever $\alpha \beta = \epsilon u$ and $i$, $j$ and $m$ are distinct. The same sort of reasoning used above, applied to the generator $1_{13} \otimes 1_{32} + 1_{42} \otimes 1_{14}$, shows that modulo $I$, $-\gamma_{1m} \otimes \alpha \beta_{mj} = a_{sj} \otimes \beta_{is} = \tilde{a}_{is} \otimes \beta_{ij}$ whenever $s$, $m$, $i$ and $j$ are mutually distinct.

We can now check that $\Psi$ is an equivariant map; e.g. if $m \in \{1, 2, 3\}$,

\[(e_{1m}(-\gamma) - 1) \cdot \Psi(\alpha, \beta) = a_{12} \otimes \gamma \beta_{2m} - \alpha_{13} \otimes \beta \gamma_{3m} \equiv 0 \mod I\]

and

\[(e_{12}(-\gamma) - 1) \cdot \Psi(\alpha, \beta) = a_{12} \otimes (\beta \gamma_{21} - \gamma_{12}^2) - \alpha \gamma_{13} \otimes \beta_{32} - \alpha_{13} \otimes \beta \gamma_{32} \equiv 0 \mod I.\]

The class of each type 10 element is in $\text{im } g$. Take $i \neq j$, $\{i, j\} \cap \{a, b\} = \emptyset$ and $\alpha \beta = \epsilon u$. Look at $(e_{21} - 1)(\alpha \gamma_{13} \otimes \beta_{32} - \alpha_{14} \otimes \gamma \beta_{42})$ and permutations to see that

\[(A.2) \quad -F_{ia}(\alpha \gamma, \beta) + F_{ja}(\alpha \gamma, \beta) + F_{ib}(\alpha, \gamma \beta) - F_{jb}(\alpha, \beta \gamma) \in I.\]

Apply $(e_{ji} - 1)$ to $e_{ia} \otimes \mu_{aj} + a_{bj} \otimes \beta_{ib}$ to see that

\[(A.3) \quad -F_{ja}(\epsilon, \mu) + F_{ja}(\epsilon, \mu) + F_{bi}(\alpha, \beta) - F_{bj}(\alpha, \beta) \in I.\]

Hence modulo $I + \text{im } \Psi$,

\[F_{12}(\epsilon, \mu) + F_{b2}(\alpha, \beta) - F_{b2}(\alpha, \beta) \equiv 0;\]

i.e., $F_{b2}(\alpha, \beta) \equiv F_{12}(\epsilon, \mu) + F_{b2}(\alpha, \beta) \equiv F_{12}(\epsilon, \mu) + F_{b2}(\alpha, \beta)$, by (A.3) so $F_{b2}(\alpha, \beta) \equiv F_{b2}(\alpha, \beta) - F_{22}(\alpha, \beta) = F_{b2}(1, \alpha \beta) - F_{ja}(1, \alpha \beta)$, by (A.2). Set $a = 1$.

The class of each type 9 element is in $\text{im } g$. As shown above $\tilde{a}_{12} \otimes \beta_{13} - \alpha_{14} \otimes \beta_{24}$

\[\equiv I.\]

Apply $(e_{31} - 1)$ to it and use the facts that $F_{31}(\alpha, \beta) + F_{14}(\alpha, \beta) - F_{34}(\alpha, \beta) \in (I + \text{im } \Psi)$, and, modulo $I$, $-\tilde{a}_{12} \otimes \beta_{31} = a_{34} \otimes \beta_{41}$. This gives $\tilde{a}_{12} \otimes \beta_{13} \equiv -a_{31} \otimes \beta_{31} \equiv 0$ in $((M_n^0 \otimes M_n^0) / I) / \text{im } \Psi$.

This completes part A.

B. Before going into the main part of the proof, we recall some facts about Hochschild homology [I].

If $S$ is any associative ring and $N$ is an $S$-bimodule, the Hochschild homology groups $HH_*(S, N)$ are defined as the homology of the complex $C_H(S, N) = ((S^{**}) \otimes N, \delta_H)$, where

\[\delta_H(s_1 \otimes \cdots \otimes s_m \otimes x) = s_1 \otimes \cdots \otimes s_{m-1} \otimes s_m x + \sum_{i=1}^{m-1} (-1)^i s_1 \otimes \cdots \otimes s_i s_{i+1} \otimes \cdots \otimes x + (-1)^m s_2 \otimes \cdots \otimes s_n \otimes x s_1.\]
The Morita equivalence $HH_\bullet(M_n^0, M_n^0) \to HH_\bullet(R, R)$ is induced by the chain equivalence

$$\text{Tr}(X_1 \otimes \cdots \otimes X_m) = \sum_{a,b,\ldots,z=1}^n \left(x_{1a}^1 \otimes x_{bc}^2 \otimes \cdots \otimes x_{yz}^{m-1} \otimes x_{za}^m\right)$$

where $X_i = \sum_{a,b=1}^n x_{ab}^i a b$. Define $f: M_n^0 \otimes M_n^0 \to HH_1(R, R) = \Omega$ by $f(a \otimes b) = \{\text{Tr}(a \otimes b)\}$. In $C_{il}(M_n^0, M_n^0)$, for any $r \in \text{SL}_n R$ and $a, b \in M_n^0$, compute

$$(rar^{-1} \otimes rbr^{-1}) = r \otimes [a, b] r^{-1} + a \otimes b + \delta_H(rar^{-1} \otimes r \otimes br^{-1} - r \otimes a \otimes br^{-1}).$$

Thus, since $\text{Tr}$ is a chain equivalence,

$$f(rar^{-1} \otimes rbr^{-1}) = f(r \otimes [a, b] r^{-1}) + f(a \otimes b).$$

As a group, $\text{St}(R, R) \cong \Omega \oplus M_n^0$. Define $\sigma: (M_n^0 \otimes M_n^0)^0 \to \text{St}(R, R)$ by

$$\sigma(a \otimes b) = (f(a \otimes b), [a, b]).$$

Check that $\sigma(1_{12} \otimes 1_{23} + 1_{43} \otimes 1_{14}) = 0$. Hence if we can show that $\sigma$ is an $\text{SL}_n R$-module map, we will have an induced map

$$\left( M_n^0 \otimes M_n^0 \right)^0 / I \to \text{St}(R, R)$$

which is inverse to the canonical map.

With the identification $\text{St}(R, R) = \Omega \oplus M_n^0$, the $\text{SL}_n R$-action is given by

$$r \cdot (h, u) = (f(r \otimes ur^{-1}) + h, rur^{-1}); \quad r \in \text{SL}_n R, h \in \Omega, u \in M_n^0$$

(to check this, one first checks that the action is well defined, then that it is compatible with the usual description of the action on $\text{St}(R, R)$ defined in terms of the generators $y_{ij}(a)$ [K4, 1.4]). Then because $\text{SL}_n R$ acts on $M_n^0$ by conjugation,

$$\sigma(r \cdot (a \otimes b)) = \sigma(rar^{-1} \otimes rbr^{-1}) = (f(rar^{-1} \otimes rbr^{-1}), r[a, b] r^{-1})$$

$$= (f \otimes [a, b] r^{-1} + f(a \otimes b), r[a, b] r^{-1})$$

$$= r \cdot (f(a \otimes b), [a, b]) = r \cdot \sigma(a \otimes b).$$

Thus $\sigma$ is an $\text{SL}_n R$-module map.

(iii) This is read from the commutative diagram:

\[ \begin{array}{ccc}
\Omega & \to & \text{im}\Psi \\
\downarrow & & \downarrow \\
(M_n^0 \otimes M_n^0)^0 / I & \to & (M_n^0 \otimes M_n^0) / I \\
\downarrow D & & \downarrow D \\
M_n^0 & \to & M_n^0
\end{array} \]

A2 Lemma. (i) $(M_n^0 \otimes M_n^0)_{\text{SL}_n R} = R$, with $\alpha \leftrightarrow \{1_{12} \otimes \alpha_{21}\}$,

(ii) $(\Lambda^2 M_n^0)_{\text{SL}_n R} = R/2 R$, and $(S^2 M_n^0)_{\text{SL}_n R} = R$,
(iii) \(((M_n^0)^{\otimes 3})_{SL_n R} = R \oplus R\) with \((\alpha, \beta) \leftrightarrow \{(1_{12} \otimes 1_{23} \otimes \alpha_{31}) + (1_{23} \otimes 1_{12} \otimes \beta_{31})\},
(iv) (\Lambda^2 M_n^0 \otimes M_n^0)_{SL_n R} = R, with \alpha \leftrightarrow \{(1_{12}1_{23} \otimes \alpha_{31})\},
(v) (\Lambda^3 M_n^0)_{SL_n R} = R, with \alpha \leftrightarrow \{(1_{12}1_{23} \otimes \alpha_{31})\},
(vi) (J_n \otimes M_n^0)_{SL_n R} = 0.

**Proof.** (i) ([Kassel K3, 3.7]) The coinvariance is detected by the equivariant map

\[ T: M_{n0} \otimes M_{n0} \rightarrow R, \quad T(\alpha \otimes \beta) = \text{Tr}(\alpha \beta). \]

(ii) \(A_2 M_n\) is the quotient of \(M_{n0} \otimes M_{n0}\) quotiented by the group generated by \(\{a \otimes b + b \otimes a; a, b \in M_{n0}\}\), so the map induced by \(T\) detects the coinvariance [K3, 3.7]. Similarly \(S^2 M_n^0\) is the quotient of \(M_n^0 \otimes M_n^0\) by the group generated by \(\{a \otimes b - b \otimes a; a, b \in M_n^0\}\); again, the map induced by \(T\) detects.

(iii) Let \(I\) be the \(SL_n R\)-submodule of \(M_{n0} \otimes M_{n0}\) generated by \(1_{12} \otimes 1_{23} + 1_{43} \otimes 1_{14}\). Lemma A.1(ii) exhibits a sequence \(\text{im}\Psi \rightarrow (M_n^0 \otimes M_n^0)/I \xrightarrow{D} M_n^0\), where \(D\) is induced by the commutator map \(\{ , \} \) of 3.2, and \(SL_n R\) acts trivially on im\(\Psi\). Since \((M_n^0)_{SL_n R} = 0\), \(D:\left((M_{n0} \otimes M_{n0})/I\right) \otimes M_{n0} \rightarrow \left(M_n^0 \otimes M_n^0\right)_{SL_n R} = R\) is an isomorphism. We will compute \((I \otimes M_n^0)_{SL_n R}\) by determining the \(SL_n R\)-covariance classes of its generators, \((\tilde{y}_{ab}(\alpha) = (e_{ac} - 1) \cdot y_{cb}(\alpha) \) (or, if \(c = b\), \(e_{ac} - 1) \cdot y_{da}\), say). Apply this sort of argument again, to conclude \(\tilde{y}_{ab}(\alpha) = 0\) unless \(\{a, b\} \cap \{2, 4\} \neq \emptyset\).

Hence we need only consider \(\{a, b\} \subset \{1, 2, 3, 4\}\).

\(P(i, j)\) is the permutation matrix \(e_{ij}e_{ji}(-1)\). Check that if \(1 \not\in \{a, b\}\), \(\hat{y}_{ab} = (e_{15} - 1) \cdot (P(5, 1) \cdot \hat{y}_{ab})\), and if \(3 \not\in \{a, b\}\), \(-y_{ab} = (e_{53} - 1) \cdot (P(5, 3) \cdot \hat{y}_{ab})\).

This eliminates classes of all the generating set of \(I \otimes M_n^0\) other than \(\{y_{31}(\alpha)\}\). However, the equivariant function on \(M_n^0 \otimes M_n^0\), \(A \otimes B \otimes C \rightarrow \text{Tr}(ABC)\), takes \(y_{31}(\alpha)\) to \(\alpha\), so there is a copy of \(R\) in \((I \otimes M_n^0)_{SL_n R}\). Thus there is a split sequence

\[ (A.7) \quad R = (I \otimes M_n^0)_{SL_n R} \xrightarrow{\epsilon} \left(M_n^0 \otimes M_n^0\right)_{SL_n R} \xrightarrow{D_*} R, \]

where \(\epsilon(\alpha)\) is the class of \(1_{12} \otimes 1_{23} \otimes \alpha_{31} + 1_{43} \otimes 1_{14} \otimes \alpha_{31}\), and \(D_*^{-1}(\alpha)\) is the class of \(1_{12} \otimes 1_{23} \otimes \alpha_{31}\).

(iv) The homology product \(M_n^0 \otimes M_n^0\) takes \(y_{31}(\alpha)\) to \(\Lambda^2 M_n^0 \otimes M_n^0\) takes \(y_{31}(\alpha)\) to

\[ \left(\otimes 1\right) y_{31}(\alpha) = 1_{12}1_{23} \otimes \alpha_{31} - 1_{43}1_{14} \otimes \alpha_{31} = -(e_{42} - 1) \cdot (1_{14}1_{23} \otimes \alpha_{31}). \]

\(D_*^{-1}(\alpha)\) maps through \(1_{12}1_{23} \otimes \alpha_{31} \in \Lambda^2 M_n^0 \otimes M_n^0\) to \(1_{13} \otimes \alpha_{31}\) in \(M_n^0 \otimes M_n^0\). Hence \((\Lambda^2 M_n^0 \otimes M_n^0)_{SL_n R}\) is as described.
(v) By (iv), the homology product induces an epimorphism \( R \to (\Lambda^3 M_n^0)_{SL_n R} \). Observe this is an isomorphism, since there is an \( SL_n R \)-invariant function in \((\Lambda^3 M_n^0)^+\), \( A \cap B \cap C \to \text{Tr}(ABC - BAC)\), \( A, B, C \in M_n^0\), which takes \( 1_{12} \cap 1_{23} \cap 1_{31} \) to \( \alpha \).

(vi) \( J_n \) is the image under the homology product of \( L_n \) (see 3.2) whereas A.1(ii) proves \((L_n/1 \equiv \ker D)\) is \( SL_n R \)-invariant. Thus, if \( I' \) is the image of \( I \) under the product, it suffices to show that \((I' \otimes M_n^0)_{SL_n R} = 0\). But \((I' \otimes M_n^0)_{SL_n R} \) is generated by the class of \((\cap \otimes 1)_{31}(\alpha)\), and this is the zero class in \((I' \otimes M_n^0)_{SL_n R} \) as well as in \((\Lambda^2 M_n^0 \otimes M_n^0)_{SL_n R}\).

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