TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 294, Number 2, April 1986

UNITARY QUASI-LIFTING: APPLICATIONS

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ABSTRACT. Let $U(3)$ be the quasi-split unitary group in three variables defined using a quadratic extension E/F of number fields. Complete local and global results are obtained for the σ-endo-(unstable) lifting from $U(2)$ to $\mathrm{GL}(3, E)$. This is used to establish quasi-(endo-)lifting for automorphic forms from $U(2)$ to $U(3)$ by means of base change from $U(3)$ to $\mathrm{GL}(3, E)$. Base change quasi-lifting is also proven. Continuing the work of [I], the exposition is elementary, and uses only a simple form of an identity of trace formulas, and base change transfer of orbital integrals of spherical functions.

1. Diagrams.

1.1. *Introduction.* In [I] we arrived at an identity of traces which appear in some trace formulas. We compared the trace formulas (in a sufficiently general special case) by matching the orbital integrals of the functions which appear. The decisive case of spherical functions suggests, by means of the Satake transform, the existence of liftings of representations according to a diagram of dual groups which we describe in this section.

There will be six arrows in our diagram. Two of them, the stable and labile base change maps b' and b'' for $U(2)$, have been studied both locally and globally in [**U(2)**]. The third map, i, is simply induction from GL(2) (viewed as a Levi subgroup of a maximal parabolic subgroup, modulo center) to GL(3). The new maps are (1) the base change map b from $U(3)$ to $\mathrm{GL}(3, E)$, which can be studied independently of the other maps (this will be done elsewhere); (2) the endo-lift e from $U(2)$ to $U(3)$; (3) the σ-endo-lift e' from $U(2)$ to $\mathrm{GL}(3, E)$.

In §4 we study the global quasi-endo-lift e in terms of almost all places. A local study of e will be given elsewhere. This is based on the complete local and global results about the lift e' obtained in §§2 and 3, which lead to a simplification of the trace identity [**I**, (4.4)]. Thus in §2 we assume the transfer of unit elements of [**I**, Lemma (3.4)], and deduce the existence of the lifting e' by means of character identities. Since we do not give here a proof of [**I**, Lemma (3.4)], we show in §3 that the eventual cancellation can be achieved without using this Lemma; moreover, we show that the image of the lift has an unstable character. Thus to study e' completely we need [**I**, Lemma (3.4)], but we do not need it for the study of e. In §4 we also show that the quasi-lifting e can be studied without using [**I**, Lemma (2.7)], but by using the (available) [**I**, Lemma (3.3)] alone. Additional comments are given

Received by the editors March 27, 1985.
1980 *Mathematics Subject Classification*. Primary 10D40.
[1] Partially supported by an NSF grant.

in §4.5. In §4.6 we recall a method which cannot be used—as it stands—to prove the existence of the unitary quasi-lifting e. In §4.7 we point out the existence of a "unitary symmetric square" quasi-lifting from SL(2) to PU(3), analogous to the symmetric square lifting [**Sym²**] from SL(2) to PGL(3).

1.2. *Notations.* We now recall the notations of [**I**], leading to the description of the diagram, and motivating the appearance of the various liftings. Thus E/F is a local or global quadratic field extension, $\underline{H}$ is $U(2)$, $\underline{G}$ is $U(3)$, $\underline{G}' = \mathrm{Res}_{E/F}\underline{G}$, $\underline{H}' = \mathrm{Res}_{E/F}\underline{H}$; H, G, G', H' are their groups of F-rational points; $\hat{H}$, $\hat{G}$, $\hat{G}'$, $\hat{H}'$ are their dual groups; $'f = \otimes 'f_v$ and $'\phi = \otimes '\phi_v$ are smooth compactly supported functions on $H(\mathbf{A})$ ($\mathbf{A}$ is for adeles); $f = \otimes f_v$ is a smooth function on $G(\mathbf{A})$, which transforms under the center $Z(\mathbf{A})$ by a fixed character ω^{-1}, and is compactly supported modulo $Z(\mathbf{A})$; $\phi = \otimes \phi_v$ is such a function on $G'(\mathbf{A}) = G(\mathbf{A}_E)$, except that ω is replaced by ω', where $\omega'(x) = \omega(x/\bar{x})$, x in $\mathbf{A}_E^\times$. The local components of $'f$, f, ϕ, $'\phi$ are taken to be matching, namely their orbital integrals are related in a certain way. At any finite place which is unramified or split in E, we can take the component to be spherical. By the Satake transform it can be viewed as a function on certain conjugacy classes in the dual group. The spherical components are related by maps dual to homomorphisms of dual groups, as follows: $('f, f)$ by the endo-map e: $\hat{H} \to \hat{G}$; (f, ϕ) by the base change map b: $\hat{G} \to \hat{G}'$; and $('\phi, \phi)$ by the σ-endo map e': $\hat{H} \to \hat{G}'$, which does not factor through the previous maps. Spherical functions, which are so related, are matching [**I**, Lemmas (2.7), (3.3), (3.4)].

All modules (= representations) are taken to be admissible of finite length if F is local, and automorphic and irreducible if F is global. ρ denotes an H-module and $\{\rho\}$ its packet [**U(2)**]. τ denotes a σ-invariant H'-module, π a G-module, Π a σ-invariant G'-module and κ a fixed character of $\mathbf{A}_E^\times/E^\times N\mathbf{A}_E^\times$, which is nontrivial on $\mathbf{A}^\times$; it appears in the dual group diagram. Induced modules are denoted by $'I(\mu)$, $I(\mu)$, $I(\eta)$, $I(\tau)$. In the local case, if E/F is unramified, then the Satake transform parametrizes the unramified modules by conjugacy classes $t \times \sigma$ in the dual group, where t is in its connected component. Hence fixing a dual group homomorphism is equivalent to a definition of lifting for such modules.

1.3. We now recall the *dual group diagram*:

$$\begin{array}{ccccc} \hat{G} & \xrightarrow{b} & \hat{G}' & & \\ e\uparrow & & i\uparrow & \nwarrow e' & \\ \hat{H} & \xrightarrow[b']{} & \hat{H}' & \xleftarrow[b'']{} & \hat{H} \end{array}$$

The dual groups are semidirect products of their connected component (denoted by a superscript 0), and the Weil group $W_{E/F}$. If C_E denotes $E^\times$ in the local case, and $\mathbf{A}_E^\times/E^\times$ in the global case, then $W_{E/F}$ is an extension of $\mathrm{Gal}(E/F)$ by C_E. $W_{E/F}$ can be realized as the group of pairs $z\sigma^i$ (z in C_E; $i = 0, 1$), where σ^2 is identified with an element in $C_F - N_{E/F}C_E$, and $\sigma z = \bar{z}\sigma$. $W_{E/F}$ acts on the connected component through its quotient $\mathrm{Gal}(E/F)$. σ maps (x, y) in $\hat{G}'^0 = \mathrm{GL}(3, \mathbf{C}) \times \mathrm{GL}(3, \mathbf{C})$ to $(\sigma y, \sigma x)$, where $\sigma x = J{}^t x^{-1} J$. (x, y) of $\hat{H}'^0 = \mathrm{GL}(2, \mathbf{C}) \times \mathrm{GL}(2, \mathbf{C})$ is mapped to $(\sigma y, \sigma x)$, where $\sigma x = w^t x^{-1} w^{-1}$,

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & & 1 \\ & 1 & \\ 1 & & 0 \end{pmatrix}.$$

Further, $\hat{G}^0 = \mathrm{GL}(3, \mathbf{C})$, $\hat{H}^0 = \mathrm{GL}(2, \mathbf{C})$, and σ maps x to σx.

In some cases it suffices to use that form of the dual group which depends only on a quotient, for example $\mathrm{Gal}(E/F)$, of $W_{E/F}$. In other cases we need groups of which $W_{E/F}$ is a quotient. For example, in the case of the stable base change maps b and b'', $(x; z\sigma^i)$ is mapped to $(x, x; z\sigma^i)$ (x in $\hat{G}^0$ or $\hat{H}^0$; z in C_E; $i = 0, 1$), and only $\mathrm{Gal}(E/F)$ is used. However, the unstable base change map b' is defined using all of $W_{E/F}$. It maps $(x; z\sigma^i)$ to $(x\kappa(z), x\kappa(z)(-1)^i; z\sigma^i)$. We shall use the study **[U(2)]** of the local and global stable and unstable lifting of H-modules to H'-modules with respect to b' and b''.

To define the other maps, we write (a, b, c) for

$$\begin{pmatrix} a & & 0 \\ & b & \\ 0 & & c \end{pmatrix},$$

$'h$ for

$$\begin{pmatrix} a & 0 & b \\ 0 & x & 0 \\ c & 0 & d \end{pmatrix} \quad \text{if } h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and

$$x(ad - bc) = 1.$$

Then the endo-map e maps $(h; z\sigma^i)$ to $('h(\kappa(z), \omega'(z)/\kappa(z)^2, \kappa(z)(-1)^i); z\sigma^i)$. Recall that κ is a character of C_E/NC_E, which is nontrivial on C_F/NC_E. ω' is our fixed character of C_E/C_F, defined by $\omega'(z) = \omega(z/\bar{z})$. The induction map i sends $(h, h'; z\sigma^i)$ to $('h(1, \omega', (-1)^i), 'h'((-1)^i, \omega', 1); z\sigma^i)$. Then the square-diagram is commutative, and we define e' so that the triangle diagram is commutative.

1.4. *Induced*. As noted above, the diagram of dual groups homomorphisms is equivalent to a diagram of liftings of unramified modules, or modules induced from unramified characters of the diagonal (minimal Levi) subgroup, when E/F, κ and ω are unramified. In the case of H, a character of the diagonal has the form $(a, \bar{a}^{-1}) \mapsto \mu(a)$ (a in $E^\times$); the corresponding (unitarily) induced module is denoted by $\rho = {'I}(\mu)$. On G, a character of the diagonal whose restriction to the center is ω is given by $(a, b, \bar{a}^{-1}) \mapsto \mu(a)(\omega/\mu)(b)$. The associated unitarily induced G-module is denoted by $I(\mu)$. $I(\eta)$ denotes the G'-module unitarily induced from the character η of the diagonal subgroup $E^\times \times E^\times \times E^\times$ of G'; the restriction of η to the center Z' is taken to be ω'. Now our diagram asserts the following. Put $\bar{\mu}$ for $\bar{\mu}(x) = \mu(\bar{x})$.

LEMMA. *b maps $I(\mu)$ to $I(\mu, \omega'\bar{\mu}/\mu, \bar{\mu}^{-1})$, e maps ${'I}(\mu)$ to $I(\kappa\mu)$, e' maps ${'I}(\mu)$ to $I(\mu, \omega'\bar{\mu}/\mu, \bar{\mu}^{-1})$, i indicates induction: the H'-module τ maps to the G'-module $I(\tau)$, b' maps ${'I}(\mu)$ to the H'-module ${'I}(\mu, \bar{\mu}^{-1}) \otimes \kappa$ and b'' maps ${'I}(\mu)$ to ${'I}(\mu, \bar{\mu}^{-1})$.*

The Lemma deals with the case where E/F, κ and ω are unramified, but the result is valid under no restriction, in the following sense. Using the definition of matching of functions, and the standard computation [**D**] of characters of induced modules (and the twisted character of $I(\eta)$ when η is a σ-invariant character), it is easy to check that when (f, ϕ), $(f, {}'f)$ and $(\phi, {}'\phi)$ are matching (see [**I**]), and (π, Π), (π, ρ) and (Π, ρ) are induced modules related by the Lemma, then $\operatorname{tr}\pi(f) = \operatorname{tr}\Pi(\phi \times \sigma)$, $\operatorname{tr}\pi(f) = \operatorname{tr}\rho({}'f)$, $\operatorname{tr}\Pi(\phi \times \sigma) = \operatorname{tr}\rho({}'\phi)$. Similar statements hold with respect to the maps b', b'', as discussed in [**U(2)**]. These relations in the induced case give a hint to be pursued in the general case.

2. The σ-endo-lifting e'.

2.1. *Quasi-lifting.* Given the above notion of local lifting in the unramified case, we can make a general definition of global lifting.

DEFINITION. Let J, J' be a pair of groups as above (H, G, etc.) for which the local notion of lifting is defined in the unramified case. If $\pi = \otimes \pi_v$ and $\pi' = \otimes \pi_v'$ are automorphic J- and J'-modules, and π_v lifts to π_v' for almost all v, then we say that π *quasi-lifts* to π'.

In some cases it is possible to define the strong notion of global lifting, in terms of all places. This has been done in [**U(2)**] in the case of the base change liftings b' and b''. The map i is simply induction. Our aim in this section is to study the local and global lifting in the case of the σ-endo-lift e'. This, or the alternative approach of §3, will be used in §4 for the study of the quasi-endo-lift e, and the base change lift b.

2.2. *Characters.* Our study of the lifting is based on the Harish-Chandra theory [**H**] of characters. The method of [**H**], although stated only in the nontwisted case, implies that if $\Pi(\phi \times \sigma)$ is the convolution operator $\int_{G'/Z'} \phi(x)\Pi(x \times \sigma)\,d'x$, then we have

PROPOSITION. *Given an admissible irreducible σ-invariant G'-module Π with central character ω', there exists a locally integrable function χ' on G', which transforms by ω' on Z', and is smooth on the σ-regular set, such that* $\operatorname{tr}\Pi(\phi \times \sigma)$ *is equal to* $\int \chi'(x)\phi(x)\,d'x$ *(x in G'/Z', $d'x$ is a Haar measure) for all ϕ.*

The obvious analogue holds in the nontwisted case, with $\Pi(\phi \times \sigma)$ replaced by $\pi(f)$. Also we use the *Weyl integration formula*. In the twisted case it asserts

$$\int_{G'/Z'} \chi'(x)\phi(x)\,d'x = \sum_T [W(T)]^{-1} \int_{T/Z} \sum_b \Delta'\chi'(t^b)F(t^b, \phi)\,dt.$$

T is taken over a set of representatives for the stable conjugacy classes of tori in G. It is then viewed as the σ-centralizer of a σ-regular element whose norm is in T, and in this sense the integral is taken. $\Delta'(t)$ means $\Delta(Nt)$; Nt in T is an element of G. The inner sum ranges over $B'(T/F)$; thus t^b is taken over a set of representatives for the σ-conjugacy classes within the stable σ-conjugacy class of t. There is no need to write out the nontwisted analogue.

2.3. *Local lifting.* We now begin the study of the lifting e'. Our first aim is to study the local lifting. For that we fix a global totally imaginary extension E/F whose completion at w is our local quadratic extension. Let ρ_w (see [**U(2)**]) be a discrete-series H_w-packet (a packet of discrete-series[2] H_w-modules). At two finite places $v = u, u'$, say u splits and u' does not split in E/F, we choose supercuspidal representations ρ_v. Let V be a finite set containing w and the places which ramify in E/F, but no infinite places. It is easy to see (using the trace formula) that there is a cuspidal H-module ρ whose components at w, u, u' are the given ones, which is unramified at all finite v outside V, and its components at the v in V are all discrete series. We choose a sequence $\{t_v;\ v \text{ outside } V\}$ so that ρ makes a contribution to the sum in the trace formula, which is associated with $'\phi$. Then the trace formula of [**I**, Proposition 4.4], asserts

$$\prod \operatorname{tr} I(\tau_v; \phi_v \times \sigma) = \prod \operatorname{tr}\{\rho_v\}('\phi_v) + 2\sum \prod \operatorname{tr} \pi_v(f_v) - \sum n('\rho) \prod \operatorname{tr}\{'\rho_v\}('f_v).$$

The products extend over the finite places in V. $\{\rho_v\}$ are the packets of the components of our ρ. By [**U(2)**], ρ lifts via the stable base change map b'' to an automorphic H'-module τ. Rigidity theorem for G' (see [**JS**]) implies that $I(\tau)$ is the only contribution to the terms involving ϕ in the Proposition of [**I**, (4.4)]. The terms $I(\mu)$ do not appear due to the condition at the split place u. Moreover, since u' is a nonsplit place, and the character of $\{\rho_{u'}\}$ (namely sum of characters of the members in the packet) is nonzero on the elliptic set, we may choose $'\phi_{u'}$ supported on the regular $H_{u'}$-elliptic set with $\operatorname{tr}\{\rho_{u'}\}('\phi_{u'}) \neq 0$. Then the matching $\phi_{u'}$ can be chosen so that its stable σ-orbital integrals are 0. Namely we can take $f_{u'} = 0$, and $'f_{u'} = 0$. Consequently

$$(*) \qquad \prod \operatorname{tr} I(\tau_v; \phi_v \times \sigma) = \prod \operatorname{tr}\{\rho_v\}('\phi_v).$$

2.4. We can repeat the same discussion with an automorphic H-module ρ' which is unramified outside V, its components at all finite $v \neq w$ in V are in the packets $\{\rho_v\}$, and at w the component is induced. In this case we obtain the identity $(*)$, in which the product extends over all finite $v \neq w$ in V. Since there are $'\phi_v$ supported on the regular set, with $\operatorname{tr}\{\rho_v\}('\phi_v) \neq 0$, we conclude the following

PROPOSITION. *Suppose that τ_w is the stable base change b'' lift* [**U(2)**] *of an irreducible H_w-module ρ_w. Then for any matching ϕ_w and $'\phi_w$, we have*

$$\operatorname{tr} I(\tau_w; \phi_w \times \sigma) = \operatorname{tr}\{\rho_w\}('\phi_w).$$

This is shown above for induced and discrete series ρ_w, when w is a nonsplit finite place. The case of the one-dimensional H_w-module follows at once, as its character is a difference of the characters of an induced and a special module. Moreover, the Proposition holds also when E_w/F_w is $\mathbf{C}/\mathbf{R}$, and $\{\rho_w\}$ is unitary. It suffices to consider discrete series ρ_w, and take $F = Q$ and an imaginary quadratic E. Repeating the proof of $(*)$, the Proposition follows in this case too.

[2] An admissible irreducible G_w-module π_w is called square-integrable, or discrete-series, if it has a coefficient $f(g) = \langle \pi_w(g)v, v' \rangle$ which is absolutely square-integrable on G_w/Z_w, where Z_w is the center of G_w.

3. Alternative approach.

3.1. *Instability.* In the proof of the Proposition in §2.4 we used only the σ-endo-transfer [**I**, Lemma (3.4)] of the unit element ϕ^0 in the Hecke algebra of G' to the unit element $'\phi^0$ in the Hecke algebra of H; and the transfer of spherical functions with respect to e': $\hat{H} \to \hat{G}'$, which follows from the statement for $(\phi^0, '\phi^0)$ by a method of Clozel [**Cl**], as in [**Sym**^{2}]. This is needed only at places where E/F, κ, ω are unramified. At the other places it suffices to transfer functions supported on the regular set, and this is easily done.

We shall now give an alternative approach, whose purpose is to show that the character of $I(\tau_w)$ is an unstable function, namely that $\operatorname{tr} I(\tau_w; \phi_w \times \sigma)$ depends only on $'\phi_w$. We shall not use [**I**, Lemma (3.4)], and conclude that complete local, and some global, results about the endo-lifting e can be obtained without using any knowledge of the σ-endo-transfer of [**I**, Lemma (3.4)]. It is useful to record the results of Keys [**Ke**] concerning the reducibility of induced G-modules.

3.2. *Reducibility.* Suppose that E/F is a nonarchimedean quadratic extension, and ν is the valuation character $\nu(x) = |x|$ on $E^\times$. μ is a unitary character, and s a real number. As $I(\mu\nu^s)$ is equivalent to $I(\bar{\mu}^{-1}\nu^{-s})$, we may assume $s \geqslant 0$. There are three cases in which an induced G-module is reducible [**Ke**]. Then the composition series has length two (since $[W(A)] = 2$). We now denote by μ a character of $E^\times$ which is trivial on $F^\times$. The cases are

(1) If $\mu^3 \neq \omega'$, then $I(\mu)$ is the direct sum of tempered non-discrete-series G-modules denoted by π^+ and π^-. Namely the condition for reducibility is that the restriction to $A \cap \mathrm{SL}(3, E)$, of the character $(a, b, \bar{a}^{-1}) \mapsto \mu(a)(\omega/\mu)(b)$ which defines $I(\mu)$ (thus $b = \bar{a}/a$), is nontrivial.

(2) $I(\mu\kappa\nu^{1/2})$ has a nontempered component $\pi_\mu^\times$ and a discrete-series component π_μ^+.

(3) If $\omega = \theta^3$, and $\mu = \theta/\bar{\theta}$ for a character θ of E^1, then $I(\mu\nu)$ has the nontempered one-dimensional component $\pi(\mu\nu)$, and the Steinberg square-integrable component $\mathrm{sp}(\mu\nu)$.

Otherwise the induced $I(\mu\nu^s)$ is irreducible.

3.3 [**U(2)**]. We shall also make use of the following result of [**U(2)**]. A local module is called *elliptic* if its character is nonzero on the elliptic regular set.

PROPOSITION. (1) *If τ is an elliptic or discrete-series σ-invariant local or global H'-module, then its central character is trivial on C_F.* (2) *Such τ is the base change lift of a unique elliptic or discrete-series H-module ρ, either through b' or through b'', but not both.*

A proof of (1) in a more general context is given in [**GL(n)**].

The second statement here implies, in the global case, that if $I(\tau)$ is the only term on the left side of [**I**, Corollary (4.4)], then precisely one of the sums involving $'f$ and $'\phi$ on the right is nonzero, and it consists of a single term.

Note that the elliptic (local) ρ are the one-dimensional, special and supercuspidal, and also the components of a reducible tempered induced H-module, which make a packet.

3.4. We shall now prove a special case of the Proposition in §2.4, but without using the transfer statement of [**I**, Lemma (3.4)].

PROPOSITION. *Let* τ_w *be the stable base change lift of the elliptic* H_w*-module* ρ_w. *Then* $\operatorname{tr} I(\tau_w; \phi_w \times \sigma) = 0$ *if* ϕ_w *matches* $'\phi_w$, *and* $'\phi_w$ *is* 0.

PROOF. We deal with the one-dimensional case first. Let ρ be a one-dimensional H-module, and τ its base change lift. Then ρ is associated with an induced $'I(\mu\nu^{1/2})$, and τ with $I(\mu\nu^{1/2}, \mu\nu^{-1/2})$. We choose $'\phi_w = 0$, so that $'\phi = 0$, and no term involving $'\phi$ appears in the trace formula [**I**, (4.4)]. We choose a sequence $\{t_v\}$ so that our $I(\tau)$ is the only contribution associated with ϕ. The only other possible terms in [**I**, (4.4)] are of the form $\operatorname{tr} \pi(f)$. The local components of any such π are almost all of the form $I(\mu\nu^{1/2})$. In any case, we conclude that for any $v \neq w$, if $\operatorname{tr} I(\tau_w; \phi_w \times \sigma) \neq 0$ then $\operatorname{tr} I(\tau_v; \phi_v \times \sigma)$ depends only on f_v. More precisely, there are G_v-modules π_v and complex constants $c(\pi_v)$ with

$$\operatorname{tr} I(\tau_v; \phi_v \times \sigma) = \sum c(\pi_v) \operatorname{tr} \pi_v(f_v)$$

for all matching ϕ_v, f_v. Taking such functions whose orbital integrals are supported on the conjugacy classes of the $(a, b, \bar{a}^{-1})$, $|a| \neq 1$, the Deligne-Casselman [**C**] theorem implies that

$$\operatorname{tr} I(\tau_v)_A(\phi_{vA} \times \sigma) = \sum c(\pi_v) \operatorname{tr} \pi_{vA}(f_{vA}),$$

where Π_A, π_A denote the Jacquet modules of Π, π (see [**Sym**^{2}, §2.4]) with respect to any parabolic with Levi subgroup A, tensored by $\delta^{-1/2}$, where $\delta(a, b, \bar{a}^{-1}) = |a|^2$ (resp. $\delta(a, b, c) = |a/c|^2$) is the modulus function on G (resp. G'), and ϕ_{vA}, f_{vA} are functions on A, A' defined by

$$f_{vA}(a, b, \bar{a}^{-1}) = |a| \int_K \int_N f_v(k^{-1}ank)\, dn\, dk,$$

$$\phi_{vA}(a, b, c) = |a/c| \int_K \int_N \phi_v(\sigma k^{-1}ank)\, dn\, dk.$$

Since the functions f_{vA}, ϕ_{vA} are arbitrary, and the Jacquet module $I(\tau_v)_A$ consists of a single (increasing) σ-invariant exponent, we conclude from the Harish-Chandra finiteness theorem [**BJ**], and linear independence of characters on A, that on the right there should be a single π_v with nonvanishing nonunitary π_{vA}, and then π_{vA} should consist of a single exponent which lifts to $I(\tau_v)_A$. Here we used the fact (see (3.2)) that if the irreducible π_v and π'_v have nonunitary characters in π_{vA} and π'_{vA} which are equal, then π_v and π'_v are equivalent. Hence our π_v is a subquotient of $I = I(\mu\nu^{1/2})$. But I is irreducible (see §3.2), hence $\pi_v = I$, and π_{vA} has two exponents, one increasing and one decaying. This contradiction establishes the proposition when ρ_w is one-dimensional, hence also when it is special.

To deal with the supercuspidal ρ_w, it suffices to construct a cuspidal ρ with this component, and a component ρ_v which is special. If $'\phi_w = 0$ we conclude as above that $\operatorname{tr} I(\tau_v; \phi_v \times \sigma)$ depends only on f_v, where τ_v is the stable base change lift of ρ_v. This contradicts the previous conclusion in the special case, as required.

It is clear that taking $F = Q$ we obtain the above conclusion also in the archimedean case.

4. The quasi-endo-lifting e.

4.1. *Cancellation*. The above results concerning the σ-endo-lifting e' can be used to simplify the identity [**I**, (4.4)] of trace formulas. First the terms $\operatorname{tr} I(\tau; \phi \times \sigma)$, where τ is a stable base change lift of an H-module ρ, are cancelled with the terms $\operatorname{tr}\{\rho\}('\phi)$. Indeed, if a discrete-series $\{\rho\}$ base-changes to a discrete-series τ, then $n(\rho) = 1$ according to [**U(2)**]. When $n(\rho) \neq 1$, it is equal to $1/2$, and ρ is of the form $\rho(\theta)$ in the notations of [**U(2)**, p. 721] (where the symbol is actually $\pi(\theta)$). According to Proposition 1 there, $\rho(\theta)$ lifts to an induced H'-module $\tau = I(\theta'\kappa, \theta''\kappa)$, where θ', θ'' are distinct characters of C_E/C_F related to the character θ (of $C_E^1 \times C_E^1$). There is no need to elaborate on this result. We simply note that the $\operatorname{tr}\{\rho\}('\phi)$ with $n(\rho) = 1/2$ cancel the $\operatorname{tr} I(\eta; \phi \times \sigma)$ with $\eta = (\kappa\theta', \kappa\theta'', \mu)$ (where $\mu\kappa^2\theta'\theta'' = \omega'$), as these appear with coefficient $1/4$.

There remains $\operatorname{tr} I(\mu, '\phi)$, which depends on $'\phi$. $I(\mu)$ lifts via e' to the G'-module $I(\mu, \mu, \omega'/\mu^2)$. If $\omega' \neq \mu^3$ then we obtain a cancellation with the term $\operatorname{tr} I((\mu, \mu', \mu); \phi \times \sigma)$, which also appears with coefficient $-1/8$. If $\omega' = \mu^3$ then we obtain a partial cancellation, which replaces the coefficient $-3/8$ by $-1/4$, in the twisted side of the formula.

4.2. *Identity*. So far we eliminated all terms which depend on $'\phi$. Let us record those terms which are left. We denote by μ any character of C_E trivial on C_F. Put

$$\Phi_1 = \sum \prod \operatorname{tr} \Pi_v(\phi_v \times \sigma), \qquad \Phi_2 = \sum \prod \operatorname{tr} I(\tau_v \otimes \kappa_v; \phi_v \times \sigma).$$

In Φ_1 the sum is over all (equivalence classes of) σ-invariant discrete-series G'-modules. In Φ_2 the sum is over the σ-invariant discrete-series H'-modules τ which are obtained by the stable base change map b'', namely $\tau \otimes \kappa$ is obtained by the unstable map b'. Further,

$$\Phi_3 = \sum \prod \operatorname{tr} I\big((\mu, \mu', \mu''); \phi_v \times \sigma\big) \quad (\text{distinct } \mu,\ \mu',\ \mu''),$$

and

$$\Phi_4 = \sum \prod \operatorname{tr} I\big((\kappa\mu, \mu', \kappa\mu); \phi_v \times \sigma\big), \quad \Phi_5 = \sum \prod \operatorname{tr} I\big((\mu, \mu, \mu); \phi_v \times \sigma\big).$$

On the other hand, we put

$$F_1 = \sum_\pi m(\pi) \prod \operatorname{tr} \pi_v(f_v).$$

The sum is over equivalence classes π in the discrete spectrum of G. They occur with finite multiplicities $m(\pi)$.

$$F_2 = \sum_{\rho \neq \rho(\theta, '\theta)} \prod \operatorname{tr}\{\rho_v\}('f_v).$$

The sum ranges over the automorphic discrete-series packets of ρ of H, which are not of the form $\rho(\theta, '\theta)$. In this case $n(\rho) = 1$ (see [**U(2)**]).

$$F_3 = \sum_{\rho = \rho(\theta, '\theta)} \prod \operatorname{tr}\{\rho_v\}('f_v).$$

Here the sum ranges over the packets $\rho = \rho(\theta, '\theta)$, where θ, θ' and $\omega/\theta'\theta$ are distinct. In this case $n(\rho) = 1/2$.

$$F_4 = \sum_{\mu} m(\mu\kappa) \prod \operatorname{tr} I(\mu_v \kappa_v, f_v) + \tfrac{1}{2} \sum \prod \operatorname{tr} 'I(\mu_v, 'f_v),$$

$$F_5 = \sum_{\mu} m(\mu) \prod \operatorname{tr} I(\mu_v, f_v) \qquad (\mu^3 = \omega'),$$

$$F_6 = \sum_{\mu} m(\mu) \prod \operatorname{tr} R(\mu_v) I(\mu_v, f_v) - \sum_{\rho} \prod \operatorname{tr}\{\rho_v\}('f_v).$$

In F_6, the first sum is over all μ with $\mu^3 \neq \omega'$. The second is over the packets $\rho = \rho(\theta, \omega/\theta^2)$, where $\theta^3 \neq \omega$. We deduce from the identity **[I**, (4.4)], of trace formulas (which follows from the computations of Arthur, and Clozel, Labesse, Langlands in the twisted case, as in **[Sym]**, the following

PROPOSITION. *The identity of trace formulas takes the form*

$$\Phi_1 + \tfrac{1}{2}\Phi_2 + \tfrac{1}{4}\Phi_3 - \tfrac{1}{8}\Phi_4 - \tfrac{1}{4}\Phi_5 = F_1 - \tfrac{1}{2}F_2 - \tfrac{1}{4}F_3 + \tfrac{1}{4}F_4 + \tfrac{1}{4}F_5 + \tfrac{1}{4}F_6.$$

4.3. *Simplification.* To simplify the formula we first note that the normalizing factor m which appears in F_4 and F_5 can be evaluated as a limit. It is equal to -1. The representations $I(\mu_v \kappa_v)$, $I(\mu_v)$ of $G(F_v)$ in F_4 and F_5 are irreducible, and the Lemma in §1.4 asserts the following. In the notations of F_4 and Φ_4 we have at each v

$$\operatorname{tr} I(\mu_v, 'f_v) = \operatorname{tr} I(\mu_v \kappa_v, f_v) = \operatorname{tr} I\big((\kappa_v \mu_v, \mu'_v, \kappa_v \mu_v); \phi_v \times \sigma\big).$$

In the case of F_5 and Φ_5 we have

$$\operatorname{tr} I(\mu_v, f_v) = \operatorname{tr} I\big((\mu_v, \mu_v, \mu_v); \phi_v \times \sigma\big).$$

Hence $\Phi_4 = -2F_4$ and $\Phi_5 = -F_5$, and these terms are cancelled in the comparison of the Proposition. Moreover, since we are working under the assumption that the orbital integral $\Phi(f_v)$ is supported on the elliptic set at some place v, we can conclude, in our case, that these terms are equal to 0. Indeed, the G-modules in F_4 and F_5 are irreducible, and their characters are supported on the split set.

The normalizing factor $m(\mu)$ of F_6 can be shown to be equal to 1, and F_6 can be shown to be equal to 0, but this will not be done here. However, it is clear from the Lemma in §1.4 that $\rho = \rho(\theta, \omega/\theta^2)$ with $\theta^3 \neq \omega$ quasi-lifts to $I(\mu)$, where $\mu = \theta \circ N_{E/F}$. In any case the trace identity takes the form

PROPOSITION. *We have*

$$\Phi_1 + \tfrac{1}{2}\Phi_2 + \tfrac{1}{4}\Phi_3 = F_1 - \tfrac{1}{2}F_2 - \tfrac{1}{4}F_3 + \tfrac{1}{4}F_6.$$

We repeat that this identity holds for matching functions $\phi_v, f_v, 'f_v$ at all v, under the assumption that the orbital integrals of some component vanishes on the regular split set. The terms consist of products over a finite set of places, and at most one of the terms on the left is nonzero, consisting of a single nonzero representation. We conclude

4.4. THEOREM. *Every discrete-series automorphic H-module ρ with an elliptic component quasi-endo-lifts to an automorphic G-module.*

As noted in [**I**], this is a sharpening of a theorem of Kudla [**Ku**], formulated in the language of modular forms. A related result is given in Gelbart and Piatetski-Shapiro [**GP**] by means of a different technique.

PROOF. It is clear from the Lemma in §1.4 that if ρ appears in F_3 then there is a nontrivial term in Φ_3, but if ρ appears in F_2 then there is a contribution in Φ_2. So we apply the identity with a function ϕ so that the suitable Φ is nonzero, and such that $'f$ is 0. Indeed, if Π_u is the component at u of the unique term Π on the left, then $\operatorname{tr}\Pi_u(\phi_u \times \sigma)$ is nonzero, and depends only on the stable orbital integral of ϕ_u, namely on the stable orbital integral of f_u, which is supported on the nonsplit set. We can take f_u with $\Phi(f_u)$ supported on the regular nonsplit set, with vanishing unstable orbital integrals. Namely the orbital integrals of $'f_u$, and consequently $'f_u$ itself, can be taken to be identically 0. Hence $'f$ is 0, so that $F_2 = F_3 = F_6 = 0$, but the left side is nonzero, hence the right side is nonzero. Hence $F_1 \neq 0$, as required.

4.5. REMARKS. Note that the same proof implies that for every π which appears in F_1 there exists a σ-invariant Π (with σ-stable components in the terminology of [**U(2)**]), so that π base change quasi-lifts to Π, and for each such Π there exists a π with this property.

One case of the Theorem which is particularly interesting is that of the one-dimensional H-module, which quasi-endo-lifts to G-modules π whose components almost everywhere are nontempered. Such π may have finitely many supercuspidal components, hence be cuspidal, and make a counterexample to the generalized Ramanujan hypothesis. This will be discussed elsewhere (see [**U(3)**]), together with the complete local and global endo-lifting and base change lifting.

The Theorem is proven here for discrete-series H-modules with at least one elliptic component. This includes the case of classical and Hilbert modular forms. The fact that we make one restriction only is due to the identity of trace formulas for global functiions with at least one component whose orbital integrals vanish on the regular split set; this is proven in [**Sym**, §4.3], by elementary means. The removal of the restriction at the last place requires additional efforts, and will not be done here. The proof of the trace identity for functions with two elliptic components is easier; see [**Sym**, §4.1].

4.6. *Different tack*. Theorem 4.4 deals with the quasi-endo-lifting e from $U(2)$ to $U(3)$. The proof is via the theory of base change, and uses in addition to the rigidity theorem for GL(3) only the local base change transfer of spherical functions from G to G'. At the remaining finite number of places we work with a function which vanishes on the (σ-) singular set. These functions are easy to transfer. We do not use the endo-transfer of [**I**, Lemma (2.7)], although this will be needed for the local lifting.

One may like to prove Theorem 4.4 by stabilizing the trace formula for $U(3)$ alone, using [**I**, Lemma (2.7)], and setting $\phi_u = 0$, namely choosing f_u with vanishing stable orbital integrals, so that the terms Φ are 0. Then, choosing discrete-series ρ, for example in F_2, one would like to assert that by the rigidity theorem for H-packets [**U(2)**], there will be a single contribution in F_2. But if $F_2 \neq 0$ then $F_1 \neq 0$, and there exists π such that ρ quasi-endo-lifts to π.

This argument does not work since there are infinitely many places where E/F splits, and there the dual-group homomorphism e takes (a, b) to $(a, 1/ab, b)$. Since only conjugacy classes matter, and $(a, 1/ab, b)$ is conjugate to $(a, b, 1/ab)$, this conjugacy class in $\hat{G} = \mathrm{GL}(3, \mathbf{C})$ is obtained also from the conjugacy class $(a, 1/ab)$ in $\hat{H} = \mathrm{GL}(2, \mathbf{C})$. Hence, using the spherical components of f at almost all v it is not possible to deduce that the components of ρ at almost all v are fixed; it is possible to say that at any split v the component ρ_v has only finitely many possibilities. This makes it a priori possible for infinitely many ρ, and we need only two, to appear in F_2. But these may cancel each other, so that one cannot deduce $F_2 \neq 0$. What makes the Theorem work is the comparison to GL(3).

4.7. *Unitary symmetric square*. Let E/F be a quadratic extension of number fields. Put $H = \mathrm{SL}(2)$. If π_0 is an automorphic $H(\mathbf{A})$-module, then for almost all v its component π_{0v} is the irreducible unramified subquotient of the H_v-module $I_0(\mu_v)$ induced from the character

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \to \mu_v(a) \quad \left(a \text{ in } F_v^\times\right).$$

For almost all v, the component Π_v of an automorphic $\mathrm{PGL}(3, \mathbf{A})$-module Π is similarly associated with the representation $I(\mu_{1v}, \mu_{2v}, \mu_{3v})$ (unitarily) induced from the unramified character $(\mu_{1v}, \mu_{2v}, \mu_{3v})$ of the upper triangular subgroup. Here $\mu_{1v}\mu_{2v}\mu_{3v} = 1$. In [**Sym**^{2}] it is shown that

LEMMA. *Given such π_0 (in fact, with an elliptic component if it is cuspidal), there exists Π as above with Π_v in $I(\mu_v, 1, \mu_v^{-1})$ for almost all v.*

Note that π_{0v} in $I_0(\mu_v)$ is represented by the conjugacy class $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$, with $a/b = \mu_v(\tilde{\omega})$ in the dual group $\hat{H} = \mathrm{PGL}(2, \mathbf{C})$, and $\Pi_v = I(\mu_{1v}, \mu_{2v}, \mu_{3v})$ by the class of the diagonal matrix $(\mu_{1v}(\tilde{\omega}), \mu_{2v}(\tilde{\omega}), \mu_{3v}(\tilde{\omega}))$ in the dual group $\hat{M} = \mathrm{SL}(3, \mathbf{C})$ of $M = \mathrm{PGL}(3)$. The lifting of the Lemma is compatible with the three-dimensional symmetric square representation Sym of $\hat{H}$ on $\hat{M}$, which maps (a, b) to $(a/b, 1, b/a)$ (see [**Sym**^{2}]). Hence we denote Π of the Lemma by $\mathrm{Sym}(\pi_0)$, and name it the symmetric square lift of π_0.

Recall that the connected component $\hat{G}^0$ of the dual group $\hat{G}$ of the projective unitary group $G = \mathrm{PU}(3)$ is also $\mathrm{SL}(3, \mathbf{C})$. Given an automorphic $H(\mathbf{A})$-module π_0, we wish to find an automorphic $G(\mathbf{A})$-module π, to be called the *unitary symmetric square* $\mathrm{US}(\pi_0)$, whose local components are defined by those of π_0, and the map Sym: $\hat{H} \to \hat{G}^0$, for almost all v. Thus, when v splits E/F, G_v is $\mathrm{PGL}(3, F_v)$, and $\mathrm{US}(\pi_{0v})$ is $I(\mu_v, 1, \mu_v^{-1})$ if π_{0v} is $I_0(\mu_v)$. If v stays prime in E, the induced unramified G_v-module $I(\mu_v)$ is parametrized by the conjugacy class of $(\mu_v(\tilde{\omega}), 1, 1) \times \sigma$ in $\hat{G} = \hat{G}^0 \times \langle\sigma\rangle$. In this case, $\pi_{0v} = I_0(\mu_v)$ determines $(\mu_v(\tilde{\omega}), 1)$ in $\hat{H}$, hence $(\mu_v(\tilde{\omega}), 1, \mu_v(\tilde{\omega})^{-1}) \times \sigma$ in $\hat{G}$, which is conjugate to $(\mu_v \circ N(\tilde{\omega}), 1, 1) \times \sigma$, and $\mathrm{US}(\pi_{0v})$ is $I(\mu_v \circ N)$. N denotes the norm map from E_v to F_v. We now assume the availability of all liftings used below under no restrictions at any component. We leave to the reader the task of specifying those cases where the proof is already complete.

PROPOSITION. *Given an automorphic $H(\mathbf{A})$-module π_0, there exists an automorphic $G(\mathbf{A})$-module $\pi = \mathrm{US}(\pi_0)$ whose component is $\mathrm{US}(\pi_{0v})$ for almost all v.*

PROOF. We follow the arrows in the following diagram:

$$\begin{array}{ccc}
\begin{array}{c} I_0(\mu) \times I_0(\mu) \text{ or } I_0(\mu \circ N) \\ \text{on } \mathrm{SL}(2, E) \end{array} & \overset{\mathrm{Sym}}{\to} & \begin{array}{c} I(\mu, 1, \mu^{-1}) \times I(\mu, 1, \mu^{-1}) \text{ or } I(\mu \circ N, 1, \mu^{-1} \circ N) \\ \text{on } \mathrm{PGL}(3, E) \end{array} \\
BC \uparrow & & \uparrow BC \\
I_0(\mu) \text{ on } \mathrm{SL}(2, F) & \overset{\mathrm{US}}{\to} & I(\mu, 1, \mu^{-1}) \text{ or } I(\mu \circ N) \text{ on } \mathrm{PU}(3)
\end{array}$$

Base change theory for GL(2) implies the existence of an automorphic $\mathrm{SL}(2, \mathbf{A}_E)$-packet π_0^E whose local components are obtained from those $I_0(\mu_v)$ of π_0 as indicated by the vertical arrow on the left (they are $I_0(\mu_v) \times I_0(\mu_v)$ when v splits, and $I_0(\mu_v \circ N)$ when v stays prime). The Lemma implies the existence of an automorphic $\mathrm{PGL}(3, \mathbf{A}_E)$-module $\mathrm{Sym}(\pi_0^E)$, whose components are as indicated by the top horizontal arrow for almost all v. If $\sigma(g) = J\,{}^t\bar{g}^{-1}J$ is the automorphism of $\mathrm{GL}(3, E)$ which defines $U(3)$, then it is clear that for almost all v we have that $\mathrm{Sym}(\pi_0^E)_v$ is σ-invariant. Hence $\mathrm{Sym}(\pi_0^E)$ is σ-invariant by the rigidity theorem for $\mathrm{GL}(n)$ of **[JS]**. The E/F-base change result for $U(3)$ noted in §4.5, implies that there exists an automorphic $G(\mathbf{A})$-module $\pi(G = \mathrm{PU}(3))$ which quasi-lifts to $\mathrm{Sym}(\pi_0^E)$. But π is the required $\mathrm{US}(\pi_0)$, as it has the desired local components for almost all v.

It will be interesting—and may have interesting applications—to verify the existence of the local unitary symmetric square lifting by means of character relations between representations of SL(2), and bar-invariant PU(3)-modules. This was the point of view which we took in a letter of 1983 to Professor Langlands. There we defined a suitable norm map of stable conjugacy classes. Further, we computed the trace formula for PU(3), twisted by the bar-automorphism $g \to \bar{g} = \sigma(\bar{g}) = J\,{}^tg^{-1}J$; this is standard, as the rank is one. The required transfer of orbital integrals of spherical functions is available (see **[Sym²]**) at a place v of F which splits in E. It is not yet available at inert v. The important case is that of the unit element of the Hecke algebra. But we have not pursued these questions.

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