CONTRIBUTIONS FROM CONJUGACY CLASSES 
OF REGULAR ELLIPTIC ELEMENTS IN HERMITIAN MODULAR GROUPS TO THE DIMENSION FORMULA 
OF HERMITIAN MODULAR CUSP FORMS$^1$

BY

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ABSTRACT. The dimension of the vector space of hermitian modular cusp forms on the hermitian upper half plane can be obtained from the Selberg trace formula; in this paper we shall compute the contributions from conjugacy classes of regular elliptic elements in hermitian modular groups by constructing an orthonormal basis in a certain Hilbert space of holomorphic functions. A generalization of the main Theorem can be applied to the dimension formula of cusp forms of $SU(p, q)$. A similar theorem was given for the case of regular elliptic elements of $Sp(n, \mathbb{Z})$ in [5] via a different method.

1. Introduction and notation. Denote by $E$ the unit matrix and by 0 the zero matrix in the matrix ring $M_n(\mathbb{C})$. Put $J = \begin{bmatrix} 0 & E \\ -E & 0 \end{bmatrix}$. The hermitian symplectic group of degree $n$, $\Omega_n$, is then defined as the group of matrices in $M_{2n}(\mathbb{C})$; it satisfies $t\bar{M}JM = J$; i.e.,

$$\Omega_n = \{ M \in M_{2n}(\mathbb{C}) | t\bar{M}JM = J \}.$$ 

Here $t\bar{M}$ is the transpose complex conjugate to $M$.

Let $\mathcal{H}_n$ be the hermitian upper half plane; specifically,

$$\mathcal{H}_n = \{ Z \in M_n(\mathbb{C}) | Z = X + iY, X = tX, Y = tY > 0 \}.$$ 

The hermitian symplectic group $\Omega_n$ operates on $\mathcal{H}_n$ transitively by the action

$$M: Z \rightarrow M(Z) = (AZ + B)(CZ + D)^{-1}, \quad M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Omega_n.$$ 

For a given imaginary quadratic number field $\mathbb{F}$, we denote by $\mathbb{K}$ its ring of integers. The hermitian modular group of degree $n$, $\Gamma_n(\mathbb{K})$, is defined as

$$\Gamma_n(\mathbb{K}) = \Omega_n \cap M_{2n}(\mathbb{K}).$$ 

An element $M$ in $\Gamma_n(\mathbb{K})$ is regular elliptic if $M$ has an isolated fixed point on $\mathcal{H}_n$, i.e. the equation $M(Z) = Z$ has a unique solution on $\mathcal{H}_n$. A similar argument as in [5] shows that the following statements are equivalent:

1. $M$ is a regular elliptic element in $\Gamma_n(\mathbb{K})$ and its characteristic polynomial $\varphi(X)$ is in $Z[X]$.

2. $M \in \Gamma_n(\mathbb{K})$ and is conjugate in $\Omega_n$ to $\text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n, \bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n]; \lambda_i$ ($i = 1, 2, \ldots, n$) are roots of unity and $\lambda_i \lambda_j \neq 1$ for all $i, j$.

Received by the editors January 7, 1985.

1980 Mathematics Subject Classification. Primary 10D20, 10D05.

$^1$Research supported by Academia Sinica and NSC of Taiwan, Republic of China.
Let \( S(k; \Gamma_n(K)) \) denote the space of holomorphic functions \( f(Z) \) on \( \mathbb{H}_n \); \( f(Z) \) satisfies the following conditions:

1. \( f(M(Z)) = \det(CZ + D)^k f(Z) \) for all \( M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) in \( \Gamma_n(K) \), \( Z \in \mathbb{H}_n \).
2. \( (\det Y)^{k/2} f(Z) \) is bounded on \( \mathbb{H}_n \), \( Z = X + iY \).

A function in \( S(k; \Gamma_n(K)) \) is called a hermitian modular cusp form of weight \( k \) and degree \( n \).

For fixed degree \( n \) and certain \( k \), the first condition may be satisfied only for \( f(Z) = 0 \). However we shall exclude these trivial cases. For example, we assume \( kn \equiv 0 \pmod{4} \) when \( K = \mathbb{Z}[i] \) and \( F = \mathbb{Q}[i] \). It is well known that \( S(k, \Gamma_n(K)) \) is a finite dimensional Hilbert space \(^6\). Furthermore, its dimension can be written as an integral of a Bergman kernel function on a certain Hilbert space over the fundamental domain in \( \mathbb{H}_n \) with respect to \( \Gamma_n(K) \), when \( k \) is sufficiently large (for example \( k > (4n - 2) \); see also \(^9\)). This is the so-called Selberg trace formula.

More precisely, let \( K(Z_1, Z_2) \) be a kernel function of the space \( H(k; \mathbb{H}_n) \) which consists of a holomorphic function on \( \mathbb{H}_n \) and satisfies

\[
\int_{\mathbb{H}_n} (\det Y)^{k/2n} |f(Z)|^2 dZ < \infty.
\]

Then

\[
d\dim \mathbb{C} S(k; \Gamma_n(K)) = \int_{\mathcal{F}_n} \sum_{\gamma \in \Gamma_n(K)} K(Z, \gamma(Z)) j(\gamma, Z)^{-k} (\det Y)^{-2n} dZ,
\]

where

1. \( \Gamma_n(K) \) is the quotient group \( \Gamma_n(K)/U \) with \( U \) the center of \( \Gamma_n(K) \),
2. \( \mathcal{F}_n \) is a fundamental domain in \( \mathbb{H}_n \) with respect to \( \Gamma_n(K) \),
3. \( j(\gamma, Z) = \det(CZ + D) \) if \( \gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_n(K) \),
4. \( Z = X + iY \) in \( \mathbb{H}_n \) and \( dZ = dXdY \) is the Euclidean measure on \( \mathbb{C}^n \).

In this paper, we shall consider the subseries with the summation ranging over all regular elliptic elements in \( \Gamma_n(K) \); or consider the contributions from conjugacy classes of regular elliptic elements to \( \dim \mathbb{C} S(k; \Gamma_n(K)) \). We shall obtain the following

**THEOREM.** Suppose \( M \in \Gamma_n(K) \) and is conjugate in \( \Omega_n \) to

\[
\text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n, \bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n]
\]

with \( \lambda_j \) (\( j = 1, 2, \ldots, n \)) roots of unity and \( \lambda_1 \lambda_j \neq 1 \) for all \( 1 \leq i, j \leq n \). Then the contribution to \( \dim \mathbb{C} S(k; \Gamma_n(K)) \) (\( k > (4n - 2) \)) of regular elliptic elements in \( \Gamma_2(K) \) which are conjugate in \( \Gamma_n(K)/U \) to \( M \) is given by

\[
N_{\{M\}} = |\mathcal{C}_{M,Z}|^{-1} \prod_{j=1}^{n} \gamma_{j}^{k} \cdot \prod_{j,k=1}^{n} (1 - \bar{\lambda}_j \bar{\lambda}_k)^{-1}.
\]

Here \( \mathcal{C}_{M,Z} \) is the centralizer of \( M \) in \( \Gamma_n(K)/U \) and \( |\mathcal{C}_{M,Z}| \) is its order.

**REMARK.** Here we shall exclude those integers \( k \) such that

\[
f(M(Z)) = \det(CZ + D)^k f(Z), \quad M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_n(K),
\]

is satisfied only for \( f(Z) = 0 \).
2. The Selberg trace formula. Since \( \mathcal{H}_n \) is mapped biholomorphically onto the bounded domain

\[ D_n: W \in M_n(C), \quad E - W^t \mathbb{W} > 0, \]

under the Cayley transform \( W = (Z - iE)(Z + iE)^{-1} \), it suffices to consider \( H(k; D_n) \) instead of \( H(k; \mathcal{H}_n) \). \( H(k; D_n) \) consists of holomorphic function \( f(W) \) satisfying

\[ \int_{D_n} \det(E - W^t \mathbb{W})^{k - 2n} |f(W)|^2 dW < \infty. \]

The Bergmann kernel function for \( H(k; D_n) \) is given by the following propositions.

**Proposition 1** [11, Theorem 3.3]. Let \( \varphi_1, \varphi_2, \ldots, \varphi_n, \ldots \) be any orthonormal basis of the Hilbert space \( H(k; D_n) \). Then the series

\[ \sum_{n=1}^{\infty} \varphi_n(W) \overline{\varphi_n(W_1)} \]

converges uniformly on each compact subset of \( D_n \times D_n \). The sum, denoted by \( K(W, W_1) \), is independent of the choice of orthonormal basis and

\[ f(W) = \int_{D_n} \det(E - W_1^t \mathbb{W}_1)^{k - 2n} K(W, W_1) f(W_1) dW_1 \]

for each \( f \in H(k; D_n) \).

**Proposition 2** [9, Lemma 2.1]. Suppose that \( k > (4n - 2) \). Then the function \( K(W, W_1) \) is given by

\[ K(W, W_1) = C(k, n) \det(E - W^t \mathbb{W}_1)^{-k} \]

with

\[ C(k, n) = \pi^{-n^2} \prod_{0 \leq i, j \leq n-1} (k - 2n + 1 + i + j). \]

**Proof.** The kernel function is a constant multiple of \( \det(E - W^t \mathbb{W}_1)^{-k} \) by arguments similar to I of [7]. The constant \( C(k, n) \) is determined by

\[ C(k, n)^{-1} = \int_{D_n} \det(E - W^t \mathbb{W})^{k - 2n} dW. \]

3. Convergence of the series. Let \( \Lambda = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n] \) be a unitary matrix. As an element of a hermitian symplectic group, the operation of \( \Lambda \) on \( D_n \) is given by

\[ \Lambda: W \rightarrow AW \Lambda, \quad W \in D_n, \]

and

\[ K(W, AW \Lambda)^{t(A^t \mathbb{W})} = C(k, n)(\det \Lambda)^{-k} \det(E - \Lambda^t W^t \mathbb{W})^{-k}. \]

Now we shall prove this function is absolutely integrable on \( D_n \) with respect to the measure \( \det(E - W^t \mathbb{W})^{k - 2n} dW \) when \( k > (2n - 1) \).
Lemma 1 [8, Theorem 1, P. 266]. If \( E - Z^t\bar{Z} \geq 0 \) and \( E - W^t\bar{W} \geq 0 \), then
\[
\det(E - Z^t\bar{Z}) \det(E - W^t\bar{W}) + |\det(Z - W)|^2 \leq |\det(E - Z^t\bar{W})|^2.
\]
Equality holds only when \( Z = W \).

Lemma 2. If \( \Lambda = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n] \) with \( \lambda_i \) \( (i = 1, 2, \ldots, n) \) roots of unity and \( 1 - \lambda_i\lambda_j \neq 0 \) for all \( i, j \), then
\[
\det(E - \Lambda W \bar{\Lambda}^t \bar{W}) \neq 0
\]
for all \( W \in \bar{D}_n \).

Proof. Applying the previous lemma with \( Z = \Lambda W \bar{\Lambda} = [\lambda_i \lambda_j w_{ij}] \), we get
\[
[\det(E - W^t\bar{W})]^2 + |\det(\Lambda W \bar{\Lambda} - W)|^2 \leq [\det(E - \Lambda W \bar{\Lambda}^t \bar{W})]^2.
\]
Now suppose \( \det(E - \Lambda W \bar{\Lambda}^t \bar{W}) = 0 \). Then it forces
\[
\det(E - W^t\bar{W}) = 0 \quad \text{and} \quad \Lambda W \bar{\Lambda} = W.
\]
From \( \Lambda W \bar{\Lambda} = W \) and our assumption on \( \Lambda \), we get \( W = 0 \), which contradicts \( \det(E - W^t\bar{W}) = 0 \). This proves our assertion.

Proposition 3. Let \( M \in \Gamma_n(\mathbf{K}) \) and be conjugate in \( \Omega_n \) to \( \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n] \), \( \lambda_i \) \( (i = 1, 2, \ldots, n) \) roots of unity and \( \lambda_i \lambda_j \neq 1 \) for all \( i, j \). Then we have
\[
\begin{align*}
(1) \int_{D_n} \det(E - W^t\bar{W})^{k-2n} |\det(E - \Lambda W \bar{\Lambda}^t \bar{W})|^{-k} \, dW &< \infty \quad \text{for} \ k > (2n - 1), \\
(2) \text{the contribution} \ N(M) \text{ in the Theorem is given by} \\
N(M) &= C(k, n) (\det \Lambda)^{-k} |C_M, Z|^{-1} \\
&\quad \times \int_{D_n} \det(E - W^t\bar{W})^{k-2n} \det(E - \Lambda W \bar{\Lambda}^t \bar{W})^{-k} \, dW.
\end{align*}
\]

Proof. (1) follows since \( \det(E - W^t\bar{W})^{k-2n} \) is a bounded measure on \( D_n \) if \( k > (2n - 1) \) and \( \det(E - \Lambda W \bar{\Lambda}^t \bar{W}) \neq 0 \) for all \( W \) in \( D_n \).

To prove (2), we let \( \{M\} \) denote the conjugacy class in \( \Gamma_n(\mathbf{K})/U \) and which can be represented by \( M \). Then we have
\[
N(M) = \int_{\mathcal{F}_n} (\det Y)^{k-2n} \sum_{\gamma \in \{M\}} K(Z, \gamma(Z)) (\bar{\gamma}, Z)^{-k} \, dZ.
\]
Note that the integral
\[
N = \int_{\mathcal{F}_n} (\det Y)^{k-2n} K(Z, M(Z)) (\bar{M}, Z)^{-k} \, dZ
\]
is transformed into
\[
(\det \Lambda)^{-k} C(k, n) \int_{D_n} \det(E - W^t\bar{W})^{k-2n} \det(E - \Lambda W \bar{\Lambda}^t \bar{W})^{-k} \, dW
\]
under the Cayley transform $W = (Z - iE)(Z + iE)^{-1}$. Now with (1), we know the integral $N$ is absolutely convergent. Hence we have

$$N = \int_{\mathbb{R}} (\det Y)^{k-2n} \sum_{\gamma \in \Gamma_n(K)} K(Z, \gamma^{-1} M \gamma(Z)) j(\gamma^{-1} M \gamma, Z)^{-k} dZ$$

$$= |C_{M,Z}| \int_{\mathbb{R}} \sum_{\gamma \in \{M\}} K(Z, \gamma(Z)) j(\gamma, Z)^{-k} dZ = |C_{M,Z}| \cdot N_{\{M\}}.$$  

This proves our assertion in (2).

REMARK. Here we use the fact that the centralizer $C_{M,Z}$ of $M$ in $\Gamma_n(K)$ is a group of finite order since it is discrete and is conjugate in $\Omega_n$ to a subgroup of a unitary group which is compact.

4. Proof of the Theorem. To prove our Theorem, by Proposition 3 it suffices to evaluate the integral

$$C(k, n) \int_{D_n} \det (E - W^t\overline{W})^{k-2n} \det (E - \overline{W} W^t)^{-k} dW.$$  

But this is not easy when $n \geq 2$. Here we shall first construct a new orthonormal basis in $H(k; D_n)$.

**LEMMA 3** [15, LEMMA 1, p. 27]. Let $S = [s_{ij}]$ be an $n \times n$ hermitian matrix, i.e. $s_{ij} = \overline{s_{ji}}$ for all $i, j$, and let $S_j (j = 1, 2, \ldots, n - 1)$ be the submatrix consisting of $j \times j$ entries on the upper left block of $S$. Then $S$ is positively definite if and only if

$$\det S > 0 \quad \text{and} \quad \det S_j > 0 \quad (j = 1, 2, \ldots, n - 1).$$

**PROPOSITION 4.** Let $\theta_{1j}, \theta_{j1} (j = 1, 2, 3, \ldots, n)$ be $2n - 1$ real numbers and $W' = [w'_{jk}] \in D_n$. Suppose $W = [w_{jk}] \in M_n(C)$ is defined by

$$\begin{aligned}
  w_{jk} &= w'_{jk} e^{i\theta_{jk}}, & j = 1 \text{ or } k = 1, i = \sqrt{-1}, \\
  w_{jk} &= w'_{jk} e^{i(\theta_{j1} + \theta_{k1} - \theta_{11})}, & j \neq 1 \text{ and } k \neq 1.
\end{aligned}$$

Then we have $W \in D_n$.

**PROOF.** Let $E - W^t\overline{W} = [a_{jk}]$ and $E - W'^t\overline{W}' = [b_{jk}]$. A direct calculation shows

1. $a_{jk} = \overline{b_{kj}}$, $b_{jk} = \overline{a_{kj}}$ for all $k, j$;
2. $a_{jj} = b_{jj} > 0$, $j = 1, 2, \ldots, j$;
3. $a_{jk} = b_{jk} e^{i(\theta_{j1} - \theta_{k1})}$ for all $j, k$.

If $W' \in D_n$, then the submatrix $W'_{n-1}$ obtained from cancellation of the $n$th row and $n$th column of $W$ is in $D_{n-1}$. Thus by Lemma 3 and an induction on $n$, it suffices to prove $\det (E - W^t\overline{W}) > 0$. But it is easy to show

$$\det (E - W^t\overline{W}) = \det (E - W'^t\overline{W}')$$

by properites (1)–(3) and elementary properties of the determinant. This proves our assertion.
For each $n^2$-tuple of nonnegative integers $\alpha = [\alpha_{jk}]$, $1 \leq j, k \leq n$; we shall let $W^\alpha$ denote the monomial
\[
\prod_{j,k=1}^n w_{jk}^{\alpha_{jk}}
\]
in the variable $W$, and let $|\alpha| = \sum_{j,k=1}^n \alpha_{jk}$ be the degree of $W^\alpha$.

PROPOSITION 5. Let $W^\alpha$ and $W^\beta$ be monomials in $w_{jk}$ $(j, k = 1, \ldots, n)$. Then
\[
\int_{D_n} \det(E - W^\alpha W^{\beta})^{-2n} W^\alpha W^{\beta} \, dW = 0
\]
unless
\[
\begin{align*}
\alpha_{11} - \beta_{11} + \sum_{j,k \geq 2} (-\alpha_{jk} + \beta_{jk}) &= 0, \\
\sum_{j=1}^n (\alpha_{jk} - \beta_{jk}) &= 0, \quad \sum_{j=1}^n (\alpha_{kj} - \beta_{kj}) = 0, \quad (k = 2, 3, \ldots, n).
\end{align*}
\]
Under the above conditions, we have $|\alpha| = |\beta|$ and
\[
\sum_{j=1}^n (\alpha_{jk} + \alpha_{kj}) = \sum_{j=1}^n (\beta_{jk} + \beta_{kj}), \quad (k = 1, 2, \ldots, n).
\]

PROOF. By the previous proposition, we can use polar coordinates on certain entries of $W$ as follows:
\[
\begin{cases}
w_{jk} = r_{jk} e^{i\theta_{jk}}, & j = 1 \text{ or } k = 1, \quad r_{jk} \geq 0, \quad 0 \leq \theta_{jk} < 2\pi; \\
w_{jk} = w_{jk} e^{i(\theta_{j1} + \theta_{1k} - \theta_{11})}, & j \neq 1 \text{ and } k \neq 1.
\end{cases}
\]
Let $D'_n$ be the subset of $D_n$ and be defined by
\[D'_n: W = [w_{jk}] \in D_n, \quad w_{j1}, w_{1j} \geq 0 \quad (j = 1, 2, \ldots, n).
\]
With these new coordinates, we have
\[
\int_{D_n} \det(E - W^\alpha W^{\beta})^{-2n} W^\alpha W^{\beta} \, dW
\]
\[
= \int_{D'_n} \det(E - W^{\alpha'} W^{\beta'})^{-2n} \prod_{j=1}^n r_{1j}^{\alpha_{1j} + \beta_{1j} + 1} \, dr_{1j}
\]
\[
\times \prod_{j=2}^n r_{j1}^{\alpha_{j1} + \beta_{j1} + 1} \, dr_{j1} \prod_{j,k \geq 2} w_{jk}^{\alpha_{jk} - \beta_{jk}} \, dw_{jk}
\]
\[
\times \int_0^{2\pi} \exp[i\theta_{11} (\alpha_{11} - \beta_{11} + \sum_{j,k \geq 2} (-\alpha_{jk} + \beta_{jk}))] \, d\theta_{11}
\]
\[
\times \prod_{k=2}^n \int_0^{2\pi} \int_0^{2\pi} \exp \left[ i\theta_{1k} \left( \sum_{j=1}^n (\alpha_{jk} - \beta_{jk}) \right) \right] \times \exp \left[ i\theta_{k1} \left( \sum_{j=1}^n (\alpha_{kj} - \beta_{kj}) \right) \right] \, d\theta_{1k} \, d\theta_{k1}.
\]
The above integral will vanish unless
\[
\begin{align*}
\alpha_{11} - \beta_{11} + \sum_{j,k \geq 2} (-\alpha_{jk} + \beta_{jk}) &= 0, \\
\sum_{j=1}^n (\alpha_{jk} - \beta_{jk}) &= 0, \quad \sum_{j=1}^n (\alpha_{kj} - \beta_{kj}) = 0, \quad (k = 2, \ldots, n).
\end{align*}
\]
This proves our first assertion. Multiplying the first equation by 2 and adding all together, we get

$$\sum_{j=1}^{n} (\alpha_{j1} + \alpha_{1j}) = \sum_{j=1}^{n} (\beta_{j1} + \beta_{1j}).$$

For $k = 2, 3, \ldots, n$, we note that

$$\sum_{j=1}^{n} (\alpha_{jk} + \alpha_{kj}) - \sum_{j=1}^{n} (\beta_{jk} + \beta_{kj})$$

$$= \sum_{j=1}^{n} (\alpha_{jk} - \beta_{jk}) + \sum_{j=1}^{n} (\alpha_{kj} - \beta_{kj}) = 0$$

and

$$2|\alpha| = \sum_{k,j=1}^{n} (\alpha_{jk} + \alpha_{kj}) = \sum_{j,k=1}^{n} (\beta_{jk} + \beta_{kj}) = 2|\beta|.$$

Thus the proof is completed.

**COROLLARY.** Suppose $\alpha, \beta$ are two $n^2$-tuples of nonnegative integers satisfying the conditions in Proposition 5. Then

$$\prod_{j,k=1}^{n} (\lambda_j \lambda_k)^{\alpha_{jk}} = \prod_{j,k=1}^{n} (\lambda_j \lambda_k)^{\beta_{jk}}$$

for any numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

**PROOF OF THE THEOREM.** Let

$$N_{\{M\}}(t\Lambda) = C(k, n) \int_{D_n} \det(E - W^t \overline{W})^{k-2n} \det(E - t^2 \Lambda W \overline{W})^{-k} dW$$

with $0 < t < 1$. If we can prove

(A) $$N_{\{M\}}(t\Lambda) = \prod_{j,k=1}^{n} (1 - t^2 \lambda_j \lambda_k)^{-1},$$

then we get

(B) $$C(k, n) \int_{D_n} \det(E - W^t \overline{W})^{k-2n} \det(E - \Lambda W \overline{W})^{-k} dW$$

$$= \prod_{j,k}^{n} (1 - \lambda_j \lambda_k)^{-1}$$

by letting $t$ approach 1. Now, we shall prove (A).

Let $S$ be the index set of all $n^2$-tuples of integers $\alpha = [\alpha_{jk}], \alpha_{jk} \geq 0$. Consider all monomials $a_\alpha(W) = W^\alpha$, $\alpha \in S$, which are arranged in such order that their degrees are nondecreasing. By an argument similar to [7, p. 188], we can prove that $\{a_\alpha(W)|\alpha \in S\}$ is a complete system in $H(k, D_n)$ in the sense that if $f \in H(k, D_n)$ and

$$\int_{D_n} \det(E - W^t \overline{W})^{k-2n} a_\alpha(W) f(W) dW = 0 \ \forall \alpha \in S,$$
then \( f(W) = 0 \). This system precisely consists of all terms in the power series expansion
\[
\prod_{j,k=1}^{n} (1 - w_{jk})^{-1} = \prod_{j,k=1}^{n} (1 + w_{jk} + \cdots + w_{jk}^{n}) \), \(|w_{jk}| < 1\).
\]

Of course, \( \{a_{\alpha}(W) = W^{\alpha} | \alpha \in S\} \) is a linear independent set in \( H(k, D_n) \). By the well-known Gram-Schmidt orthogonalization process, we can construct an orthonormal basis \( \{\psi_{\alpha}(W) | \alpha \in S\} \) from \( \{a_{\alpha}(W) | \alpha \in S\} \). Proposition 5 and its corollary then imply that the basis \( \{\psi_{\alpha}(W) | \alpha \in S\} \) has the following properties:

1. \( \psi_{\alpha}(W) \) is a finite linear combination of monomials of degree \( |\alpha| \).
2. \( \psi_{\alpha}(t^{2\overline{\alpha}}W\overline{\alpha}) = t^{2|\alpha|} \prod_{j,k=1}^{n} (\overline{\lambda}_{j}\overline{\lambda}_{k})^{\alpha_{jk}} \cdot \psi_{\alpha}(W) \).

Choose \( \{\psi_{\alpha}(W) | \alpha \in S\} \) as an orthonormal basis of \( H(k, D_n) \) and note that
\[
C(k, n) \det(E - W^{t\overline{W}})^{-k} = K(W, W^{1})
\]
is a kernel function of \( H(k, D_n) \). By Proposition 1 we then have
\[
C(k, n) \det(E - t^{2\overline{\alpha}}W\overline{\alpha})^{-k} = \sum_{\alpha \in S} \psi_{\alpha}(t^{2\overline{\alpha}}W\overline{\alpha})\psi_{\alpha}(W)
\]
\[
= \sum_{\alpha \in S} t^{2|\alpha|} \prod_{p,q=1}^{n} (\overline{\lambda}_{p}\overline{\lambda}_{q})^{\alpha_{pq}} \psi_{\alpha}(W)\psi_{\alpha}(W).
\]

Multiply both sides with \( \det(E - W^{t\overline{W}})^{k-2n} \) and integrate on \( D_n \) to get
\[
N_{(M)}(tA) = \sum_{\alpha \in S} t^{2|\alpha|} \prod_{j,k=1}^{n} (\overline{\lambda}_{j}\overline{\lambda}_{k})^{\alpha_{jk}}
\]
\[
= \prod_{j,k=1}^{n} (1 - t^{2\overline{\lambda}_{j}\overline{\lambda}_{k}})^{-1}
\]
by the orthonormality of \( \{\psi_{\alpha}(W) | \alpha \in S\} \). This proves our assertion in (A) and hence completes our proof.

**Remark 1.** Note that for \( 0 < t < 1 \), the integrand in \( N_{(M)}(tA) \) is absolutely integrable and it can be integrated term by term after its decomposition as a Bergmann kernel function. However, it is not permissible for the integrand of \( N_{(M)}(A) \) to do so.

**Remark 2.** The Gram-Schmidt orthogonalization process is applied to monomials of the same degree since monomials of different degrees are orthogonal to each other by Proposition 5. Furthermore, we assume \( \psi_{\alpha}(W) \) is the function obtained from \( W^{\alpha} \) by this process.

**5. Generalizations and applications.** We shall generalize the evaluation of the integral is our Theorem to cases as follows:

1. The integrand \( \det(E - W^{t\overline{W}})^{-k} \) is changed into a general form \( \det(E - A_{1}W\overline{A}_{2}^{t}\overline{W})^{-k} \) with \( A_{1}, A_{2} \) in \( U(n) \), the unitary group.
2. The domain \( D_{n} \) is changed into the hyperbolic space of \( p \times q \) matrices defined by \( D_{p,q}: W \in M_{p,q}(C), \quad E - t^{i\overline{W}W} > 0. \)
Here $M_{p,q}(C)$ is the set of all $p \times q$ matrices over $C$ and $E_q$ is the unit matrix of $M_q(C)$.

For the first generalization, we then have the following

**Proposition 6.** Let $\Lambda_1 = \text{diag}[\lambda_1, \ldots, \lambda_n]$, $\Lambda_2 = \text{diag}[\lambda_{n+1}, \ldots, \lambda_{2n}]$, with $\lambda_j$ $(j = 1, 2, \ldots, 2n)$ roots of unity and $\lambda_j \lambda_{n+k} \neq 1$ for all $1 \leq j, k \leq n$, and let

$$I = C(k, n) \int_{D_n} \det(E - W^t\overline{W})^{k-2n} \det(E - \overline{\Lambda_1}W\overline{\Lambda_2}^t\overline{W})^{-k} dW \quad (k > 4n - 2).$$

Then

$$I = \prod_{j,k=1}^n (1 - \overline{\lambda_j} \overline{\lambda_{n+k}})^{-1}.$$

**Proof.** The proof follows from a slight change in our proof of the Theorem. Conditions in Proposition 5 imply

$$\sum_{j=1}^n (\alpha_{jk} - \beta_{jk}) = \sum_{j=1}^n (\alpha_{kj} - \beta_{kj}) = 0 \quad (k = 1, 2, \ldots, n).$$

Let $\psi_\alpha(W)$, $\alpha \in S$, be the function obtained from $W^\alpha$ with a Gram-Schmidt orthogonalization process. Then we have

$$\psi_\alpha(\overline{\Lambda_1}W\overline{\Lambda_2}) = \prod_{j=1}^n \overline{\lambda_j}^{a(j)} \prod_{k=1}^n \overline{\lambda_{n+k}}^{b(k)} \psi_\alpha(W) = \prod_{j,k=1}^n (\overline{\lambda_j} \overline{\lambda_{n+k}})^{a_{jk}} \psi_\alpha(W)$$

with $a(j) = \sum_{k=1}^n \alpha_{jk}$ and $b(k) = \sum_{j=1}^n \alpha_{jk}$. It follows

$$I = \lim_{t \to 1} \sum_{\alpha \in S} (t^2 \overline{\lambda_j} \overline{\lambda_{n+k}})^{a_{jk}}$$

$$= \lim_{t \to 1} \prod_{j,k=1}^n (1 - t^2 \overline{\lambda_j} \overline{\lambda_{n+k}})^{-1} = \prod_{j,k=1}^n (1 - \overline{\lambda_j} \overline{\lambda_{n+k}})^{-1}.$$

Now we consider the second generalization. Let $F$ be any imaginary quadratic field and define an algebraic group $G_{p,q}$ over $Q$ as follows:

$$(G_{p,q})_Q = \left\{ M \in SL_{p+q}(F) | \ MRM = R, \ R = \begin{bmatrix} E_p & 0 \\ 0 & -E_q \end{bmatrix} \right\},$$

and $G_{p,q}_R = SU(p, q)$. The group $SU(p, q)$ operates on the bounded domain $D_{p,q}$ by the action

$$M : Z \to M(Z) = (AZ + B)(CZ + D)^{-1}, \quad M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ in } SU(p, q).$$

Here $M$ is so decomposed that $A, B, C$ and $D$ are $p \times p, p \times q, q \times p$ and $q \times q$ matrices respectively.

Let $\Gamma$ be a discrete subgroup of $G_R$ such that $\Gamma \backslash G_R$ has definite volume with respect to the invariant measure $\det(E_q - t\overline{W}W)^{-p-q} dW$. For positive integer
$k$, we let $S(k; \Gamma)$ be the vector space of the holomorphic function $f(W)$ on $D_{p,q}$ satisfying the conditions:

(1) $f(\gamma(W)) = \det(CW + D)^k f(W)$ for all $\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma$.

(2) $[\det(E_q - tWW)]^{k/2} f(W)$ is bounded in $D_{p,q}$.

A function $f$ in $S(k; \Gamma)$ is called a cusp form of weight $k$. A standard argument [6] shows that if $k > 2(p + q - 1)$, then $S(k; \Gamma)$ is a finite dimensional vector space. Furthermore, its dimension can be calculated via the Selberg trace formula.

An element $M$ in $\Gamma$ is regular elliptic if $M$ is conjugate in $G_\mathbb{R}$ to $\Lambda_p \times \bar{\Lambda}_q = \text{diag}[\lambda_1, \ldots, \lambda_p, \bar{\lambda}_{p+1}, \ldots, \bar{\lambda}_{p+q}] \in S(U_p \times U_q)$ and $\lambda_j \lambda_{p+r} \neq 1$ for all $1 \leq j \leq p$, $1 \leq r \leq q$. This is equivalent to saying that $M$ has an isolated fixed point on $D_{p,q}$.

With these preparations, we now have the following

**PROPOSITION 7.** Suppose $M \in \Gamma$ and is conjugate in $G_\mathbb{R}$ to $\Lambda_p \times \bar{\Lambda}_q = \text{diag}[\lambda_1, \ldots, \lambda_p, \bar{\lambda}_{p+1}, \ldots, \bar{\lambda}_{p+q}] \in S(U_p \times U_q)$ with $\lambda_j \lambda_{p+r} \neq 1$ for all $1 \leq j \leq p$, $1 \leq r \leq q$. Then the contribution of elements in $\Gamma$ which are conjugate in $\Gamma$ to $M$, to $\dim_C S(k; \Gamma)$ ($k > 2(p + q - 1)$), is given by

$$N(M) = |C_{M,Z}|^{-1} \prod_{s=1}^{q} \lambda_{p+s}^k \prod_{j=1}^{p} \prod_{r=1}^{q} (1 - \bar{\lambda}_j \bar{\lambda}_{p+r})^{-1}.$$ 

Here $|C_{M,Z}|$ is the order of $C_{M,Z}$ which is the centralizer of $M$ in $\Gamma$, the quotient of $\Gamma$ by its center.

**PROOF.** Let $H(k; D_{p,q})$ be the vector space of holomorphic functions which are square integrable on $D_{p,q}$ with respect to the measure $\det(E_q - tWW)^{k-p-q} dW$. From the argument of [7] or the explicit formula given in [9], we get that

$$K(W_1, W_2) = C(k; p, q) \det(E_q - tW_2 W_1)^{-k}$$

with

$$C(k; p, q) = \pi^{-pq} \prod_{j=0}^{p-1} \prod_{r=0}^{q-1} (k - p - q + 1 + j + r),$$

the kernel function of $H(k; D_{p,q})$. Also we note that the set of monomials in $W = [w_{jr}]$ ($j = 1, \ldots, p$, $r = 1, \ldots, q$) is an independent set as well as a complete system in $H(k; D_{p,q})$. Hence we can apply the Gram-Schmidt orthogonalization process to this set and get an orthonormal basis of $H(k; D_{p,q})$. The orthogonal relations in Proposition 5 still exist if we introduce the same coordinates for $D_{p,q}$ as we have done for $D_n$ in Proposition 4. Consequently, we prove that

$$I_{p,q} = C(k; p, q) \int_{D_{p,q}} \det(E_q - tWW)^{k-p-q} \det(E_q - \bar{\lambda}_k W \bar{\lambda}_{p} W)^{-k} dW$$

$$= \prod_{j=1}^{p} \prod_{r=1}^{q} (1 - \bar{\lambda}_j \bar{\lambda}_{p+r})^{-1}.$$
On the other hand, a standard argument (to change the order of integration and summation) [13] shows that the contribution $N_{(M)}$ is given by

$$N_{(M)} = |C_{M,Z}|^{-1} (\det \Lambda_q)^{-k} I_{p,q}$$

$$= |C_{M,Z}|^{-1} \prod_{s=1}^{q} \lambda_{p+s}^{-k} \cdot \prod_{j=1}^{p} \prod_{r=1}^{q} (1 - \lambda_j \lambda_{p+r})^{-1}.$$ 

This proves our assertion.

REMARK. Proposition 6 can be applied to cases which may be left out by our main Theorem.

REFERENCES