THE COMPLEX EQUILIBRIUM MEASURE OF A SYMMETRIC CONVEX SET IN $\mathbb{R}^n$

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ABSTRACT. We give a formula for the measure on a convex symmetric set $K$ in $\mathbb{R}^n$ which is the Monge-Ampere operator applied to the extremal plurisubharmonic function $L_K$ for the convex set. The measure is concentrated on the set $K$ and is absolutely continuous with respect to Lebesgue measure with a density which behaves at the boundary like the reciprocal of the square root of the distance to the boundary. The precise asymptotic formula for $x \in K$ near a boundary point $x_0$ of $K$ is shown to be of the form $c(x_0)/[\text{dist}(x, \partial K)]^{-1/2}$, where the constant $c(x_0)$ depends both on the curvature of $K$ at $x_0$ and on the global structure of $K$.

1. Introduction. Let us denote the family of plurisubharmonic (psh) functions on $\mathbb{C}^n$ of minimal growth by

$$L = \{v \text{ psh on } \mathbb{C}^n, v(z) \leq \log(1 + |z|) + O(1)\}.$$ 

For $K$ a compact subset of $\mathbb{C}^n$ the extremal function $L^*_K$ for $K$ with logarithmic singularity at infinity is defined by setting

$$L_K(z) = \sup\{v(z): v \in L, v \leq 0 \text{ on } K\}$$

and

$$L^*_K(z) = \limsup_{\varsigma \to z} L_K(\varsigma)$$

(cf. Siciak [13] and Zaharjuta [17]). The function $L^*_K$ is in general not smooth on $\mathbb{C}^n \setminus K$ when $n > 1$ and, in particular, it is not harmonic. It is a theorem of Siciak that either $L^*_K \equiv +\infty$, in which case the set $K$ is pluripolar, or else $L^*_K \in L$. If $L_K$ is continuous on $\mathbb{C}^n$, then $L_K \equiv L^*_K \in L$.

The extremal function $L^*_K$ satisfies the complex Monge-Ampere equation

$$(dd^c L^*_K)^n = 0$$

in a generalized sense on $\mathbb{C}^n \setminus K$ [2, Corollary 9.4]. Thus, for nonpluripolar sets $K$, (1.1)

$$\lambda_K := (dd^c L^*_K)^n$$

is a positive Borel measure supported on $K$. It has total mass equal to $(2\pi)^n$ (cf. [16]). We will call $\lambda_K$ the complex equilibrium measure for $K$.

Here we consider compact sets $K \subset \mathbb{R}^n \subset \mathbb{C}^n$. In this case $K$ is polynomially convex, and so $L_K > 0$ on $\mathbb{C}^n \setminus K$. There are rather reasonable hypotheses on

Received by the editors May 7, 1985.
1980 Mathematics Subject Classification. Primary 32F05, 31C10.
Key words and phrases. Plurisubharmonic function, Monge-Ampere operator, extremal function.
1 This research was supported in part by the NSF while the second author was visiting Indiana University.

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K that will insure that $L_K$ is continuous on $\mathbb{C}^n$, such as:

for each $x_0 \in \partial K$, there exists a real analytic curve

t \to \gamma(t) \in \mathbb{R}^n, -1 < t < +1, such that \gamma(t) \in \text{int}(K)

for $-1 < t < 0$, and $\gamma(0) = x_0$.

(See Plesniak [9] and Sadullaev [12].)

Our first result concerning the extremal measure for subsets $K \subset \mathbb{R}^n$ is that on compact subsets of the interior of $K$, $\lambda_K$ is equivalent to $n$-dimensional Lebesgue measure.

**THEOREM 1.** If $K \subset \mathbb{R}^n$ is compact, then there exist constants $0 < c_1 < c_2 < \infty$ such that

$$
\int_E c_1 \, dx \leq \lambda_K(E) \leq \int_E \frac{c_2 \, dx}{[\text{dist}(x, \partial K)]^n}
$$

holds for any Borel set $E \subset \text{int}(K)$. We can take

$$
c_1 = 2^n[\text{diam}(K)]^{-n}, \quad c_2 = (2\sqrt{n})^n.
$$

In case $K \subset \mathbb{R}^n$ is convex and symmetric about the origin, the extremal function $L_K$ may be given in a relatively simple form. Starting with the case $n = 1$, we note that a convex $K \subset \mathbb{R}$ is an interval. The extremal function for $K = [-1, +1] \subset \mathbb{C}$ is given by the familiar Green function

$$
L_{[-1,+1]}(z) = \psi(z) := \log |z + \sqrt{z^2 - 1}|.
$$

For convex $K \subset \mathbb{R}^n$, we have the support function

$$
\rho_K(\xi) := \sup_{x \in K} \xi \cdot x
$$

defined for $\xi \in \mathbb{R}^n$. It is a theorem of Lundin that the extremal function may be obtained as

$$
L_K(z) = \sup_{|\xi| = 1, \xi \in \mathbb{R}^n} \frac{z \cdot \xi}{\rho_K(\xi)}
$$

where $z \cdot \xi = z_1 \xi_1 + \cdots + z_n \xi_n$ is a projection to the complexification of the $\xi$-axis. Dividing by the support function $\rho_K(\xi)$ is a normalization so that $K$ projects to the interval $[-1, +1]$. The representation (1.4) was given by Lundin in [6].

For the case of convex symmetric sets, the extremal measure can also be computed in an explicit form.

**THEOREM 1.2.** Let $K \subset \mathbb{R}^n$ be a compact convex set, symmetric about the origin and with nonempty interior. Then $\lambda_K = n! \cdot \lambda(x) \, dx$, where $\lambda(x)$ is the $n$-dimensional volume of the convex hull of $S_x(K^*)$, where $K^*$ is the convex set dual to $K$, and $S_x : \mathbb{R}^n \to \mathbb{R}^n$ is the map

$$
S_x(\eta) = \eta/(1 - (x \cdot \eta)^2)^{1/2}.
$$

In case $S_x(K^*)$ is convex, then

$$
\lambda(x) = \frac{1}{n} \int_{|\xi| = 1} \frac{d\sigma(\xi)}{[\rho_K^2(\xi) - (x \cdot \xi)^2]^{n/2}}, \quad x \in K
$$

($d\sigma$ denotes surface area measure on the unit sphere).
This theorem extends a result of Lundin [7], who has shown that when $K = B_n$ is the unit ball in $\mathbb{R}^n$, then

$$L_{B_n}(z) = \sinh^{-1}[\frac{1}{2}(|z|^2 - 1 + |z^2 - 1|)]^{1/2}$$

and that

$$\lambda_{B_n} = \frac{c_n}{(1 - |z|^2)^{1/2}} dx, \quad \text{where} \quad c_n = 2^n \cdot \Gamma \left( \frac{n + 1}{2} \right) \cdot \pi^{(n-1)/2}.$$

A less precise but more general result is as follows.

**Theorem 1.3.** If $K \subset \mathbb{R}^n$ is compact and convex with nonempty interior and smooth boundary, then there are constants $0 < c_1 < c_2$ such that

$$c_1 dx \leq \lambda_K \leq c_2 dx \frac{[\text{dist}(x, \partial K)]^{1/2}}{[\text{dist}(x, \partial K)]^{1/2}}.

We also compute the asymptotic behavior of $\lambda_K(x)$ at $\partial K$.

**Theorem 1.4.** If $K$ is a smoothly bounded symmetric convex set with nonvanishing curvature, then there is a smooth function $c(x)$ on $\partial K$ such that

$$\lambda_K(x) \approx c(\hat{x}) |x - \hat{x}|^{-1/2}$$

where $\hat{x} \in \partial K$ is the boundary point closest to $x$.

This theorem is proved in §4, where a geometric construction is given for the quantity $c(\hat{x})$. It turns out that $c(\hat{x})$ is given by the volume of a certain set in $\mathbb{R}^n$ (see Figure 3), which depends both on the curvature of $\partial K$ at $\hat{x}$ and a global geometric envelope, the “ellipsoidal hull”, of $K$.

Our interest in knowing $\lambda_K$ more precisely in these specific cases arises from the connection between $L_K$ and some problems concerning polynomials. Theorem 1.3 gives a more concrete statement of the Leja polynomial condition in the case of a smoothly bounded convex set $K \subset \mathbb{R}^n$. The condition is as follows (cf. [14]). If $\mathcal{F} = \{P_\alpha : \alpha \in A\}$ is any family of polynomials and if

$$S_\mathcal{F} = \left\{ x \in K : \sup_{\alpha \in A} |p_\alpha(x)| = +\infty \right\},$$

then the following are equivalent:

for any $\lambda > 1$ there exists an open set $U \supset K$ and an $M < \infty$ such that for all polynomials in the family $\mathcal{F}$

$$\sup_U |p_\alpha| \leq M \cdot \lambda^{\deg(p_\alpha)}$$

and

the fine interior of $S_\mathcal{F}$ has Lebesgue measure zero.

Here the “fine interior” is taken with respect to the plurifine topology induced from $\mathbb{C}^n$, cf. [3].
It was shown by Siciak [14] that if a measure $\mu$ satisfies the Leja polynomial condition, in the sense that (1.6) holds whenever $\mu(S_T) = 0$, then it also satisfies a version of the Bernstein-Markov condition:

for $0 < s < +\infty, \lambda > 1$, there exist an open set $U \supset K$, and a constant $C_s < \infty$ such that for any polynomial $p$,

$$
(1.7) \quad \sup_{U} |p| \leq C_s \cdot \lambda^{\deg(p)} \left[ \int |p|^s \, d\mu \right]^{1/s}.
$$

This inequality is used in polynomial approximation and analytic continuation. It follows from the results above that any measure $\mu$ with $dx|_K \ll \mu$ will satisfy (1.7). There is evidence, however, that the measure $\lambda_K$ should be in some sense optimal in these problems. For instance, it is conjectured in [15] that $\lambda_K$ should reflect the asymptotic behavior of the extremal points of $K$.

2. Equivalence of $\lambda_K$ and Lebesgue measure. In this section we will prove a comparison result which is closely related to a theorem of Levenberg [5]. This will lead to a proof of Theorem 1.1.

We will denote by $P(\Omega)$ the space of all psh functions on the domain $\Omega$ in $\mathbb{C}^n$.

**Lemma 2.1.** Let $u_1, u_2 \in P(\Omega)$ be given locally bounded functions, and let $S \subset \Omega \cap \mathbb{R}^n$ be a closed set containing the supports of the Borel measures $(dd^c u_1)^n$ and $(dd^c u_2)^n$. If the sets $\{u_1 = 0\}$ and $\{u_2 = 0\}$ differ from $S$ by at most a pluripolar set, and if $0 \leq u_1 \leq u_2$ on $\Omega$, then $(dd^c u_1)^n \leq (dd^c u_2)^n$.

**Proof.** Without loss of generality, we may assume that $B = \{|z| < 1\}$ is a compact subset of $\Omega$. Let $u_j^\varepsilon = u_j \ast \chi_\varepsilon$ be smoothings decreasing to $u_j$ as $\varepsilon \downarrow 0$. Let $v_j$ denote the restriction of $u_j$ to $\partial B$. Similarly, let $v_j^\varepsilon$ denote the restriction of $u_j^\varepsilon + j \cdot \varepsilon$ to $\partial B$, so that $0 < v_j^\varepsilon < v_j$ on $\partial B$. Given a subset $S$ of the unit ball $B$ and a function $v$ on $\partial B$, let $\mathcal{F}(S, v)$ denote the family of all psh functions $w$ on the unit ball $B$ such that $w \leq 0$ on $S$, and lim $\sup_{z \to x} w(\zeta) \leq v(\zeta)$ for all $z \in \partial B$. Let

$$
S^\varepsilon = \{z \in \mathbb{R}^n \cap \Omega : \text{dist}(x, \partial \Omega) \geq \varepsilon \text{ and } \text{dist}(x, S) \leq \delta\}
$$

and let

$$
U^\varepsilon_{S,j}(z) := \sup\{w(z) : w \in \mathcal{F}(S^\varepsilon, v^\varepsilon_j)\}, \quad z \in B.
$$

The set $S^\varepsilon$ is easily seen to be regular, so $U^\varepsilon_{S,j}$ is psh and continuous on the closure $\overline{B}$ of $B$. It therefore follows from a theorem of Levenberg [5] that

$$
(dd^c U^\varepsilon_{S,j})^n \leq (dd^c U^\varepsilon_{S,2})^n.
$$

As $\delta \downarrow 0$, the functions $U^\varepsilon_{S,j}$ increase almost everywhere to the functions $[U^\varepsilon_{S,j}]^*$, the uppersemicontinuous regularizations of the envelope functions for the families $\mathcal{F}(S, v^\varepsilon_j)$. Because the operator $(dd^c)^n$ is continuous on monotone limits of locally bounded psh functions [2], we can pass to the limit in the last inequality to obtain

$$
(dd^c U^\varepsilon_{S,1})^n \leq (dd^c U^\varepsilon_{S,2})^n.
$$

Now, let $\varepsilon \downarrow 0$. The functions $U^\varepsilon_{S,j}$ clearly decrease. And, they decrease to the uppersemicontinuous regularization of the envelope function of the family $\mathcal{F}(S, v_j)$, $U^\varepsilon_{S,j}$. Thus, we have

$$
(dd^c U^\varepsilon_{S,1})^n \leq (dd^c U^\varepsilon_{S,2})^n.
$$
But, it is evident that $U_{S,j}^* \geq u_j$. On the other hand, we have $u_j \geq U_{S,j}^*$ by the domination principle (see [2, Corollary 4.5]). Thus, $u_j = U_{S,j}$ which proves the lemma.

**Proposition 2.2.** Let $E_1, \ldots, E_n$ be compact subsets of $\mathbb{C}$. Then

$$\lambda_{E_1 \times \cdots \times E_n} = \lambda_{E_1} \otimes \cdots \otimes \lambda_{E_n}.\]

**Proof.** By a result of Siciak [13] we know that

$$L^*_{E_1 \times \cdots \times E_n}(z_1, \ldots, z_n) = \max_{1 \leq j \leq n} L^*_{E_j}(z_j).$$

Let us assume that $E_j$ is a Jordan domain in $\mathbb{C}$ with real analytic boundary. Then $L_{E_j}$ may be extended from $\mathbb{C} \setminus E_j$ to a function $\tilde{L}_j$ which is harmonic in a neighborhood of $\partial E_j$. Thus we may write

$$L^*_{E_1 \times \cdots \times E_n}(z_1, \ldots, z_n) = \max_{1 \leq j \leq n} \{\tilde{L}_1, \ldots, \tilde{L}_n, 0\}.\]

However,

$$(d \sigma_j)^n = d^c \tilde{L}_1 \wedge \cdots \wedge d^c \tilde{L}_n |_{M}$$

where $M = \{\tilde{L}_1 = \cdots = \tilde{L}_n = 0\}$ is given the orientation of $d\tilde{L}_1 \wedge \cdots \wedge d\tilde{L}_n$ (see the remark on p. 7 of [1]). Since

$$d^c \tilde{E}_j = \frac{\partial}{\partial n_j} (\tilde{L}_j) \, d\sigma_j$$

where $d\sigma_j$ is the arclength measure of $\partial E_j$ and $dn_j$ is the outward normal, we see that $d^c \tilde{L}_j$ may be naturally identified with $\lambda_{E_j}$. This proves the proposition in the smooth case.

For the general case, we take smooth $E_j^\varepsilon$,

$$E_j \subset E_j^\varepsilon \subset \{z \in \mathbb{C} : \text{dist}(z, E_j) \leq \varepsilon\}.\]

If we take $E_j^\varepsilon \subset E_j^\delta$ for $\varepsilon \leq \delta$, then $L^*_{E_1^\varepsilon \times \cdots \times E_n^\varepsilon}$ increases a.e. to $L^*_{E_1 \times \cdots \times E_n}$. Thus, by the convergence theorem of [2],

$$\lim_{\varepsilon \to 0} \lambda_{E_1^\varepsilon \times \cdots \times E_n^\varepsilon} = \lambda_{E_1 \times \cdots \times E_n},$$

and the proposition is proved.

In the case of the interval $[-1, +1]$, it is well known that

$$\lambda_{[-1,+1]} = (2 \cdot dx)/(1 - x^2)^{1/2},$$

and so a product of intervals,

$$E = \{-\delta \leq x_1 \leq \delta, \ldots, -\delta \leq x_n \leq \delta\}$$

has extremal measure

$$\lambda_E = \frac{2^n \cdot dx_1 \cdots dx_n}{\delta^n \cdot [1 - (x_1/\delta)^2]^{1/2} \cdots [1 - (x_n/\delta)^2]^{1/2}}.$$
generality, we can assume that \( x_0 = 0 \). Let \( \Omega = \{ z \in \mathbb{C}^n : |z| < \eta \} \) and \( S = \Omega \cap \mathbb{R}^n \). It follows that \( S = \{ L^*_K = 0 \} \cap \Omega = \{ L^*_r = 0 \} \cap \Omega \) and so by Lemma 2.1,

\[
\lambda_K |S| \geq \frac{2^n \cdot dx_1 \cdots dx_n}{r^n \cdot |1 - (x_1/r)^2|^{1/2} \cdots |1 - (x_n/r)^2|^{1/2}} |S|
\]

For the other inequality, we note that \( \{ x : -\eta/n \leq x_j \leq +\eta/n \} \subset \text{int}(K) \), so again by Lemma 2.1

\[
\frac{\left[2^{n} \cdot dx_{1} \cdots dx_{n}\right]}{(\text{dist}(x, \partial K)^{n/2}} \geq \lambda_K |\text{int}(K)|.
\]

This completes the proof.

**Proof of Theorem 1.3.** First let us assume that \( \partial K \) is real analytic. Thus, it is pluripolar and \( \lambda_K (\partial K) = 0 \). Since \( \partial K \) is \( C^2 \), we may touch any \( x_0 \in K \) by an internally tangent ball of fixed radius \( r \). Thus, they may use an internally tangent ball and an externally tangent cube to obtain (1.5) as in Theorem 1.1, because there is no mass on \( \partial K \).

If \( \partial K \) is merely \( C^2 \), we may consider a sequence \( \{ K_j \} \) of convex, symmetric domains with real analytic boundary such that \( \partial K_j \) approaches \( \partial K \) in \( C^2 \). Thus, we obtain (1.5) for each \( \lambda_{K_j} \). Further, the constants \( c_1 \) and \( c_2 \) depend only on the diameter of \( K_j \) and the radius of an internally tangent ball. Thus, we may be chosen independent of \( j \). It follows from the dominated convergence theorem, then, that \( \lambda_K \) puts no mass on \( \partial K \), so that (1.5) holds.

**3. Representation of \((dd^c L_K)^n\).** In this section we will use Lundin's representation (1.4) of \( L_K \) and give the basic form of \((dd^c L_K)^n\) for convex sets in \( \mathbb{R}^n \). We first give a geometrical construction of an auxiliary function that will be used to describe \((dd^c L_K)^n\).

![Figure 1](https://example.com/image.png)

\[ z = x + iy \quad \beta = h(x, y) \]
For $\beta > 0$, let $E_\beta$ denote the ellipse

$$E_\beta : \quad s^2 + t^2 / \beta^2 = 1.$$  

Let $K^*$ denote the dual convex set to $K$. That is,

$$K^* = \{ \eta \in \mathbb{R}^n : x \cdot \eta \leq 1 \text{ for all } x \in K \}.$$  

It is readily checked that when $K$ is convex and symmetric about the origin, so is $K^*$ and

$$K^* = \{ \eta = r \cdot \alpha \in \mathbb{R}^n : |\alpha| = 1, r \leq 1 / \rho(\alpha) \}.$$  

We will use the function $h = h_K$ defined for $z = x + i \cdot y$ with $x \in \text{int}(K)$, $y \in \mathbb{R}^n$ by

$$h_K(z) = h(x,y) = \inf_{\beta : T_z(K^*) \subset E_\beta} \{ \beta \}$$  

(see Figure 1) where $T_z$ is the linear map from $\mathbb{R}^n$ to $\mathbb{R}^2$ defined by $\zeta = s + i \cdot t = T_z(\xi) = z \cdot \xi$, or, more precisely by

$$\begin{bmatrix} s \\ t \end{bmatrix} = T_z(\xi) = \begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{bmatrix} \cdot \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}.$$  

The function $h(x,y) = h_K(x,y)$ has the following properties.

**Proposition 3.1.** If $K^*$ is the dual convex set to the compact convex set $K$, and if $K^*$ is compact, then for $h(x,y)$ defined in (3.2) we have

(i) $h(x,y)$ is continuous for $x \in \text{int}(K)$ and $y \in \mathbb{R}^n$;

(ii) $h(x, \lambda \cdot y) = \lambda \cdot h(x, y)$ for $\lambda > 0$;

(iii) $y \rightarrow h(x,y)$ is convex on $\mathbb{R}^n$ for each fixed $x \in \text{int}(K)$;

(iv) $y \rightarrow h(x,y)$ is the support function of the convex hull of the set, $S_x(K^*)$, where $S_x$ is the (nonlinear) transformation of $\mathbb{R}^n$ defined by

$$S_x(\xi) = \xi / (1 - (x \cdot \xi)^2)^{1/2}.$$  

**Proof.** First, note that $h$ is defined and finite when $x \in \text{int}(K)$ because then $\xi \cdot x < 1$ for all $\xi \in K^*$, so the set $T_z(K^*)$ is contained in $E_\beta$ for some $\beta > 0$. That $h$ is continuous is clear because $K^*$ is compact. Assertion (ii) follows from the obvious scaling property

$$T_{x+i \cdot \lambda \cdot y}(K^*) \subset E_\beta \iff T_{x+i \cdot y}(K^*) \subset E_{\beta / \lambda}.$$  

Both assertions also follow directly from (iv), as does assertion (iii). We will therefore prove assertion (iv).

The number $\beta = h(x,y)$ is defined as the smallest number $\beta \geq 0$ such that

$$(x \cdot \xi)^2 + (y \cdot \xi)^2 / \beta^2 \leq 1, \quad \xi \in K^*,$$

with equality holding for at least one $\xi$. Solving the inequality for $\beta$ yields

$$\beta \geq y \cdot \xi / (1 - (x \cdot \xi)^2)^{1/2} = y \cdot S_x(\xi), \quad \xi \in K^*,$$

with equality for at least one $\xi$. In other words, $h(x,y)$ is the maximum of the right-hand side of (3.3) over $K^*$, which is exactly assertion (iv). This completes the proof of the proposition.
For $f$ a convex function of $y \in \mathbb{R}^n$, let $M(f)$ denote the real Monge-Ampere operator applied to $f$. That is, if $f$ is smooth, then $M(f)$ is the determinant of the Hessian of $f$,

$$M(f) = \det[\partial^2 f / \partial y_i \partial y_k].$$

The operator extends by continuity to be defined as a nonnegative Borel measure on the class of convex functions. The measure of a Borel set $E$ is the volume of the set of all direction vectors $\lambda$ of hyperplanes, $\lambda \cdot y + \text{const.}$, which support the graph of $f$ at a point $x \in E$. See e.g. [11].

We will use two facts about this operator. First, if the convex function $f(y)$ defined on $\mathbb{R}^n$ is thought of as a psh function on $\mathbb{C}^n$, then we have the formula

$$(dd^c f)^n = n! \cdot M(f) \otimes dx.$$  

Second, if $f$ is the function $y \rightarrow h(x, y)$, then because $h$ is homogeneous of degree 1, $M(f)$ must be supported at the origin. Further, this set of direction vectors which support the graph of the function is clearly the convex hull of the set $S_x(K^*)$. Thus, if $x_0$ is a fixed point of $\text{int}(K)$, then

$$L_K(x_0 + i \cdot y) = \lim_{c \to 0^+} (dd^c h(x_0, y))^n = n! \cdot \text{vol}\{\text{ch} S_x(K^*)\} \cdot dx.$$  

We next show the relation between the function $h = h_K$ and the extremal function $L_K$.

**Theorem 3.2.** Let $K$ be a convex compact symmetric set in $\mathbb{R}^n$ with $0 \in \text{int}(K)$. Then

$$\lim_{\varepsilon \to 0^+} \frac{L_K(x + i \cdot \varepsilon \cdot y)}{\varepsilon} = h_K(x, y)$$

uniformly on compact subsets of $\text{int}(K) \oplus i\mathbb{R}^n \subset \mathbb{C}^n$. In particular, if $x_0 \in \text{int}(K)$ and $\varepsilon > 0$ is given, there exists $\delta > 0$ such that, for all $|x - x_0| \leq \delta$, $|y| \leq \delta$,

$$h_K(x_0, y) - \varepsilon \cdot |y| \leq L_K(x_0 + i \cdot y) \leq h_K(x_0, y) + \varepsilon \cdot |y|.$$
THEOREM 1.2. First we note that it may be argued as in the proof of Theorem 1.3 to see that \( \lambda_K \) puts no mass on \( \partial K \). Now the formula for \( \lambda(x) \) follows directly from formula (3.4), inequality (3.6) of Theorem 3.2, and the comparison theorem of §2. The integral formula may be seen as follows. The set \( S_x(K^*) \) is given by \( \{ y \in \mathbb{R}^n : (y \cdot \xi)^2 + (x \cdot \xi)^2 \leq \rho_K^2(\xi) \} \) for all \( \xi \in \mathbb{R}^n \). This completes the proof.

**Remark.** We note that \( \text{ch } S_x(K^*) \) is the dual of the set

\[ \{ y \in \mathbb{R}^n : (y \cdot \xi)^2 + (x \cdot \xi)^2 \leq \rho_K^2(\xi) \text{ for all } \xi \in \mathbb{R}^n \}. \]

Since the total mass of \( \lambda_K \) is independent of \( K \), it follows that the constant

\[ c_n = \int \text{vol}(\text{ch } S_x(K^*)) \, dx \]

is the same for all symmetric convex sets \( K \subset \mathbb{R}^n \). [\( c_n = (2\pi)^n/n! \]. It would be interesting to know if there is a purely geometrical proof of this fact.
4. Asymptotic behavior of $\lambda_K$. Let $K \subset \mathbb{R}^n$ be smoothly bounded, convex, and symmetric. We will discuss the asymptotic behavior of $\lambda(x)$, and, in the process, prove Theorem 1.4. By Theorem 1.2, we will need to estimate the asymptotic behavior of the volume of the convex hull of $S_x(K^*)$ as $x$ approaches $x_0 \in \partial K$.

First we establish some geometric notation. Let $\eta_0$ be a normal vector to $\partial K$ at $x_0$, normalized so that $\eta_0 \cdot x_0 = 1$. Any ellipsoid symmetric about $0 \in \mathbb{R}^n$ and tangent to $\partial K$ at $x_0$ has the form

$$\left(\eta_0 \cdot x\right)^2 + R(x) \leq 1$$

where $R(x)$ is a positive semidefinite quadratic form on $\mathbb{R}^n$ such that $R(x_0) = 0$. By abuse of notation, $R$ may be identified with a positive semidefinite quadratic form on the hypersurface $x_0$ orthogonal to $x_0$.

We define the ellipsoidal hull of $K$ at $x_0$, written $EH(K, x_0)$, as the intersection of all the ellipsoids of the form (4.1) which contain $K$. Similarly, we define the ellipsoidal core of $K$ at $x_0$, written $EC(K, x_0)$, as the union of all of the ellipsoids of the form (4.1) which are contained in $K$. The ellipsoidal hull and core are related as follows, which is proved in [4].

PROPOSITION 4.1. If $K$ and $K^*$ are both smoothly bounded, then $EC(K, x_0)^* = EH(K^*, \eta_0)$.

Next, we relate the ellipsoidal hull to the mapping

$$S_{x_0}(\eta) = \eta \cdot (1 - (x_0 \cdot \eta)^2)^{-1/2}.$$ 

LEMMA 4.2. $\text{ch} S_{x_0}(K^*) = S_{x_0}(\{EH(K^*, \eta_0)\})$.

PROOF. We note that $\xi = S_{x_0}(\eta)$ carries the degenerate ellipsoid $(\xi_0 \cdot \eta)^2 + (x_0 \cdot \eta)^2 \leq 1$ to the “strip” $|\xi_0 \cdot \xi| \leq 1$. If $\xi_0 \perp \eta_0$, then these degenerate ellipsoids are tangent to $K^*$ at $\eta_0$, and thus the set of these ellipsoids with $\xi_0 \perp \eta_0$ generates the ellipsoidal hull. Thus,

$$S_{x_0}(\{EH(K^*, \eta_0)\}) = \bigcap \{|\xi_0 \cdot \xi| \leq 1\}$$

where the $\xi_0$ are chosen so that $\xi_0 \perp \eta_0$ and the corresponding degenerate ellipsoid contains $K^*$. Thus, $\{|\xi_0 \cdot \xi| \leq 1\} \supset S_{x_0}(K^*)$, and so $\text{ch} S_{x_0}(K^*) \subset S_{x_0}(\{EH(K^*, \eta_0)\})$.

On the other hand, $x_0 \cdot \eta_0 = 1$, and so $S_{x_0}(K^*)$ contains the line $\{t\eta_0: t \in \mathbb{R}\}$. It follows that $\text{ch} S_{x_0}(K^*)$ is an intersection of half-spaces of the form $\{|\xi_0 \cdot \xi| \leq 1\}$ with $\xi_0 \cdot \eta_0 \neq 0$. This proves the lemma.

LEMMA 4.3. For $x = tx_0$, $0 < t < 1$, let us set

$$\omega_t = (\text{ch} S_x(K^*)) \cap (\eta_0^\perp).$$

Then $\omega_t$ increases to $S_{x_0}(\{EH(K^*, \eta_0)\}) \cap (\eta_0^\perp)$ as $t$ increases to 1.

PROOF. This follows from Lemma 4.2 since $\omega_t$ is clearly increasing in $t$.

Now we extend $\eta_0$ to an orthogonal set of coordinate axes, $(\eta_0, \eta_1, \ldots, \eta_{n-1})$ such that $\eta_1, \ldots, \eta_{n-1}$ are the principal curvature directions of $\partial K$ at $x_0$, and we let $\kappa_1, \ldots, \kappa_{n-1}$ be the principal curvatures of $\partial K$ at $x_0$. Let $(\xi_0, \ldots, \xi_{n-1})$ denote coordinates with respect to these new axes, and let $W_t: \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation given by

$$W_t(\xi) = ((1 - t^2)^{1/2} \cdot \xi_0, \xi_1, \ldots, \xi_{n-1}).$$
Let $Q$ be the quadratic form on $\mathbb{R}^n$ which in $\xi$-coordinates is

\[
Q(\eta) = \sum_{j=1}^{n-1} \frac{\xi_j^2}{\kappa_j}
\]

and define the ellipsoid

\[
E = \{ \eta \in \mathbb{R}^n : \xi_0^2 + Q(\xi) \leq 1 \}.
\]

**Lemma 4.4.**

\[
\lim_{t \to 1} W_t[S_{tx_0}(K^*) \cap \{ \mathbb{R}^n \setminus \eta_0 \}] = E.
\]

**Proof.** Since $\pm \eta_0$ are the only points of $\partial K^*$ which get mapped to infinity under $S_{x_0}$, it follows that the left-hand side of (4.3) is determined only by a neighborhood of $\pm \eta_0$ in $\partial K^*$. That is, if $\tilde{\eta} \in \partial K^*$ and $\tilde{\eta} \neq \pm \eta_0$, then $\lim_{t \to 1} W_t S_{tx_0}(\tilde{\eta}) \subset \eta_0$. Thus, as $t \to 1$, we must consider smaller and smaller neighborhoods in order to find the left-hand side of (4.3).

Consider the ellipsoid which is symmetric about 0 and which is tangential to $\partial K^*$ at $\pm \eta_0$. The normal to $K^*$ at $\eta_0$ is $x_0$, so this ellipsoid has the form $(x_0 \cdot \eta)^2 + R(\eta) \leq 1$, as in (4.1). Since $\eta_0 \cdot x_0 = 1$ and the curvatures of $\partial K^*$ at $\eta_0$ are $\kappa_1^{-1}, \ldots, \kappa_{n-1}^{-1}$, it follows that this ellipsoid coincides with

\[
\tilde{E} = \{ \eta \in \mathbb{R}^n : (\eta \cdot x_0)^2 + Q(\eta) \leq 1 \}.
\]

Without loss of generality, we may assume that $\partial K^*$ coincides with $\partial \tilde{E}$ in a small neighborhood of $\pm \eta_0$. Now it suffices to show that

\[
\lim_{t \to 1} W_t[S_{tx_0}(\tilde{E})] = E.
\]

To see this we consider first the ellipsoid

\[
\tilde{E}_c = \{ \eta : (\eta \cdot x_0)^2/c^2 + Q(\eta) \leq 1 \}.
\]

We see that

\[
S_{tx_0}(\tilde{E}_c) = \{ \eta : (\eta \cdot x_0)^2/c^2 + Q(\eta) \leq 1 - t^2(\eta \cdot x_0)^2 \}
\]

is equal to $\tilde{E}$ if and only if $c^2 = (1 - t^2)^{-1}$. Thus

\[
W_t[S_{tx_0}(\tilde{E})] = W_t[\tilde{E}_c]
\]

with $c^2 = (1 - t^2)^{-1}$, and so we have, with $\eta = \xi_0 \eta_0 + \cdots + \xi_{n-1} \eta_{n-1}$,

\[
W_t[S_{tx_0}(\tilde{E})] = \left\{ \eta \in \mathbb{R}^n : (1 - t^2) \left[ \frac{\xi_0}{\sqrt{1 - t^2}} + \sum_{j=1}^{n-1} \xi_j (\eta_j \cdot x_0) \right]^2 + Q(\eta) \leq 1 \right\}.
\]

Taking the limit as $t \to 1$, we obtain $E$.

Now we define $\omega = (\eta_0^\perp) \cap \text{EH}(K^*, \eta_0)$, $\Omega = \text{ch}(\omega \cup E)$ (see Figure 3), and

\[
c(x_0) = n\text{-dimensional volume of } \Omega.
\]

**Lemma 4.5.** With the above notation

\[
\text{volume}(\text{ch } S_{tx_0}(K^*)) \approx \frac{|\eta_0|}{\sqrt{1 - t^2}} \cdot \text{vol}(\Omega)
\]

holds as $t \to 1$. 

PROOF. We will show that

$$\lim_{t \to 1} W_t(ch_{t_{x_0}}(K^*)) = ch(\omega \cup E_1).$$

It follows, then, that the volume of $ch_{t_{x_0}}(K^*)$ is $(1 - t^2)^{-1/2}$ times

$$|\eta_0| \cdot \text{vol}(ch(\omega \cup E_1)),$$

where the factor $|\eta_0|$ enters because $|\eta_0|$ is the unit of length on the $\eta_0$-axis. Thus, the lemma follows from (4.4).

To show (4.4), we use the notation $\omega_t = (\eta_0^t) \cap S_{t_{x_0}}(K^*)$. From Lemma 4.3 we have

$$W_t(ch_{t_{x_0}}(K^*) \cap (\eta_0^t)) \supset \omega_t,$$

and so

$$\lim_{t \to 1} W_t(ch_{t_{x_0}}(K^*) \cap (\eta_0^t)) \supset ch(\omega_t \cup E).$$

Thus, $\supset$ holds in (4.4).

For the reverse inclusion, we note that for any $\varepsilon > 0$,

$$S_{t_{x_0}}(K^*) \subset S_{x_0}(K^* \cap \{ |\eta - \eta_0| > \varepsilon \}) \cup S_{t_{x_0}}(K^* \cap \{ |\eta - \eta_0| > \varepsilon \}) = \overline{K}_1 \cup \overline{K}_2(t).$$

Thus,

$$\lim_{t \to 1} W_t(ch_{t_{x_0}}(K^*)) \subset \lim_{t \to 1} W_t(ch(\overline{K}_1 \cup \overline{K}_2(t)))$$

$$= \lim_{t \to 1} ch(W_t\overline{K}_1 \cup W_t\overline{K}_2(t)) \subset ch(\omega \cup E) = \Omega.$$
where the next-to-last inclusion follows from Lemma 4.4.

**Proof of Theorem 1.4.** We will show that the quantity

\[ c(x_0) = \frac{\text{vol}(\Omega)}{\sqrt{2} \cdot \rho_K(x_0)} \]

is the asymptotic value in Theorem 1.4. First, we note that, with the notation of Theorem 1.4, \( |x - \bar{x}| = \text{dist}(x, \partial K) \) and for fixed \( x_0 \in \partial K \)

\[ \text{dist}(tx_0, \partial K) \approx |1 - t|x_0 \cdot \frac{\eta_0}{|\eta_0|} = \frac{1 - t}{|\eta_0|} \]

holds for \( t \to 1 \). Now by Lemma 4.5 we have, with \( \Omega = \Omega_{x_0} \),

\[ \lambda(tx_0) = \text{vol} \text{ch} S_{tx_0}(K^*) \approx \frac{|\eta_0| \text{vol}(\Omega)}{\sqrt{1 - t^2}} \approx \frac{|\eta_0| \text{vol}(\Omega)}{\sqrt{2} \sqrt{1 - t}} = \frac{|\eta_0|^{1/2} \text{vol}(\Omega)}{\sqrt{2} (\text{dist}(tx_0, \partial K))^{1/2}} \]

since \( \eta_0 \) was normalized so that \( x_0 \cdot \eta_0 = 1 \); i.e., \( \rho_K(x_0) = |\eta_0|^{-1} \).

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