

## THE COMPLEX EQUILIBRIUM MEASURE OF A SYMMETRIC CONVEX SET IN $\mathbf{R}^n$

BY

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ABSTRACT. We give a formula for the measure on a convex symmetric set  $K$  in  $\mathbf{R}^n$  which is the Monge-Ampere operator applied to the extremal plurisubharmonic function  $L_K$  for the convex set. The measure is concentrated on the set  $K$  and is absolutely continuous with respect to Lebesgue measure with a density which behaves at the boundary like the reciprocal of the square root of the distance to the boundary. The precise asymptotic formula for  $x \in K$  near a boundary point  $x_0$  of  $K$  is shown to be of the form  $c(x_0)/[\text{dist}(x, \partial K)]^{-1/2}$ , where the constant  $c(x_0)$  depends both on the curvature of  $K$  at  $x_0$  and on the global structure of  $K$ .

**1. Introduction.** Let us denote the family of plurisubharmonic (psh) functions on  $\mathbf{C}^n$  of minimal growth by

$$\mathcal{L} = \{v \text{ psh on } \mathbf{C}^n, v(z) \leq \log(1 + |z|) + O(1)\}.$$

For  $K$  a compact subset of  $\mathbf{C}^n$  the extremal function  $L_K^*$  for  $K$  with logarithmic singularity at infinity is defined by setting

$$L_K(z) = \sup\{v(z) : v \in \mathcal{L}, v \leq 0 \text{ on } K\}$$

and

$$L_K^*(z) = \limsup_{\zeta \rightarrow z} L_K(\zeta)$$

(cf. Siciak [13] and Zaharjuta [17]). The function  $L_K^*$  is in general not smooth on  $\mathbf{C}^n \setminus K$  when  $n > 1$  and, in particular, it is not harmonic. It is a theorem of Siciak that either  $L_K^* \equiv +\infty$ , in which case the set  $K$  is pluripolar, or else  $L_K^* \in \mathcal{L}$ . If  $L_K$  is continuous on  $\mathbf{C}^n$ , then  $L_K \equiv L_K^* \in \mathcal{L}$ .

The extremal function  $L_K^*$  satisfies the complex Monge-Ampere equation

$$(dd^c L_K^*)^n = 0$$

in a generalized sense on  $\mathbf{C}^n \setminus K$  [2, Corollary 9.4]. Thus, for nonpluripolar sets  $K$ ,

$$(1.1) \quad \lambda_K := (dd^c L_K^*)^n$$

is a positive Borel measure supported on  $K$ . It has total mass equal to  $(2\pi)^n$  (cf. [16]). We will call  $\lambda_K$  the *complex equilibrium measure* for  $K$ .

Here we consider compact sets  $K \subset \mathbf{R}^n \subset \mathbf{C}^n$ . In this case  $K$  is polynomially convex, and so  $L_K > 0$  on  $\mathbf{C}^n \setminus K$ . There are rather reasonable hypotheses on

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$K$  that will insure that  $L_K$  is continuous on  $\mathbf{C}^n$ , such as:

for each  $x_0 \in \partial K$ , there exists a real analytic curve  
 $t \rightarrow \gamma(t) \in \mathbf{R}^n$ ,  $-1 < t < +1$ , such that  $\gamma(t) \in \text{int}(K)$   
 for  $-1 < t < 0$ , and  $\gamma(0) = x_0$ .

(See Plesniak [9] and Sadullaev [12].)

Our first result concerning the extremal measure for subsets  $K \subset \mathbf{R}^n$  is that on compact subsets of the interior of  $K$ ,  $\lambda_K$  is equivalent to  $n$ -dimensional Lebesgue measure.

**THEOREM 1.1.** *If  $K \subset \mathbf{R}^n$  is compact, then there exist constants  $0 < c_1 < c_2 < \infty$  such that*

$$(1.2) \quad \int_E c_1 dx \leq \lambda_K(E) \leq \int_E \frac{c_2 dx}{[\text{dist}(x, \partial K)]^n}$$

holds for any Borel set  $E \subset \text{int}(K)$ . We can take

$$c_1 = 2^n [\text{diam}(K)]^{-n}, \quad c_2 = (2\sqrt{n})^n.$$

In case  $K \subset \mathbf{R}^n$  is convex and symmetric about the origin, the extremal function  $L_K$  may be given in a relatively simple form. Starting with the case  $n = 1$ , we note that a convex  $K \subset \mathbf{R}$  is an interval. The extremal function for  $K = [-1, +1] \subset \mathbf{C}$  is given by the familiar Green function

$$L_{[-1,+1]}(z) = \psi(z) := \log |z + \sqrt{z^2 - 1}|.$$

For convex  $K \subset \mathbf{R}^n$ , we have the support function

$$(1.3) \quad \rho_K(\xi) := \sup_{x \in K} \xi \cdot x$$

defined for  $\xi \in \mathbf{R}^n$ . It is a theorem of Lundin that the extremal function may be obtained as

$$(1.4) \quad L_K(z) = \sup_{|\xi|=1, \xi \in \mathbf{R}^n} \psi \left( \frac{z \cdot \xi}{\rho_K(\xi)} \right)$$

where  $z \cdot \xi = z_1 \xi_1 + \dots + z_n \xi_n$  is a projection to the complexification of the  $\xi$ -axis. Dividing by the support function  $\rho_K(\xi)$  is a normalization so that  $K$  projects to the interval  $[-1, +1]$ . The representation (1.4) was given by Lundin in [6].

For the case of convex symmetric sets, the extremal measure can also be computed in an explicit form.

**THEOREM 1.2.** *Let  $K \subset \mathbf{R}^n$  be a compact convex set, symmetric about the origin and with nonempty interior. Then  $\lambda_K = n! \cdot \lambda(x) dx$ , where  $\lambda(x)$  is the  $n$ -dimensional volume of the convex hull of  $S_x(K^*)$ , where  $K^*$  is the convex set dual to  $K$ , and  $S_x: \mathbf{R}^n \rightarrow \mathbf{R}^n$  is the map*

$$S_x(\eta) = \eta / (1 - (x \cdot \eta)^2)^{1/2}.$$

In case  $S_x(K^*)$  is convex, then

$$\lambda(x) = \frac{1}{n} \int_{|\xi|=1} \frac{d\sigma(\xi)}{[\rho_K^2(\xi) - (x \cdot \xi)^2]^{n/2}}, \quad x \in K$$

( $d\sigma$  denotes surface area measure on the unit sphere).

This theorem extends a result of Lundin [7], who has shown that when  $K = \mathbf{B}^n$  is the unit ball in  $\mathbf{R}^n$ , then

$$L_{\mathbf{B}^n}(z) = \sinh^{-1}[\frac{1}{2}(|z|^2 - 1 + |z^2 - 1|)]^{1/2}$$

and that

$$\lambda_{\mathbf{B}^n} = \frac{c_n}{(1 - |x|^2)^{1/2}} dx, \quad \text{where } c_n = 2^n \cdot \Gamma\left(\frac{n+1}{2}\right) \cdot \pi^{(n-1)/2}.$$

A less precise but more general result is as follows.

**THEOREM 1.3.** *If  $K \subset \mathbf{R}^n$  is compact and convex with nonempty interior and smooth boundary, then there are constants  $0 < c_1 < c_2$  such that*

$$(1.5) \quad \frac{c_1 dx}{[\text{dist}(x, \partial K)]^{1/2}} \leq \lambda_K \leq \frac{c_2 dx}{[\text{dist}(x, \partial K)]^{1/2}}.$$

We also compute the asymptotic behavior of  $\lambda_K(x)$  at  $\partial K$ .

**THEOREM 1.4.** *If  $K$  is a smoothly bounded symmetric convex set with non-vanishing curvature, then there is a smooth function  $c(x)$  on  $\partial K$  such that*

$$\lambda_K(x) \approx c(\bar{x})|x - \bar{x}|^{-1/2}$$

where  $\bar{x} \in \partial K$  is the boundary point closest to  $x$ .

This theorem is proved in §4, where a geometric construction is given for the quantity  $c(\bar{x})$ . It turns out that  $c(\bar{x})$  is given by the volume of a certain set in  $\mathbf{R}^n$  (see Figure 3), which depends both on the curvature of  $\partial K$  at  $\bar{x}$  and a global geometric envelope, the “ellipsoidal hull”, of  $K$ .

Our interest in knowing  $\lambda_K$  more precisely in these specific cases arises from the connection between  $L_K$  and some problems concerning polynomials. Theorem 1.3 gives a more concrete statement of the Leja polynomial condition in the case of a smoothly bounded convex set  $K \subset \mathbf{R}^n$ . The condition is as follows (cf. [14]). If  $\mathcal{F} = \{P_\alpha : \alpha \in A\}$  is any family of polynomials and if

$$S_{\mathcal{F}} = \left\{ x \in K : \sup_{\alpha \in A} |p_\alpha(x)| = +\infty \right\},$$

then the following are equivalent:

for any  $\lambda > 1$  there exists an open set  $U \supset K$  and an  $M < \infty$  such that for all polynomials in the family  $\mathcal{F}$

$$(1.6) \quad \sup_U |p_\alpha| \leq M \cdot \lambda^{\text{deg}(p_\alpha)}$$

and

the fine interior of  $S_{\mathcal{F}}$  has Lebesgue measure zero.

Here the “fine interior” is taken with respect to the plurifine topology induced from  $\mathbf{C}^n$ , cf. [3].

It was shown by Siciak [14] that if a measure  $\mu$  satisfies the Leja polynomial condition, in the sense that (1.6) holds whenever  $\mu(S_{\mathcal{F}}) = 0$ , then it also satisfies a version of the Bernstein-Markov condition:

for  $0 < s < +\infty, \lambda > 1$ , there exist an open set  $U \supset K$ , and a constant  $C_s < \infty$  such that for any polynomial  $p$ ,

$$(1.7) \quad \sup_U |p| \leq C_s \cdot \lambda^{\deg(p)} \left[ \int |p|^s d\mu \right]^{1/s}.$$

This inequality is used in polynomial approximation and analytic continuation. It follows from the results above that any measure  $\mu$  with  $dx|_K \ll \mu$  will satisfy (1.7). There is evidence, however, that the measure  $\lambda_K$  should be in some sense optimal in these problems. For instance, it is conjectured in [15] that  $\lambda_K$  should reflect the asymptotic behavior of the extremal points of  $K$ .

**2. Equivalence of  $\lambda_K$  and Lebesgue measure.** In this section we will prove a comparison result which is closely related to a theorem of Levenberg [5]. This will lead to a proof of Theorem 1.1.

We will denote by  $P(\Omega)$  the space of all psh functions on the domain  $\Omega$  in  $\mathbf{C}^n$ .

LEMMA 2.1. *Let  $u_1, u_2 \in P(\Omega)$  be given locally bounded functions, and let  $S \subset \Omega \cap \mathbf{R}^n$  be a closed set containing the supports of the Borel measures  $(dd^c u_1)^n$  and  $(dd^c u_2)^n$ . If the sets  $\{u_1 = 0\}$  and  $\{u_2 = 0\}$  differ from  $S$  by at most a pluripolar set, and if  $0 \leq u_1 \leq u_2$  on  $\Omega$ , then  $(dd^c u_1)^n \leq (dd^c u_2)^n$ .*

PROOF. Without loss of generality, we may assume that  $B = \{|z| < 1\}$  is a compact subset of  $\Omega$ . Let  $u_j^\varepsilon = u_j * \chi_\varepsilon$  be smoothings decreasing to  $u_j$  as  $\varepsilon \downarrow 0$ . Let  $v_j$  denote the restriction of  $u_j$  to  $\partial B$ . Similarly, let  $v_j^\varepsilon$  denote the restriction of  $u_j^\varepsilon + j \cdot \varepsilon$  to  $\partial B$ , so that  $0 < v_1^\varepsilon < v_2^\varepsilon$  on  $\partial B$ . Given a subset  $S$  of the unit ball  $B$  and a function  $v$  on  $\partial B$ , let  $\mathcal{F}(S, v)$  denote the family of all psh functions  $w$  on the unit ball  $B$  such that  $w \leq 0$  on  $S$ , and  $\limsup_{\zeta \rightarrow z} w(\zeta) \leq v(z)$  for all  $z \in \partial B$ . Let

$$S_\delta^\varepsilon = \{x \in \mathbf{R}^n \cap \Omega : \text{dist}(x, \partial\Omega) \geq \varepsilon \text{ and } \text{dist}(x, S) \leq \delta\}$$

and let

$$U_{\delta,j}^\varepsilon(z) := \sup\{w(z) : w \in \mathcal{F}(S_\delta^\varepsilon, v_j^\varepsilon)\}, \quad z \in B.$$

The set  $S_j^\varepsilon$  is easily seen to be regular, so  $U_{\delta,j}^\varepsilon$  is psh and continuous on the closure  $\bar{B}$  of  $B$ . It therefore follows from a theorem of Levenberg [5] that

$$(dd^c U_{\delta,1}^\varepsilon)^n \leq (dd^c U_{\delta,2}^\varepsilon)^n.$$

As  $\delta \downarrow 0$ , the functions  $U_{\delta,j}^\varepsilon$  increase almost everywhere to the functions  $[U_{S,j}^\varepsilon]^*$ , the uppersemicontinuous regularizations of the envelope functions for the families  $\mathcal{F}(S, v_j^\varepsilon)$ . Because the operator  $(dd^c)^n$  is continuous on monotone limits of locally bounded psh functions [2], we can pass to the limit in the last inequality to obtain

$$(dd^c U_{S,1}^\varepsilon)^n \leq (dd^c U_{S,2}^\varepsilon)^n.$$

Now, let  $\varepsilon \downarrow 0$ . The functions  $U_{S,j}^\varepsilon$  clearly decrease. And, they decrease to the uppersemicontinuous regularization of the envelope function of the family  $\mathcal{F}(S, v_j)$ ,  $U_{S,j}^*$ . Thus, we have

$$(dd^c U_{S,1}^*)^n \leq (dd^c U_{S,2}^*)^n.$$

But, it is evident that  $U_{S,j}^* \geq u_j$ . On the other hand, we have  $u_j \geq U_{S,j}^*$  by the domination principle (see [2, Corollary 4.5]). Thus,  $u_j = U_{S,j}^*$  which proves the lemma.

PROPOSITION 2.2. *Let  $E_1, \dots, E_n$  be compact subsets of  $\mathbf{C}$ . Then*

$$\lambda_{E_1 \times \dots \times E_n} = \lambda_{E_1} \otimes \dots \otimes \lambda_{E_n}.$$

PROOF. By a result of Siciak [13] we know that

$$L_{E_1 \times \dots \times E_n}^*(z_1, \dots, z_n) = \max_{1 \leq j \leq n} L_{E_j}^*(z_j).$$

Let us assume that  $E_j$  is a Jordan domain in  $\mathbf{C}$  with real analytic boundary. Then  $L_{E_j}$  may be extended from  $\mathbf{C} \setminus E_j$  to a function  $\tilde{L}_j$  which is harmonic in a neighborhood of  $\partial E_j$ . Thus we may write

$$L_{E_1 \times \dots \times E_n}^*(z_1, \dots, z_n) = \max_{1 \leq j \leq n} \{\tilde{L}_1, \dots, \tilde{L}_n, 0\}.$$

However,

$$(dd^c L_{E_1 \times \dots \times E_n})^n = d^c \tilde{L}_1 \wedge \dots \wedge d^c \tilde{L}_n|_M$$

where  $M = \{\tilde{L}_1 = \dots = \tilde{L}_n = 0\}$  is given the orientation of  $d\tilde{L}_1 \wedge \dots \wedge d\tilde{L}_n$  (see the remark on p. 7 of [1]). Since

$$d^c \tilde{E}_j = \frac{\partial}{\partial n_j} (\tilde{L}_j) d\sigma_j$$

where  $d\sigma_j$  is the arclength measure of  $\partial E_j$  and  $dn_j$  is the outward normal, we see that  $d^c \tilde{L}_j$  may be naturally identified with  $\lambda_{E_j}$ . This proves the proposition in the smooth case.

For the general case, we take smooth  $E_j^\varepsilon$ ,

$$E_j \subset E_j^\varepsilon \subset \{z \in \mathbf{C} : \text{dist}(z, E_j) \leq \varepsilon\}.$$

If we take  $E_j^\varepsilon \subset E_j^\delta$  for  $\varepsilon \leq \delta$ , then  $L_{E_1^\varepsilon \times \dots \times E_n^\varepsilon}^*$  increases a.e. to  $L_{E_1 \times \dots \times E_n}^*$ . Thus, by the convergence theorem of [2],

$$\lim_{\varepsilon \rightarrow 0} \lambda_{E_1^\varepsilon \times \dots \times E_n^\varepsilon} = \lambda_{E_1 \times \dots \times E_n},$$

and the proposition is proved.

In the case of the interval  $[-1, +1]$ , it is well known that

$$\lambda_{[-1,+1]} = (2 \cdot dx)/(1 - x^2)^{1/2},$$

and so a product of intervals,

$$E = \{-\delta \leq x_1 \leq \delta, \dots, -\delta \leq x_n \leq \delta\}$$

has extremal measure

$$\lambda_E = \frac{2^n \cdot dx_1 \dots dx_n}{\delta^n \cdot [1 - (x_1/\delta)^2]^{1/2} \dots [1 - (x_n/\delta)^2]^{1/2}}.$$

PROOF OF THEOREM 1.1. If  $r$  is the diameter of  $K$ , then without loss of generality we may assume that  $K \subset (r \cdot I)^n \subset \mathbf{R}^n$  where  $r \cdot I = [-r, +r]$ . Thus,  $L_K^* \geq L_{(r \cdot I)^n}^*$ . If  $x_0 \in \text{int}(K)$ , then  $\text{dist}(x_0, \partial K) = \eta > 0$ . Without loss of

generality, we can assume that  $x_0 = 0$ . Let  $\Omega = \{z \in \mathbf{C}^n : |z| < \eta\}$  and  $S = \Omega \cap \mathbf{R}^n$ . It follows that  $S = \{L_K^* = 0\} \cap \Omega = \{L_{(r,I)^n}^* = 0\} \cap \Omega$  and so by Lemma 2.1,

$$\begin{aligned} \lambda_K|_S &\geq \frac{2^n \cdot dx_1 \cdots dx_n}{r^n \cdot [1 - (x_1/r)^2]^{1/2} \cdots [1 - (x_n/r)^2]^{1/2}} \Big|_S \\ &\geq \frac{2^n \cdot dx_1 \cdots dx_n}{r^n} \Big|_S. \end{aligned}$$

For the other inequality, we note that  $\{x: -\eta/\sqrt{n} \leq x_j \leq +\eta/\sqrt{n}\} \subset \text{int}(K)$ , so again by Lemma 2.1

$$\frac{[2\sqrt{n}]^n \cdot dx_1 \cdots dx_n}{(\text{dist}(x, \partial K))^{n/2}} \geq \lambda_K|_{\text{int}(K)}.$$

This completes the proof.

PROOF OF THEOREM 1.3. First let us assume that  $\partial K$  is real analytic. Thus, it is pluripolar and  $\lambda_K(\partial K) = 0$ . Since  $\partial K$  is  $C^2$ , we may touch any  $x_0 \in K$  by an internally tangent ball of fixed radius  $r$ . Thus, they may use an internally tangent ball and an externally tangent cube to obtain (1.5) as in Theorem 1.1, because there is no mass on  $\partial K$ .

If  $\partial K$  is merely  $C^2$ , we may consider a sequence  $\{K_j\}$  of convex, symmetric domains with real analytic boundary such that  $\partial K_j$  approaches  $\partial K$  in  $C^2$ . Thus, we obtain (1.5) for each  $\lambda_{K_j}$ . Further, the constants  $c_1$  and  $c_2$  depend only on the diameter of  $K_j$  and the radius of an internally tangent ball. Thus, we may be chosen independent of  $j$ . It follows from the dominated convergence theorem, then, that  $\lambda_K$  puts no mass on  $\partial K$ , so that (1.5) holds.

**3. Representation of  $(dd^c L_K)^n$ .** In this section we will use Lundin's representation (1.4) of  $L_K$  and give the basic form of  $(dd^c L_K)^n$  for convex sets in  $\mathbf{R}^n$ . We first give a geometrical construction of an auxiliary function that will be used to describe  $(dd^c L_K)^n$ .

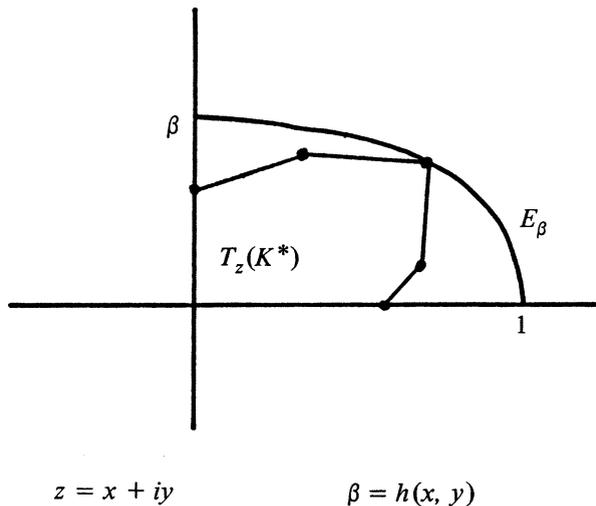


FIGURE 1

For  $\beta > 0$ , let  $E_\beta$  denote the ellipse

$$(3.1) \quad E_\beta: \quad s^2 + t^2/\beta^2 = 1.$$

Let  $K^*$  denote the dual convex set to  $K$ . That is,

$$K^* = \{\eta \in \mathbf{R}^n: x \cdot \eta \leq 1 \text{ for all } x \in K\}.$$

It is readily checked that when  $K$  is convex and symmetric about the origin, so is  $K^*$  and

$$K^* = \{\eta = r \cdot \alpha \in \mathbf{R}^n: |\alpha| = 1, r \leq 1/\rho(\alpha)\}.$$

We will use the function  $h = h_K$  defined for  $z = x + i \cdot y$  with  $x \in \text{int}(K)$ ,  $y \in \mathbf{R}^n$  by

$$(3.2) \quad h_K(z) = h(x, y) = \inf\{\beta: T_z(K^*) \subset E_\beta\}$$

(see Figure 1) where  $T_z$  is the linear map from  $\mathbf{R}^n$  to  $\mathbf{R}^2$  defined by  $\zeta = s + i \cdot t = T_z(\xi) = z \cdot \xi$ , or, more precisely by

$$\begin{bmatrix} s \\ t \end{bmatrix} = T_z(\xi) = \begin{bmatrix} x_1 & & x_n \\ & \cdots & \\ y_1 & & y_n \end{bmatrix} \cdot \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}.$$

The function  $h(x, y) = h_K(x, y)$  has the following properties.

PROPOSITION 3.1. *If  $K^*$  is the dual convex set to the compact convex set  $K$ , and if  $K^*$  is compact, then for  $h(x, y)$  defined in (3.2) we have*

- (i)  $h(x, y)$  is continuous for  $x \in \text{int}(K)$  and  $y \in \mathbf{R}^n$ ;
- (ii)  $h(x, \lambda \cdot y) = \lambda \cdot h(x, y)$  for  $\lambda > 0$ ;
- (iii)  $y \rightarrow h(x, y)$  is convex on  $\mathbf{R}^n$  for each fixed  $x \in \text{int}(K)$ ;
- (iv)  $y \rightarrow h(x, y)$  is the support function of the convex hull of the set,  $S_x(K^*)$ , where  $S_x$  is the (nonlinear) transformation of  $\mathbf{R}^n$  defined by

$$S_x(\xi) = \xi/(1 - (x \cdot \xi)^2)^{1/2}.$$

PROOF. First, note that  $h$  is defined and finite when  $x \in \text{int}(K)$  because then  $\xi \cdot x < 1$  for all  $\xi \in K^*$ , so the set  $T_z(K^*)$  is contained in  $E_\beta$  for some  $\beta > 0$ . That  $h$  is continuous is clear because  $K^*$  is compact. Assertion (ii) follows from the obvious scaling property

$$T_{x+i \cdot \lambda \cdot y}(K^*) \subset E_\beta \quad \text{iff} \quad T_{x+i \cdot y}(K^*) \subset E_{\beta/\lambda}.$$

Both assertions also follow directly from (iv), as does assertion (iii). We will therefore prove assertion (iv).

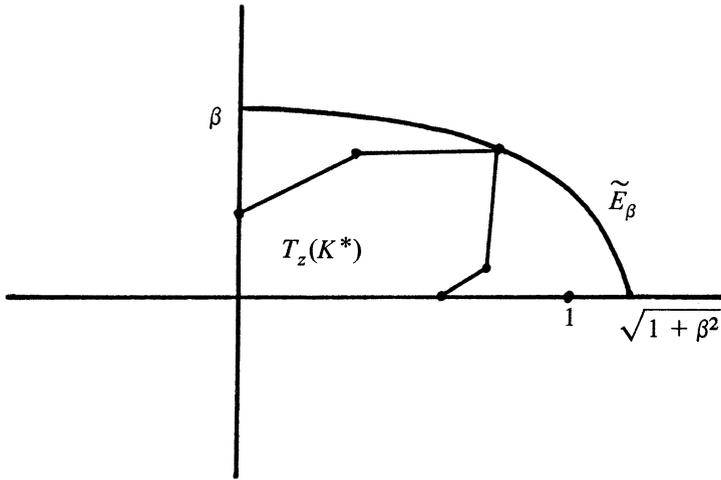
The number  $\beta = h(x, y)$  is defined as the smallest number  $\beta \geq 0$  such that

$$(x \cdot \xi)^2 + (y \cdot \xi)^2/\beta^2 \leq 1, \quad \xi \in K^*,$$

with equality holding for at least one  $\xi$ . Solving the inequality for  $\beta$  yields

$$(3.3) \quad \beta \geq y \cdot \xi / (1 - [x \cdot \xi]^2)^{1/2} = y \cdot S_x(\xi), \quad \xi \in K^*,$$

with equality for at least one  $\xi$ . In other words,  $h(x, y)$  is the maximum of the right-hand side of (3.3) over  $K^*$ , which is exactly assertion (iv). This completes the proof of the proposition.



$$z = x + iy \quad \log(\beta + \sqrt{1 + \beta^2}) = L_K(z)$$

FIGURE 2

For  $f$  a convex function of  $y \in \mathbf{R}^n$ , let  $M(f)$  denote the real Monge-Ampere operator applied to  $f$ . That is, if  $f$  is smooth, then  $M(f)$  is the determinant of the Hessian of  $f$ ,

$$M(f) = \det[\partial^2 f / \partial y_j \partial y_k].$$

The operator extends by continuity to be defined as a nonnegative Borel measure on the class of convex functions. The measure of a Borel set  $E$  is the volume of the set of all direction vectors  $\lambda$  of hyperplanes,  $\lambda \cdot y + \text{const.}$ , which support the graph of  $f$  at a point  $x \in E$ . See e.g. [11].

We will use two facts about this operator. First, if the convex function  $f(y)$  defined on  $\mathbf{R}^n$  is thought of as a psh function on  $\mathbf{C}^n$ , then we have the formula

$$(dd^c f)^n = n! \cdot M(f) \otimes dx.$$

Second, if  $f$  is the function  $y \rightarrow h(x, y)$ , then because  $h$  is homogeneous of degree 1,  $M(f)$  must be supported at the origin. Further, this set of direction vectors which support the graph of the function is clearly the convex hull of the set  $S_x(K^*)$ . Thus, if  $x_0$  is a fixed point of  $\text{int}(K)$ , then

$$(3.4) \quad (dd^c h(x_0, y))^n = n! \cdot \text{vol}\{\text{ch } S_x(K^*)\} \cdot dx.$$

We next show the relation between the function  $h = h_K$  and the extremal function  $L_K$ .

**THEOREM 3.2.** *Let  $K$  be a convex compact symmetric set in  $\mathbf{R}^n$  with  $0 \in \text{int}(K)$ . Then*

$$(3.5) \quad \lim_{\varepsilon \rightarrow 0^+} \frac{L_K(x + i \cdot \varepsilon \cdot y)}{\varepsilon} = h_K(x, y)$$

*uniformly on compact subsets of  $\text{int}(K) \oplus i\mathbf{R}^n \subset \mathbf{C}^n$ . In particular, if  $x_0 \in \text{int}(K)$  and  $\varepsilon > 0$  is given, there exists  $\delta > 0$  such that, for all  $|x - x_0| \leq \delta$ ,  $|y| \leq \delta$ ,*

$$(3.6) \quad h_K(x_0, y) - \varepsilon \cdot |y| \leq L_K(x_0 + i \cdot y) \leq h_K(x_0, y) + \varepsilon \cdot |y|.$$

PROOF. Recall that the level sets of the function

$$g(\zeta) := |\zeta + \sqrt{\zeta^2 - 1}|$$

are the confocal ellipses

$$\tilde{E}_\beta: \quad s^2/(1 + \beta^2) + t^2/\beta^2 = 1, \quad \zeta = s + i \cdot t, \quad \beta > 0,$$

and  $g = \beta + \sqrt{1 + \beta^2}$  on  $\tilde{E}_\beta$ . According to Lundin's formula, (1.4), choosing  $\beta = \beta(z)$  such that  $L_K(z) = \log[\beta + \sqrt{1 + \beta^2}]$  is the same as

$$\beta + \sqrt{1 + \beta^2} = \sup\{g(z \cdot \xi/\rho(\xi)) : |\xi| = 1\}.$$

Equivalently,  $\beta = \beta(z)$  is defined by

$$(3.7) \quad \beta(z) = \inf\{\beta : T_z(K^*) \subset \tilde{E}_\beta\}$$

(see Figure 2). Note also that  $d\beta/dg = 1$  at  $g = 1, \beta = 0$ . The limit in (3.5) is therefore equivalent to calculating

$$(3.8) \quad \lim_{\epsilon \rightarrow 0^+} \frac{\beta(x + i\epsilon y)}{\epsilon}.$$

We will calculate this limit by comparing it with the function  $h$  defined in terms of the ellipses  $E_\beta$  in (3.2). Notice that  $E_\beta \subset \tilde{E}_\beta$  for all  $\beta > 0$  so that for all  $z = x + iy \in C^n$ ,

$$(3.9) \quad h(x, y) \geq \beta(x + iy).$$

On the other hand, if  $x \in \text{int}(K)$  then for sufficiently small  $y \in \mathbf{R}^n$ , the first components,  $s = x \cdot \xi$ , of  $T_z(K^*)$  are contained in  $[-r, +r]$  for some  $r < 1$ . However, if  $\lambda \geq 1$ , then  $\lambda \cdot E_\beta \cap \{(s, t) : |s| \leq r\} \subset E_{a \cdot \beta}$  for  $a \geq [(\lambda^2 - r^2)/(1 - r^2)]^{1/2}$ . Further,  $\tilde{E}_\beta \subset \sqrt{1 + \beta^2} \cdot E_\beta$ . Consequently, if  $T_z(K^*) \subset \tilde{E}_\beta$ , then

$$T_z(K^*) = T_z(K^*) \cap \{(s, t) : |s| \leq r\} \subset E_{a \cdot \beta}$$

for  $a = [1 + \beta^2/(1 - r^2)]^{1/2} \approx 1 + \text{const} \cdot \beta^2$ . That is,

$$(3.10) \quad \beta(x + iy) \geq h(x, y) \cdot [1 + \text{const} \cdot \beta^2]^{-1/2}.$$

From (3.9), (3.10), and the definition (3.2) of  $h$ , it therefore follows that the limit in (3.5) is equal to  $h(x, y)$ . All the set inclusions used above vary continuously with  $x$ , so it is easy to see that the limit is uniform on compact subsets of  $\text{int}(K) \oplus i\mathbf{R}^n$ . This completes the proof.

PROOF OF THEOREM 1.2. First we note that it may be argued as in the proof of Theorem 1.3 to see that  $\lambda_K$  puts no mass on  $\partial K$ . Now the formula for  $\lambda(x)$  follows directly from formula (3.4), inequality (3.6) of Theorem 3.2, and the comparison theorem of §2. The integral formula may be seen as follows. The set  $S_x(K^*)$  is given by  $\{\eta = r \cdot \xi : 0 \leq r \leq ([\rho_K(\xi)]^2 - [x \cdot \xi]^2)^{-1/2}\}$ , and so the integral formula gives the volume of this region in polar coordinates.

REMARK. We note that  $\text{ch } S_x(K^*)$  is the dual of the set

$$\{y \in \mathbf{R}^n : (y \cdot \xi)^2 + (x \cdot \xi)^2 \leq \rho_K^2(\xi) \text{ for all } \xi \in \mathbf{R}^n\}.$$

Since the total mass of  $\lambda_K$  is independent of  $K$ , it follows that the constant

$$c_n = \int \text{vol}(\text{ch } S_x(K^*)) \, dx$$

is the same for all symmetric convex sets  $K \subset \mathbf{R}^n$ . [ $c_n = (2\pi)^n/n!$ .] It would be interesting to know if there is a purely geometrical proof of this fact.

**4. Asymptotic behavior of  $\lambda_K$ .** Let  $K \subset \mathbf{R}^n$  be smoothly bounded, convex, and symmetric. We will discuss the asymptotic behavior of  $\lambda(x)$ , and, in the process, prove Theorem 1.4. By Theorem 1.2, we will need to estimate the asymptotic behavior of the volume of the convex hull of  $S_x(K^*)$  as  $x$  approaches  $x_0 \in \partial K$ .

First we establish some geometric notation. Let  $\eta_0$  be a normal vector to  $\partial K$  at  $x_0$ , normalized so that  $\eta_0 \cdot x_0 = 1$ . Any ellipsoid symmetric about  $0 \in \mathbf{R}^n$  and tangent to  $\partial K$  at  $x_0$  has the form

$$(4.1) \quad (\eta_0 \cdot x)^2 + R(x) \leq 1$$

where  $R(x)$  is a positive semidefinite quadratic form on  $\mathbf{R}^n$  such that  $R(x_0) = 0$ . By abuse of notation,  $R$  may be identified with a positive semidefinite quadratic form on the hypersurface  $x_0^\perp$  orthogonal to  $x_0$ .

We define the *ellipsoidal hull* of  $K$  at  $x_0$ , written  $\text{EH}(K, x_0)$ , as the intersection of all the ellipsoids of the form (4.1) which contain  $K$ . Similarly, we define the *ellipsoidal core* of  $K$  at  $x_0$ , written  $\text{EC}(K, x_0)$ , as the union of all of the ellipsoids of the form (4.1) which are contained in  $K$ . The ellipsoidal hull and core are related as follows, which is proved in [4].

**PROPOSITION 4.1.** *If  $K$  and  $K^*$  are both smoothly bounded, then  $\text{EC}(K, x_0)^* = \text{EH}(K^*, \eta_0)$ .*

Next, we relate the ellipsoidal hull to the mapping

$$S_{x_0}(\eta) = \eta \cdot (1 - (x_0 \cdot \eta)^2)^{-1/2}.$$

**LEMMA 4.2.**  $\text{ch } S_{x_0}(K^*) = S_{x_0}\{\text{EH}(K^*, \eta_0)\}$ .

**PROOF.** We note that  $\xi = S_{x_0}(\eta)$  carries the degenerate ellipsoid  $(\xi_0 \cdot \eta)^2 + (x_0 \cdot \eta)^2 \leq 1$  to the ‘‘strip’’  $|\xi_0 \cdot \xi| \leq 1$ . If  $\xi_0 \perp \eta_0$ , then these degenerate ellipsoids are tangent to  $K^*$  at  $\eta_0$ , and thus the set of these ellipsoids with  $\xi_0 \perp \eta_0$  generates the ellipsoidal hull. Thus,

$$S_{x_0}\{\text{EH}(K^*, \eta_0)\} = \bigcap \{|\xi_0 \cdot \xi| \leq 1\}$$

where the  $\xi_0$  are chosen so that  $\xi_0 \perp \eta_0$  and the corresponding degenerate ellipsoid contains  $K^*$ . Thus,  $\{|\xi_0 \cdot \xi| \leq 1\} \supset S_{x_0}(K^*)$ , and so  $\text{ch } S_{x_0}(K^*) \subset S_{x_0}\{\text{EH}(K^*, \eta_0)\}$ .

On the other hand,  $x_0 \cdot \eta_0 = 1$ , and so  $S_{x_0}(K^*)$  contains the line  $\{t\eta_0 : t \in \mathbf{R}\}$ . It follows that  $\text{ch } S_{x_0}(K^*)$  is an intersection of half-spaces of the form  $\{\xi_0 \cdot \xi \leq 1\}$  with  $\xi_0 \cdot \eta_0 = 0$ . This proves the lemma.

**LEMMA 4.3.** *For  $x = tx_0$ ,  $0 < t < 1$ , let us set*

$$\omega_t = (\text{ch } S_x(K^*)) \cap (\eta_0^\perp).$$

*Then  $\omega_t$  increases to  $S_{x_0}(\text{EH}(K^*, \eta_0)) \cap (\eta_0^\perp)$  as  $t$  increases to 1.*

**PROOF.** This follows from Lemma 4.2 since  $\omega_t$  is clearly increasing in  $t$ .

Now we extend  $\eta_0$  to an orthogonal set of coordinate axes,  $(\eta_0, \eta_1, \dots, \eta_{n-1})$  such that  $\eta_1, \dots, \eta_{n-1}$  are the principal curvature directions of  $\partial K$  at  $x_0$ , and we let  $\kappa_1, \dots, \kappa_{n-1}$  be the principal curvatures of  $\partial K$  at  $x_0$ . Let  $(\xi_0, \dots, \xi_{n-1})$  denote coordinates with respect to these new axes, and let  $W_t: \mathbf{R}^n \rightarrow \mathbf{R}^n$  be the linear transformation given by

$$W_t(\xi) = ((1 - t^2)^{1/2} \cdot \xi_0, \xi_1, \dots, \xi_{n-1}).$$

Let  $Q$  be the quadratic form on  $\mathbf{R}^n$  which in  $\xi$ -coordinates is

$$Q(\eta) = \sum_{j=1}^{n-1} \frac{\xi_j^2}{\kappa_j}$$

and define the ellipsoid

$$(4.2) \quad E = \{\eta \in \mathbf{R}^n : \xi_0^2 + Q(\xi) \leq 1\}.$$

LEMMA 4.4.

$$(4.3) \quad \lim_{t \rightarrow 1} W_t[S_{tx_0}(K^*) \cap \{\mathbf{R}^n \setminus \eta_0^\perp\}] = E.$$

PROOF. Since  $\pm\eta_0$  are the only points of  $\partial K^*$  which get mapped to infinity under  $S_{x_0}$ , it follows that the left-hand side of (4.3) is determined only by a neighborhood of  $\pm\eta_0$  in  $\partial K^*$ . That is, if  $\tilde{\eta} \in \partial K^*$  and  $\tilde{\eta} \neq \pm\eta_0$ , then  $\lim_{t \rightarrow 1} W_t S_{tx_0}(\tilde{\eta}) \in \eta_0^\perp$ . Thus, as  $t \rightarrow 1$ , we must consider smaller and smaller neighborhoods in order to find the left-hand side of (4.3).

Consider the ellipsoid which is symmetric about 0 and which is tangential to  $\partial K^*$  at  $\pm\eta_0$ . The normal to  $K^*$  at  $\eta_0$  is  $x_0$ , so this ellipsoid has the form  $(x_0 \cdot \eta)^2 + R(\eta) \leq 1$ , as in (4.1). Since  $\eta_0 \cdot x_0 = 1$  and the curvatures of  $\partial K^*$  at  $\eta_0$  are  $\kappa_1^{-1}, \dots, \kappa_{n-1}^{-1}$ , it follows that this ellipsoid coincides with

$$\tilde{E} = \{\eta \in \mathbf{R}^n : (\eta \cdot x_0)^2 + Q(\eta) \leq 1\}.$$

Without loss of generality, we may assume that  $\partial K^*$  coincides with  $\partial \tilde{E}$  in a small neighborhood of  $\pm\eta_0$ . Now it suffices to show that

$$\lim_{t \rightarrow 1} W_t S_{tx_0}(\tilde{E}) = E.$$

To see this we consider first the ellipsoid

$$\tilde{E}_c = \{\eta : (\eta \cdot x_0)^2/c^2 + Q(\eta) \leq 1\}.$$

We see that

$$S_{tx_0}^{-1}(E_c) = \{\eta : (\eta \cdot x_0)^2/c^2 + Q(\eta) \leq 1 - t^2(\eta \cdot x_0)^2\}$$

is equal to  $\tilde{E}$  if and only if  $c^2 = (1 - t^2)^{-1}$ . Thus

$$W_t[S_{tx_0}(\tilde{E})] = W_t[\tilde{E}_c]$$

with  $c^2 = (1 - t^2)^{-1}$ , and so we have, with  $\eta = \xi_0\eta_0 + \dots + \xi_{n-1}\eta_{n-1}$ ,

$$W_t[S_{tx_0}(\tilde{E})] = \left\{ \eta \in \mathbf{R}^n : (1 - t^2) \left[ \frac{\xi_0}{\sqrt{1 - t^2}} + \sum_{j=1}^{n-1} \xi_j(\eta_j \cdot x_0) \right]^2 + Q(\eta) \leq 1 \right\}.$$

Taking the limit as  $t \rightarrow 1$ , we obtain  $E$ .

Now we define  $\omega = (\eta_0^\perp) \cap \text{EH}(K^*, \eta_0)$ ,  $\Omega = \text{ch}(\omega \cup E)$  (see Figure 3), and

$$c(x_0) = n\text{-dimensional volume of } \Omega.$$

LEMMA 4.5. *With the above notation*

$$\text{volume}(\text{ch } S_{tx_0}(K^*)) \approx \frac{|\eta_0|}{\sqrt{1 - t^2}} \cdot \text{vol}(\Omega)$$

holds as  $t \rightarrow 1$ .

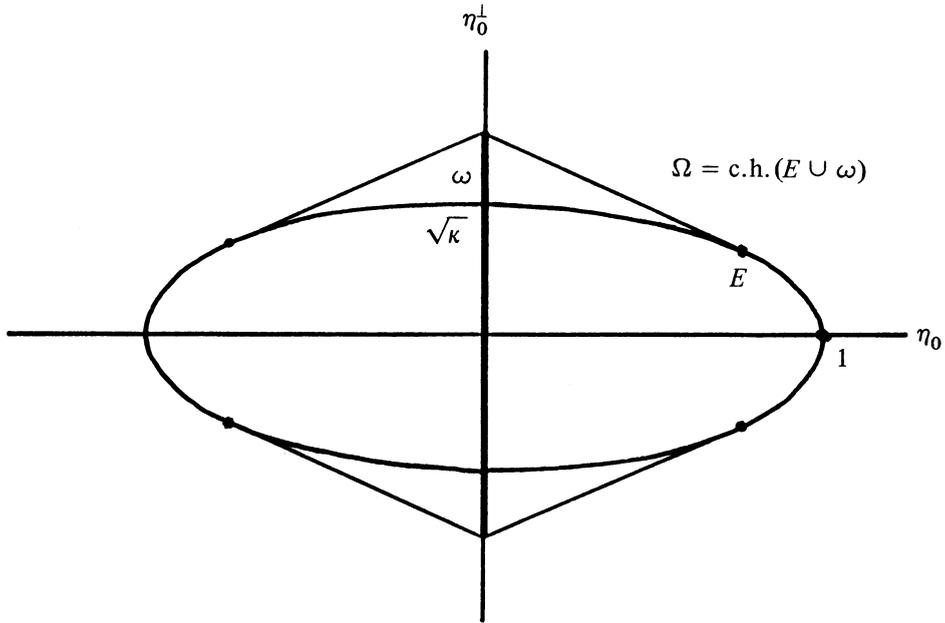


FIGURE 3

PROOF. We will show that

$$(4.4) \quad \lim_{t \rightarrow 1} W_t(\text{ch } S_{tx_0}(K^*)) = \text{ch}(\omega \cup E_1).$$

It follows, then, that the volume of  $\text{ch } S_{tx_0}(K^*)$  is  $(1 - t^2)^{-1/2}$  times

$$|\eta_0| \cdot \text{vol}(\text{ch}(\omega \cup E_1)),$$

where the factor  $|\eta_0|$  enters because  $|\eta_0|$  is the unit of length on the  $\eta_0$ -axis. Thus, the lemma follows from (4.4).

To show (4.4), we use the notation  $\omega_t = (\eta_0^\perp) \cap S_{tx_0}(K^*)$ . From Lemma 4.3 we have

$$W_t(\text{ch } S_{tx_0}(K^*) \cap (\eta_0^\perp)) \supset \omega_t,$$

and so

$$\lim_{t \rightarrow 1} W_t(\text{ch } S_{tx_0}(K^*) \cap (\eta_0^\perp)) \supset \text{ch}(\omega_t \cup E).$$

Thus,  $\supset$  holds in (4.4).

For the reverse inclusion, we note that for any  $\varepsilon > 0$ ,

$$\begin{aligned} S_{tx_0}(K^*) &\subset S_{x_0}(K^* \cap \{|\eta - \eta_0| > \varepsilon\}) \cup S_{tx_0}(K^* \cap \{|\eta - \eta_0| > \varepsilon\}) \\ &= \bar{K}_1 \cup \bar{K}_2(t). \end{aligned}$$

Thus,

$$\begin{aligned} \lim_{t \rightarrow 1} W_t(\text{ch } S_{tx_0}(K^*)) &\subset \lim_{t \rightarrow 1} W_t(\text{ch}(\bar{K}_1 \cup \bar{K}_2(t))) \\ &= \lim_{t \rightarrow 1} \text{ch}(W_t \bar{K}_1 \cup W_t \bar{K}_2(t)) \subset \text{ch}(\omega \cup E) = \Omega \end{aligned}$$

where the next-to-last inclusion follows from Lemma 4.4.

PROOF OF THEOREM 1.4. We will show that the quantity

$$(4.5) \quad c(x_0) = (\text{vol}(\Omega)) / \sqrt{2 \cdot \rho_K(x_0)}$$

is the asymptotic value in Theorem 1.4. First, we note that, with the notation of Theorem 1.4,  $|x - \bar{x}| = \text{dist}(x, \partial K)$  and for fixed  $x_0 \in \partial K$

$$\text{dist}(tx_0, \partial K) \approx |1 - t|x_0 \cdot \frac{\eta_0}{|\eta_0|} = \frac{1 - t}{|\eta_0|}$$

holds for  $t \rightarrow 1$ . Now by Lemma 4.5 we have, with  $\Omega = \Omega_{x_0}$ ,

$$\begin{aligned} \lambda(tx_0) &= \text{vol ch } S_{tx_0}(K^*) \approx \frac{|\eta_0| \text{vol}(\Omega)}{\sqrt{1 - t^2}} \approx \frac{|\eta_0| \text{vol}(\Omega)}{\sqrt{2} \sqrt{1 - t}} = \frac{|\eta_0|^{1/2} \text{vol}(\Omega)}{\sqrt{2} (\text{dist}(tx_0, \partial K))^{1/2}} \\ &= \frac{\text{vol}(\Omega_{x_0})}{(2 \cdot \rho_K(x_0) |tx_0 - \bar{x}|)^{1/2}} \approx \frac{\text{vol}(\Omega_{x_0})}{(2 \cdot \rho_K(\bar{x}) \cdot |tx_0 - \bar{x}|)^{1/2}} \end{aligned}$$

since  $\eta_0$  was normalized so that  $x_0 \cdot \eta_0 = 1$ ; i.e.,  $\rho_K(x_0) = |\eta_0|^{-1}$ .

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