TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 294, Number 2, April 1986

GLOBAL BOUNDEDNESS FOR A DELAY DIFFERENTIAL EQUATION[1]

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ABSTRACT. The inequality $(\partial_t u - \Delta u)(t,x) \leq u(t,x)(1 - u(t-\tau,x))$ is investigated. It is shown that nonnegative solutions of the Dirichlet problem in a bounded interval remain bounded as time goes to infinity, whereas in a more dimensional domain, in general, this holds only if the delay is not too large.

0. Introduction. The subject of this article is the population dynamics governed by Hutchinson's equation, i.e. the net reproduction rate is modelled by $f(u(t), u(t-\tau)) = u(t)(1-u(t-\tau))$. Here all constants except the delay are normalized to one. For a detailed description see [**2**]. The question is, does such a law by itself limit the total size of a population if that population is allowed to diffuse.

Suppose that we ask the question a little bit differently, demanding that the population u reproduce not with the exact rate f but rather with a rate not exceeding f, unspecified death terms possibly also existing. Then there is the complete answer (for slightly more general f).

THEOREM. *Suppose u fulfills*

$$(0)\qquad \begin{gathered} u \geq 0, \partial_t u - \Delta u \leq f(u, u(\cdot - \tau, \cdot)) \quad \text{in } \mathbf{R}^+ \times \Omega, \\ u = u_0 \quad \text{in } [-\tau, 0] \times \Omega, \end{gathered}$$

where f is such that

$$f(u,v) < u, \quad \lim_{\lambda\to\infty} \frac{1}{\lambda} f(\lambda u, \lambda v) = -\infty \qquad \text{f.a. } u, v \in \mathbf{R}^+.$$

(a) *If $\Omega = [0, L]$ and on $\partial\Omega$, $u <$ const or $\partial_v u <$ const, then there exists $K(f, L, \text{const})$ independent of u_0 with*

$$\varlimsup_{t\to\infty} \|u(t,\cdot)\|_\infty < K.$$

(b) *If $n = \dim(\Omega) \geq 2$ and $u < g$ in $\partial\Omega$, $g \in H_2^1(\Omega)$, there exists $\tau(n)$ such that for $\tau < \tau(n)$ and $\int_{-\tau}^0 \int_\Omega u_0^2 < \infty$ there is again a constant K independent of u_0 such that*

$$\varlimsup_{t\to\infty} \int_\Omega u^2(t,\cdot) \leq K.$$

Received by the editors June 10, 1985.
1980 *Mathematics Subject Classification*. Primary 35B35; Secondary 35K55, 34G20, 34K25.

[1]Diese Arbeit ist mit Unterstützung der DFG im SFB 123 und über ein Forschungsstipendium entstanden.

(b′) *If in the above case* $\partial_v u < g$ *holds in* $\partial\Omega$ *instead of* $u <$ const, *if* $g \in L_2(\partial\Omega)$ *and* Ω *is locally uniformly Lipschitz—i.e. there is* $\alpha > 0$ *such that for* $x, y \in \partial\Omega$ *with* $|x - y| < \alpha$, $v_x \cdot v_y \geq \alpha - 1$ *holds—then there exists* $\tau(\alpha, n)$ *such that for* $\tau < \tau(\alpha, n)$ *one again gets a constant* K *independent of* u_0 *such that*

$$\int_{-\tau}^{0}\int_{\Omega} u_0^2 < \infty \quad \textit{implies} \quad \overline{\lim_{t\to\infty}} \int_{\Omega} u^2 < K.$$

(c) *On the other hand, given* $n \geq 2$, $\gamma < 1$, $f(u, v) = u(1 - v)$ *there are* $\tau, R > 0$ *and solutions* u *of* (0) *with* $\operatorname{supp} u \Subset B_R$ *such that*

$$\|u(t, \cdot)\| > e^{\gamma t}. \qquad \square$$

It should be emphasized that the counterexample (c) shows to what extent the comparison principle fails, since for radially symmetric solutions of (0) in B_R the one-dimensional result applies, as becomes clear from the proof. So all symmetric solutions stay bounded; still there is also an unbounded solution insofar as the situation differs completely from the otherwise similar equation in [**3**].

The same kind of theorem can be stated for distributed delays, as becomes clear from the first step of the proof.

The proofs of parts (a) and (b) of the theorem are in two steps. First, in Lemma 1 boundedness for equation (0) is derived once one knows decay estimates for the special equation (1) below, where the support is moving according to the law

$$u(t, x) \cdot u(t - \tau, x) \equiv 0.$$

The appropriate decay estimates are then verified in Lemmas 2 and 3 respectively. The counterexample is also constructed as a function with moving support in Lemma 4.

ACKNOWLEDGEMENTS. I would like to thank R. Pego and B. Fiedler for helpful discussions.

1. Reduction to a problem with undetermined free boundary. Here the problem of getting bounds on solutions of (1) is reduced to getting decay estimates for a simpler equation.

LEMMA 1. (a) *Suppose* Ω *uniformly locally Lipschitz*; *i.e. there exists* $\varepsilon > 0$ *such that for* $x, y \in \partial\Omega$ *with* $|x - y| < \varepsilon$, $v_x \geq \varepsilon - 1$ *holds*; *let* $\tau \in \mathbf{R}^+$ *and suppose that for any solution of*

$$\begin{aligned} \partial_t u - \Delta u \leq u, 0 \leq u \quad & \textit{in } (\tau, T) \times \Omega, \\ u(t, x)u(t - \tau, x) \equiv 0 \quad & \textit{in } (0, T) \times \Omega, \\ \partial_v u \leq 0 \quad & \textit{in } (0, T) \times \delta\Omega, \end{aligned} \tag{1}$$

the inequality

$$\alpha \int_{\Omega} u^2(0, x)\, dx \geq \int_{\Omega} u^2(T, x)\, dx$$

holds with some $T > \tau$, $\alpha < 1$. *Then any solution of*

$$\begin{gathered} \partial_t u - \Delta u \leq f(u, u(\cdot - \tau, \cdot)),\ 0 \leq u \quad \textit{in } \mathbf{R}^+ \times \Omega, \\ \partial_v u \leq g \quad \textit{in } \mathbf{R}^+ \times \partial\Omega, \end{gathered} \tag{0}$$

with

$$\int_\Omega u^2(0,x)\,dx < \infty, \qquad g \in L_2(\partial\Omega),$$

satisfies

$$\varlimsup_{t\to\infty} \int_\Omega u^2(t,x)\,dx \le K$$

where $K = K(f, \varepsilon, \alpha, \tau, \int_{\partial\Omega} g^2)$ *does not depend on the initial values.*

(b) *If* Ω, *an arbitrary domain, and* $\tau \in \mathbf{R}^+$ *are such that only for solutions of the Dirichlet problem*

$$(1')\qquad \begin{aligned} \partial_t u - \Delta u \le u,\ u \ge 0 \quad & \text{in } (0,T)\times\Omega, \\ u(t,x)\cdot u(t-\tau,x) \equiv 0 \quad & \text{in } (\tau,T)\times\Omega, \\ u = 0 \quad & \text{in } (0,T)\times\partial\Omega, \end{aligned}$$

the decay estimate

$$\alpha \int_\Omega u^2(0,x) \ge \int_\Omega u^2(T,x)$$

holds with some $T > \tau$, $\alpha > 1$, *then for any solution*

$$(0')\qquad \begin{aligned} \partial_t u - \Delta u \le f(u, u(\cdot-\tau,\cdot)), 0 \le u \quad & \text{in } \mathbf{R}^+\times\Omega, \\ u \le g \quad \text{in } \mathbf{R}^+ \times \partial\Omega, & \end{aligned}$$

with $\int_\Omega u^2(0,x) < \infty$, $g \in H_2'(\Omega)$ *we have*

$$\varlimsup_{t\to\infty} \int_\Omega u^2(t,x)\,dx \le K$$

with $K = K(f, \alpha, T, \|g\|_{1,2})$.

PROOF OF (a). First of all we have for any solution u of (0),

$$\frac{1}{2}\partial_t \int_\Omega u^2 + \int_\Omega |\nabla u|^2 \le \int_\Omega u^2 + \sqrt{\int_{\partial\Omega} g^2}\sqrt{\int_{\partial\Omega} u^2}$$

and

$$\int_{\partial\Omega} u^2 \le c_\Omega \left[\int_\Omega |\nabla u|^2 + \int_\Omega |u|^2\right]$$

so that

$$(3)\qquad \partial_t \int_\Omega u^2 \le 4\int_\Omega u^2 + \overline{c_\Omega}\int_{\partial\Omega} g^2.$$

Consequently if the theorem does not hold, there has to be a sequence u_i of solutions of (0) in $(0,T)\times\Omega$ such that

$$\int_\Omega u_i^2(0,x)\,dx =: \lambda_i^2 \uparrow \infty$$

and

$$\int_\Omega u_i^2(T,x)\,dx \ge \frac{1+\alpha}{2}\lambda_i^2.$$

Consider $v_i := u_i/\lambda_i$. We have

$$\partial_t v_i - \Delta v_i \le v_i \quad \text{in } (0,T)\times\Omega,$$
$$\partial_\nu v_i \le \frac{g}{\lambda_i} \quad \text{in } (0,T)\times\Omega.$$

As a sequence of positive subsolutions of a parabolic equation, v_i is compact in L^2; more precisely, we have $v_i = v_0/\lambda_i + \tilde v_i$ where

$$\partial_t v_0 - \Delta v_0 = v_0 \quad \text{in } (0,T)\times\Omega,$$
$$v_0(0,x)\equiv 0 \quad \text{in } \Omega, \qquad \partial_\nu v_0 = g \quad \text{in } (0,T)\times\partial\Omega$$

and $\tilde v$ fulfills

$$\tilde v_i(t,x) \le e^s[E(s)\tilde v_i(t-s,\cdot)](x) \quad (\forall 0\le s\le t)$$

where E is the evolution operator for the heat equation with Neumann boundary values. So for $\tilde v_i$ we have the inequality

$$\int_{\tau+\delta}^{T+\tau}\int_\Omega \tilde v_i^2 \le e^\delta \int_{\tau+\delta}^{T+\tau} dt\int_\Omega dx\left(\frac{2}{\delta}\int_{\delta/2}^{\delta} dx\, E(s)(\tilde v_i(t-s,\cdot))^2(x)\right)$$
$$\le e^\delta \int_\tau^{T+\tau-\delta}\int_\Omega \tilde v_i^2$$

where a compact operator of $\tilde v_i$ appears in the middle. As a result $\tilde v_i$ is a compact sequence in $L^2((\partial,T)\times\Omega)$. Let $v=\lim_i v_i$ for a subsequence. By (3) we have

$$1 \ge \lim_{t\downarrow\tau}\int_\Omega v(t)^2, \quad \frac{1+\alpha}{2}\le \lim_{t\uparrow T+\tau}\int_\Omega v(t)^2. \tag{4}$$

Clearly also

$$\partial_t v - \Delta v \le 0, v\ge 0 \quad \text{in } (0,T)\times\Omega,$$
$$\partial_\nu v \le 0 \quad \text{in } (0,T)\times\partial\Omega,$$

$$\frac{1}{2}\int_\Omega \tilde v_i^2(T) + \int_\tau^T\int_\Omega |\nabla \tilde v_i^2|$$
$$\le \int_\Omega \tilde v_i^2(\tau) + \int_\tau^T\int_\Omega \tilde v_i^2 + \int_\tau^T\int_{\partial\Omega} g\tilde v_i - \int_\tau^T\int_\Omega \tilde v_i\frac{1}{\lambda_i}(1-f(\lambda_i\tilde v_i,\lambda_i\tilde v_i(\cdot-\tau,\cdot)))$$

so that

$$-\int_\tau^T\int_\Omega \frac{1}{\lambda_i} f(\lambda_i\tilde v_i(t),\lambda_i\tilde v_i(t-\tau))\tilde v_i \le \text{const} < \infty$$

and by going to the limit a.e.

$$\tilde v(t)\tilde v(t-\tau)\equiv 0,$$

a contradiction to (4) in view of the assumption.

The proof of (b) is exactly the same, except that the test function to be used in $(1')$ is $(u-g)_+$ instead of u.

2. Decay estimate in the interval. By the reduction effected in Lemma 1, part (a) of the theorem is now a consequence of the following estimate.

LEMMA 2. *Let u be a solution of*

$$(1)\qquad \begin{aligned} \partial_t u - \partial_x^2 u \le u, u \ge 0 & \quad \textit{in } (-\tau, T) \times (0, L), \\ u(t,x) \cdot u(t-\tau, x) \equiv 0 & \quad \textit{in } (0,T) \times (0,L), \\ \partial_\nu u \le 0 & \quad \textit{in } (-\tau, T) \times (\{0\} \cup \{L\}), \end{aligned}$$

then

$$\int_0^L u^2(T) \Big/ \int_0^L u^2(0) \le \exp\left(2T\left(1 - \frac{\pi^2}{4L^2}\frac{T^2}{(\tau \wedge T)^2}\right)\right).$$

PROOF. First in one space dimension we approximate u by a Lipschitz function u_ε, satisfying the same inequalities as u, and such that $\partial(\operatorname{supp}(u_\varepsilon))$ is a smooth manifold. We continue u to $(1-\varepsilon, L+\varepsilon)$, by reflection and define

$$\tilde{u}_{\varepsilon,\delta} = \frac{1}{\varepsilon\delta}\int dy \int ds\, \varsigma\left(\frac{t-s}{\delta}\right)\varsigma\left(\frac{x-y}{\varepsilon}\right) u(s,y)$$

and

$$\tilde{u}_\varepsilon(t,x) = \frac{1}{\varepsilon}\int dy\, \varsigma\left(\frac{x-y}{\varepsilon}\right) u(t,y)$$

where $\varsigma \in C_0^\infty(\mathbf{R}, \mathbf{R}^+)$, $\int \varsigma = 1$, $\operatorname{supp}(\varsigma) \subset B_1(0)$. For $\tilde{u}_\varepsilon$ we have

$$\partial_t \tilde{u}_\varepsilon(t,x) \le c\varepsilon^{-5/2}\sqrt{\int_{B_\varepsilon(x)} u^2(t,y)\,dy}.$$

Also

$$u_\varepsilon(t,x) \wedge u_\varepsilon(t-\tau,x) \le \frac{1}{\varepsilon}\|\varsigma\|_\infty \min\left[\int_{B_\varepsilon(x)} u(t,y), \int_{B_\varepsilon(x)} u(t-\tau,y)\right].$$

Since $u(t,\cdot)\cdot u(t-\tau,\cdot) \equiv 0$, for s equal to t or s equal to $(t-\tau)$, $u(s,\cdot)$ has a zero in $B_\varepsilon(x)$, so by Poincaré's inequality

$$\int_{B_\varepsilon(x)} u^2(s,\cdot) < 4\frac{\varepsilon^2}{\pi^2}\int_{B_\varepsilon(x)} (\partial_x u)^2(s,\cdot).$$

Consequently

$$\tilde{u}_\varepsilon(t,x) \wedge \tilde{u}_\varepsilon(t-\tau,x) \le c \cdot \varepsilon^{1/2}$$

and

$$\tilde{u}_{\varepsilon,\delta}(t,x) \wedge \tilde{u}_{\varepsilon,\delta}(t-\tau,x) \le c\cdot(\varepsilon^{1/2} + \delta\varepsilon^{-5/2}).$$

Now choose $\delta = \varepsilon^3$, and also choose an $\tilde{\varepsilon}$ between $2c\sqrt{\varepsilon}$ and $4c\sqrt{\varepsilon}$ which is a regular value for $\tilde{u}_{\varepsilon,\varepsilon^3}$. Then for

$$u_\varepsilon := \max(0, \tilde{u}_{\varepsilon,\varepsilon^3 - \tilde{\varepsilon}})$$

we have indeed that u_ε is Lipschitz, $u_\varepsilon(t-\tau,\cdot)\cdot u_\varepsilon(t,\cdot) \equiv 0$, and $\partial(\operatorname{supp}(u_\varepsilon))$ is a smooth manifold.

It clearly suffices to show the estimate for u_ε.

Take a connected component C of the interior of $\operatorname{supp}(u_\varepsilon)$ in $(0,T)\times(0,L)$. We have $C \cap C - (\tau, 0) = \emptyset$ by assumption, and since C is bounded and connected in $\mathbf{R}^2$ it follows that also $C \cap C - n(\tau,0) = \emptyset$.

(The easiest way to see this is as follows: Without loss of generality assume $\overline{C} \cap \overline{C} - (\tau, 0) = \emptyset$. Take $(t_1, x_1) \in \overline{C}$ with $x_1 = \min\{x, \text{there is } (t,x) \in \overline{C}\}$ and

$(t_2, x_2) \in \overline{C}$ with $x_2 = \max\{x$, there is $(t,x) \in \overline{C}\}$. Define $\tilde{C} = \overline{C} \cup (t_1, x_1 - \mathbf{R}^+) \cup (t_2, x_2 + \mathbf{R}^+)$. $\mathbf{R}^2 \times \tilde{C}$ has two unbounded components $U^+, U^-, \tilde{C} \cap \tilde{C} - (\tau, 0) = \varnothing$, so $\tilde{C} - (\tau,0) \subset U^-$, $U^- - (\tau, 0) \subset U^-$, and $\tilde{C} - n(\tau,0) \subset U^- - (n-1)(\tau,0) \subset U^-$.)

Defining $C_t = \{x, (t,x) \in C\}$ we have $C_t \cap C_{t-n\tau} = \varnothing$ for all $n \in \mathbf{N}$ or

$$\int_0^T |C_t| \leq \tau \cdot L \quad \text{if } \tau < T.$$

We use this inequality, only true in one space dimension, in the Sobolev estimate

$$\frac{1}{2}\partial_t \left(\int_{C_t} u^2 \right) \leq \int_{C_t} (u^2 - |\nabla_u|^2 \leq \left(\int_{C_t} u^2 \right) \left(1 - \frac{\pi^2}{4|C_t|^2} \right).$$

So we have Jensen's inequality

$$\int_{C_t} u^2 \Big/ \int_{C_0} u^2 \leq \exp\left(2T \fint_0^T \left(1 - \frac{\pi^2}{4|C|^2} \right) \right) \leq \exp\left(2T \left(1 - \frac{\pi^2}{4(\fint_0^T |C_t|)^2} \right) \right)$$

and finally

$$\int_{C_T} u^2 \Big/ \int_{C_0} u^2 \leq \exp\left(2T \left(1 - \frac{\pi^2}{4L^2} \frac{T^2}{\tau^2} \right) \right)$$

if $\tau < T$.

This proves the lemma.

3. Decay estimate for arbitrary space dimension.

LEMMA 3. (a) *Suppose*

$$\begin{aligned} \partial_t u - \Delta u \leq u, 0 \leq u \quad & \textit{in } (0,T) \times \Omega, \\ u(t,x) \cdot u(t-\tau, x) \equiv 0 \quad & \textit{in } (\tau, T) \times \Omega, \\ \partial_\nu u \leq 0 \quad & \textit{in } (0,T) \times \Omega. \end{aligned} \tag{1}$$

Suppose $\partial\Omega$ is locally uniformly Lipschitz, i.e. there exists $\alpha > 0$ such that for $(x,y) \in \partial\Omega$ with $|x - y| < \alpha$ we have $\nu_x \cdot \nu_y \geq \alpha - 1$. Then there exists a constant $\tau(n, \alpha) > 0$ such that for $\tau < \tau(n,\alpha)$,

$$e^{\gamma t} \int_\Omega u^2(t) < \text{const}$$

with $\gamma = \gamma(\tau, n, \alpha) > 0$.

(b) *If Ω is arbitrary but $u = 0$ in $\partial\Omega$, the conclusion holds with a constant $\tau(n)$ depending on dimension only.*

PROOF. We use the following Sobolev estimate in $\mathbf{R}^n$. If $|\{u = 0\} \cap B_\rho| \cdot |B_\rho|^{-1} > \varepsilon > 0$, then

$$\int_{B_\rho} u^2 < C(\varepsilon, n)\rho^2 \int_{B_\rho} |\nabla u|^2. \tag{5}$$

Consider first case (b); then for any $t, x \in \Omega$ either

$$|\{u(t,\cdot) = 0\} \cap B_\rho(x)|\, |B_\rho|^{-1} > \tfrac{1}{2}$$

or

$$|\{u(t-\tau,\cdot) = 0\} \cap B_\rho(x)|\, |B_\rho|^{-1} > \tfrac{1}{2}.$$

On the other hand,

$$\int_{B_\rho(x)} u^2(t) - \int_{B_{2\rho}(x)} u^2(t-\tau) \le \left(\frac{1}{\rho^2}+1\right)\int_{t-\tau}^t \int_{B_{2\rho}} |u|^2.$$

So

$$\int_{B_\rho(x)} u^2(t) \le \rho^2 C\left(\frac{1}{4},n\right)\int_{B_{2\rho(x)}} (|\nabla u|^2(t) + |\nabla u|^2(t-\tau)) + \left(\frac{1}{\rho^2}+1\right)\int_{t-\tau}^t \int_{B_{2\rho}} u^2. \tag{6}$$

Integrating over x gives

$$\begin{aligned}|B_\rho| &\int_\Omega u^2(t) \\ &\le |B_{2\rho}|\left[\rho^2 C\left(\frac{1}{4},n\right)\int_\Omega (|\nabla u|^2(t) + |\nabla u|^2(t-\tau)) + \left(\frac{1}{\rho^2}+1\right)\int_{t-\tau}^t\int_\Omega u^2\right],\end{aligned}$$

$$\begin{aligned}\int_\Omega e^{\gamma t}u^2(t) = \int_\Omega e^{\gamma\tau}u^2(\tau) &\le (\gamma+2)\int_\tau^t e^{\gamma s}\int_\Omega u^2(s) - 2\int_\tau^t e^{\gamma s}\int_\Omega |\nabla u|^2(s) \\ &\le -2\int_\tau^t e^{\gamma s}\int_\Omega \nabla u^2 + \frac{\gamma+2}{1-2^n\tau e^{\gamma\tau}(1/\rho^2+1)} \\ &\quad\cdot\left[2^{n+1}\rho^2 C\left(\frac{1}{4},n\right)e^{\gamma\tau}\int_\tau^t e^{\gamma s}\int_\Omega |\nabla u|^2\right. \\ &\quad\left. + e^{\gamma\tau}\int_0^\tau\int_\Omega\left(2^n\rho^2 C\left(\frac{1}{4},n\right)|\nabla u^2| + 2^n\left(\frac{1}{\rho^2}+1\right)\tau e^{\gamma\tau}|u|^2\right)\right].\end{aligned}$$

So if one can choose ρ such that

$$\frac{(\gamma+2)2^{n+1}\rho^2 e^{\gamma\tau}C(\frac{1}{4},n)}{1-2^n\tau e^{\gamma\tau}(1/\rho^2+1)} < 2,$$

one gets

$$\int_\Omega e^{\gamma\tau}u^2(t) < \text{const}.$$

It is easily seen that this is the case if $\tau < 1/2^n e^{\gamma\tau}$ and

$$8C\tau\left(\frac{\gamma}{2}+1\right) < \left(\frac{1}{2^n e^{\gamma\tau}} - \tau\right)^2.$$

In case (a) the proof changes only insofar as we have to use the Sobolev estimate (5) in Ω instead of $\mathbf{R}$.

If $\rho < \alpha$,

$$|\{u=0\}\cap B_\rho \cap \Omega|\,|B_\rho\cap\Omega|^{-1} > \varepsilon > 0;$$

then

$$\int_{B_\rho} u^2 < C(\varepsilon,\alpha,n)\rho^2\int_{B_\rho}|\nabla u|^2,$$

and the volume fractions are estimated by

$$|B_{2\rho}(x)\cap\Omega|\,|B_\rho(x)\cap\Omega|^{-1} \le C(\alpha,n) \quad \text{for } \rho<\alpha$$

in inequality (6).

4. The counterexample. As in the decay estimates, for the counterexample we also work with the condition $u(t-\tau,x)u(t,x)\equiv 0$, using the obvious fact that any solution of (1) is also a solution of (0) for $f(u,v)=u(1-v)$.

LEMMA 4. *Let $\gamma<1$, $u\geq 2$. Then there exists a ball $B_R\subset \mathbf{R}^N$ and a delay $\tau\in\mathbf{R}^+$ such that a solution of*

$$\begin{aligned} \partial_t u-\Delta u\leq u, 0\leq u \quad & \text{in } \mathbf{R}^+\times B_R,\\ u(t-\tau,x)\cdot u(t,x)\equiv 0 \quad & \text{in } (\tau,\infty)\times B_R \end{aligned}$$

with $\operatorname{supp}(u)\Subset B_R$ *can be constructed which grows like* $\int_{B_R} u>e^{\gamma t}$.

PROOF. The idea is to prescribe first $\operatorname{supp}(u)$, which will be a ball B_r moving spirally in B_R, and then to solve the Dirichlet problem for $\partial_t u-\Delta u=u$ in that moving domain. If r is large enough and the movement slow enough, the first eigenvalue of that problem will be as close to 1 as we please. Starting then with the first eigenfunction as initial value and extending u by zero to all of B_R, one gets the desired counterexample.

So take a point x_0 in B_R with coordinates $(\rho\cos\varphi,\rho\sin\varphi,0,\dots,0)$ and move it with constant angular velocity c. Solving $\partial_t u-\Delta u=u$ in $B_r(x(t))$ is then the same as solving

$$\partial_t u-c\partial_1 u x_2+c\partial_x u x_1-\Delta u=u$$

in $B_r(x_0)$. For $c\rho$ small and r large the first eigenvalue of this operator is arbitrary near 1. But that makes τ large as the condition $u(t,x)\cdot u(t-\tau,x)$ identically zero is fulfilled for $1-\cos(c\tau)\geq r/\rho$.

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