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S^1-EQUIVARIANT FUNCTION SPACES AND CHARACTERISTIC CLASSES

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ABSTRACT. We determine the structure of the homology of the Becker-Schultz space $SG(S^1) \simeq Q(\mathbf{C}P^\infty_+ \wedge S^1)$ of stable S^1-equivariant self-maps of spheres (with standard free S^1-action) as a Hopf algebra over the Dyer-Lashof algebra. We use this to compute the homology of $BSG(S^1)$. Along the way, we give a fresh account of the partially framed transfer construction and the Becker-Schultz homotopy equivalence. We compute the effect in homology of the "S^1-transfers" $\mathbf{C}P^\infty_+ \wedge S^1 \to Q((B\mathbf{Z}_{p^n})_+)$, $n \geq 0$, and of the equivariant J-homomorphisms $SO \to Q(\mathbf{R}P^\infty_+)$ and $U \to Q(\mathbf{C}P^\infty_+ \wedge S^1)$. By composing, we obtain $U \to QS^0$ in homology, answering a question of J. P. May.

Introduction. Let H be a compact Lie group admitting a finite-dimensional orthogonal representation W such that H acts freely on the unit sphere sW. H must thus be S^1, S^3, the normalizer of S^1 in S^3, or one of a known list [**13**] of finite groups with periodic cohomology, including (as subgroups of S^3) the cyclic and generalized quaternion groups. Let $\mathrm{End}_H(sW)$ denote the space of H-equivariant continuous self-maps of sW. By joining with the identity map we obtain inclusions $\mathrm{End}_H(s(nW)) \subset \mathrm{End}_H(s((n+1)W))$, and we write $G(H)$ for the direct limit. The homotopy type of $G(H)$ was determined by J. C. Becker and R. E. Schultz [**2**], and turns out to be independent of W. If we write $SG(H)$ for the component of $G(H)$ containing the identity map (so $SG(H) = G(H)$ if H is connected), then in [**3**], Becker and Schultz (see also [**9**] in case H is finite) enrich the composition product in $SG(H)$ to an infinite-loop space structure. The classifying space $BSG(H)$ classifies oriented spehrical fibrations with a fiber-preserving H-action modelled on $s(nW)$, stabilized by forming fiberwise joins with the trivial H-fibration with fiber sW.

In this paper we determine the mod p homology of $SG(S^1)$ and of $BSG(S^1)$ as Hopf algebras over the Dyer-Lashof algebra. Along the way, we compute the effect in homology of the "forgetful" maps $SG(S^1) \to SG(\mathbf{Z}_{p^n})$ and of the equivariant J-homomorphisms $J_{\mathbf{Z}_2}: SO \to SG(\mathbf{Z}_2)$ and $j_{S^1}: U \to SG(S^1)$.

The starting point for our analysis is the study of certain "transfer" maps. §1 is devoted to an account of the construction and general properties of these maps. In §2 we study certain transfers t associated to an inclusion $K \subset H$ of compact Lie groups. If E is a smooth principal H-space, then

$$t\colon (E/H)^{\varsigma_H} \to Q((E/K)^{\varsigma_K}),$$

where ς_H is the vector-bundle obtained by mixing $E \downarrow E/H$ with the adjoint representation of H on its Lie algebra, the superscript denotes formation of the Thom space and QX is the enveloping infinite loop space of X.

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Then in §3 we describe a map

$$\gamma\colon \mathrm{End}_H(E) \to Q((E/H)^{\varsigma_H})$$

generalizing a variant of a construction of Becker and Schultz [**2**]. We show that if $K \subset H$, then the inclusion $\mathrm{End}_H(E) \subset \mathrm{End}_K(E)$ corresponds under γ to the transfer t. If H is a "periodic" Lie group as above, then we obtain from γ a map

$$\bar{\gamma}\colon G(H) \to Q(BH^{\varsigma})$$

which was shown by Becker and Schultz to be a homotopy equivalence. In particular, $SG(S^1) = G(S^1)$ is homotopy-equivalent to $Q(\mathbf{C}P^{\infty}_{+} \wedge S^1)$.

§4 is devoted to the evaluation in mod p homology of the transfer

$$t_n\colon \mathbf{C}P^{\infty}_{+} \wedge S^1 \to Q(B\mathbf{Z}^{+}_{p^n}), \qquad n \geq 0,$$

associated to the inclusion $\mathbf{Z}_{p^n} \subset S^1$. This is one of our principal technical results. We give the statement here for $n = 1$. Let $\bar{a}_r \in H_{2r+1}(\mathbf{C}P^{\infty}_{+} \wedge S^1)$ and $e_r \in h_r(B\mathbf{Z}_p)$ be the standard generators (see §4), and let χ be the canonical antiautomorphism on $H_*(QX)$.

THEOREM A. *For $p = 2$,*

$$t_{1*}\bar{a}_r = \sum_s e_{2s+1} * \chi e_{2(r-s)} + \sum_t Q^{t+1} e_t * (\chi e_{r-t})^{*2}.$$

For $p > 2$,

$$t_{1*}\bar{a}_r = \sum_s e_{2s+1} * \chi e_{2(r-s)}.$$

Since transfers compose well, and the transfer associated to $1 \subset \mathbf{Z}_p$ essentially defines the Dyer-Lashof operations, it is easy to deduce from Theorem A the effect of t_0 in homology; the formulae are given in Theorme C, and in Theorems 4.4 and 4.5. An easy filtration argument (carried out in §7) results in the

COROLLARY. *For $p = 2$,*

$$t_{0*}\colon H_*(Q(\mathbf{C}P^{\infty}_{+} \wedge S^1)) \to H_*(QS^0)$$

is injective. For p odd,

$$t_{1*}\colon H_*(Q(\mathbf{C}P^{\infty}_{+} \wedge S^1)) \to H_*(QB\mathbf{Z}^{+}_{p})$$

is injective, while t_{0*} is not.

§5 is dedicated to a study of the equivariant J-homomrphisms

$$j_{\mathbf{Z}_2}\colon SO \to SG(\mathbf{Z}_2) \simeq Q\mathbf{R}P^{\infty}_{+},$$
$$j_{S^1}\colon U \to SG(S^1) \simeq Q(\mathbf{C}P^{\infty}_{+} \wedge S^1).$$

Let

$$\lambda_{\mathbf{C}}\colon \mathbf{C}P^{n-1}_{+} \wedge S^1 \to U(n)$$

send a pair (l, z), where $l \subset \mathbf{C}^n$ is a complex line and $z \in \mathbf{C}$ has $|z| = 1$, to the unitary transformation which is the identity on $l^{\perp}$ and multiplies by z on l. Then we have the following theorem.

THEOREM B. *The composite*

$$\mathbf{C}P_+^\infty \wedge S^1 \xrightarrow{\lambda_{\mathbf{C}}} U \xrightarrow{j_{S^1}} Q(\mathbf{C}P_+^\infty \wedge S^1)$$

is homotopic to the standard inclusion.

There is a real analogue; see Theorem 5.1. Theorems A and B have as a corollary the determination of the image of the generators of $H_*(U)$ under the classical complex J-homomorphism $J_{\mathbf{C}}: U \to QS^0$ in terms of the loop-structure in $H_*(QS^0)$. For p odd, this resolves a problem left open by J. P. May in [**5**, pp. 121–123].

THEOREM C. *The image of $\bar{a}_r \in H_{2r+1}(U; \mathbf{Z}_p)$ under $J_{\mathbf{C}^*}$ is*

$$\begin{aligned} J_{\mathbf{C}^*}\bar{a}_r = &\sum_s Q^{2s+1}[1] * \chi Q^{2(r-s)}[1] * [1] \\ &+ \sum_t Q^{t+1}Q^t[1] * (\chi Q^{r-t}[1])^{*2} * [1] \end{aligned}$$

for $p = 2$, and

$$\begin{aligned} J_{\mathbf{C}^*}\bar{a}_r &= (-1)^k \textstyle\sum_s \beta Q^s[1] * \chi Q^{k-s}[1] && \text{if } r = (p-1)k - 1 \\ &= 0 && \text{otherwise} \end{aligned}$$

for p odd.

The infinite-loop structure of $SG(H)$ is very different from the natural infinite-loop structure on $Q(BH^{\varsigma_H})$. In [**11**], R. E. Schultz described the composition product in terms of the transfer associated to the diagonal embedding $\Delta: H \to H \times H$. For H finite, this theorem is recovered and extended in [**8**]; and by the Corollary to Theorem A, this case suffices for present purposes. In §6 we combine this with Theorem A to prove the following result, which completely determines the composition product $\circ$ in $H_*(SG(S^1))$.

THEOREM D. *For $p = 2$,*

$$\begin{aligned} \bar{a}_q \circ \bar{a}_r = \bar{a}_q * \bar{a}_r &+ \sum_s (q-s, r-s) Q^{2(q+r-s)+1}\bar{a}_s \\ &+ \sum_t (q-2t, r-2t) Q^{q+r+1}Q^{q+r-2t}\bar{a}_t. \end{aligned}$$

For p odd,

$$\bar{a}_q \circ \bar{a}_r = \bar{a}_q * \bar{a}_r + \sum_t c(q,r,t)\beta Q^t \bar{a}_{q+r+1-(p-1)t},$$

where

$$c(q,r,t) = \sum_k (-1)^{r+k} \binom{(p-1)k-1}{r} \binom{q+r+1-(p-1)t}{t-k}.$$

We then appeal to results of [**8**], relating $G(H)$ for H finite to the multiplicative structure in a certain E_∞-ring space $A(H)$, the "Burnside space" of H, introduced (as a space) by G. Segal [**12**]. By the Corollary to Theorem A, the relationship between the Dyer-Lashof operations in $SG(S^1)$ and the $*$-product and loop Dyer-Lashof operations in $Q(\mathbf{C}P_+^\infty \wedge S^1)$ are implied by analogous relationships for $\mathbf{Z}_p$. These formulae show how the Dyer-Lashof action on $\bar{a}_r$ (given, in view of Theorem B

and the fact that i_{S^1} is an infinite-loop map, by Kochman's formula [**7**]) determine the action on all of $H_*(SG(S^1))$.

We turn next to a "global" analysis of $H_*(SG(S^1))$ by means of a "weight valuation". In §7, we prove

THEOREM E. *$H_*(SG(S^1);\mathbf{Z}_p)$ is a primitively generated Hopf algebra. For $p=2$*

$$H_*(SG(S^1)) \cong H_*(U) \otimes P[\bar{a}_r^{*2} : r \geq 0] \\ \otimes H_*(Q(\mathbf{C}P_+^\infty \wedge S^1))//P[\bar{a}_r : r \geq 0],$$

and for p odd,

$$H_*(SG(S^1)) \cong H_*(Q(\mathbf{C}P_+^\infty \wedge S^1))$$

as Hopf algebras.

Finally, we follow [**10**, **5** and **8**] in using the classifying-space spectral sequence and Dyer-Lashof operations to prove (using the usual Dyer-Lashof notation [**5**, p. 16])

THEOREM F. *As Hopf algebras, for $p=2$*

$$H_*(BSG(S^1)) \cong H_*(BU) \otimes E[\sigma Q^{2r+1}\bar{a}_r : r \geq 0] \\ \otimes P[\sigma Q^{2r+2}\bar{a}_r : r \geq 0] \\ \otimes P[\sigma Q^I \bar{a}_r : I \text{ admissible},\ l(I) \geq 1,\ e(I) > 2r+2,\ r \geq 0],$$

and for p odd,

$$H_*(BSG(S^1)) \cong H_*(BU) \otimes S[\sigma\beta^\varepsilon Q^{r+1}\bar{a}_r : \varepsilon = 0 \text{ or } 1,\ r \geq 0] \\ \otimes S[\sigma Q^I \bar{a}_r : I \text{ admissible},\ l(I) \geq 1,\ e(I)+b(I) > 2r+2,\ r \geq 0].$$

Here S denotes the free commutative algebra.

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1. Generalities on the transfer. In this section we recall the definition of the transfer, generalized slightly to allow twisting by a vector bundle. We then catalogue various properties which will be used later. All manifolds will be smooth, and we write $\tau(M)$ for the tangent bundle of M.

Let $\pi\colon E \to B$ be a smooth map between closed manifolds. Choose an embedding $j\colon E \to \underline{\underline{k}}_B$ of E into the trivial vector bundle over B of fiber-dimension k, such that

$$\begin{tikzcd} E \arrow[rr, "j"] \arrow[dr, "\pi"'] & & \underline{\underline{k}}_B \arrow[dl, "\mathrm{pr}"] \\ & B & \end{tikzcd} \tag{1.1}$$

commutes. (For example, embed E into $\mathbf{R}^k$ by i, and let j have components (π, i).) Write $\nu(j)$ for the normal bundle of j. The Pontrjagin-Thom construction then yields a map of Thom spaces:

$$B^+ \wedge S^k \cong B^{\underline{\underline{k}}} \to E^{\nu(j)}. \tag{1.2}$$

Given any vector bundle ς over B, we may compose j with the axis embedding $\underline{\underline{k}}_B \to \varsigma \oplus \underline{\underline{k}}_B$; the Pontrjagin-Thom construction then gives a "twisted" form of (1.2):

$$B^\varsigma \wedge S^k \cong B^{\varsigma \oplus \underline{\underline{k}}} \to E^{\pi^*\varsigma \oplus \nu(j)}. \tag{1.3}$$

Given also a bundle ξ over E, a *relative framing* of π, ξ, ς is a bundle isomorphism

$$\phi\colon \pi^*\varsigma \oplus \nu(j) \cong \xi \oplus \underline{\underline{k}}_E \tag{1.4}$$

(or, rather, an equivalence class of pairs (j, φ)). We then obtain a *stable* map

$$t_\pi = t_{\pi,\phi}\colon B^\varsigma \to E^\xi \tag{1.5}$$

called the *transfer associated to* π, ξ, ς (with (j, ϕ) understood).

If π is a *submersion*, we have a natural exact sequence of vector bundles

$$0 \to \tau(\pi) \to \tau(E) \to \pi^*\tau(B) \to 0 \tag{1.6}$$

in which $\tau(\pi)$ is the bundle of *tangent along the fiber* of π. For any embedding j as in (1.1), there is a natural exact sequence

$$0 \to \tau(\pi) \to \underline{\underline{k}}_E \to \nu(j) \to 0.$$

Since $\underline{\underline{k}}_E$ has a natural metric, we have a natural splitting:

$$\tau(\pi) \oplus \nu(j) \cong \underline{\underline{k}}_E. \tag{1.7}$$

A *relative trivialization* of π, ξ, ς is a short exact sequence

$$0 \to \xi \to \pi^*\varsigma \to \tau(\pi) \to 0. \tag{1.8}$$

A choice of metric in $\pi^*\varsigma$ splits (1.8) and gives an isomorphism

$$\pi^*\varsigma \cong \xi \oplus \tau(\pi). \tag{1.9}$$

Combining this with (1.7), we have defined a relative framing

$$\phi\colon \pi^*\varsigma \oplus \nu(j) \cong \xi \oplus \tau(\pi) \oplus \nu(j) \cong \xi \oplus \underline{\underline{k}}_E. \tag{1.10}$$

The homotopy class of the associated transfer $t_{\pi,\phi}\colon B^\varsigma \to E^\xi$ is independent of choice of metric in $\pi^*\varsigma$, because the homotopy class of the homeomorphism

$$E^{\pi^*\varsigma \oplus \nu(j)} \cong E^{\xi \oplus \tau(\pi) \oplus \nu(j)}$$

is. All the relative framings we deal with here arise in this way.

We now note several features of the transfer.

Note 1.11. *Fundamental classes.* In case B is a single point and $\varsigma = 0$, the map (1.3) gives a stable homotopy class

$$[E] \in \tilde{S}_n(E^{\nu(E)}),$$

where $\nu(E)$ is the zero dimensional virtual normal bundle of E and n is the dimension of E. This is the *stable homotopy fundamental class* of E, with twisted coefficients. A framing of E gives rise to a class $[E] \in S_n(E)$; and pinching E to a point produces a class in the stable homotopy of spheres, $S_n(*)$—namely, the usual Pontrjagin-Thom class of the framed manifold E.

Note 1.12. *Fundamental class of a framed map.* In case $\xi = 0$ and $\dim \varsigma = \dim E - \dim B = n$, the transfer (1.5) composed with the pinch map $E \to *$ gives a stable cohomotopy class

$$e(\pi, \phi) \in \tilde{S}^{-n}(B^{\varsigma - \underline{\underline{n}}}).$$

This is the *stable fundamental class* of π, with twisted coefficients. A framing $\varsigma \oplus \underline{\underline{k}} \cong \underline{\underline{n}} \oplus \underline{\underline{k}}$ gives rise to a class $e \in S^{-n}(B)$, the Pontrjagin-Thom class of the framed map π.

Note 1.13. *Composition with the projection.* Suppose that π is a fibration with fiber F, and that we are given a trivialization $\nu(j) \cong (\underline{\underline{k-n}})_E$, $n = \dim F$. Then F has a well-defined framing, and so defines $[F]\colon S^k \to S^{k-n}$. Then the diagram

$$\begin{array}{ccc} B^{\varsigma \oplus \underline{\underline{k}}} & \xrightarrow{t} & E^{\pi^* \varsigma \oplus (\underline{\underline{k-n}})} \\ \| & & \downarrow \pi \\ B^{\varsigma} \wedge S^k & \xrightarrow{1 \wedge [F]} & B^{\varsigma} \wedge S^{k-n} \end{array}$$

commutes.

Note 1.14. *Products.* Let $\pi'\colon E' \to B', \xi', \varsigma'$ be another relatively framed map. Then $\pi \times \pi', \xi \times \xi', \varsigma \times \varsigma'$ has a natural relative framing, and the diagram

$$\begin{array}{ccc} (B \times B)^{\varsigma \times \varsigma'} & \xrightarrow{t_{\pi \times \pi'}} & (E \times E')^{\xi \times \xi'} \\ \| & & \| \\ B^{\varsigma} \wedge B'^{\varsigma'} & \xrightarrow{t_{\pi} \wedge t_{\pi'}} & E^{\xi} \wedge E'^{\xi'} \end{array}$$

commutes.

Note 1.15. *Compositions.* Let $\pi_1\colon E_1 \to E, \xi_1, \xi$ be a relatively framed map to E. Then $\pi \circ \pi_1\colon E_1 \to B, \xi_1, \varsigma$ has a canonical relative framing, and the associated transfer is given by the composite:

$$t_{\pi \circ \pi_1} = t_{\pi_1} \circ t_{\pi}.$$

Note 1.16. *Pull-backs.* Let $f\colon B' \to B$ be a smooth map transverse to π, and let $\pi'\colon E' \to B', \xi', \varsigma'$ be the pull-back of π, ξ, ς along f. Then π', ξ', ς' has a canonical relative framing; and the diagram

$$\begin{array}{ccc} B'^{\varsigma'} & \xrightarrow{f} & B^{\varsigma} \\ {\scriptstyle t_{\pi'}} \downarrow & & \downarrow {\scriptstyle t_{\pi}} \\ E'^{\xi'} & \xrightarrow{f} & E^{\xi} \end{array}$$

commutes.

Note 1.17. *Reframings.* The transfer can be interesting even in case $E = B, \pi = \mathrm{id}$, and $\xi = \varsigma$. One then has the *canonical framing* $\phi\colon \varsigma \to \varsigma$ by the identity. Then a smooth map $\lambda\colon B \to O(k)$ induces a new relative framing

$$1 \oplus \hat{\lambda}\colon \varsigma \oplus \underline{\underline{k}} \to \varsigma \oplus \underline{\underline{k}},$$

where $\hat{\lambda}(b, v) = (b, \lambda(b)v)$, with an associated transfer $t_{1,\hat{\lambda}}\colon B^{\varsigma} \to B^{\varsigma}$. If $J\colon O(k)^+ \to S^0$ is the stable map adjoint to the inclusion $O(k) \subset O \subset QS^0$, then $t_{1,\hat{\lambda}}$ is the composite

$$t_{1,\hat{\lambda}}\colon B^{\varsigma} \xrightarrow{\Delta} B^{\varsigma} \wedge B^+ \xrightarrow{1 \wedge J\lambda^+} B^{\varsigma} \wedge S^0 \cong B^{\varsigma}.$$

Using Note 1.15, the transfer associated to reframings of more general maps may also be expressed in terms of J.

2. Transfers associated to Lie groups. Let H be a compact Lie group and E a compact smooth principal right H-space. H acts also from the left on its Lie algebra $\mathfrak{H}$ by the adjoint action; and E/H thus supports a canonical vector bundle

$$\varsigma = \varsigma_H = E \times_H \mathfrak{H},$$

called the *adjoint bundle.* In this section we show how the transfer endows the formation of the Thom space E^ς with new functoriality, and apply the notes of §1 to verify some properties which will be used in §4.

To begin, one may easily check the following lemma.

LEMMA 2.1. *There is a natural short exact sequence of vector bundles over* E/H:

$$0 \to \varsigma_H \to \tau(E)/H \to \tau(E/H) \to 0. \qquad \square$$

Now let $K \subset H$ be a closed subgroup, so that $\pi\colon E/K \to E/H$ is a fiber bundle with fiber H/K. We have a natural commutative diagram of vector bundles over E/K, with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \tau(\pi) & & \\
 & & & & & & \downarrow & & \\
0 & \to & \varsigma_K & \to & \tau(E)/K) & \to & \tau(E/K) & \to & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \to & \pi^*\varsigma_H & \to & \pi^*(\tau(E)/H) & \to & \pi^*\tau(E/H) & \to & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

The serpent lemma then provides a natural short exact sequence

$$0 \to \varsigma_K \to \pi^*\varsigma_H \to \tau(\pi) \to 0, \tag{2.2}$$

i.e., a relative trivialization (1.8) of $\pi, \varsigma_K, \varsigma_H$. There results a stable "transfer" map

$$t\colon (E/H)^{\varsigma_H} \to (E/K)^{\varsigma_K}.$$

If, for example, K is the trivial subgroup, then

$$t\colon (E/H)^{\varsigma_H} \to E^+. \tag{2.3}$$

These maps provide the construction $(E/H)^{\varsigma_H}$ with the following extended functionality. Consider the category whose objects are pairs (E, H), where H is a Lie group and E a compact smooth principal H-space, in which a morphism from (E, H') to (E, H) is a closed inclusion $H' \supseteq H$ and an H-equivariant smooth map $E' \to E$. Then $(E, H) \mapsto (E/H)^{\varsigma_H}$ describes a functor into the stable homotopy category.

By approximating BH by manifolds, we may contruct a stable map

$$t\colon (BH)^{\varsigma_H} \to (BK)^{\varsigma_K} \tag{2.4}$$

from a closed inclusion $K \subseteq H$ which is well defined up to weak homotopy. This gives the suspension spectrum of $(BH)^{\varsigma_H}$ a contravariant functoriality on the cateogry of Lie groups and closed inclusions.

We now prove three lemmas which will be useful later.

Let $K \subset H$ be a closed *normal* subgroup, and E a smooth principal H-space. Then

$$\begin{array}{ccc} E/K \times H/K & \xrightarrow{\alpha} & E/K \\ \downarrow \mathrm{pr}_1 & & \downarrow \pi \\ E/K & \xrightarrow{\pi} & E/H \end{array} \tag{2.5}$$

is a pull-back diagram if α is the H/K-action map $\alpha(xK, h) = xhK$. This action lifts to make ς_K an H/K-equivariant vector bundle over E/K; so we have a canonical bundle isomorphism

$$\alpha^* \varsigma_K \cong \varsigma_K \times \underline{\underline{0}}$$

over $E/K \times H/K$. Thus the pull-back property (Note 1.16) of the transfer asserts that the diagram

$$\begin{array}{ccc} (E/K)^{\varsigma_K \oplus \tau(\pi)} & \xrightarrow{\pi} & (E/H)^{\varsigma_H} \\ \downarrow t_{\mathrm{pr}_1} & & \downarrow t_\pi \\ (E/K \times H/K)^{\varsigma_K \times \underline{\underline{0}}} & \xrightarrow{\bar{\alpha}} & (E/K)^{\varsigma_K} \end{array} \tag{2.6}$$

is homotopy-commutative. Now π is a principal H/K-bundle, so a choice of orientation for H/K determines a natural trivialization for $\tau(\pi)$. Now using Notes 1.14 and 1.11, the diagram (2.6) leads to the following lemma.

LEMMA 2.7. *Let $K \subset H$ be a closed normal subgroup of codimension n, and let E be a compact smooth principal H-space. Then, with the above notations, the following diagram is homotopy commutative.*

$$\begin{array}{ccc} (E/K)^{\varsigma_K} \wedge S^n & \xrightarrow{\pi} & (E/H)^{\varsigma_H} \\ \downarrow 1 \wedge [H/K] & & \downarrow t_\pi \\ (E/K)^{\varsigma_K} \wedge (H/K)^+ & \xrightarrow{\bar{\alpha}} & (E/K)^{\varsigma_K} \end{array}$$

Here $[H/K]$ is the class of H/K with the chosen orientation and the corresponding right-invariant framing. $\square$

this is a degenerate form of a "double coset formula", and its proper generalization should be of great interest.

LEMMA 2.8. *Let H be a compact Lie group of dimension n, and let E be a compact smooth principal H-space with base point $*$ and projection $\pi\colon E \to E/H$. The diagonal inclusion $\Delta\colon H \to H \times H$ iduces $\overline{\Delta}\colon (E \times E)/H \to E/H \times E/H$. Let $j\colon E \to (E \times E)/H$ be the map $e \to (e, *)H$ and let $i\colon E/H \to E/H \times E/H$ be the inclusion on the first factor. Note that $j^*(\varsigma_H) \simeq \underline{\underline{m}}$ and $i^*(\varsigma_H \times \varsigma_H) \simeq \varsigma_H \oplus \underline{\underline{n}}$. Under these identifications*

$$\begin{array}{ccc} (E/H)^{\varsigma_H} \wedge S^n & \xrightarrow{i} & (E/H)^{\varsigma_H} \wedge (E/H)^{\varsigma_H} \\ \downarrow t_\pi \wedge 1 & & \downarrow t_{\overline{\Delta}} \\ E^+ \wedge S^n & \xrightarrow{j} & ((E \times E)/H)^{\varsigma_H} \end{array}$$

commutes up to homotopy.

PROOF. The diagram

$$\begin{array}{ccc} E & \xrightarrow{j} & (E \times E)/H \\ \downarrow \pi & & \downarrow \bar{\Delta} \\ E/H & \xrightarrow{i} & E/H \times E/H \end{array}$$

is a pull-back, so Note 1.16 gives the result. □

Now let H be an Abelian compact Lie group with multiplication $\mu\colon H \times H \to H$. Since μ is a group-homomorphism, the classifying space BH inherits a product $\bar{\mu}$, and the inversion $x \mapsto x^{-1}$ in H induces a self-map χ of BH. H acts trivially on its Lie algebra, so the bundle ς has a canonical trivialization. Furthermore, μ is covered by a bundle map $\bar{\mu}\colon \underline{\underline{0}} \times \varsigma \to \varsigma$, including an action map

$$\bar{\mu}\colon BH^+ \wedge BH_\varsigma \to BH^\varsigma.$$

Let $t\colon BH^\varsigma \to S^0$ be the transfer associated to the inclusion of the trivial subgroup. The following lemma (due to R. Schultz [**11**] in the case H is finite) computes the transfer t_Δ, associated to the diagonal subgroup $H \subset H \times H$, in terms of t.

LEMMA 2.9. *With the above notation, the following diagram is weak homotopy-commutative*:

$$\begin{array}{ccccc} BH^\varsigma \wedge BH^\varsigma & & \xrightarrow{\cong} & & B(H \times H)^{\varsigma_{H \times H}} \\ \downarrow \Delta\wedge 1 & & & & \\ BH^\varsigma \wedge BH^+ \wedge BH^\varsigma & & & & \\ \downarrow 1\wedge\chi^+\wedge 1 & & & & \downarrow t_\Delta \\ BH^\varsigma \wedge BH^+ \wedge BH^\varsigma & & & & \\ \downarrow 1\wedge\bar{\mu} & & & & \\ BH^\varsigma \wedge BH^\varsigma & \xrightarrow{1\wedge t} & BH^\varsigma \wedge S^0 & \xrightarrow{\cong} & BH^\varsigma \end{array}$$

PROOF. Apply the composition property (Note 1.15) of the transfer to the commutative diagram

$$\begin{array}{ccc} BH & \xrightarrow{\Delta} & BH \times BH \\ {\scriptstyle i_1} \searrow & & \downarrow \sigma \\ & & BH \times BH \end{array}$$

where $\sigma(x, y) = (x, \chi(x)y)$ and $i_1(x) = (x, 1)$. Of course the transfer associated to the diffeomorphism σ is just the inverse of the map $\bar{\sigma}$ induced on Thom spaces; so

$$t_\Delta = t_{i_1} \circ \bar{\sigma}.$$

Now the left leg of Lemma 2.9 is $\bar{\sigma}$, and the bottom composite, by Note 1.14, is t_{i_1}. □

3. The Becker-Schultz map. We now describe the "graph" construction

$$\gamma\colon \mathrm{End}_H(E) \to Q((E/H)^\varsigma),$$

which is a variant of a construction of Becker and Schultz [**2**]. Here H is a Lie group, E a compact smooth principal H-space, $\mathrm{End}_H(E)$ denotes the space of H-equivariant conditions self-maps of E, and QX is the enveloping infinite loop space of the space X.

Choose an embedding of $B = E/H$ in a Euclidean space $\mathbf{R}^s$, and let ν denote the normal bundle. We have the usual Pontrjagin-Thom collapse

$$c\colon S^s \to B^\nu. \tag{3.1}$$

Let $f\colon E \to E$ be an equivariant map, and form the "graph" $f'\colon E \to E^2$ by $f'(e) = (e, f(e))$. This is equivariant, and we get a map $\bar{f}'\colon B \to E^2/H$ of orbit spaces. If $\overline{\mathrm{pr}}_1\colon E^2/H \to B$ is induced from projection to the first factor, then $\overline{\mathrm{pr}}_1 \circ \bar{f}' = 1$. Thus $\bar{f}'$ is covered by a bundle map $\nu \to \overline{\mathrm{pr}}_1^*\nu$, and we get a map

$$\bar{f}'\colon B^\nu \to (E^2/H)^{\overline{\mathrm{pr}}_1^*\nu}$$

on Thom spaces. This construction depends only on continuity of f, and depends continuously on f, so we get a map

$$\mathrm{End}_H(E)^+ \wedge B^\nu \to (E^2/H)^{\overline{\mathrm{pr}}_1^*\nu}. \tag{3.2}$$

The reduced diagonal $\overline{\Delta}\colon B \to E^2/H$ has normal bundle isomorphic to $\tau(E)/H$, which, by Lemma 2.1, is isomorphic to $\tau(B) \oplus \varsigma$. Since $\overline{\Delta}^*\overline{\mathrm{pr}}_1^*\nu = \nu$, we get a transfer

$$t_{\overline{\Delta}}\colon (E^2/H)^{\overline{\mathrm{pr}}_1^*\nu} \to B^{\nu\oplus\tau(B)\oplus\varsigma} \cong B^\varsigma \wedge S^s. \tag{3.3}$$

The composite of $\mathrm{End}_H(E)^+\wedge$ (3.1), (3.2), and (3.3), gives a pointed map

$$\hat{\gamma}_s\colon \mathrm{End}_H(E)^+ \wedge S^s \to B^\varsigma \wedge S^s.$$

The map γ is obtained by composing the adjoint of $\hat{\gamma}_s$ with the inclusion $\Omega^s\Sigma^s B^\varsigma \to QB^\varsigma$, and is, up to homotopy, independent of choices. We may also compose with a map $B \to BH$ classifying $E \to B$, to obtain a map

$$\bar{\gamma}\colon \mathrm{End}_H(E)^+ \to QBH^\varsigma.$$

The following naturality result is due essentially to Becker and Schultz [**2**, 5.16].

THEOREM 3.4. *Let K be a closed subgroup of the Lie group H, and let E be a compact smooth principal H-space. Then*

$$\begin{array}{ccc} \mathrm{End}_H(E) & \xrightarrow{\gamma_H} & Q((E/H)^{\varsigma_H}) \\ \downarrow \mathrm{inc} & & \downarrow \bar{t} \\ \mathrm{End}_K(E) & \xrightarrow{\gamma_K} & Q((E/K)^{\varsigma_K}) \end{array}$$

commutes up to homotopy if $\bar{t}$ is the infinite-loop extension of the transfer t associated to the canonical framing (2.2) *of $\pi\colon E/K \to E/H$.*

PROOF. By Notes 1.12, 1.15, and 1.16, we have a homotopy-commutative diagram (in which the subscripts indicate the groups involved)

$$\begin{array}{ccccccc} S^s & \xrightarrow{c_H} & (E/H)^{\nu_H} & \xrightarrow{\bar{f}'} & (E^2/H)^{\overline{\mathrm{pr}}_1^*\nu_H} & \xrightarrow{t_\Delta} & (E/H)^{\varsigma_H} \wedge S^s \\ {\scriptstyle c_K}\searrow & & \downarrow t & & \downarrow t & & \downarrow t \\ & & (E/K)^{\nu_K} & \xrightarrow{\bar{f}'} & (E^2/K)^{\mathrm{pr}_1^*\nu_K} & \xrightarrow{t_\Delta} & (E/K)^{\varsigma_K} \wedge S^s \end{array}$$

from which the theorem follows. $\square$

We will usually study γ by mapping some compact smooth manifold M into $\mathrm{End}_H(E)$, in such a way that the reuslting equivariant map $f\colon M \times E \to E$ is

smooth. Moreover we will arrange that the reduced graph $\bar{f}'\colon M \times B \to E^2/H$ given by $\bar{f}'(m,e) = (f(m,e),e)$ is transverse to $\overline{\Delta}\colon B \to E^2/H$. Then the pull-back property (Note 1.16), applied to the pull-back diagram

$$\begin{tikzcd} \Gamma \arrow[r, "g"] \arrow[d] \arrow[dd, bend right=60, "p"'] & B \arrow[d, "\overline{\Delta}"] \\ M \times B \arrow[r, "\bar{f}'"] \arrow[d, "\mathrm{pr}_1"] & E^2/H \\ M & \end{tikzcd} \tag{3.5}$$

shows that

$$\begin{tikzcd} M^+ \arrow[r, "t_p"] \arrow[d] & Q\Gamma^{g*\varsigma} \arrow[d, "Qg"] \\ \mathrm{End}_H(E)^+ \arrow[r, "\gamma"] & QB \end{tikzcd}$$

commutes up to homotopy.

In [**2**], Becker and Schultz consider principal H-spaces arising as the unit sphere sV of an orthogonal representation V of H. They construct a map

$$\lambda\colon \mathrm{End}_H(sV)^+ \to Q(sV/H).$$

The description [**2**, 5.6] of λ shows that it is identical to γ except that Δ is replaced by the antidiagonal $\Delta^-(u) = (u,-u)$; the identification (Theorem 3.4) is essentially the transfer $t_{\overline{\Delta}^-}$. Let $A(u) = -u$ denote the antipodal map. We have a commutative diagram, for any $f \in \mathrm{End}_H(sV)$,

$$\begin{tikzcd} & (E^2/H)^{\overline{\mathrm{pr}}_1^*\nu} \arrow[dr, "t_{\overline{\Delta}^-}"] \arrow[dd, "\overline{1\times A}"] & \\ B^\nu \arrow[ur, "\bar{f}'"] \arrow[dr, "\overline{Af}^-"'] & & B^\varsigma \wedge S^s \\ & (E^2/H)^{\overline{\mathrm{pr}}_1^*\nu} \arrow[ur, "t_{\overline{\Delta}}"'] & \end{tikzcd} \tag{3.6}$$

so we find that

$$\begin{tikzcd} \mathrm{End}_H(sV)^+ \arrow[dr, "\lambda"] \arrow[dd, "(A\circ)"] & \\ & QB^\varsigma \\ \mathrm{End}_H(sV)^+ \arrow[ur, "\gamma"'] & \end{tikzcd} \tag{3.7}$$

commutes up to homotopy.

When H admits a finite dimensional orthogonal representation V such that the unit sphere sV is a principal H-space, the maps $\bar{\gamma}$ are compatible under joining with the antipodal map, and given, in the notation of the introduction, a map

$$\bar{\gamma}\colon G(H) \to Q(BH^\varsigma). \tag{3.8}$$

THEOREM 3.9 (BECKER-SCHULTZ [**2**]). *$\bar{\gamma}$ is a homotopy-equivalence.*

We refer the reader to [**2**] for the proof.

THEOREM 3.10. *Let $H = \mathrm{id}$. If $\dim V$ is even, then $\bar{\gamma}(f) = 1 - [f]$ whereas if $\dim V$ is odd, then $\bar{\gamma}(f) = 1 + [f]$.*

PROOF. [**2**, 6.13] shows $\bar{\lambda}(f) = 1 - [f]$. (There is a mistake in the original proof of this fact in [**2**] but Becker (private communication) has given a correct proof.) The corrected proof shows that $\bar{\lambda}(f) = 1 + [f]$ when $\dim V$ is odd. The even dimensonal case follows directly from (3.7) and [**2**, 6.13].

Finally, we recall R. E. Schultz's expression for the composition pairing $\circ\colon G(H) \times G(H) \to G(H)$. Let $\Delta\colon H \to H \times H$ be the diagonal inclusion, and let

$$\#\colon QBH^{\varsigma} \times QBH^{\varsigma} \xrightarrow{\wedge} Q(B(H \times H)^{\varsigma_{H \times H}}) \xrightarrow{\bar{t}_{\Delta}} QBH^{\varsigma},$$

where t_{Δ} is the transfer associated to Δ. Then

THEOREM 3.11 (SZHULTZ [**11**]). *The composition pairing $\circ$ is homotopic (under the identification $\bar{\gamma}$) to the composite*

$$(QBH^{\varsigma})^2 \xrightarrow{\Delta^2} (QBH^{\varsigma})^4 \xrightarrow{1 \times \# \times 1} (QBH^{\varsigma})^3 \xrightarrow{*} QBH^{\varsigma}.$$

4. The homology of the $U(1)$-transfer. Let $U(1)$ act by multiplication on $S^{2m-1} \subset \mathbf{C}^m$. There results (2.3) a stable transfer map

$$\mathbf{C}P_+^{m-1} \wedge S^1 \to S_+^{2m-1},$$

using the canonical trivialization of the adjoint bundle to identify the source. Pinching S^{2m-1} to a point gives an "Euler class" $\mathbf{C}P_+^{m-1} \wedge S^1 \to S^0$. These are compatible over m, and give a stable map

$$\mathbf{C}P_+^{\infty} \wedge S^1 \to S^0. \tag{4.1}$$

This is precisely the map considered by K. Knapp in [**6**]. We will compute the homology of its adjoint, the pointed map

$$t_0\colon \mathbf{C}P_+^{\infty} \wedge S^1 \to QS^0. \tag{4.2}$$

Actually, the same work yields a computation of the homology of the transfer associated to an inclusion $\mathbf{Z}_{p^n} \subset U(1)$. If L_n denotes $B\mathbf{Z}_{p^n}$, then this is a pointed map

$$t_n\colon \mathbf{C}P_+^{\infty} \wedge S^1 \to QL_n^+. \tag{4.3}$$

To state the results, let H_* denote $\mathrm{mod}\, p$ homology, p any prime. Let $x \in H^2(\mathbf{C}P^{\infty})$ be the canonical generator, let $a_r \in H_{2r}(\mathbf{C}P^{\infty})$ be dual to x^r, and let $\bar{a}_r \in \overline{H}_{2r+1}(\mathbf{C}P_+^{\infty} \wedge S^1)$ be the suspension of a_r. Let $u \in H^1(L_n)$ and $v = -\beta_n u \in H^2(L_n)$ be the natural generators, and let $e_r \in H_r(L_n)$ be the dual of the monomial in dimension r. Embed $\mathbf{Z}_{p^n}$ into S^1 so that the resulting map $\pi\colon L_n \to \mathbf{C}P^{\infty}$ pulls x back to v; then $\pi_* e_{2n} = a_n$.

THEOREM 4.4. *Let $p = 2$ and let t_n be as in (4.2) and (4.3). Then*
(a) $t_{0*}\bar{a}_r = \sum_s Q^{2s+1}[1] * \chi Q^{2(r-s)}[1] + \sum_t Q^{t+1}Q^t[1] * (\chi Q^{r-t}[1])^{*2}$.
(b) $t_{1*}\bar{a}_r = \sum_s e_{2s+1} * \chi e_{2(r-s)} + \sum_t Q^{t+1} e_t * (\chi e_{r-t})^{*2}$.
(c) *For $n \geq 2$,* $t_{n*}\bar{a}_r = \sum_s e_{2s+1} * \chi e_{2(r-s)} + \sum_t Q^{2t+1} e_{2t} * (\chi e_{r-2t})^{*2}$.

THEOREM 4.5. *Let p be odd and let t_n be as in (4.2) and (4.3). Then*
(a) $t_{0*}\bar{a}_r = (-1)^k \sum_t \beta Q^t[1] * \chi Q^{k-t}[1]$ *for $r = (p-1)k - 1$, and $= 0$ otherwise.*
(b) *For $n \geq 1$,* $t_{n*}\bar{a}_r = \sum_s e_{2s+1} * \chi e_{2(r-s)}$.

PROOF. We apply Lemma 2.7 to $\mathbf{Z}_{p^n} \subset U(1)$ with $E = S^{2m-1}$. As a framed manifold, the quotient group $U(1)/\mathbf{Z}_{p^n}$ is S^1 with its nonbounding framing. The lemma thus adjoints to give (taking $m = \infty$ henceforth) a homotopy-commutative diagram:

$$\begin{array}{ccc} L_n^+ \wedge S^1 & \xrightarrow{\pi^+\wedge 1} & \mathbf{C}P_+^\infty \wedge S^1 \\ {\scriptstyle 1\wedge[S^1]}\downarrow & & \\ L_n^+ \wedge QS_+^1 & & \downarrow {\scriptstyle t_n} \\ \wedge\downarrow & & \\ Q((L_n \wedge S^1)_+) & \xrightarrow{Q\alpha_+} & QL_n^+ \end{array} \tag{4.6}$$

Here $\wedge$ is the usual smash product map, and α is the action map. Since $\pi_* e_{2r} = a_r$, we can compute $t_{n*}\bar{a}_r$ from (4.6).

To begin with, let

$$h\colon \pi_* QS_+^1 \to \overline{H}_* QS_+^1$$

be the Hurewicz homomorphism. Then $h[S^1] \in \overline{H}_1 QS_+^1$ is characterized by three properties:

(1) It is a coalgebra-primitive and is killed by all Steenrod operations.

(2) It reduces in $\overline{H}_1 QS^0$ to the Hurewicz image of $\eta \in \pi_1 QS^0$, by Note 1.11. This is $Q^1[1] * [-2]$ for $p = 2$ and 0 for p odd.

(3) It reduces in $\overline{H}_1 S^1$ to the fundamental class σ.

It follows that

$$h[S^1] = \sigma * [-1] + Q^1[1] * [-2] \quad \text{for } p = 2, \tag{4.7}$$

$$h[S^1] = \sigma * [-1] \quad \text{for } p \text{ odd}. \tag{4.8}$$

We now restrict attention to $p = 2$, leaving the rather degenerate case of p odd to the reader. We first treat the case $n = 1$, so that the diagonal and Steenrod action in $H_* L_1 = H_* \mathbf{R}P^\infty$ are given by

$$\Delta e_r = \sum_{s+t=r} e_s \otimes e_t, \tag{4.9}$$

$$\mathrm{Sq}_*^t e_r = \binom{r-t}{t} e_{r-t}. \tag{4.10}$$

Recall the distributivity formulae [**5**, p. 15]:

$$x \wedge (y * z) = \sum (x' \wedge y) * (x'' \wedge z), \tag{4.11}$$

$$x \wedge Q^s y = \sum_t Q^{s+t}((\mathrm{Sq}_*^t x) \wedge y), \tag{4.12}$$

where $\Delta x = \Sigma x' \otimes x''$. From (4.7) and (4.11),

$$\begin{aligned} e_r \wedge h[S^1] &= e_r \wedge (\sigma * [-1] + Q^1[1] * [-2]) \\ &= \sum_s (e_s \wedge \sigma) * (e_{r-s} \wedge [-1]) + \sum_s (e_s \wedge Q^1[1]) * (e_{r-s} \wedge [-2]). \end{aligned}$$

From (4.11), (4.12) and (4.10),

$$\begin{aligned} e_{r-s} \wedge [-1] &= \chi e_{r-s}, \\ e_{r-s} \wedge [-2] &= (\chi e_t)^{*2} && \text{if } r-s = 2t, \\ &= 0 && \text{if } r-s \text{ is odd}, \\ e_s \wedge Q^1[1] &= te_t^{*2} && \text{if } s = 2t-1, \\ &= Q^{t+1}e_t && \text{if } s = 2t. \end{aligned}$$

Thus (replacing r by $2r$)

$$e_{2r} \wedge h[S^1] = \sum_s (e_s \otimes \sigma) * \chi e_{2r-s} + \sum_t Q^{t+1}e_t * (\chi e_{r-t})^{*2}. \tag{4.13}$$

Finally, in cohomology

$$\alpha^* x = x \otimes 1 + 1 \otimes x,$$

so dualizing,

$$\alpha_*(e_r \otimes 1) = e_r, \qquad \alpha_*(e_r \otimes \sigma) = (r+1)e_{r+1}. \tag{4.14}$$

Theorem 4.4(b) now follows by applying α_* to (4.13) and recalling the diagram (4.6).

The proof for $p = 2, n > 1$, is completely analogous, using the different structure of H_*L_n, namely

$$\begin{aligned} \Delta e_{2r} &= \sum_{s+t=r} e_{2s} \otimes e_{2t}, \\ \Delta e_{2r+1} &= \sum_{s+t=2r+1} e_s \otimes e_t, \\ \mathrm{Sq}_*^t e_r &= \binom{r-t}{t} e_{r-t} \quad \text{for } t \text{ even}, \\ &= 0 \qquad\qquad\qquad \text{for } t \text{ odd}. \end{aligned}$$

Part (a) follows from part (b), since transfers compose well (Note 1.15), while the transfer $t\colon \mathbf{R}P_+^\infty \to QS^0$ is given in homology by

$$t_* e_r = Q^r[1]$$

by definition of the Dyer-Lashof operation Q^r. This completes the proof. □

REMARK 4.15. The *odd* cells in $B\mathbf{Z}_{p^n}^+$ map to 0 in $\mathbf{C}P_+^\infty$, and hence in $QB\mathbf{Z}_{p^n}^+$ and in QS^0. This results in the following amusing identities for all $n \geq 1$ and $r \geq 0$:

(4.16) In $H_*(QB\mathbf{Z}_{2^n}^+; \mathbf{Z}_2)$,

$$\sum_s e_{2s+1} * \chi e_{2(r-s)+1} + \sum_t e_{2t+1}^{*2} * (\chi e_{r-2t})^{*2} = 0.$$

(4.17) In $H_*(QS^0; \mathbf{Z}_2)$

$$\sum_s Q^{2s+1}[1] * \chi Q^{2(r-s)+1}[1] + \sum_t (Q^{2t+1}[1])^{*2} * (\chi Q^{r-2t}[1])^{*2} = 0.$$

(4.18) In $H_*(QB\mathbf{Z}_{p^n}^+; \mathbf{Z}_p)$ for p odd,

$$\sum_s e_{2s+1} * \chi e_{2(r-s)+1} = 0.$$

(4.19) In $H_*(QS^0; \mathbf{Z}_p)$ for p odd,

$$\sum_t \beta Q^t[1] * \chi\beta Q^{r-t}[1] = 0.$$

REMARK 4.20. A similar analysis may be applied to compute the transfer

$$t\colon (\mathbf{H}P^\infty)^\varsigma \to Q\mathbf{R}P^\infty_+$$

in mod 2 homology, where $\mathbf{H}P^\infty$ is quaternionic projective space and $\mathbf{Z}_2 \subset \mathrm{Sp}(1)$ is the center. The Thom complex $(\mathbf{H}P^\infty)^\varsigma$ is then James' "quasiprojective space". The computation is substantially more tedious here, however, since by the same argument as above, the Hurewicz image of the stable homotopy fundamental class of $\mathrm{Sp}(1)/\mathbf{Z}_2 = SO(3)$ in $H_3Q(SO(3)^+)$ has nine terms:

$$\begin{aligned} h[SO(3)] = {} & e_3 + e_1 * e_2 * e_0^{-1} + Q^2 e_1 * e_0^{-1} \\ & + Q^3 e_0 * e_0^{-1} + e_1 * Q^1 e_1 * e_0^{-2} + Q^2 Q^1 e_0 * e_0^{-3} \\ & + Q^1 e_0 * Q^1 e_1 * e_0^{-3} + Q^1 e_0 * Q^2 e_0 * e_0^{-3} + Q^1 e_0 * Q^1 Q^1 e_0 * e_0^{-7}, \end{aligned}$$

where e_n generates $H_n SO(3)$.

REMARK 4.21. The restriction map $G(S^1) \to G(\mathbf{Z}_{p^n})$, regarded as a map $Q(\mathbf{C}P^\infty_+ \wedge S^1) \to QB\mathbf{Z}^+_{p^n}$, is by Theorem 3.10 the infinite loop extension of the transfer $t_n\colon \mathbf{C}P^\infty_+ \wedge S^1 \to QB\mathbf{Z}^+_{p^n}$. In homology it therefore commutes with loop products and loop Dyer-Lashof operations. Since these generate $H_*(Q(\mathbf{C}P^\infty_+ \wedge S^1))$ from $\overline{H}_*(\mathbf{C}P^\infty_+ \wedge S^1)$, we have completely analyzed the homology of the restriction maps $G(S^1) \to G(\mathbf{Z}_{p^n})$ and $G(S^1) \to G(1)$.

5. Equivariant J-homomorphisms. We shall study maps

$$j_{\mathbf{Z}_2}\colon O \to Q\mathbf{R}P^\infty_+, \qquad j_{S^1}\colon U \to Q(\mathbf{C}P^\infty_+ \wedge S^1)$$

arising from the fact that orthogonal transformations are $\mathbf{Z}_2$-equivariant and unitary transformations are S^1-equivariant. Define a map $\lambda_{\mathbf{R}}\colon \mathbf{R}P^{n-1} \to O(n)$ by sending a real line $l \subset \mathbf{R}^n$ to the reflection through the hyperplane perpendicular to l. Define a map $\mathbf{C}P^{n-1}_+ \wedge S^1 \to U(n)$ by sending (l, z), where $l \subset \mathbf{C}^n$ is a complex line and $z \in \mathbf{C}$ has $|z| = 1$, to the unitary transformation which is the identity on $l^\perp$ and multiplication by z on l. Also, let $\lambda\colon \mathbf{R}P^{n-1} \to SO(n)$ be $\lambda_{\mathbf{R}}$ composed with the reflection R through the hyperplane $x_1 = 0$ (if $x_1, \ldots, x_n$ are the coordinates in $\mathbf{R}^n$). These maps are compatible as n varies, and give maps

$$\lambda_{\mathbf{R}}\colon \mathbf{R}P^\infty \to O, \quad \lambda\colon \mathbf{R}P^\infty \to SO, \quad \lambda_{\mathbf{C}}\colon \mathbf{C}P^\infty_+ \wedge S^1 \to U.$$

Let $i\colon S^0 \to \mathbf{R}P^\infty_+$ be induced by the inclusion $1 \subset \mathbf{Z}_2$, and let $t\colon \mathbf{R}P^\infty_+ \to QS^0$ be the associated transfer. Let $t_\lambda\colon \mathbf{R}P^\infty_+ \to Q\mathbf{R}P^\infty_+$ be the transfer associated to the identity map of $\mathbf{R}P^\infty$ with the framing twisted by λ (as in Note 1.17). Any infinite loop space has an involution χ obtained by smashing with $-1 \in QS^0$. Let $\iota\colon X \to QX$ denote the standard inclusion, and $\Delta\colon X \to X \times X$ the diagonal map, for any space X.

THEOREM 5.1. *With these notations,*
(a) $j_{\mathbf{Z}_2} \circ \lambda_{\mathbf{R}} \simeq \chi\iota\colon \mathbf{R}P^\infty_+ \to Q\mathbf{R}P^\infty_+$.
(b) *The following three maps are homotopic.*

(i) $\mathbf{R}P^\infty_+ \xrightarrow{\lambda} SO \xrightarrow{j_{\mathbf{Z}_2}} Q\mathbf{R}P^\infty_+$,

(ii) $\mathbf{R}P^\infty_+ \xrightarrow{\Delta\iota} Q\mathbf{R}P^\infty_+ \times Q\mathbf{R}P^\infty_+ \xrightarrow{i\bar{t}*\chi} Q\mathbf{R}P^\infty_+ \xrightarrow{*(-1)} Q\mathbf{R}P^\infty_+$,

(iii) $\mathbf{R}P^\infty_+ \xrightarrow{t\lambda} Q\mathbf{R}P^\infty_+ \xrightarrow{*(-1)} Q\mathbf{R}P^\infty_+$.

(c) $j_{S^1} \circ \lambda_{\mathbf{C}} \simeq \iota\colon \mathbf{C}P^\infty_+ \wedge S^1 \to Q(\mathbf{C}P^\infty_+ \wedge S^1)$.

Before giving the proofs, we note some corollaries.

COROLLARY 5.2. *In the stable category,* $\mathbf{C}P^\infty_+ \wedge S^1$ *is a retract of* U. □

COROLLARY 5.3. (a) *In* mod 2 *homology,*

$$j_{\mathbf{Z}_2^*} e_r = \sum_{s+t=r} Q^s e_0 * \chi e_t * e_0^{-1}.$$

(b) *In homology with any coefficients,*

$$j_{S^{1*}} \bar{a}_r = \bar{a}_r. \qquad \square$$

Here we are writing e_r for $\lambda_* e_r$ and $\bar{a}_r$ for $\lambda_{\mathbf{C}^*} \bar{a}_r$.

PROOF OF 5.1(a) Let $-\lambda_{\mathbf{R}}\colon \mathbf{R}P^{2n-1} \to O(2n)$ be $\lambda_{\mathbf{R}}$ composed with the antipodal map $A \in SO(2n)$. Since $SO(2n)$ is connected, $-\lambda_{\mathbf{R}}$ is homotopic to $\lambda_{\mathbf{R}}$. We apply (3.5) to the map

$$f = j_{\mathbf{Z}_2} \circ (-\lambda_{\mathbf{R}})\colon \mathbf{R}P^{2n-1} \to \mathrm{End}_{\mathbf{Z}_2}(S^{2n-1}).$$

The adjoint map $f\colon \mathbf{R}P^{2n-1} \times S^{2n-1} \to S^{2n-1}$ is smooth, and the associated map f' is transverse to the diagonal $\overline{\Delta}$ (see (3.5)), with pull-back $S^{2n-1}/\mathbf{Z}_2$:

$$\begin{tikzcd}
S^{2n-1}/\mathbf{Z}_2 \arrow[r, "="] \arrow[d, "\Delta"] \arrow[dd, bend right=60, "p"'] & S^{2n-1}/\mathbf{Z}_2 \arrow[d, "\overline{\Delta}"] \\
\mathbf{R}P^{2n-1} \times S^{2n-1}/\mathbf{Z}_2 \arrow[r, "f'"] \arrow[d, "\mathrm{pr}_1"] & (S^{2n-1} \times S^{2n-1})/\mathbf{Z}_2 \\
\mathbf{R}P^{2n-1} &
\end{tikzcd}$$

The composite p is clearly the identity. The framing, however, is nontrivial; it is twisted because $\nu(\overline{\Delta})$ is pulled back across the degree -1 map f'. Thus

$$j_{\mathbf{Z}_2} \circ \lambda_{\mathbf{R}} \simeq \chi\iota$$

by Note 1.17.

The proof of 5.1(c) is similar, except that the analogue of f' has degree $+1$.

PROOF OF 5.1(b). By Lemma 2.8, we have a homotopy-commutative diagram

$$\begin{tikzcd}
QS^0 \times Q\mathbf{R}P^\infty_+ \arrow[r, "i\times 1"] \arrow[d, "1\wedge\bar{t}"] & Q\mathbf{R}P^\infty_+ \times Q\mathbf{R}P^\infty_+ \arrow[d, "\#"] \\
QS^0 \arrow[r, "i"] & Q\mathbf{R}P^\infty_+
\end{tikzcd}$$

where $\#$ is the diagonal transfer as in Theorem 3.11. Thus the map $(-1)\#\colon Q\mathbf{R}P^\infty_+ \to Q\mathbf{R}P^\infty_+$ is homotopic to $i\bar{t}\chi$. By Theorem 3.11, therefore, the lower right-hand

box in the following diagram commutes.

$$\begin{tikzcd}
\mathbf{R}P^\infty_+ \arrow[r, "\iota"] \arrow[d, "\lambda_{\mathbf{R}}"] \arrow[dd, bend right=60, "\lambda"'] & Q\mathbf{R}P^\infty_+ \arrow[r, "\Delta"] \arrow[d, "\chi"] & Q\mathbf{R}P^\infty_+ \times Q\mathbf{R}P^\infty_+ \arrow[d, "\chi\times\chi"] \arrow[dd, bend left=60, "i\bar{t}*\chi"] \\
O \arrow[r, "j_{\mathbf{Z}_2}"] \arrow[d, "R\circ"] & Q\mathbf{R}P^\infty_+ \arrow[r, "\Delta"] \arrow[d, "(-1)\circ"] & Q\mathbf{R}P^\infty_+ \times Q\mathbf{R}P^\infty_+ \arrow[d, "i\bar{t}\chi*1"] \\
O \arrow[r, "j_{\mathbf{Z}_2}"] & Q\mathbf{R}P^\infty_+ & Q\mathbf{R}P^\infty_+ \arrow[l, "*(-1)"']
\end{tikzcd}$$

The upper left-hand box commutes by (5.1)(a), and the rest of the diagram commutes for trivial reasons. This proves (i)$\simeq$(ii).

To see that (ii)$\simeq$(iii), we recall that E. H. Brown has proven [**4**] that t_λ is homotopic to the composite

$$\mathbf{R}P^\infty_+ \xrightarrow{\iota} Q\mathbf{R}P^\infty_+ \xrightarrow{\Delta} Q\mathbf{R}P^\infty_+ \times Q\mathbf{R}P^\infty_+ \xrightarrow{i\bar{t}*\chi} Q\mathbf{R}P^\infty_+.$$

The result follows. $\square$

Theorem 3.10 implies that

$$\begin{tikzcd}
U \arrow[r, "j_{S^1}"] \arrow[d, "J_{\mathbf{C}}"] \arrow[dd, bend right=60, "j_{\mathbf{C}}"'] & Q(\mathbf{C}P^\infty_+ \wedge S^1) \arrow[d] \\
Q_1(S^0) \arrow[d, "*(-1)"] & \\
Q_0(S^0) \arrow[r, "\mathrm{id}"] & Q(S^0)
\end{tikzcd}$$

commutes up to homotopy. Adjoint to this is the stable diagram:

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "j_{\mathbf{C}}"] & \mathbf{C}P^\infty_+ \wedge S^1 \arrow[d, "t"] \\
S^0 \arrow[r, "\mathrm{id}"] & S^0
\end{tikzcd} \tag{5.4}$$

Theorem C of the Introduction follows from this and Theorems 4.4 and 4.5. Also, (5.4) together with 5.1(c) yield the homotopy commutative stable diagram:

$$\begin{tikzcd}
\mathbf{C}P^\infty_+ \wedge S^1 \arrow[r, "\lambda"] \arrow[d, "t"] & U \arrow[d, "j_{\mathbf{C}}"] \\
S^0 \arrow[r, "\mathrm{id}"] & S^0
\end{tikzcd}$$

6. The local structure of $H_*(SG(S^1);\mathbf{Z}_p)$. In this section we describe formulae sufficient to characterize the R-Hopf algebra structure of $H_*(SG(S^1);\mathbf{Z}_p)$. We rely on [**8**]. There we construct, for a *finite* group H, an E_∞-ring space $A(H)$, called the *Burnside space* of H. There is a homotopy equivalence of infinite loop spaces

$$\prod_K Q(BW^+)K) \simeq A(H), \tag{6.1}$$

where K ranges over a set of subgroups of H representing its conjugacy classes of subgroups, and W_k is the quotient by K of the normalizer of K in H. There is thus a map

$$i\colon QBH^+ \to A(H), \tag{6.2}$$

which is up to homotopy a split monomorphism. The ring-structure map μ in $A(H)$ extends the transfer $\#$ induced as in Theorem 3.11 by the diagonal inclusion $\Delta\colon H \to H \times H$; that is to say, the diagram

$$\begin{array}{ccc} QBH^+ \times QBH^+ & \xrightarrow{\#} & QBH^+ \\ \downarrow i\times i & & \downarrow i \\ A(H) \times A(H) & \xrightarrow{\mu} & A(H) \end{array} \tag{6.3}$$

is homotopy-commutative.

If H admits a finite-dimensional orthogonal representation W such that the unit sphere sW is a free H-space, then a map $\rho\colon G(H) \to A(H)$ is constructed. This map carries the infinite loop-space structure [**3**, **8**] on $SG(H)$ to the multiplicative structure in $A(H)$. Furthermore, if $*(-1)$ denotes the evident component-shifting self-map of $A(H)$, then

$$\begin{array}{ccc} G(H) & \xrightarrow{\gamma} & QBH^+ \\ \rho\downarrow & & \downarrow i \\ A(H) & \xrightarrow{*(-1)} & A(H) \end{array}$$

is homotopy-commutative. In particular, ρ is (up to homotopy) a split monomorphism.

It follows that relations between the composition structure and the loop structure in $H_*(SG(H))$ can be obtained by translating the usual distributivity and mixed Cartan and Adem relations valid in $H_*(A(H))$. To describe the translation, let

$$\#\colon H_*(A(H)) \otimes H_*(A(H)) \to H_*(A(H))$$

be the multiplicative product μ_*; on elements in the image of $H_*(QBH^+)$ it is induced by the diagonal transfer. Let $\hat{Q}^r$ be the multiplicative Dyer-Lashof operation in $H_*(A(H))$, while $*$ and Q^r denote the additive (loop) product and Dyer-Lashof operation. Let $\circ$ and $\tilde{Q}^r$ denote the (composition) product and (composition) Dyer-Lashof operation in $SG(H)$. Then

LEMMA 6.5. *Omitting "ρ_*",*

$$x \circ y = \sum (-1)^{|x''|\,|y'|} x' * y' * (x''\#y''),$$

$$\tilde{Q}^r(x * y) = \sum_{i+j+k=r} \tilde{Q}^i x' * Q^j(x''\#y') * \tilde{Q}^k(y''),$$

where $\Delta x = \Sigma X' \otimes x''$ *and* $\Delta y = \Sigma y' \otimes y''$.

PROOF. The first expression is due (in the more general case of a periodic compact Lie group) to R. E. Schultz [**11**]; it follows from his Theorem 3.11 above. It follows also from (6.3) and Hopf-ring distributivity:

$$\begin{aligned}(x \circ y) * [1] &= (x * [1])\#(y * [1]) \\ &= \sum (-1)^{|x''|\,|y'|} x' * y' * (x''\#y'') * [1].\end{aligned}$$

The second expression follows similarly from (6.3), the mixed Cartan formula, and the fact that $\hat{Q}^u[1] = 0$ for $u > 0$. $\square$

In principle, the structural formulae valid in the homology of an E_∞-ring space could now be translated into formulae relating $\circ$ and $\tilde{Q}^r$ to $*$ and Q^r. These would determine the structure of $H_*(SG(H))$, give $x \circ y$ and $\tilde{Q}^r x$ for $x, y \in H_*(BH)$. In view of the complexity of the mixed Adem relation, however, we shall not complete this exercise here.

We turn now to S^1.

Since the infinite loop map $i\colon SG(S^1) \to SG(\mathbf{Z}_p)$ is injective in homology, the formulae considered above hold also in $H_*(SG(S^1))$. Indeed, since i_* was computed in §4, while the R-Hopf algebra structure of $H_*(SG(\mathbf{Z}_2))$ was computed in [**8**], we "know" the R-Hopf algebra structure of $H_*(SG(S^1); \mathbf{Z}_2)$; and similarly, for p odd, we "know" the Hopf algebra structure of $H_*(SG(S^1); \mathbf{Z}_p)$ and some information about its R-module structure. We wish to be more explicit, however; and, by the indicated formulae, it will suffice to compute $\bar{a}_q \circ \bar{a}_r$ and $\tilde{Q}^q \bar{a}_r$, where as in §4, $\bar{a}_r$ is the canonical generator of $H_{2r+1}(\mathbf{C}P^\infty_+ \wedge S^1; \mathbf{Z}_p)$.

THEOREM 6.6. *Let $p = 2$. In $H_*(SG(S^1)) \cong H_*(Q(\mathbf{C}P^\infty_+ \wedge S^1))$,*

(a)
$$\bar{a}_q \circ \bar{a}_r = \bar{a}_q * \bar{a}_r + \sum_s (q-s, r_s) Q^{2(q+r-s)+1} \bar{a}_s + \sum_t (q-2t, r-2t) Q^{q+r+1} Q^{q+r-2t} \bar{a}_t.$$

(b)
$$\tilde{Q}^{2q} \bar{a}_r = \binom{q-1}{r} \bar{a}_{q+r}, \qquad \hat{Q}^{2q+1} \bar{a}_r = 0.$$

THEOREM 6.7. *Let p be odd. In $H_*(SG(S^1)) \cong H_*(Q(\mathbf{C}P^\infty_+ \wedge S^1))$,*

(a)
$$\bar{a}_q \circ \bar{a}_r = \bar{a}_q * \bar{a}_r + \sum_t c(q,r,t) \beta Q^t \bar{a}_{q+r+1-(p-1)t},$$

where

$$c(q,r,t) = \sum_k (-1)^{r+k} \binom{(p-1)k-1}{r} \binom{q+r+1-(p-1)t}{t-k}.$$

(b)
$$\tilde{Q}^q \bar{a}_r = (-1)^{q+r+1} \binom{q-1}{r} \bar{a}_{r+q(p-1)}.$$

PROOF OF THEOREM 6.6. We apply Lemma 2.9 with $H = S^1$. Note that $\chi\colon H_*(\mathbf{C}P^\infty) \to H_*(\mathbf{C}P^\infty)$ is the identity (with mod 2 coefficients), and that $\bar{\mu}_*\colon H_*(\mathbf{C}P^\infty_+) \otimes H_*(\mathbf{C}P^\infty_+ \wedge S^1) \to H_*(\mathbf{C}P^\infty_+ \wedge S^1)$ is given by

$$\bar{\mu}_*(a_u \otimes \bar{a}_r) = (u,r) \bar{a}_{r+u}.$$

Thus,

$$\bar{a}_q \# \bar{a}_r = \sum_u (u,r) \bar{a}_{q-u} \wedge t_* \bar{a}_{r+u}. \tag{6.8}$$

According to Theorem 4.4, $t_*\bar{a}_{r+u}$ is a sum of two sorts of terms. For the first, we use

$$\Delta\bar{a}_{q-u} = \bar{a}_{q-u} \otimes 1 + 1 \otimes \bar{a}_{q-u},$$
$$1 \wedge \chi Q^t[1] = \delta_0^t,$$
$$\mathrm{Sq}^{2t}\bar{a}_{q-u} = \binom{q-u-t}{t}\bar{a}_{q-u-t},$$

to find

$$\bar{a}_{q-u} \wedge (Q^{2s+1}[1] * \chi Q^{2(u+r-s)}[1]) = \delta_s^{u+r} \sum_t \binom{q-u-t}{t} Q^{2s+2t+1}\bar{a}_{q-u-t}. \tag{6.9}$$

Similarly, for the second we find

$$\bar{a}_{q-u} \wedge (Q^{s+1}Q^s[1] * (\chi Q^{u+r-s}[1])^{*2}) = \delta_s^{u+r} \sum_{i,t} \binom{q-u-i}{i}\binom{q-u-i-t}{t} Q^{s+2i+1}Q^{s+2t}\bar{a}_{q-u-i-t}.$$

Now Q^{s+2i+1} kills this dimension unless $2i \geq q-u$, while the first binomial coefficient is zero unless $2i \leq q-u$. The expression is thus trivial unless $q-u$ is even, and then

$$\bar{a}_{2v} \wedge (Q^{s+1}Q^s[1] * (\chi Q^{u+r-s}[1])^{*2}) = \delta_s^{u+r} \sum_t \binom{v-t}{t} Q^{s+q-u+1}Q^{s_2t}\bar{a}_{v-t}. \tag{6.10}$$

Substituting (6.9) and (6.10) into (6.8),

$$\bar{a}_q\#\bar{a}_r = \sum_{u,t}(u,r)\binom{q-u-t}{t} Q^{2(u+r+t)+1}\bar{a}_{q-u-t} + \sum_{v,t}(q-2v,r)\binom{v-t}{t} Q^{q+r+1}Q^{q+r-2(v+t)}\bar{a}_{v-t}.$$

Now, for fixed s,

$$\sum_{u+t=q-s}(u,r)\binom{q-u-t}{t} = (q-s, r-s),$$
$$\sum_{v-t=s}(q-2v,r)\binom{v-t}{t} = (q-2s, r-2s),$$

so we conclude that

$$\bar{a}_q\#\bar{a}_r = \sum_s (q-s,r-s)Q^{2(q+r-s)+1}\bar{a}_s + \sum_s (q-2s,r-2s)Q^{q+r+1}Q^{q+r-2s}\bar{a}_s.$$

Now $\bar{a}_q$ and $\bar{a}_r$ are primitive, so by Lemma 6.5,

$$\bar{a}_q \circ \bar{a}_r = \bar{a}_q * \bar{a}_r + \bar{a}_q \# \bar{a}_r,$$

and part (a) of the theorem follows

Part (b) follows from Kochman's formula [**7**] and the fact that $j_{S^1} : U \to SG(S^1)$ is an infinite-loop map [**3**]. □

The proof of Theorem 6.7 is analogous and is left to the reader. □

REMARK 6.11. From (6.6) we find that

$$\bar{a}_r \circ \bar{a}_r = \bar{a}_r * \bar{a}_r + Q^{2r+1}\bar{a}_r = 0$$

which is consistent with Corollary 5.3. Indeed, it can be used to prove that $j_{S^1*}\bar{a}_r = \bar{a}_r$ at $p = 2$.

7. The global structure of $H_*(SG(S^1))$ and of $H_*(BSG(S^1))$. Recall the *weight valuation*

$$w\colon H_*(QS^0; \mathbf{Z}_p) \to \overline{\mathbf{N}},$$

where $\overline{\mathbf{N}} = \{0, 1, \ldots, \infty\}$. It is the smallest function with this source and target satisfying

$$w(x * y) \geq w(x) + w(y), \tag{7.1}$$

$$\begin{aligned} &w(x + y) \geq \min\{w(x), w(y)\}, \\ &w(Q^I[1]) = p^{l(I)} \quad \text{for } I \text{ admissible with } e(I) > 0. \end{aligned} \tag{7.2}$$

(We refer to [**5**, pp. 16, 42] for definitions of length, admissibility, and excess, for Dyer-Lashof operations.) Thus in particular $w(0) = \infty$ and $w([n]) = 0$.

Similarly, we define

$$w\colon H_*(QB\mathbf{Z}_p^+; \mathbf{Z}_p) \to \overline{\mathbf{N}},$$

respectively

$$w\colon H_*(Q(\mathbf{C}P_+^\infty \wedge S^1); \mathbf{Z}_p) \to \overline{\mathbf{N}},$$

to be the smallest such functions satisfying (7.1), (7.2), and, for I admissible,

$$\begin{aligned} w(Q^I e_r) &= p^{l(I)+1} \quad \text{for } e(I) > r \geq 0, \text{ and } l(I) > 0, \\ w(e_r) &= p \qquad\qquad \text{for } r > 0, \\ w(e_0) &= 0, \end{aligned}$$

respectively,

$$w(Q^I \bar{a}_r) = p^{l(I)+1} \quad \text{for } e(I) > 2r + 1 \geq 1.$$

In all three cases it is clear that

$$w(x_1 * \cdots * x_r) = w(x_1) + \cdots + w(x_r) \tag{7.3}$$

if $x_1 * \cdots * x_r$ is nonzero and each x_i is such an element, $Q^I[1], Q^I e_r$, or $Q^I \bar{a}_r$.

It is easy to check, using Theorems 4.4 and 4.5, that the transfer map

$$\begin{tikzcd} H_*(Q(\mathbf{C}P_+^\infty \wedge S^1)) \arrow[r, "t_{1*}"] \arrow[dr, "t_{0*}"'] & H_*(QB\mathbf{Z}_p^+) \arrow[d, "t_*"] \\ & H_*(QS^0) \end{tikzcd}$$

preserve weight: $w(\theta_*(x)) \geq w(x)$ for $\theta = t_0, t_1$, or t. Thus it makes sense to compute them modulo higher weight, and we have the following lemma.

LEMMA 7.4. *For any p,*

$$t_{1*}Q^I\bar{a}_r \equiv Q^I e_{2r+1} * e_0^{-p^{l(I)+1}},$$

and for $p = 2$,

$$t_{0*}Q^I\bar{a}_r \equiv Q^I Q^{2r+1}[1] * [-2^{l(I)+1}],$$

mod *higher weight.*

PROOF. For I empty, these are immediate from Theorems 4.4 and 4.5. The Adem relation for Q^nQ^0 implies that for any $n > 0$ and any $k \in \mathbf{Z}$, $w(Q^n[pk]) \geq p^2$. It follows by induction that

$$Q^I(Q^{2r+1}[1] * [-p]) \equiv Q^I Q^{2r+1}[1] * [-p^{l(I)+1}]$$

mod higher weight. The assertion for t_{0*} follows, and the other case is similar. □

As a corollary, we have the injectivity needed in §6:

COROLLARY 7.5. *For $p = 2$,*

$$t_{0*}\colon H_*(Q(\mathbf{C}P_+^\infty \wedge S^1)) \to H_*(QS^0)$$

is injective, and for p arbitrary,

$$t_{1*}\colon H_*(Q(\mathbf{C}P_+^\infty \wedge S^1)) \to H_*(QB\mathbf{Z}_p^+)$$

is injective. □

Next we note the behavior of the weight valuation with respect to the composition product in $SG(S^1) \simeq Q(\mathbf{C}P_+^\infty \wedge S^1)$.

LEMMA 7.6. *In $H_*(Q(\mathbf{C}P_+^\infty \wedge S^1))$,*

(a) *$w(x\#y) \geq w(x) + w(y)$, with equality if and only if $p = 2$ and $w(x) = 2 = w(y)$; i.e., $x = \bar{a}_s, y = \bar{a}_t$.*

(b) *$w(x \circ y) \geq w(x) + w(y)$.*

(c) *$x \circ y \equiv x * y$ modulo higher weight unless $p = 2, x = \bar{a}_s, y = \bar{a}_t$.*

PROOF. (a) By the mixed Cartan formula, $Q^I\bar{a}_s\#Q^J\bar{a}_t$ is a sum of terms of the form $Q^K(\bar{a}_i\#\bar{a}_j)$, where $l(K) = l(I) + l(J)$. Substituting in the value of $\bar{a}_i\#\bar{a}_j$ from Theorems 6.6 and 6.7 we find that each term in this sum has weight at least $p^{l(I)+l(J)+2}$. Now

$$p^{l(I)+l(J)+2} \geq p^{l(I)+1} + p^{l(J)+2},$$

and the inequality is strict unless $p = 2$ and $l(I) = 0 = l(J)$. This proves (a) for such elements. If $x_1, \ldots, x_q, y_1, \ldots, y_r$, are such elements, then each is primitive, so the distributivity formula implies that

$$\begin{aligned}(x_1 * \cdots * x_q)\#(y_1 * \cdots * y_r) &= 0 \quad \text{if } q \neq r \\ &= \sum_{\sigma\in\Sigma_q} \pm(x_1\#y_{\sigma(1)}) * \cdots * (x_q\#y_{\sigma(q)}) \quad \text{if } q = r\end{aligned}$$

and the result holds by (7.3).

(b) and (c). Since $Q^I\bar{a}_s$ and $Q^J\bar{a}_t$ are primitive,

$$Q^I\bar{a}_s \circ Q^J\bar{a}_t = Q^I\bar{a}_s * Q^J\bar{a}_t + Q^I\bar{a}_s\#Q^J\bar{a}_t$$

by Lemma 6.5. By (a), this is congruent to $Q^I\bar{a}_s * Q^J\bar{a}_t$ mod higher weight, with the noted exceptions. Again, the extension to monomials is easy. □

As a corollary, we have

THEOREM 7.7. *$H_*(SG(S^1))$ is a primitively generated Hopf algebra. For $p = 2$,*

$$H_*(SG(S^1)) \cong H_*(U) \otimes P[\bar{a}_r^{*2} : r \geq 0] \\ \otimes H_*(Q(\mathbf{C}P_+^\infty \wedge S^1))//P[\bar{a}_r : r \geq 0],$$

and for p odd,

$$H_*(SG(S^1)) \cong H_*(Q(\mathbf{C}P_+^\infty \wedge S^1)),$$

as Hopf algebras. □

We now study $H_*(BSG(S^1))$ by means of the classifying space spectral sequence

$$\mathrm{Tor}_{s,t}^{H_*(SG(S^1))}(\mathbf{Z}_p, \mathbf{Z}_p) \Rightarrow H_*(BSG(S^1)). \tag{7.8}$$

This is a first quadrant homology spectral sequence of Hopf algebras. Consider first, among the generators with $s > 1$, the divided powers of the suspension $\sigma\bar{a}_r$ of $\bar{a}_r$. The equivariant J-homomorphism $j_{S^1} : U \to SG(S^1)$ maps onto these elements, by Corollary 5.3. Since the spectral sequence

$$\mathrm{Tor}^{H_*(U)}(\mathbf{Z}_p, \mathbf{Z}_p) \Rightarrow H_*(BU)$$

collapses at E^2, we conclude that these generators are permanent cycles.

For $p = 2$, there are no further generators with $s > 1$, so the spectral sequence (7.8) collapses at E^2. For $p \neq 2$, each odd generator in $H_*(SG(S^1))$ leads to a divided power sequence in E^2. These are connected by the universal differential [**5**, p. 125]

$$d^{p-1}\gamma_{p+j}(\sigma x) = -(\sigma\beta\tilde{Q}_1 x)\gamma_j(\sigma x).$$

Here, if $2s = |x| + 1$, then $\tilde{Q}_1 x = u\tilde{Q}^s x$ for some unit $u \in \mathbf{Z}_p$. To compute $\tilde{Q}_1$, we have

LEMMA 7.9. *In $H_*(SG(S^1))$, for I admissible with $e(I) > 2r+1$ and $l(I) > 0$,*

$$\tilde{Q}_1 Q^I \bar{a}_r \equiv Q_1 Q^I \bar{a}_r$$

mod *higher weight.*

This is a routine exercise with the weight valuation, and is described in more detail in [**9**], so its proof is omitted here. This lemma results in

$$E^p \cong \Gamma[\sigma\bar{a}_r : r \geq 0] \otimes E[\sigma\beta Q^s \bar{a}_r : s > r \geq 0] \\ \otimes D[\sigma Q^I a_r : l(I) \geq 1,\ e(I) > 2r+1,\ r \geq 0],$$

where D denotes the free commutative algebra truncated at height p. No further differentials are now possible: $E^p = E^\infty$.

Lemma 7.9, together with the equivariant J-homomorphism, and the next lemma when $p = 2$, determines the multiplicative extension, and Theorem F of the Introduction results.

LEMMA 7.10. *For* $p = 2$

$$\tilde{Q}^{4r+3}Q^{2r+1}\bar{a}_r = 0$$

in $H_*(SG(S^1))$.

PROOF. Using Lemma 6.5 and the fact that $\bar{a}_r$ is primitive, we expand

$$\tilde{Q}^{4r+3}Q^{2r+1}\bar{a}_r = \tilde{Q}^{4r+3}(\bar{a}_r^{*2})$$

into a sum in which each term appears exactly twice. Since we are working mod 2, the lemma follows. □

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