ON ALGEBRAS WITH CONVOLUTION STRUCTURES
FOR LAGUERRE POLYNOMIALS

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ABSTRACT. In this paper we treat the convolution algebra connected with
Laguerre polynomials which was constructed by Askey and Gasper [1]. For
this algebra, we study the maximal ideal space, Wiener's general Tauberian
theorem, spectral synthesis and Helson sets. We also study Sidon sets and
idempotent measures for the algebras with dual convolution structures.

1. Introduction and preliminary results. Let $\alpha > -1$ and let n be a nonnegative integer. Let $L_n^\alpha(x)$ denote the Laguerre polynomial defined by

$$L_n^\alpha(x) = \frac{e^x x^{-\alpha}}{n!} \left(\frac{d}{dx} \right)^n (e^{-x} x^{n+\alpha}).$$

Laguerre polynomials have the following properties:

$$L_n^\alpha(0) = \binom{n+\alpha}{n},$$

$$\int_0^\infty L_m^\alpha(x) L_n^\alpha(x) e^{-x} x^\alpha \, dx = \Gamma(\alpha+1) \binom{n+\alpha}{n} \delta_{mn},$$

where $\binom{p}{q} = p(p-1)\cdots(p-q+1)/q!$ and δ_{mn} is Kronecker's symbol. Denote by $R_n^\alpha(x)$ the normalized Laguerre polynomial so that

$$R_n^\alpha(x) = L_n^\alpha(x)/L_n^\alpha(0).$$

The purpose of this paper is to study some structures of convolution algebras connected with Laguerre polynomials, e.g., maximal ideal spaces, Helson sets, idempotent measures, spectral synthesis of the set of one point, etc.

Askey and Gaspar [1] proved the following:

(A) [1, Theorem 2] If $\alpha \geq -1/2$ and $\tau \geq 2$ or if $\alpha \geq \alpha_0 = (-5 + (17)^{1/2})/2$ and $\tau \geq 1$, then

$$(1) \qquad e^{-\tau x} R_m^\alpha(x) R_n^\alpha(x) = \sum_{k=0}^\infty D_k^\alpha(m, n; \tau) R_k^\alpha(x), \qquad x \geq 0,$$

with $D_k^\alpha(m, n; \tau) \geq 0$, where the series $\sum_{k=0}^\infty D_k^\alpha(m, n; \tau) R_k^\alpha(x)$ converges for every $x \geq 0$.

They also constructed a convolution algebra as follows. Let l be the space of absolutely convergent sequences $a = \{a_n\}_{n=0}^{\infty}$, $\sum_{n=0}^{\infty} |a_n| < \infty$ with norm $\|a\| = \sum_{n=0}^{\infty} |a_n|$. For a and b in l, define the convolution $a * b$ by

$$(2) \qquad (a * b)_k = \sum_{m,n=0}^{\infty} a_m b_n D_k^{\alpha}(m, n; \tau), \qquad k = 0, 1, 2, \ldots.$$

Then $\|a * b\| \leq \|a\| \|b\|$, since $\sum_{k=0}^{\infty} |D_k(m, n; \tau)| = 1$ by (A). Denote by $l^{(\alpha,\tau)}$ the algebra l with the convolution $*$. Then $l^{(\alpha,\tau)}$ is a commutative Banach algebra.

We define the function $\hat{a}(x)$ on $[0, \infty)$ by

$$\hat{a}(x) = \sum_{n=0}^{\infty} a_n R_n^{\alpha}(x) e^{-\tau x}$$

for every $a = \{a_n\}_{n=0}^{\infty}$ in $l^{(\alpha,\tau)}$. Since $|R_n^{\alpha}(x) e^{-x}| \leq 1$, $x \geq 0$ for $\alpha \geq 1/2$ [1, (5.9)], the function $\hat{a}(x)$ is continuous on $[0, \infty)$ and $\lim_{x \to \infty} \hat{a}(x) = 0$ for every a in $l^{(\alpha,\tau)}$. We put

$$A^{(\alpha,\tau)} = \{\hat{a}; a \text{ in } l^{(\alpha,\tau)}\}.$$

In §2, first we will shw that $A^{(\alpha,\tau)}$ is a Banach algebra which is isomorphic and isometric to $l^{(\alpha,\tau)}$ if we introduce the product of pointwise multiplication of functions and the norm $\|\hat{a}\| = \|a\|$ to $A^{(\alpha,\tau)}$. Next we will determine the maximal ideal space of $A^{(\alpha,\tau)}$ and have a result analogous to Wiener's general Tauberian theorem. In §3, we will study problems of Helson sets and spectral synthesis for $A^{(\alpha,\tau)}$ which have originally arisen from the algebra $A(\mathbf{T})$ of absolutely convergent Fourier series (cf. Kahane [8]). By means of our results in §3, we may obtain that the structure of $A^{(\alpha,\tau)}$ is simpler than that of $A(\mathbf{T})$ and is similar to that of the algebra of absolutely convergent Jacobi polynomial series or the algebra of Hankel transforms. See Igari and Uno [7] and Schwartz [11].

We will also consider the algebras with dual convolution structures for Laguerre polynomials.

Görlich and Markett [5] introduced the spaces $L_{W(\alpha)}^p$, $\alpha \geq 0$, of measurable functions on $[0, \infty)$ which are suitable for defining convolution structures for Laguerre polynomials $L_n^{\alpha}(x)$, $\alpha \geq 0$. We will deal with the space $L_{W(\alpha)}^1$ and denote it briefly by L_{α};

$$L_{\alpha} = \{f; \|f\| = \int_0^{\infty} |f(x)| e^{-x/2} x^{\alpha} \, dx < \infty\}, \qquad \alpha \geq 0.$$

For f and g in L_{α}, the convolution is defined by

$$f * g(t) = \int_0^{\infty} T_t^{\alpha}(f; x) g(x) e^{-x} x^{\alpha} \, dx, \qquad t \geq 0,$$

where $T_t^{\alpha}(f; x)$ is the Laguerre translation of f given by

$$(3) \quad T_t^{\alpha}(f; x) = \frac{\Gamma(\alpha + 1) 2^{\alpha}}{(2\pi)^{1/2}} \int_0^{\pi} f(x + t + 2(xt)^{1/2} \cos \theta) \exp(-(xt)^{1/2} \cos \theta)$$

$$\cdot \frac{J_{\alpha-1/2}((xt)^{1/2} \sin \theta)}{((xt)^{1/2} \sin \theta)^{\alpha-1/2}} \sin^{2\alpha} \theta \, d\theta$$

for $x, t > 0$, $T_t^{\alpha}(f; 0) = f(t)$ for $t > 0$, $T_0^{\alpha}(f; x) = f(x)$ for $x \geq 0$.

(B) [5, Theorem 1(ii)] Let $\alpha \geq 0$. Then L_α is a commutative Banach algebra; that is, $\|f * g\| \leq \|f\| \, \|g\|$ for f and g in L_α.

Let $M[0, \infty)$ be the space of all bounded regular Borel measures on $[0, \infty)$. Then the space $M[0, \infty)$ is a Banach space with total variation norm $\|\mu\| = \int_0^\infty d|\mu|(x)$ for μ in $M[0, \infty)$.

In §4, we will introduce a convolution structure to $M[0, \infty)$ so that L_α is included in $M[0, \infty)$ as a closed ideal by the mapping $f \mapsto \mu_f$ of L_α into $M[0, \infty)$, where $d\mu_f(x) = f(x)e^{-x/2}x^\alpha \, dx$, and denote by M_α the algebra $M[0, \infty)$ with this convolution structure. We will determine the maximal ideal spaces of L_α and M_α. In §5, we will study idempotent measures in M_α and Sidon sets for L_α. Although the Laguerre translation is not positive, our results show that the structures of M_α and L_α are similar to those of the algebras for ultraspherical polynomials or Jacobi polynomials with positive convolution structures. See Dunkl [2], Gasper [4], Igari [6], and also [11].

2. Algebras $l^{(\alpha, \tau)}$ and $A^{(\alpha, \tau)}$. Let C_c^∞ be the space of infinitely differentiable functions with compact support in $[0, \infty)$. First, we prove two lemmas.

LEMMA 1. *Let $\alpha > -1$ and let p be a positive integer. Then there is a constant C depending only on α and p such that*

$$\left| \int_0^\infty f(x) L_n^\alpha(x) e^{-x} x^\alpha \, dx \right| \leq C n^{(\alpha - p)/2} \sup_{0 \leq x} \left| \left(\frac{d}{dx} \right)^p f(x) \right|$$

for $f \in C_c^\infty$ and $n \geq p$.

PROOF. We use the identity

$$(4) \qquad n! \left(\frac{d}{dx} \right)^m (L_n^\beta(x) e^{-x} x^\beta) = (m+n)! L_{m+n}^{\beta - m}(x) e^{-x} x^{\beta - m}$$

(Erdélyi et al. [3, 10.12(28)]) and the inequality

$$\begin{aligned}
(5) \qquad |L_n^\alpha(x)| e^{-x/2} x^{\alpha/2} &\leq K n^\alpha x^{\alpha/2} \quad \text{(for } 0 \leq x < 1/n) \\
&\leq K n^{\alpha/2} x^{-1/4} (|\nu - x| + \nu^{1/3})^{-1/4} \\
&\qquad \qquad \qquad \text{(for } 1/n \leq x < 2\nu) \\
&\leq K n^{\alpha/2} e^{-Lx} \quad \text{(for } 2\nu \leq x),
\end{aligned}$$

where $\nu = 4n + 2\alpha + 2$, and K and L are positive constants not depending on n and x (cf. Muckenhoupt [9, (2.13)]). Put $A = \int_0^\infty f(x) L_n^\alpha(x) e^{-x} x^\alpha \, dx$. Then by (4) and integration by parts, we have

$$\begin{aligned}
A &= \frac{(n-p)!}{n!} \int_0^\infty f(x) \left(\frac{d}{dx} \right)^p (L_{n-p}^{\alpha+p}(x) e^{-x} x^{\alpha+p}) \, dx \\
&= \frac{(n-p)!(-1)^p}{n!} \int_0^\infty \left\{ \left(\frac{d}{dx} \right)^p f(x) \right\} L_{n-p}^{\alpha+p}(x) e^{-x} x^{\alpha+p} \, dx
\end{aligned}$$

since f has a compact support and

$$\left(\frac{d}{dx} \right)^{p-j} (L_{n-p}^{\alpha+p}(x) e^{-x} x^{\alpha+p})|_{x=0} = 0$$

for $j = 1, 2, 3, \ldots, p$. Thus

$$|A| \leq \frac{(n-p)!}{n!} \cdot \int_0^\infty |L_{n-p}^{\alpha+p}(x)| e^{-x} x^{\alpha+p} \, dx \cdot \sup_{0 \leq x} \left| \left(\frac{d}{dx} \right)^p f(x) \right|.$$

We write

$$\int_0^\infty |L_{n-p}^{\alpha+p}(x)| e^{-x} x^{\alpha+p} \, dx$$

$$= \left(\int_0^{1/(n-p)} + \int_{1/(n-p)}^{2\nu_p} + \int_{2\nu_p}^\infty \right) |L_{n-p}^{\alpha+p}(x)| e^{-x} x^{\alpha+p} \, dx$$

$$= I_1 + I_2 + I_3,$$

where $\nu_p = 4(n-p) + 2(\alpha+p) + 2$. By (5), we have

$$I_1 \leq K(n-p)^{\alpha+p} \int_0^{1/(n-p)} e^{-x/2} x^{\alpha+p} \, dx,$$

$$I_2 \leq K(n-p)^{(\alpha+p)/2} \int_0^\infty e^{-x/2} x^{(\alpha+p-1/2)/2} \, dx,$$

$$I_3 \leq K(n-p)^{(\alpha+p)/2} \int_0^\infty e^{-(L+1/2)x} x^{(\alpha+p)/2} \, dx,$$

and these inequalities complete the proof.

LEMMA 2. *Let $\alpha \geq -1/2$ and let τ be a real number.*
(i) *Let f be in C_c^∞ and put*

$$a_n = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty f(x) e^{\tau x} L_n^\alpha(x) e^{-x} x^\alpha \, dx.$$

Then the sequence $\{a_n\}_{n=0}^\infty$ belongs to l, and

$$f(x) = \sum_{n=0}^\infty a_n R_n^\alpha(x) e^{-\tau x}$$

for every $x \geq 0$.
(ii) *The sequences $\{a_n\}_{n=0}^\infty$ of all f in C_c^∞ are dense in l.*

PROOF. By Lemma 1, we have

$$|a_n| \leq \Gamma(\alpha+1)^{-1} C n^{(\alpha-p)-2} \sup_{0 \leq x} \left| \left(\frac{d}{dx} \right)^p (f(x) e^{\tau x}) \right|.$$

Since p is arbitrary, we have $\sum_{n=0}^\infty |a_n| < \infty$. By the equiconvergence theorem for Laguerre series for $x > 0$ and the summability theorem for Laguerre series at $x = 0$ (cf. Szegö [12, Theorems 9.1.5 and 9.1.7]), we have $f(x) = \sum_{n=0}^\infty a_n R_n^\alpha(x) e^{-\tau x}$ for every $x \geq 0$.

To prove (ii), it is enough to show that, for every $j = 0, 1, 2, \ldots, c(j)$ is approximated by sequences $\{a_n\}_{n=0}^\infty$ of functions in C_c^∞, where $c(j)_n = 0$ for $n \neq j$ and $c(j)_n = 1$ for $n = j$. Let $h(x)$ be a function of C_c^∞ such that $h(x) = 1$ for $0 \leq x \leq 1$, $0 < h(x) < 1$ for $1 < x < 2$ and $h(x) = 0$ for $2 \leq x$. Put

$f_k(x) = h(x/k)R_j^\alpha(x)e^{-\tau x}$ for every $k = 1, 2, 3, \ldots$. Then f_k belongs to C_c^∞. Define the sequence $a^{(k)} = \{a_n^{(k)}\}_{n=0}^\infty$ by

$$a_n^{(k)} = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty f_k(x)e^{\tau x}L_n^\alpha(x)e^{-x}x^\alpha\, dx.$$

We will show that $a^{(k)}$ converges to $c(j)$ in l as k tends to infinity. By Lemma 1, we have

$$|a_n^{(k)} - c(j)_n| = \frac{1}{\Gamma(\alpha+1)} \left| \int_0^\infty \left(h\left(\frac{x}{k}\right) - 1 \right) R_j^\alpha(x)L_n^\alpha(x)e^{-x}x^\alpha\, dx \right|$$

$$\leq \frac{1}{\Gamma(\alpha+1)} Cn^{(\alpha-p)/2} \sup_{0\leq x} \left| \left(\frac{d}{dx}\right)^p \left(h\left(\frac{x}{k}\right) - 1 \right) R_j^\alpha(x) \right|$$

for $n \geq p$, and thus $|a_n^{(k)} - c(j)_n| \leq C'n^{(\alpha-p)/2}$ for $n \geq p$ with a constant C' not depending on n and k. Since $a_n^{(k)} - c(j)_n$ converges to 0 as k tends to infinity and the series $\sum n^{(\alpha-p)/2}$ converges for large p, we have

$$\lim_{k\to\infty} \sum_{n=0}^\infty |a_n^{(k)} - c(j)_n| = \sum_{n=0}^\infty \lim_{k\to\infty} |a_n^{(k)} - c(j)_n| = 0.$$

This completes the proof.

Let $\alpha \geq -1/2$ and $\tau \geq 2$ or let $\alpha \geq \alpha_0 = (-5 + (17)^{1/2})/2$ and $\tau \geq 1$. By (1) and (2), we have

$$\hat{a}(x)\hat{b}(x) = \sum_{m,n} a_m b_n R_m^\alpha(x)R_n^\alpha(x)e^{-2\tau x}$$

$$= \sum_k \left\{ \sum_{m,n} a_m b_n D_k^\alpha(m, n; \tau) \right\} R_k^\alpha(x)e^{-\tau x}$$

$$= \sum_k (a * b)_k R_k^\alpha(x)e^{-\tau x} = (a * b)\hat{}(x)$$

for every $x \geq 0$. This shows that $A^{(\alpha,\tau)}$ is an algebra with the product of pointwise multiplication of functions. Let a be in $l^{(\alpha,\tau)}$ and suppose $\hat{a}(x) = 0$ on $[0, \infty)$. Then, for all f in C_c^∞, we have

$$0 = \int_0^\infty f(x)\hat{a}(x)e^{\tau x}e^{-x}x^\alpha\, dx$$

$$= \Gamma(\alpha+1) \sum_{n=0}^\infty \binom{n+\alpha}{n}^{-1} a_n b_n,$$

where

$$b_n = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty f(x)L_n^\alpha(x)e^{-x}x^\alpha\, dx.$$

Since the sequence $\{b_n\}_{n=0}^\infty$ is dense in l by Lemma 2(ii), we have $a_n = 0$ for all n. This enables us to define the norm of $\hat{a}$ in $A^{(\alpha,\tau)}$ by $\|\hat{a}\| = \|a\|$. Then we have the following.

PROPOSITION 1. *Let $\alpha \geq -1/2$ and $\tau \geq 2$, or let $\alpha \geq \alpha_0 = (-5 + (17)^{1/2})/2$ and $\tau \geq 1$. Then $A^{(\alpha,\tau)}$ is the commutative semisimple Banach algebra with no unit which is isomorphic and isometric to $l^{(\alpha,\tau)}$ by the transform $\hat{}$. Moreover, $A^{(\alpha,\tau)}$ consists of continuous functions on $[0,\infty)$ vanishing at infinity, and includes C_c^{∞} as a dense subset.*

Let x be in $[0,\infty)$ and define the mapping χ_x of $A^{(\alpha,\tau)}$ into the complex numbers by

$$\chi_x \colon f \mapsto f(x), \quad f \text{ in } A^{(\alpha,\tau)}.$$

Then χ_x is a multiplicative linear functional on $A^{(\alpha,\tau)}$. We denote by $\mathcal{M}(A^{(\alpha,\tau)})$ the maximal ideal space of $A^{(\alpha,\tau)}$ and define the mapping ι of $[0,\infty)$ into $\mathcal{M}(A^{(\alpha,\tau)})$ by $\iota \colon x \to \chi_x$.

THEOREM 1. *Let $\alpha \geq -1/2$ and $\tau \geq 2$ or let $\alpha \geq \alpha_0$ and $\tau \geq 1$. Then the maximal ideal space $\mathcal{M}(A^{(\alpha,\tau)})$ is homeomorphic to the interval $[0,\infty)$, and the Gelfand transform of f in $A^{(\alpha,\tau)}$ is given by f itself.*

PROOF. Clearly, $\chi_x \neq \chi_y$ if $x \neq y$. Since both spaces $[0,\infty)$ and $\mathcal{M}(A^{(\alpha,\tau)})$ are locally compact Hausdorff spaces, it is enough to show that the mapping ι is surjective. Let χ be a multiplicative linear functional on $A^{(\alpha,\tau)}$. Suppose that f belongs to $A^{(\alpha,\tau)}$. Then

$$f(x) = \sum_{n=0}^{\infty} a_n R_n^{\alpha}(x) e^{-\tau x}, \quad \sum_{n=0}^{\infty} |a_n| < \infty.$$

If we define $c(n)$, $n = 0, 1, 2, \ldots$, by $c(n)_k = 0$ for $k \neq n$ and $c(n)_k = 1$ for $k = n$, then we have

(6) $c(n)\hat{}(x) = R_n^{\alpha}(x) e^{-\tau x}, \quad \|c(n)\| = 1$ for all n.

To complete the proof, it is enough to show that

$$\chi(c(n)\hat{}) = R_n^{\alpha}(x_0) e^{-\tau x_0}, \qquad n = 0, 1, 2, \ldots,$$

for some x_0 in $[0,\infty)$.

By the recurrence formula

(7) $(n+\alpha)R_n^{\alpha}(x) = (-x + 2n + \alpha - 1)R_{n-1}^{\alpha}(x) - (n-1)R_{n-2}^{\alpha}(x),$

$n = 2, 3, 4, \ldots$, $R_0^{\alpha}(x) = 1$ and $R_1^{\alpha}(x) = 1 - x/(\alpha+1)$ [12, (5.1.10)], we have

$$(\alpha+1)c(1)\hat{}(x)c(n-1)\hat{}(x) = (n-1)c(n-2)\hat{}(x)c(0)\hat{}(x)$$
$$- 2(n-1)c(n-1)\hat{}(x)c(0)\hat{}(x)$$
$$+ (n+\alpha)c(n)\hat{}(x)c(0)\hat{}(x), \qquad x \geq 0,$$

and thus

(8) $(\alpha+1)\chi(c(1)\hat{})\chi(c(n-1)\hat{}) = (n-1)\chi(c(n-2)\hat{})\chi(c(0)\hat{})$
$$-2(n-1)\chi(c(n-1)\hat{})\chi(c(0)\hat{}) + (n+\alpha)\chi(c(n)\hat{})\chi(c(0)\hat{}).$$

Now we claim that $\chi(c(0)\hat{}) \neq 0$. For, since

$$\chi(f) = \chi((f/c(0)\hat{})c(0)\hat{}) = \chi(f/c(0)\hat{})\chi(c(0)\hat{})$$

for all f in C_c^∞ which is dense in $A^{(\alpha,\tau)}$, we have that χ is trivial if $\chi(c(0)\hat{\ }) = 0$. Put $g_n = \chi(c(0)\hat{\ })^{-1}c(n)\hat{\ }$, $n = 0, 1, 2, \ldots$, and choose the unique complex number x_0 so that $\chi(g_1) = 1 - x_0/(\alpha + 1)$. Then, by (8), we have

$$(n + \alpha)\chi(g_n) = (-x_0 + 2n + \alpha - 1)\chi(g_{n-1}) - (n - 1)\chi(g_{n-2}),$$

$n = 2, 3, 4, \ldots$, $\chi(g_0) = 1$ and $\chi(g_1) = 1 - x_0/(\alpha + 1)$. This shows that the sequence $\chi(g_n)$, $n = 0, 1, 2, \ldots$, satisfies the recurrence formula (7) with $x = x_0$. Thus we have

$$(9) \qquad\qquad \chi(g_n) = R_n^\alpha(x_0), \qquad n = 0, 1, 2, \ldots.$$

Since the norm of a multiplicative linear functional is at most one, it follows that $|R_n^\alpha(x_0)| \le 1$ for all n by (6). If x_0 does not belong to $[0, \infty)$, then $\lim_{n\to\infty} |R_n^\alpha(x_0)| = \infty$ by Perron's formula in the complex domain [**12**, Theorem 8.22.3]. This shows that x_0 belongs to $[0, \infty)$. By (1) and (9), we have

$$\chi(c(0)\hat{\ }) = \sum_{k=0}^{\infty} D_k^\alpha(0, 0; \tau)\chi(g_k)$$

$$= \sum_{k=0}^{\infty} D_k^\alpha(0, 0; \tau) R_n^\alpha(x_0) = e^{-\tau x_0},$$

and therefore

$$\chi(c(n)\hat{\ }) = R_n^\alpha(x_0) e^{-\tau x_0}.$$

This completes the proof.

By the theorem and Lemma 2(ii), we have the following.

COROLLARY 1. *The semisimple Banach algebra $A^{(\alpha,\tau)}$ is regular.*

Let E be a compact subset of $[0, \infty)$ and define $I(E) = \{f \in A^{(\alpha,\tau)}; f = 0 \text{ on } E\}$. Then $I(E)$ is a closed ideal in $A^{(\alpha,\tau)}$. The application of the usual Banach algebra proof of the Wiener-Lévy theorem to the quotient algebra $A^{(\alpha,\tau)}/I(E)$ yields the following.

COROLLARY 2. *Let f be in $A^{(\alpha,\tau)}$. Suppose that $f \ne 0$ on a compact subset E of $[0, \infty)$. Then there is a function g in $A^{(\alpha,\tau)}$ such that $f(x)g(x) = 1$ for x in E.*

This corollary and Lemma 2(ii) yield the following result, analogous to Wiener's general Tauberian theorem (cf. Rudin [**10**, 7.2]).

COROLLARY 3. *Let f be in $A^{(\alpha,\tau)}$ and suppose that $f(x) \ne 0$ for all x in $[0, \infty)$. Then f is contained in no proper closed ideal in $A^{(\alpha,\tau)}$.*

PROOF. Let h be in C_c^∞ and let E be the support of h. By Corollary 2, we have a function g in $A^{(\alpha,\tau)}$ such that $fg = 1$ on E. It follows from $hfg = h$ that an ideal containing f includes C_c^∞. By Lemma 2(ii), a closed ideal containing f coincides with $A^{(\alpha,\tau)}$.

3. Spectral synthesis and Helson sets. Let E be a closed subset of $[0, \infty)$, and let $I(E)$ be the closed ideal of f in $A^{(\alpha,\tau)}$ such that $f = 0$ on E. Denote by $J(E)$ the ideal of f in $A^{(\alpha,\tau)}$ such that $f = 0$ on a neighborhood of E. If $J(E)$ is dense in $I(E)$, then E is called a *set of spectral synthesis* for $A^{(\alpha,\tau)}$. By an argument similar to that used for Schwartz's example in the Euclidean space of three dimension (cf. [**10**, 7.3]), we have the following.

THEOREM 2. *Let $\tau \geq 1$. If $\alpha \geq 1/2$ and x_0 is in the open interval $(0, \infty)$, then the singleton $\{x_0\}$ is not a set of spectral synthesis for $A^{(\alpha, \tau)}$.*

PROOF. Let k be the greatest integer not exceeding $\alpha + 1/2$. It follows from the identity $(d/dx)L_n^\alpha(x) = -L_{n-1}^{\alpha+1}(x)$ [**12**, (5.1.14)] that there is a positive constant C and a neighborhood V of x_0 such that $|(d/dx)^j(R_n^\alpha(x)e^{-\tau x})| \leq C$ on V for $j = 1, 2, 3, \ldots, k$ and all n. This implies that every f in $A^{(\alpha, \tau)}$ is k-times continuously differentiable on $(0, \infty)$. Let $I_1 = \{f \in A^{(\alpha, \tau)}; f(x_0) = 0\}$ and $I_2 = \{f \in A^{(\alpha, \tau)}; f(x_0) = (df/dx)(x_0) = 0\}$. Then I_1 and I_2 are distinct closed ideals for $\alpha \geq 1/2$. The proof is complete.

A closed subset E of $[0, \infty)$ will be called a *Helson set* with respect to the algebra $A^{(\alpha, \tau)}$ if, for every continuous function g on E vanishing at infinity, there is a function f in $A^{(\alpha, \tau)}$ such that $f = g$ on E.

The following theorem is suggested by the characterization of Helson sets with respect to the algebra of absolutely convergent Jacobi polynomial series [**7**, Theorem 2].

THEOREM 3. *Let $\alpha > -1/2$ and $\tau \geq 2$, or let $\alpha \geq \alpha_0$ and $\tau \geq 1$. Then every Helson set with respect to $A^{(\alpha, \tau)}$ is finite.*

PROOF. Every finite set is a Helson set with respect to $A^{(\alpha, \tau)}$ since $A^{(\alpha, \tau)}$ is regular.

Conversely, let E be a Helson set with respect to $A^{(\alpha, \tau)}$. Suppose that E is infinite. Then there are a sequence $\{x_j\}_{j=1}^\infty$ in $E \cap (0, \infty)$, such that $x_i \neq x_j$ for $i \neq j$, and a point x_0 such that $\lim_{j \to \infty} x_j = x_0$ and $0 \leq x_0 \leq \infty$. Let $Q(E)$ be the quotient algebra $A^{(\alpha, \tau)}/I(E)$ with quotient norm $\| \cdot \|_{Q(E)}$, and let $C_0(E)$ be the Banach algebra of continuous functions on E vanishing at infinity with uniform norm $\| \cdot \|_{C_0(E)}$. Since E is a Helson set with respect to $A^{(\alpha, \tau)}$, it follows that $Q(E)$ is isomorphic to $C_0(E)$ and the norms in $Q(E)$ and in $C_0(E)$ are equivalent. Let g_k be a function in $C_0(E)$ such that $g_k(x_{2j}) = 1$ and $g_k(x_{2j-1}) = 0$ for $j = 1, 2, 3, \ldots, k$, $g_k(x_j) = 0$ for $j = 2k+1, 2k+2, \ldots$ and $\|g_k\|_{C_0(E)} = 1$. By the norm equivalence and the definition of quotient norm, we can choose a function f_k in $A^{(\alpha, \tau)}$ for every k so that $f_k(x) = g_k(x)$ on E and $\|f_k\| \leq C$ with a constant C not depending on k. Since $A^{(\alpha, \tau)}$ can be regarded as the dual of the space c_0 of sequences vanishing at infinity, the sequence $\{f_k\}_{k=1}^\infty$ has a subsequence, say also $\{f_k\}_{k=1}^\infty$, converging to a function f in $A^{(\alpha, \tau)}$ in the weak* topology $\sigma(A^{(\alpha, \tau)}, c_0)$. If $\alpha > -1/2$, then $R_n^\alpha(x) \to 0$ as $n \to \infty$ for every x in $(0, \infty)$ [**12**, (7.6.8)]. Thus we have $f_k(x) \to f(x)$ as $k \to \infty$ for every x in $(0, \infty)$ by the definition of weak* topology $\sigma(A^{(\alpha, \tau)}, c_0)$, and in particular $f(x_{2j}) = 1$ and $f(x_{2j-1}) = 0$ for all j. This contradicts the continuity of f on $[0, \infty)$ in the case x_0 in $[0, \infty)$ and contradicts the vanishing at infinity of f in the case $x_0 = \infty$. The proof is complete.

4. Algebras L_α and M_α. Let $\alpha > -1/2$. If we put $z = x + t + 2(xt)^{1/2}\cos\theta$, $0 \leq \theta \leq \pi$ in (3), then we have the following for $x, t > 0$ (cf. [**5**]):

$$(xt)^{1/2}\sin\theta = \{2(xt + xz + tz) - x^2 - t^2 - z^2\}^{1/2}/2$$
$$= \rho(x, t, z), \quad \text{say.}$$
$$T_t^\alpha(f; x) = \int_0^\infty f(z)K(x, t, z)e^{-z}z^\alpha \, dz,$$

$$K(x,t,z) = \begin{cases} \dfrac{\Gamma(\alpha+1)2^{\alpha-1}}{(2\pi)^{1/2}(xtz)^\alpha} e^{(x+t+z)/2} \cdot J_{\alpha-1/2}(\rho(x,t,z))\rho(x,t,z)^{\alpha-1/2}, \\ 0, \end{cases}$$

where the first value is assigned only if $2(xt + tz + zx) - x^2 - t^2 - z^2 > 0$.

The kernel $K(x,t,z)$ satisfies the following.

(C) [**5**, Lemma 1(ii)] Let $\alpha \geq 0$. Then

$$\int_0^\infty |K(x,t,z)| e^{-z/2} z^\alpha \, dz \leq e^{(x+t)/2}$$

for $x, t > 0$.

Let M' be the space $\{\mu \in M[0,\infty); |\mu|(\{0\}) = 0\}$, and let μ and ν be in M'. We define the convolution $\mu * \nu$ for $\alpha \geq 0$ by

$$(10) \qquad \mu * \nu(E) = \int_0^\infty \int_0^\infty \left\{ \int_E K(x,t,z) e^{-z/2} z^\alpha \, dz \right\} e^{-(x+t)/2} \, d\mu(x) \, d\nu(t)$$

for every Borel set E of $(0,\infty)$. Then $\mu * \nu$ belongs to M' and $\|\mu * \nu\| \leq \|\mu\| \|\nu\|$ by (C). We denote by M'_α the algebra M' with this convolution. Every μ in $M[0,\infty)$ has the unique decomposition $\mu = \nu + \mu(\{0\})\delta_0$, where ν is in M'_α and δ_0 is the measure with the unit mass at the point 0. We extend the convolution to all of $M[0,\infty)$ by treating δ_0 as a unit. This algebra is a commutative Banach algebra with a unit and is denoted by M_α. We identify L_α with its image by the mapping $f \mapsto \mu_f$, $d\mu_f(x) = f(x)e^{-x/2}x^\alpha \, dx$. Then L_α is a closed subalgebra of M'_α since $\mu_f * \mu_g = f * g$ and $\|\mu_f\| = \|f\|$.

The Fourier Laguerre coefficient $\hat{f}(n)$ of f in L_α is defined by

$$\hat{f}(n) = \int_0^\infty f(x) R_n^\alpha(x) e^{-x} x^\alpha \, dx$$

for every $n = 0, 1, 2, \ldots$. We define the Fourier-Stieltjes Laguerre coefficient $\hat{\mu}(n)$ of μ in M_α by

$$\hat{\mu}(n) = \int_0^\infty R_n^\alpha(x) e^{-x/2} \, d\mu(x)$$

for every n. Then we have $\hat{\mu}_f(n) = \hat{f}(n)$, and

$$(11) \qquad (f * g)\hat{\ }(n) = \hat{f}(n)\hat{g}(n), \quad (\mu * \nu)\hat{\ }(n) = \hat{\mu}(n)\hat{\nu}(n)$$

for f, g in L_α and μ, ν in M_α by the identity

$$T_t^\alpha(R_n^\alpha; x) = R_n^\alpha(x)R_n^\alpha(t), \qquad \alpha > -1/2 \quad (\text{Watson [13]}).$$

LEMMA 3. *Let $\alpha \geq 0$.*

(i) $|\hat{\mu}(n)| \leq \|\mu\|$ *for μ in M_α.*

(ii) *If a measure μ in M_α satisfies that $\hat{\mu}(n) = 0$ for all n, then $\mu = 0$.*

PROOF. The inequality $|R_n^\alpha(x)e^{-x/2}| \leq 1$, $\alpha \geq 0$ [**3**, 10.18(14)] implies (i). To prove (ii), it is enough to show that $\int_0^\infty f(x) \, d\mu(x) = 0$ for every f in C_c^∞. By Lemma 2(i), we have $f(x) = \sum_{n=0}^\infty a_n R_n^\alpha(x) e^{-x/2}$ with $\sum_{n=0}^\infty |a_n| < \infty$. Thus we have $\int_0^\infty f(x) \, d\mu(x) = \sum_{n=0}^\infty a_n \hat{\mu}(n) = 0$ by the Lebesgue convergence theorem since

$$\sum_{n=0}^\infty \int_0^\infty |a_n R_n^\alpha(x) e^{-x/2}| \, d|\mu|(x) = \sum_{n=0}^\infty |a_n| \|\mu\| < \infty.$$

Now we consider the maximal ideal space of L_α. For every $n = 0, 1, 2, \ldots$, we define the functional χ_n by

$$\chi_n(f) = \hat{f}(n), \quad f \text{ in } L_\alpha.$$

Then χ_n is a multiplicative linear functional on L_α by (11).

PROPOSITION 2. *Let $\alpha \geq 0$. Then the maximal ideal space of L_α is the space $\{\chi_n; n = 0, 1, 2, \ldots\}$ with the discrete topology.*

PROOF. Let χ be a multiplicative linear functional on L_α. We have an integer $n_0 \geq 0$ such that $\chi(R_{n_0}^\alpha) \neq 0$. For, if $\chi(R_n^\alpha) = 0$ for all n, then we have $\chi(f) = 0$ for all f in L_α since the Cesàro mean of order $\delta > \alpha + 1/2$ of the Fourier Laguerre series converges in L_α by [**5**, (3.7)]. It follows from the orthogonality of Laguerre polynomials that n_0 is unique, and also $f * R_{n_0}^\alpha = \hat{f}(n_0) R_{n_0}^\alpha$ for f in L_α. This implies that $\chi(f)\chi(R_{n_0}^\alpha) = \hat{f}(n_0)\chi(R_{n_0}^\alpha)$, and thus $\chi(f) = \hat{f}(n_0)$. Since the Gelfand topology is clearly discrete, the proof is complete.

We denote by $\mathcal{M}(M_\alpha)$ and $\mathcal{M}(M_\alpha')$ the maximal ideal spaces of M_α and M_α', respectively. The algebra M_α has the direct sum decomposition $M_\alpha = M_\alpha' \oplus C\delta_0$. Here, $C\delta_0$ is the closed subalgebra of M_α generated by δ_0. It follows from the definition of the convolution that M_α' is a closed ideal in M_α. Thus $\mathcal{M}(M_\alpha)$ is the one point compactification $\{\chi_\infty\} \cup \mathcal{M}(M_\alpha')$ of $\mathcal{M}(M_\alpha')$, where χ_∞ is the multiplicative linear functional on M_α such that $\chi_\infty(\mu) = \mu(\{0\})$ for μ in M_α. The next two lemmas are essential to the following discussion. See [**2**] and also [**11**].

LEMMA 4. *Let $\alpha \geq 0$. If μ and ν are in M_α', then $\mu * \nu$ is in L_α.*

PROOF. Let E be a Borel set in $(0, \infty)$ such that $\int_E e^{-x/2} x^\alpha \, dx = 0$. For all $x, t > 0$, the braces $\{\ \}$ in the definition (10) of the convolution of μ and ν are zero since $K(x, t, z)$ are integrable with respect to the measure $e^{-z/2} z^\alpha \, dz$ by (C). Thus we have $\mu * \nu(E) = 0$; that is, $\mu * \nu$ is absolutely continuous with respect to $e^{-z/2} z^\alpha \, dz$. This shows that $\mu * \nu$ in L_α.

LEMMA 5. *Let $\alpha \geq 0$. If μ is in M_α', then $\lim_{n \to \infty} \hat{\mu}(n) = 0$. In particular, $\chi_\infty(\mu) = \lim_{n \to \infty} \hat{\mu}(n)$ for μ in M_α.*

PROOF. Since $\mu * \mu$ is in L_α for μ in M_α' by Lemma 4, it is enough to show that $\lim_{n \to \infty} \hat{f}(n) = 0$ for every f in L_α. The usual argument in the Riemann-Lebesgue theorem implies that $\lim_{n \to \infty} \hat{f}(n) = 0$ by Lemma 3(i) and the convergence of the Cesàro mean of order $\delta > \alpha + 1/2$ in L_α.

We can determine the maximal ideal spaces $\mathcal{M}(M_\alpha')$ and $\mathcal{M}(M_\alpha)$ by using Lemma 4.

THEOREM 4. *Let $\alpha \geq 0$.*
(i) *The maximal ideal space $\mathcal{M}(M_\alpha')$ of M_α' is the space $\{\chi_n; n = 0, 1, 2, \ldots\}$ with the discrete topology, where $\chi_n(\mu) = \hat{\mu}(n)$ for μ in M_α'.*
(ii) *The maximal ideal space of $\mathcal{M}(M_\alpha)$ of M_α is the space $\{\chi_n; n = 0, 1, 2, \ldots\} \cup \{\chi_\infty\}$ with topology of the one point compactification of $\{\chi_n; n = 0, 1, 2, \ldots\}$.*

PROOF. Every χ_n is a multiplicative linear functional on M_α' by (11). Conversely, let χ be a multiplicative linear functional on M_α'. Let ν be a measure of M_α' such that $\chi(\nu) \neq 0$. Since $\nu * \nu$ is in L_α by Lemma 4, the restriction of χ to L_α

is nontrivial. Thus there is a unique element χ_n of $\mathcal{M}(L_\alpha)$ such that $\chi(f) = \chi_n(f)$ for all f in L_α. Since $\mu * \nu$ in L_α for every μ in M'_α by Lemma 4, we have

$$\chi(\mu * \nu) = \chi_n(\mu * \nu) = (\mu * \nu)\widehat{\ }(n) = \hat{\mu}(n)\hat{\nu}(n),$$

and thus $\chi(\mu) = \hat{\mu}(n)$. Clearly, the Gelfand topology is discrete. Therefore (i) is proved. Since $\mathcal{M}(M_\alpha)$ is the one point compactification $\mathcal{M}(M'_\alpha) \cup \{\chi_\infty\}$ of $\mathcal{M}(M'_\alpha)$, (i) implies (ii). The proof is complete.

5. Idempotent measures and Sidon sets. For μ in M_α, if $\mu * \mu = \mu$, it is called an *idempotent measure* in M_α. It follows from Lemma 3(ii) that a measure μ in M_α is idempotent if and only if $\hat{\mu}(n) = 0$ or 1 for every n. The following consideration is motivated by characterizations of the idempotent measures related to ultraspherical polynomials [2] or Jacobi polynomials [6].

Let μ be an idempotent measure in M_α and decompose μ to $\mu = \nu + \mu(\{0\})\delta_0$, ν in M'_α. By the convolution equation $\mu * \mu = \mu$, we have

$$\nu + \mu(\{0\})\delta_0 = \nu * \nu + 2\mu(\{0\})\nu + \mu(\{0\})^2\delta_0.$$

Thus $\mu(\{0\}) = \mu(\{0\})^2$, and so $\mu(\{0\}) = 0$ or 1. If $\mu(\{0\}) = 0$, then $\lim_{n\to\infty} \hat{\mu}(n) = \lim_{n\to\infty} \hat{\nu}(n) = 0$ by Lemma 5. Since $\hat{\mu}(n)$ takes only values 0 or 1, we have $\hat{\mu}(n) = 0$ except for a finite number of n's. Thus we have

$$d\mu(x) = \frac{1}{\Gamma(\alpha+1)} \sum L_n^\alpha(x)e^{-x/2}x^\alpha \, dx,$$

where $\sum$ is a finite sum. If $\mu(\{0\}) = 1$, then $\hat{\nu}(n)$ takes only values 0 or -1. Since $\lim_{n\to\infty} \hat{\nu}(n) = 0$, we have

$$d\mu(x) = -\frac{1}{\Gamma(\alpha+1)} \sum L_n^\alpha(x)e^{-x/2}x^\alpha \, dx + d\delta_0(x),$$

where $\sum$ is a finite sum. Therefore we have the following.

THEOREM 5. *Let $\alpha \geq 0$. If μ is an idempotent measure in M_α, then μ has the form*

$$d\mu(x) = \frac{1}{\Gamma(\alpha+1)} \sum a_n L_n^\alpha(x)e^{-x/2}x^\alpha \, dx + a_\infty d\delta_0(x),$$

where $a_n = 1$ or -1, $a_\infty = 0$ or 1 and $\sum$ is a finite sum.

A subset E of the nonnegative integers will be called a *Sidon set* with respect to L_α if every sequence $\{a_n\}_{n \in E}$ on E vanishing at infinity is the restriction of the Fourier Laguerre coefficients of a function f in L_α to E; that is, $a_n = \hat{f}(n)$ for all n in E. The concept of Sidon sets with respect to L_α is dual to that of Helson sets with respect to $A^{(\alpha,\tau)}$. We have a theorem which is dual to Theorem 3 (cf. [2 and 10, 5.7]).

THEOREM 6. *Let $\alpha \geq 0$ and let E be a subset of the nonnegative integers. Then the following are equivuvalent:*

(i) *E is a Sidon set with respect to L_α.*

(ii) *For every bounded sequence $\{a_n\}_{n \in E}$ on E, there is a measure μ in M_α such that $a_n = \hat{\mu}(n)$ for all n in E.*

(iii) *E is finite.*

PROOF. By the usual Sidon set argument, we have that (i) implies (ii) (cf. [**10**, Proof of 5.7.3]). We will show that (ii) implies (iii). Suppose that E is infinite and put $E = \{n_j\}_{j=1}^{\infty}$, $n_1 < n_2 < n_3 < \cdots$. Define the sequence $\{a_n\}_{n \in E}$ by $a_{n_j} = 0$ for odd j and $a_{n_j} = 1$ for even j. If there is a measure μ in M_α such that $\hat{\mu}(n) = a_n$ for all n in E, we have a contradiction since $\lim_{j \to \infty} \hat{\mu}(n_j)$ exists by Lemma 5. If E is finite, then

$$f(x) = \frac{1}{\Gamma(\alpha + 1)} \sum_{n \in E} a_n L_n^\alpha(x)$$

belongs to L_α, and thus (iii) imples (i). The proof is complete.

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