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INSUFFICIENCY OF TORRES' CONDITIONS FOR TWO-COMPONENT CLASSICAL LINKS

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ABSTRACT. Torres has given necessary conditions for a polynomial to be the Alexander polynomial of a two component link. For certain links, additional conditions are necessary. Hillman gave one example for linking number 6. Here we give examples for all other linking numbers except 0, ± 1, and ± 2.

1. Introduction. In 1953, Torres [T] gave necessary conditions for a polynomial to be the Alexander polynomial of a link. More recently, in the case of two component links with linking number b, Bailey [**B**] showed equivalently that the Alexander polynomial of the link can be expressed in the form

$$\Delta(x, y) = \frac{1-(xy)^b}{1-xy} A(x, y) - (1-x)(1-y)\left(\frac{1-(xy)^{b-1}}{1-xy}\right) B(x, y),$$

where $A(x, y)$ and $B(x, y)$ satisfy certain conditions.

Using Bailey's result, Hillman [**H**] gave an additional condition on the Alexander polynomial of certain two component links whose linking number is divisible by at least two distinct primes. In §3 of this paper, a similar result is given for prime power linking numbers in

(3.7) THEOREM. *Let L be a two-component link with linking number, p^α, where p is a prime. Let $\lambda(x) = a(x + x^{-1}) + (1 - 2a)$, where $\lambda(-1)$ is square-free, and let $\Delta(x, y)$ be the Alexander polynomial of L. If the knot polynomial*

$$(1-x)^{-1}(1-x^{p^\alpha})\Delta(x, 1) = \lambda(x)$$

and if ω is a primitive p^βth root of unity for some $\beta \leqslant \alpha$, then the $\mathbf{Z}[\omega]/\lambda(\omega)$-ideal generated by $B(\omega, 1) \pmod{\lambda(\omega)}$ is of the form $J\bar{J}$ for some ideal J.

It should be noted that the ideal $J\bar{J}$ depends only on $\Delta(x, y)$ and not on the expansion given above.

Following (3.7), we show how to realize counterexamples to Torres' condition for two-component links, provided the linking number of the components is not 0, ± 1, or ± 2. It should be noted that the Torres conditions do suffice if $b = 0$ or ± 1. Hence, only the case when $b = 2$ remains unsettled. Finally, a counterexample to Torres' conditions for m-component links ($m > 3$) is given.

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2. Definitions. A classical link of multiplicity m is a collection, $L = L_1 \cup \cdots \cup L_m$ of oriented smooth simple closed curves in S^3 satisfying $L_i \cap L_j = \varnothing$ if $i \neq j$. The number m denotes the number of components of the link. If $m = 1$, then L is a knot. The number $b = \operatorname{lk}(L_i, L_j)$ is the linking number of the ith component and the jth component. A link is trivial if it is the boundary of m disjoint 2-disks in S^3.

The complement of the link is the space

$$X = S^3 - \bigcup_{i=1}^{m} \nu(L_i),$$

where $\nu(L_i)$ is a small open tubular neighborhood of L_i. The neighborhoods $\nu(L_i)$ can be chosen so small that $\nu(L_i) \cap \nu(L_j) = \varnothing$ if $i \neq j$. The basepoint of X is denoted by $*$.

For each i, let m_i be a small circle linking the ith component L_i of L with $\operatorname{lk}(m_i, L_i) = 1$, let l_i be a translate of L_i into X whose basepoint coincides with that of m_i and such that $\operatorname{lk}(l_i, L_i) = 0$ and let γ_i be a path in X from $*$ to the basepoint of m_i. The elements α_i of $\pi_1 X$ represented by $\gamma_i * m_i * \gamma_i^{-1}$ are called meridians of the link. The elements β_i of $\pi_1 X$ represented by $\gamma_i * l_i * \gamma_i^{-1}$ are called longitudes of the link. The pair (α_i, β_i) is determined up to simultaneous conjugation by the element of $\pi_1 X$.

An orientation of a link consists of an ordering of the components together with an orientation of each component.

By Alexander duality, $H_1(X) \cong \mathbf{Z}^m$, where m is the multiplcity of the link. A canonical basis of $H_1(X)$, defined by any choice of meridians of L, allow the identification of $\mathbf{Z}[H_1(X)]$ with $\Lambda_m = \mathbf{Z}[t_1, t_1^{-1}, \ldots, t_m, t_m^{-1}]$, the identification depending only on the orientation of the link. There is a natural involution of Λ_m (denoted with an overbar) which maps $t_i \to t_i^{-1}$. The augmentation of Λ_m is given by $\varepsilon\colon \Lambda_m \to \mathbf{Z}$, where $\varepsilon(t_i) = 1$.

The canonical homomorphism $h\colon \pi_1 X \to H_1(X)$ defines a regular covering space $p\colon \tilde{X} \to X$ with $\mathbf{Z}^m$ as the group of covering transformations. The space $\tilde{X}$ is called the universal abelian cover of X. The Alexander module of L is $H_1(\tilde{X}, \tilde{*})$ considered as a module over Λ_m. The module of L is $H_1(\tilde{X})$ considered as a module over Λ_m.

These modules are related by the exact sequence

$$0 \to H_1(\tilde{X}) \to H_1(\tilde{X}, \tilde{*}) \xrightarrow{\Phi} M \to 0,$$

where Φ is the boundary homomorphism, $\Phi\colon H_1(\tilde{X}, \tilde{*}) \to H_0(\tilde{*})$, and M is the augmentation ideal of Λ_m generated by $t_1 - 1, \ldots, t_m - 1$ **[L-3]**.

Given a presentation matrix for $H_1(\tilde{X}, \tilde{*})$ as a Λ_m-module, the sequence of elementary ideals, or Fitting invariants, is $\tilde{E}_i(L)$, where $\tilde{E}_i(L)$ is the ideal of Λ_m generated by the $(n - i)$-order minors of the presentation matrix **[F]**. Let $\tilde{\Delta}_i(L)$ be the greatest common divisor or $\tilde{E}_i(L)$. A sequence of ideals $E_i(L)$ and polynomials $\Delta_i(L)$ can be defined the same way from a presentation matrix for $H_1(\tilde{X})$. Although $\tilde{E}_i(L) \neq E_{i-1}(L)$ in general, $\tilde{\Delta}_i(L) = \Delta_{i-1}(L)$ **[L-1]**. The Alexander polynomial of L is

$$\tilde{\Delta}_1(L) = \Delta_L(t_1, \ldots, t_m).$$

3. Two-component links.

A. Suppose $L = K_1 \cup K_2$ is a two-component link with linking number b. Under these circumstances Torres **[T]** has shown that the Alexander polynomial of L can be chosen to have the following properties:

$$\text{(3.1)} \qquad \begin{aligned} &(1) \quad \Delta(x, y) = x^{b-1}y^{b-1}\Delta(x^{-1}, y^{-1}), \\ &(2) \quad \Delta(x, 1) = (1 - x^b)(1 - x)^{-1}\Delta_2(x), \\ &\qquad\ \Delta(1, y) = (1 - y^b)(1 - y)^{-1}\Delta_1(y), \end{aligned}$$

where $\Delta_2(x)$ and $\Delta_1(y)$ are knot polynomials. In fact, $\Delta_2(x)$ and $\Delta_1(y)$ are the Alexander polynomials of the component knots corresponding to the meridians x and y, respectively.

Bailey **[B]** has characterized the module of L, $H_1(\tilde{X})$, as a Λ_2-module having a presentation matrix with a certain symmetry condition. Bailey's main result is the following theorem.

(3.2) THEOREM (BAILEY). *A Λ_2-module is a link module if and only if it has a presentation matrix of the form*

$$M_L = \begin{bmatrix} \left(1 - (xy)^b\right)(1 - xy)^{-1} & -\left[(1 - x)(1 - y)\left(1 - (xy)^{b-1}\right)(1 - xy)^{-1}\beta(x, y)\right] \\ \beta^{\mathrm{tr}}(x^{-1}, y^{-1}) & \mathbf{A}(x, y) \end{bmatrix}$$

where $\beta(x, y)$ is a row matrix, $\mathbf{A}(x, y)$ is a square matrix with entries in Λ_2, satisfying $\mathbf{A}(x, y) = \mathbf{A}^{\mathrm{tr}}(x^{-1}, y^{-1})$ and $\mathbf{A}(1, 1) = \mathrm{diag}(\pm 1, \ldots, \pm 1)$. Furthermore, $\mathbf{A}(x, 1)$ (resp. $\mathbf{A}(1, y)$) is a presentation matrix for the first (resp. second) component of the link and b is the linking number of the components.

One corollary of Bailey's theorem is that the Alexander polynomial of a two-component link has the form

$$\text{(3.3)} \qquad \begin{aligned} D(x, y) = {} & \left(1 - (xy)^b\right)(1 - xy)^{-1}A(x, y) \\ & -(1 - x)(1 - y)(1 - xy)^{-1}\left(1 - (xy)^{b-1}\right)B(x, y), \end{aligned}$$

where $A(x, y) = A(x^{-1}, y^{-1})$, $B(x, y) = B(x^{-1}, y^{-1})$, and $A(x, 1)$ and $A(1, y)$ are knot polynomials.

For instance, one may take

$$A(x, y) = \det \mathbf{A}(x, y), \quad B(x, y) = \det\begin{bmatrix} 0 & \beta(x, y) \\ \beta^{\mathrm{tr}}(x^{-1}, y^{-1}) & \mathbf{A}(x, y) \end{bmatrix}.$$

Moreover, Bailey showed that a polynomial in Λ_2 has this form if and only if it satisfies (3.1).

Using Bailey's result, Hillman has proven the following theorem.

(3.4) THEOREM (HILLMAN). *Let L be a two-component link with linking number $b > 1$ and with Alexander polynomial $\Delta(x, y)$. If the knot polynomial $(1 - x^b) \cdot (1 - x)^{-1}\Delta(x, 1)$ is (up to units) the d-cyclotomic polynomial, $\Phi_d(x)$, for some $d > 1$ dividing b and if ω is a primitive dth root of unity, then the $\mathbf{Z}[\omega]$-ideal generated by $B(\omega, 1)$ is of the form $J\bar{J}$ for some J.*

The hypothesis of the theorem is vacuous unless d is divisible by at least two distinct primes. Hillman's theorem suggests two questions:

(3.5) QUESTION 1. Do counterexamples to (3.1) exist whenever b is a nonprime power number?

(3.6) QUESTION 2. Do counterexamples to (3.1) exist if b is a power of a prime?

B. To answer (3.5), suppose d is a nonprime power number, $\Phi_d(x)$ is the d-cyclotomic polynomial and a is an integer. Let

$$D(x, y) = \left(1 - (xy)^d\right)(1 - xy)^{-1}\Phi_d(x) - (1 - x)(1 - y)\left(1 - (xy)^{d-1}\right)(1 - xy)^{-1}(a).$$

By direct computation, one finds that $\Phi_d(x)$ is a knot polynomial; hence $D(x, y)$ satisfies (3.1). If ω is a primitive dth root of unity for $d \neq p^\alpha$, one may ask, in view of Hillman's theorem, if there is an integer, a, such that the ideal generated by a does not factor as $J\bar{J}$ in $\mathbf{Z}[\omega]$?

Suppose that q is a prime, $q \nmid d$, and Q is a prime of $\mathbf{Z}[\omega]$ lying over q. The prime q is unramified since the only ramified primes are those dividing d. The Galois group, $\mathrm{Gal}(\mathbf{Q}[\omega]/\mathbf{Q})$ is isomorphic to $(\mathbf{Z}/d)^\times$. The decomposition group of q, $D(Q|q)$, is the (cyclic) subgroup of $\mathrm{Gal}(\mathbf{Q}[\omega]/\mathbf{Q})$ generated by $\omega \to \omega^q$, which corresponds to the subgroup of $(\mathbf{Z}/d)^\times$ generated by q.

Suppose that complex conjugation, σ, is an element of the decomposition group, in other words, that $Q = \bar{Q}$. This will happen, for instance, if $q \equiv -1 \pmod d$, and by Dirichlet's density theorem there are infinitely many such primes. Now, any such prime q factors in $\mathbf{Z}[\omega]$ as ΠQ_i with each Q_i distinct and $Q_i = \bar{Q}_i$. In particular, $(q) \neq J\bar{J}$ for any ideal J of $\mathbf{Z}[\omega]$. One may then take $a = q$ and

$$D(x, y) = \left(1 - (xy)^d\right)(1 - xy)^{-1}\Phi_d(x) - (1 - x)(1 - y)\left(1 - (xy)^{d-1}\right)(1 - xy)^{-1}(q).$$

C. In order to answer (3.6) one uses an argument similar to that in **[H]**. Suppose the linking number is a prime power, say $b = p^\alpha$. Suppose further that in (3.3) $A(x, 1) = \lambda(x) = a(x + x^{-1}) + (1 - 2a)$ and $\lambda(-1) = 1 - 4a$ is square-free. Then $\lambda(x)$ is a knot polynomial and $R = \Lambda_2/(\lambda(x), y - 1)$ is a Dedekind domain **[L-2]**. (Note that $R = \mathbf{Z}[\alpha, \alpha^{-1}]$, where α is a root of $\lambda(x)$ and that the image of $A(x, 1)$ in

R is 0.) Let q be a prime ideal of R such that $q = \bar{q}$ and consider the localizations

$$\begin{tikzcd}
\Lambda_2 \arrow[r] \arrow[d] \arrow[rr, bend left, "f"] & \Lambda_2/(1-y) \arrow[r] \arrow[d] & R \arrow[d] \\
(\Lambda_2)_{f^{-1}(q)} \arrow[r, "\rho"] \arrow[rr, bend right, "f_q"'] & (\Lambda_2)_{f^{-1}(q)}/(1-y) \arrow[r] & R_q
\end{tikzcd}$$

Since $f(A) = f(\det \mathbf{A}) = 0$, $f_q(A) = 0$. R_q is a Euclidean domain, so the rows of $f_q(\mathbf{A})$ are linearly dependent. Hence, the first row of $f_q(\mathbf{A})$ can be reduced to zero by elementary row operations. By performing the conjugate column operations, the first column of $f_q(\mathbf{A})$ can be reduced to zero as well. An elementary f_q-matrix can be lifted to an elementary $(\Lambda_2)_{f^{-1}(q)}$-matrix, so there is an elementary $(\Lambda_2)_{f^{-1}(q)}$-matrix, $\mathbf{P}$, such that $f_q(\mathbf{P}\mathbf{A}\bar{\mathbf{P}}^{\mathrm{tr}})$ has first row and column zero (here, bar denotes $x \to x^{-1}$). Let $\mathbf{Q} = 1 \oplus \mathbf{P}$. Then $\mathbf{Q}\mathbf{B}\bar{\mathbf{Q}}$ has the form

$$\begin{bmatrix} 0 & \beta_1 & \gamma \\ \bar{\beta}_1 & a\lambda(x) + (y-1)b & \lambda(x)\mu + \nu(y-1) \\ \bar{\gamma}^{\mathrm{tr}} & \lambda(x^{-1})\bar{\mu}^{\mathrm{tr}} + (y^{-1} - 1)\bar{\nu}^{\mathrm{tr}} & \mathbf{C} \end{bmatrix}$$

where γ, μ, ν are row matrices with entries in $(\Lambda_2)_{f^{-1}(q)}$, $\mathbf{C} = \bar{\mathbf{C}}^{\mathrm{tr}}$ is a square matrix with entries in $(\Lambda_2)_{f^{-1}(q)}$, a, b and β_1 are elements of $(\Lambda_2)_{f^{-1}(q)}$, and $\ker f_q = (y-1, \lambda(x))$.

Since $A(x,1) = \lambda(x)$, $\ker f_q = (y-1, \lambda(x))$ and the matrix $\mathbf{P}\mathbf{A}\bar{\mathbf{P}}^{\mathrm{tr}}$ in $(\Lambda_2)_{f^{-1}(q)}/(y-1)$ has the form

$$\begin{bmatrix} a\lambda(x) & \lambda(x)\mu \\ \lambda(x)\bar{\mu}^{\mathrm{tr}} & \mathbf{C} \end{bmatrix}.$$

Hence,

$$\lambda(x) = \rho(\det \mathbf{A}) = \rho(\det \mathbf{P}\mathbf{A}\bar{\mathbf{P}}^{\mathrm{tr}}) = a\lambda(x)\rho(\det \mathbf{C}) \pmod{\lambda^2}.$$

That is,

$$1 = a\rho(\det \mathbf{C}) \pmod{\lambda},$$

so $f_q(\det \mathbf{C})$ is a unit in R_q. Therefore, the ideal $(f_q(\det \mathbf{B})) = (f_q(\beta_1))(\overline{f_q(\beta_1)})$. Since R_q is a discrete valuation ring, let $v_q(I)$ be defined by $I_q = q^{v_q(I)}$ for each ideal I of R. Thus, if $q = \bar{q}$, $v_q(f_q(\det \mathbf{B})) = 2v_q(f_q(\beta_1)) = 2w_q$. If $q \neq \bar{q}$, $v_q(f_q(\det \mathbf{B})) = v_{\bar{q}}(f_{\bar{q}}(\det \mathbf{B}))$ since $f_q(\det \mathbf{B}) = f_{\bar{q}}(\det \mathbf{B})$ for all q. Let $z_q = v_q(f_q(\det \mathbf{B}))$ in this case. Let $S = \{q \neq \bar{q} \mid z_q > 0\}$ and let $T \subset S$ contain exactly one representative of each conjugate pair. Let

$$J = \prod_{r \in T} r^{z_q} \prod_{q = \bar{q}} q^{w_q}.$$

Then $v_q(J\bar{J}) = v_q(f_q(\det \mathbf{B}))$ for all primes q of R (i.e., $(J\bar{J})_q = f_q(\det \mathbf{B}) \forall q$). Thus, $f(\det \mathbf{B}) = J\bar{J}[S]$.

Now let ω be a primitive p^βth root of unity, $\beta \leqslant \alpha$ ($d = p^\beta$ where d divides b). Consider $R \xrightarrow{g} \mathbf{Z}[\omega]/\lambda(\omega)$, where g is defined by evaluation. The ideal generated by the image of B in $\mathbf{Z}[\omega]/\lambda(x)$ is of the form $J\bar{J}$ for some ideal J since the involution in Λ_2 is compatible with complex conjugation in $\mathbf{Z}[\omega]$. Thus one has

(3.7) THEOREM. *Let L be a two-component link with linking number, p^α, where p is a prime. Let $\lambda(x) = a(x + x^{-1}) + (1 - 2a)$, where $\lambda(-1)$ is square-free and let $\Delta(x, y)$ be the Alexander polynomial of L. If the knot polynomial*

$$(1 - x)^{-1}(1 - x^{p^\alpha})\Delta(x, 1) = \lambda(x)$$

and if ω is a primitive p^βth root of unity for some $\beta \leqslant \alpha$, then the $\mathbf{Z}[\omega]/\lambda(\omega)$-ideal generated by $\mathbf{B}(\omega, 1)$ (mod $\lambda(\omega)$) is of the form $J\bar{J}$ for some ideal J.

Question 2 can now be specialized as follows.

(3.8) QUESTION 2′. Let

$$D(x, y) = \left(1 - (xy)^{p^\alpha}\right)(1 - xy)^{-1}\left(a(x + x^{-1}) + 1 - 2a\right)$$
$$- (1 - x)(1 - y)\left(1 - (xy)^{p^\alpha - 1}\right)(1 - xy)^{-1}(c).$$

Is it possible to choose a and c so that

(i) $4a - 1$ is square-free,

(ii) c does not generate an ideal of the form $J\bar{J}$ in $\mathbf{Z}[\omega]/\lambda(\omega)$?

The answer to the question is yes, provided $p \neq 2$.

D. Let ω be a primitive pth root of unity and let $\theta = \omega + \omega^{-1} - 2$. Then $\lambda(\omega) = 1 + a\theta$.

Consider the diagram

$$\begin{array}{rcc} L = \mathbf{Q}[\omega] & \supset & \mathbf{Z}[\omega] \\ \text{degree } 2 \downarrow & & \downarrow \\ K = \mathbf{Q}[\theta] & \supset & \mathbf{Z}[\theta] \\ \text{degree}(p-1)/2 \downarrow & & \downarrow \\ \mathbf{Q} & \supset & \mathbf{Z} \end{array}$$

The following properties are easily established [**La-1, M**].

(i) $\mathbf{Z}[\omega + \omega^{-1}] = \mathbf{Z}[\theta]$.

(ii) $f(a) = N_{K/\mathbf{Q}}(\lambda(\omega))$ splits over $\mathbf{Z}[\theta]$ into factors which are linear in a.

(iii) $\theta = (\omega^{-1} - 1)(1 - \omega)$ and $N_{K/\mathbf{Q}}(\theta) = (-1)^{[(p-1)/2]}p$.

(iv) If $q \in \mathbf{Z}$ is a prime such that $q \equiv -1 \pmod p$, then q splits into $r = (p - 1)/2$ distinct primes in $\mathbf{Z}[\theta]$. Furthermore, the decomposition group of q is $D = \langle\sigma\rangle$ where σ is complex conjugation, so $Q_i = \bar{Q}_i$ for each Q_i dividing q in $\mathbf{Z}[\omega]$.

Now fix $q \equiv -1 \pmod p$ and let Q be a prime dividing q.

(3.9) LEMMA. *If $1 + a\theta \in Q$, then for any Q' ($\neq Q$) dividing q, $1 - a\theta \notin Q'$. Hence, if $1 + a\theta \in Q^n$, then $1 + a\theta \notin (Q')^n$.*

PROOF. If $p = 3$, $\mathbf{Q}[\theta] = \mathbf{Q}$, and there is nothing to prove. If $p > 3$, let $(q) = Q_1 \cdots Q_{(p-1)/2}$ be the splitting of q in $\mathbf{Z}[\theta]$. WLOG $Q = Q_1$ and $Q' = Q_2$. Suppose $1 + a\theta \in Q_1$ and $1 + a\theta \in Q_2$. There is $\tau \in \mathrm{Gal}(\mathbf{Q}[\theta] : \mathbf{Q})$ such that $\tau(Q_1) = Q_2$.

Now $\tau(\theta) = \theta'$ for some $\theta \neq \theta'$ since θ is primitive. Hence,

$$\tau(1 + a\theta) = 1 + a\theta'.$$

Therefore

$$(1 + a\theta) - (1 + a\theta') = a(\theta - \theta') \in Q_2.$$

However, $a \notin Q_2$ since $a \in Q_2$ implies $1 \in Q_2$ and $\theta - \theta' \notin Q_2$ since $\theta - \theta'$ is only divisible by primes lying over p. This cannot happen since Q_2 is a prime ideal and $q \neq p$.

(3.10) LEMMA. *There is an* $a \in \mathbf{Z}$ *such that* $1 + a\theta \in Q^2$. *Hence*, $f(a) \equiv 0 \pmod{q^2}$ *has an integral solution, and these conditions on a are equivalent.*

PROOF. $\mathbf{Z}[\theta]/Q \simeq \mathbf{Z}/q$. Let $g(a) = 1 + a\theta$. There is a solution to $g(a) = 0$ in $\mathbf{Z}[\theta]/Q$ since $\mathbf{Z}[\theta]/Q$ is a field and θ is nonzero in $\mathbf{Z}[\theta]/Q$. For a, one takes the corresponding element of $\mathbf{Z}/q$. For this choice of a, $f(a) \equiv 0 \pmod q$. If $f(a) \not\equiv 0 \pmod{q^2}$, then a can be modified $\pmod q$ so that $f(a) \equiv 0 \pmod{q^2}$. This follows because

$$\begin{aligned} f(a + kq) &\equiv f(a) + kqf'(a) \pmod{q^2} \\ &\equiv qr + kqf'(a) \pmod{q^2} \quad (\text{since } f(a) \equiv 0 \bmod q) \\ &\equiv q\big(r + kf'(a)\big) \pmod{q^2}. \end{aligned}$$

f and f' are relatively prime, so $f'(a) \not\equiv 0 \pmod q$. Hence, one seeks k such that

$$f(a + kq) \equiv q\big(r + kf'(a)\big) \equiv 0 \pmod{q^2}.$$

Equivalently,

$$r + kf'(a) \equiv 0 \pmod q.$$

But $h(k) = r + f'(a)k$ is a linear polynomial in k; hence it has a root in $\mathbf{Z}/q$. Let $a' = a + kq$. Then $g(a') \equiv 0 \pmod{Q^2}$ and $f(a') \equiv 0 \pmod{q^2}$. Similarly,

(3.11) LEMMA. $1 - 4a \not\equiv 0 \pmod q$.

Finally, if $1 - 4a$ is not a prime, Dirichlet's density theorem allows modification of $a \pmod{q^2}$ so that $1 - 4a$ is a prime. That is, a can be chosen within a given residue class mod q^2 so that $1 - 4a$ is prime.

In summary, for $\lambda(x) = a(x + x^{-1}) + (1 - 2a)$ there is a prime $q = -1 \pmod p$ and an integer a such that $1 - 4a$ is prime and such that $\lambda(\omega) \in Q^2$, where Q is a prime in K dividing q. All that remains is to show that q does not factor as $J\bar{J}$ in $\mathbf{Z}[\omega]/\lambda(\omega)$.

The condition $(q) \neq J\bar{J}$ in $\mathbf{Z}[\omega]/\lambda(\omega)$ is equivalent to

$$(q) + (\lambda(\omega)) \neq J\bar{J} + (\lambda(\omega)) \quad \text{in } \mathbf{Z}[\omega].$$

Suppose $\lambda(\omega) = \prod Q_i^{e_i} \prod P_j^{f_j}$, $P_j \neq Q_i$, is a factorization of $\lambda(\omega)$ in $\mathbf{Z}[\omega]$. (It is possible that some of the e_i s are zero.) Then

$$(q) + (\lambda(\omega)) = \prod_{i=1}^{(p-1)/2} Q_i^{\min(1, e_i)},$$

$$J\bar{J} + (\lambda(\omega)) = \prod_{i=1}^{(p-1)/2} Q_i^{h_i} \prod P_j^{g_j},$$

where $h_i = e_i$ or some even integer less than e_i. (If Q_i^b is the Q_i-factor of J, then Q_i^{2b} is the Q_i-factor of $J\bar{J}$.) Thus, for inequality, it is sufficient that some e_i be at least 2. By Lemmas 3.9 and 3.10 the first time this situation occurs is when $f(a) \equiv 0 \pmod{q^2}$.

Thus, to realize a counterexamples to (3.1) for a given linking number b, one needs to fix a prime p dividing b and consider $f(a)$, the norm of $1 + a\theta$. If $f(a) \equiv 0 \pmod q$ with $q = -1 \pmod p$, then a can be modified $(\operatorname{mod} q)$ so that $f(a) \equiv 0 \pmod{q^2}$. If $1 - 4a$ is not square-free, then $1 - 4a$ can be modified $(\operatorname{mod} q^2)$ so that $1 - 4a$ is square-free (in fact, $1 - 4a$ can be modified $(\operatorname{mod} q^2)$ so that it is a prime). Then

$$\begin{aligned} D(x, y) = &\left(1 - (xy)^b\right)(1 - xy)^{-1}\left(a(x + x^{-1}) + (1 - 2a)\right) \\ &- (1 - x)(1 - y)\left(1 - (xy)^{b-1}\right)(1 - xy)^{-1}(q) \end{aligned}$$

satisfies (3.1) but is not the Alexander polynomial for any two-component link with linking number b.

E. *Comment on the case $p^\alpha = 2^\alpha$.* If $\alpha = 1$, then condition (i) requires that $\lambda(-1)$ be square-free. Also, $\omega = -1$ in this case. Hence, $\lambda(\omega) = \lambda(-1)$. Condition (ii) then requires $\lambda(-1)$ to have a square factor. Hence, this technique will not yield a counterexample to (3.1) because (i) and (ii) cannot be satisfied simultaneously.

If $b = 4$, then $\mathbf{Z}[\omega] = \mathbf{Z}[i]$ and Question 2′ (3.8) can be specialized in the following easily realizable conditions.

(i) $1 - 4a$ is square free.

(ii) -1 is not a square $(\operatorname{mod} q)$ and $\lambda(i) = 1 - 2a \equiv 0 \pmod{q^2}$.

Condition (ii) assures that q is a prime in $\mathbf{Z}[i]$. *If* $1 - 2a \equiv 0 \pmod{q^2}$, then the argument of the previous section shows that $(q) \neq J\bar{J}$ in $\mathbf{Z}[i]/\lambda(i)$. Hence, in (3.8), one takes $c = q$ and

$$\begin{aligned} D(x, y) = &\left(1 - (xy)^4\right)(1 - xy)^{-1}\left(a(x + x^{-1}) + 1 - 2a\right) \\ &- (1 - x)(1 - y)\left(1 - (xy)^3\right)(1 - xy)^{-1} \cdot q. \end{aligned}$$

Similar counterexamples exist for any $b = 2^\alpha$ with $\alpha > 1$.

F. *Examples.* The case $p = 3$.

In this case $\mathbf{Q}[\theta] = \mathbf{Q}$, $\mathbf{Q}[\omega] = \mathbf{Q}[\sqrt{-3}]$, $\lambda(\omega) = 1 - 3a$ and $q \equiv -1 \pmod 3$ reduces to q remains prime in $\mathbf{Z}[\omega]$ (since $r = 1$). The conditions can be reformulated as follows.

(i) $1 - 4a$ is square-free.

(ii) -3 is not a square $(\operatorname{mod} q)$ ($q = 2$ is excluded) and $1 - 3a \equiv 0 \pmod{q^2}$. For instance, one may take $q = 5$ and $a = 17$.

For $p = 5$, $q = 19$ and $a = 9 - 19^2$. Then, $f(a) = 19^2 \cdot 1721$ and $1 - 4a = 1409$, so the conditions are met.

4. A generalization for m-component links. Let $B = (b_{ij})$ be a matrix of linking number **[L-1]**. The entry in row i and column j, b_{ij}, is $\operatorname{lk}(L_i, L_j)$. The diagonal entries are undefined. If B is an $(m \times m)$ matrix, a splitting, S, of B is a proper subset of the integers $\{1, \dots, m\}$ such that $b_{ij} = 0$ whenever $i \in S$ and $j \notin S$. If there is a splitting of B, B is said to be splittable. Levine **[L-2]** has shown that any link with an unsplittable linking matrix has nonzero Alexander polynomial and that the zero polynomial is allowed if and only if B is splittable. Let B_i denote the matrix obtained from B by deleting the ith row and column.

For convenience, let $b_{ii} = 0$ and consider

$$B = \begin{bmatrix} 0 & b & 0 & & 0 \\ b & 0 & 1 & & \\ 0 & 1 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{bmatrix}.$$

Consider, also, the polynomial

$$f_m(t_1, \dots, t_m) = (t_2 - 1) \cdots (t_{m-1} - 1)\Delta(t_1, t_2)$$

where $\Delta(t_1, t_2)$ is one of the examples from 3B or 3C, and hence cannot be the Alexander polynomial of a two-component link with linking number b. (Thus, $b \neq 0, 1, 2$.) In Theorem 4.3 it will be shown that f_m cannot be the Alexander polynomial of any m-component link with linking matrix B.

For $m > 2$, the Torres conditions are

T1 $\qquad \Delta(t_1, \dots, t_m) = (-1)^m t_1^{a_1} \cdots t_m^{a_m} \Delta(t_1^{-1}, \dots, t_m^{-1})$ for some a_i.

T2 $\qquad \Delta\Big(t_1, \dots, \underset{i}{1}, \dots, t_m\Big) = \big(t_1^{b_{1i}} \cdots \hat{t}_i \cdots t_m^{b_{mi}} - 1\big)\Gamma_i$

where Γ_i satisfies Torres' conditions for $m - 1$ variables.

(4.1) Lemma. *The polynomial $f_3(t_1, t_2, t_3)$ determines the linking matrix B.*

Proof. Suppose $B = (b_{ij})$. We use Torres' second conditions on $f(t_1, t_2, t_3)$ to determine the b_{ij}.

$$f_3(t_1, t_2, t_3) = (t_2 - 1)\left[\frac{1 - (t_1 t_2)^b}{t - t_1 t_2}\lambda(t_1) - (1 - t_1)(1 - t_2)\left(\frac{1 - (t_2 t_2)^{b-1}}{1 - t_1 t_2}\right) m\right],$$

where $\lambda(t_1) = \lambda(t_1^{-1})$, $\lambda(1) = 1$ and $m \in \mathbf{Z}$. Then

(i) by substituting $t_1 = 1$, we get

$$t_2^b - 1 = \big(t_2^{b_{12}} t_3^{b_{13}} - 1\big)\Gamma_1,$$

for Γ_1 satisfying Torres' conditions for two variables. Clearly $b_{13} = 0$, and $b_{12} \neq 0$ since $b \neq 0$. There are two choices for b_{12} which are considered below.

(ii) by substituting $t_2 = 1$, we get $f(t_1, 1, t_3) = 0$. Hence $\Gamma_2 = 0$, or $b_{12} = b_{23} = 0$, which is impossible.

(iii) by substituting $t_3 = 1$, we get

$$(t_2 - 1)\left[\frac{1-(t_1t_2)^b}{1-t_1t_2}\lambda(t_1) - (1-t_1)(1-t_2)\left(\frac{1-(t_1t_2)^{b-1}}{1-t_1t_2}\right)m\right] = \left(t_2^{b_{23}} - 1\right)\Gamma_3$$

for some Γ_3 satisfying Torres' conditions for two variables. (Recall $b_{13} = 0$.)

Now consider the cases for b_{12}.

Case I. b_{12} is a proper divisor of b or $b_{12} = 1$. Then $\Gamma_1 = 1 + t_2^{b_{12}} + \cdots + t_2^{b-b_{12}}$, so $\Gamma_1(1,1) = b_{23} > 1$. Then in (iii)

$$\left(t_2^{b_{23}} - 1\right)\Gamma_3 = (t_2 - 1)\left[\frac{1-(t_1t_2)^b}{1-t_1t_2}\lambda(t_1) - (1-t_1)(1-t_2)\left(\frac{1-(t_1t_2)^{b-1}}{1-t_1t_2}\right)m\right].$$

That is,

$$\begin{aligned}&\left(1 + t_2 + \cdots + t_2^{b_{23}-1}\right)\Gamma_3\\ &\quad= \left[\frac{1-(t_1t_2)^b}{1-t_1t_2}\lambda(t_1) - (1-t_1)(1-t_2)\left(\frac{1-(t_1t_2)^{b-1}}{1-t_1t_2}\right)m\right].\end{aligned}$$

Thus, $1 + t_2 + \cdots + t_2^{b_{23}-1}$ divides the right-hand side.

Let $t_2 = 1$; then b_{23} divides $1 + t_1 + \cdots + t_1^{b-1}$, which is impossible since $b_{23} > 1$.

Case II. $b_{12} = b$. Then $\Gamma_1 = 1$ and $\Gamma_1(1,1) = b_{23} = 1$. Thus $(t_2 - 1)$ divides $f(t_1, t_2, 1)$, which is clearly true, and

$$B = \begin{bmatrix} 0 & b & 0 \\ b & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

(4.2) LEMMA. *Up to a permutation of* $\{3, 4, \ldots, m-1\}$, *the polynomial* $f_m(t_1, \ldots, t_m)$ *determines the linking matrix* ($m \geqslant 3$).

PROOF. (Induction on m. The first step is (4.1).)

(i) Substituting $t_1 = 1$ gives

$$\left(t_2^b - 1\right)(t_3 - 1)\cdots(t_{m-1} - 1) = \left(t_2^{b_{12}} \cdots t_m^{b_{1m}} - 1\right)\Gamma_1.$$

Clearly $b_{1m} = 0$. The choices for b_{1j}, $2 \leqslant j \leqslant m-1$, are considered below.

(ii) Substituting $t_j = 0$, $2 \leqslant j \leqslant m-1$, gives

$$f_m(t_1, 1, t_3, \ldots, t_m) = \cdots = f_m(t_1, \ldots, t_{m-2}, 1, t_m) = 0.$$

(iii) Substituting $t_m = 1$ gives

$$\begin{aligned}f_m(t_1, \ldots, t_{m-1}, 1) &= (t_2 - 1)\cdots(t_{m-1} - 1)\Delta(t_1, t_2)\\ &= \left(t_2^{b_{2m}} \cdots t_{m-1}^{b_{m-1,m}} - 1\right)\Gamma_m.\end{aligned}$$

Now consider the choices for b_{1j}, $2 \leqslant j \leqslant m-1$.

Case I. $b_{12} = 0$. By permuting $\{3, \ldots, m-1\}$ if necessary, we may assume $b_{1,m-1} = 1$, $b_{ij} = 0$, $3 \leqslant j \leqslant m-2$, the first row of B is (0____0 1 0) and

$$\Gamma_1(t_2, \ldots, t_m) = \left(t_2^b - 1\right)(t_3 - 1)\cdots(t_{m-2} - 1).$$

Then

$$\Gamma_1(t_2,\dots,t_{m-1},1) = \left(t_2^b - 1\right)(t_3 - 1)\cdots(t_{m-2} - 1)$$
$$= \left(t_2^{b_{2m}} \cdots t_{m-1}^{b_{m-1,1}} - 1\right)\Gamma_{1m}.$$

Hence $b_{m-1,1} = 0$. There are three choices for b_{2m}. If $b_{2m} = b$, then $b_{jm} = 0$, $3 \leqslant j \leqslant m - 2$, and in (iii) above $f_m(t_1,\dots,t_{m-1},1)$ must be divisible by $(t_2^b - 1)$ which cannot happen. If $b_{2m} = 1$, then $b_{jm} = 0$, $3 \leqslant j \leqslant m$, the last column of B is $(0 \quad 1 \quad 0\_\_\_\_0)^{\mathrm{tr}}$ and

$$\Gamma_{1m}(t_2,\dots,t_{m-1}) = (t_3 - 1)\cdots(t_{m-2} - 1)\left(1 + t_2 + \cdots + t_2^{b-1}\right).$$

Note that Γ_{1m} is the same as Γ_1 of Case II with one fewer variable. Following the procedure of Case II, one finds that $b_{m-2,m-1} = b$ which leads to a contradiction, namely that $(t_{m-2}^b - 1)$ divides $\Gamma_{1m}(1,t_2,\dots,t_{m-1})$.

Case II. $b_{12} = 1$ or b_{12} is a proper divisor of b. Then $b_{1j} = 0$ for $3 \leqslant j \leqslant m - 1$, the first row of B is $(0 \quad b_{12} \quad 0\_\_\_\_0)$ and

$$\Gamma_1(t_2,\dots,t_m) = \left(1 + t_2^{b_{12}} + \cdots + t_2^{b-b_{12}}\right)(t_3 - 1)\cdots(t_{m-1} - 1).$$

Thus

$$\Gamma(1,t_3,\dots,t_m) = \frac{b}{b_{12}}(t_3 - 1)\cdots(t_{m-1} - 1)$$
$$= \left(t_3^{b_{23}} \cdots t_m^{b_{2m}} - 1\right)\Gamma_{12}.$$

Clearly $b_{2m} = 0$. By permuting $\{3,\dots,m-1\}$ if necessary, we may assume $b_{23} = 1$ and $b_{2j} = 0$, $4 \leqslant j \leqslant m - 1$, and the second row of B is $(1 \quad 0 \quad 1 \quad 0\_\_\_\_0)$. Continuing this way

$$\Gamma_{12}(t_3,\dots,t_m) = \frac{b}{b_{12}}(t_4 - 1)\cdots(t_{m-1} - 1)$$
$$\vdots$$
$$\Gamma_{12\cdots m-2}(t_{m-1},t_m) = \frac{b}{b_{12}} \quad \text{so} \quad b_{m-1,m} = \frac{b}{b_{12}} > 1$$

and

$$B = \begin{bmatrix} 0 & 1 & & & \\ 1 & \ddots & \ddots & & \frac{b}{b_{12}} \\ & \ddots & \ddots & 1 & \\ & & 1 & \ddots & \\ & \frac{b}{b_{12}} & & & 0 \end{bmatrix}.$$

But then

$$\Gamma_1(t_2,\dots,t_{m-1},1) = (t_3 - 1)\cdots(t_{m-1} - 1)\left(1 + t_2^{b_{12}} \cdots t_2^{b-b_{12}}\right)$$
$$= \left(t_{m-1}^{b/b_{12}} - 1\right)\Gamma_{1m}$$

(since $b_{jm} = 0$, $j < m - 1$). This is impossible (since $b/b_{12} > 1$ so $(t_{m-1}^{b/b_{12}} - 1)$ does not divide $(t_3 - 1)\cdots(t_{m-1} - 1)(1 + t_2^{b_{12}} \cdots t_2^{b-b_{12}})$.

Case III. $b_{12} = b$. Then, $b_{ij} = 0$, $3 \leqslant j \leqslant m - 1$. Hence the first row of B is $(0 \quad b \quad 0 \_\_\_\_ 0)$, and

$$\begin{aligned}\Gamma_1(t_2, \ldots, t_m) &= (t_3 - 1)\cdots(t_{m-1} - 1)\\ &= \left(t_2^{b_{2m}} \cdots t_{m-1}^{b_{m-1,m}} - 1\right)\Gamma_{1m}.\end{aligned}$$

Clearly $b_{2m} = 0$. By permuting $\{3, \ldots, m - 1\}$ if necessary, we may assume $b_{m-1,m} = 1$ and $b_{jm} = 0$, $3 \leqslant j \leqslant m - 2$. Hence, the last column of B is $(0 \_\_\_\_ 0 \quad 1 \quad 0)^{\mathrm{tr}}$, and $\Gamma_{1m} = (t_3 - 1)\cdots(t_{m-2} - 1)$. Continuing in this way we find

$$B = \begin{bmatrix} 0 & b & 0 & & & \\ b & 0 & 1 & & & \\ 0 & 1 & 0 & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & 0 & \end{bmatrix},$$

and in (iii) above $\Gamma_m = f_{m-1}(t_1, \ldots, t_{m-1})$, and in (ii) $\Gamma_2 = \cdots = \Gamma_{m-1} = 0$.

(4.3) Theorem. *With the conditions as above $f_m(t_1, \ldots, t_m)$ cannot be the Alexander polynomial of an m-component link with linking matrix B.*

Proof (Induction on m). For $m = 3$, $f_3(t_1, t_2, 1) = (t_2 - 1)\Delta(t_1, t_2)$. Hence $\Gamma_3 = \Delta(t_1, t_2)$, which is not allowed by Torres' second condition.

For $m > 3$,

$$f_m(t_1, \ldots, t_{m-1}, 1) = (t_{m-1} - 1)\left[\prod_{i=2}^{m-2}(t_i - 1)\right]\Delta(t_1, t_2).$$

Hence

$$\Gamma_m = \prod_{i=2}^{m-2}(t_i - 1)\Delta(t_1, t_2) = f_{m-1}(t_1, \ldots, t_{m-1}),$$

which is not allowed by induction.

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