

INSUFFICIENCY OF TORRES' CONDITIONS FOR TWO-COMPONENT CLASSICAL LINKS

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ABSTRACT. Torres has given necessary conditions for a polynomial to be the Alexander polynomial of a two component link. For certain links, additional conditions are necessary. Hillman gave one example for linking number 6. Here we give examples for all other linking numbers except 0, ± 1 , and ± 2 .

1. Introduction. In 1953, Torres [T] gave necessary conditions for a polynomial to be the Alexander polynomial of a link. More recently, in the case of two component links with linking number b , Bailey [B] showed equivalently that the Alexander polynomial of the link can be expressed in the form

$$\Delta(x, y) = \frac{1 - (xy)^b}{1 - xy} A(x, y) - (1 - x)(1 - y) \left(\frac{1 - (xy)^{b-1}}{1 - xy} \right) B(x, y),$$

where $A(x, y)$ and $B(x, y)$ satisfy certain conditions.

Using Bailey's result, Hillman [H] gave an additional condition on the Alexander polynomial of certain two component links whose linking number is divisible by at least two distinct primes. In §3 of this paper, a similar result is given for prime power linking numbers in

(3.7) THEOREM. *Let L be a two-component link with linking number, p^α , where p is a prime. Let $\lambda(x) = a(x + x^{-1}) + (1 - 2a)$, where $\lambda(-1)$ is square-free, and let $\Delta(x, y)$ be the Alexander polynomial of L . If the knot polynomial*

$$(1 - x)^{-1}(1 - x^{p^\alpha})\Delta(x, 1) = \lambda(x)$$

and if ω is a primitive p^β th root of unity for some $\beta \leq \alpha$, then the $\mathbf{Z}[\omega]/\lambda(\omega)$ -ideal generated by $B(\omega, 1) \pmod{\lambda(\omega)}$ is of the form $J\bar{J}$ for some ideal J .

It should be noted that the ideal $J\bar{J}$ depends only on $\Delta(x, y)$ and not on the expansion given above.

Following (3.7), we show how to realize counterexamples to Torres' condition for two-component links, provided the linking number of the components is not 0, ± 1 , or ± 2 . It should be noted that the Torres conditions do suffice if $b = 0$ or ± 1 . Hence, only the case when $b = 2$ remains unsettled. Finally, a counterexample to Torres' conditions for m -component links ($m > 3$) is given.

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2. Definitions. A classical link of multiplicity m is a collection, $L = L_1 \cup \cdots \cup L_m$ of oriented smooth simple closed curves in S^3 satisfying $L_i \cap L_j = \emptyset$ if $i \neq j$. The number m denotes the number of components of the link. If $m = 1$, then L is a knot. The number $b = \text{lk}(L_i, L_j)$ is the linking number of the i th component and the j th component. A link is trivial if it is the boundary of m disjoint 2-disks in S^3 .

The complement of the link is the space

$$X = S^3 - \bigcup_{i=1}^m \nu(L_i),$$

where $\nu(L_i)$ is a small open tubular neighborhood of L_i . The neighborhoods $\nu(L_i)$ can be chosen so small that $\nu(L_i) \cap \nu(L_j) = \emptyset$ if $i \neq j$. The basepoint of X is denoted by $*$.

For each i , let m_i be a small circle linking the i th component L_i of L with $\text{lk}(m_i, L_i) = 1$, let l_i be a translate of L_i into X whose basepoint coincides with that of m_i and such that $\text{lk}(l_i, L_i) = 0$ and let γ_i be a path in X from $*$ to the basepoint of m_i . The elements α_i of $\pi_1 X$ represented by $\gamma_i * m_i * \gamma_i^{-1}$ are called meridians of the link. The elements β_i of $\pi_1 X$ represented by $\gamma_i * l_i * \gamma_i^{-1}$ are called longitudes of the link. The pair (α_i, β_i) is determined up to simultaneous conjugation by the element of $\pi_1 X$.

An orientation of a link consists of an ordering of the components together with an orientation of each component.

By Alexander duality, $H_1(X) \cong \mathbf{Z}^m$, where m is the multiplicity of the link. A canonical basis of $H_1(X)$, defined by any choice of meridians of L , allow the identification of $\mathbf{Z}[H_1(X)]$ with $\Lambda_m = \mathbf{Z}[t_1, t_1^{-1}, \dots, t_m, t_m^{-1}]$, the identification depending only on the orientation of the link. There is a natural involution of Λ_m (denoted with an overbar) which maps $t_i \rightarrow t_i^{-1}$. The augmentation of Λ_m is given by $\epsilon: \Lambda_m \rightarrow \mathbf{Z}$, where $\epsilon(t_i) = 1$.

The canonical homomorphism $h: \pi_1 X \rightarrow H_1(X)$ defines a regular covering space $p: \tilde{X} \rightarrow X$ with \mathbf{Z}^m as the group of covering transformations. The space \tilde{X} is called the universal abelian cover of X . The Alexander module of L is $H_1(\tilde{X}, \tilde{*})$ considered as a module over Λ_m . The module of L is $H_1(\tilde{X})$ considered as a module over Λ_m .

These modules are related by the exact sequence

$$0 \rightarrow H_1(\tilde{X}) \rightarrow H_1(\tilde{X}, \tilde{*}) \xrightarrow{\Phi} M \rightarrow 0,$$

where Φ is the boundary homomorphism, $\Phi: H_1(\tilde{X}, \tilde{*}) \rightarrow H_0(\tilde{*})$, and M is the augmentation ideal of Λ_m generated by $t_1 - 1, \dots, t_m - 1$ [L-3].

Given a presentation matrix for $H_1(\tilde{X}, \tilde{*})$ as a Λ_m -module, the sequence of elementary ideals, or Fitting invariants, is $\tilde{E}_i(L)$, where $\tilde{E}_i(L)$ is the ideal of Λ_m generated by the $(n - i)$ -order minors of the presentation matrix [F]. Let $\tilde{\Delta}_i(L)$ be

the greatest common divisor or $\tilde{E}_i(L)$. A sequence of ideals $E_i(L)$ and polynomials $\Delta_i(L)$ can be defined the same way from a presentation matrix for $H_1(\tilde{X})$. Although $\tilde{E}_i(L) \neq E_{i-1}(L)$ in general, $\tilde{\Delta}_i(L) = \Delta_{i-1}(L)$ [L-1]. The Alexander polynomial of L is

$$\tilde{\Delta}_1(L) = \Delta_L(t_1, \dots, t_m).$$

3. Two-component links.

A. Suppose $L = K_1 \cup K_2$ is a two-component link with linking number b . Under these circumstances Torres [T] has shown that the Alexander polynomial of L can be chosen to have the following properties:

$$(3.1) \quad \begin{aligned} (1) \quad \Delta(x, y) &= x^{b-1}y^{b-1}\Delta(x^{-1}, y^{-1}), \\ (2) \quad \Delta(x, 1) &= (1 - x^b)(1 - x)^{-1}\Delta_2(x), \\ \Delta(1, y) &= (1 - y^b)(1 - y)^{-1}\Delta_1(y), \end{aligned}$$

where $\Delta_2(x)$ and $\Delta_1(y)$ are knot polynomials. In fact, $\Delta_2(x)$ and $\Delta_1(y)$ are the Alexander polynomials of the component knots corresponding to the meridians x and y , respectively.

Bailey [B] has characterized the module of L , $H_1(\tilde{X})$, as a Λ_2 -module having a presentation matrix with a certain symmetry condition. Bailey's main result is the following theorem.

(3.2) THEOREM (BAILEY). *A Λ_2 -module is a link module if and only if it has a presentation matrix of the form*

$$M_L = \begin{bmatrix} (1 - (xy)^b)(1 - xy)^{-1} & -[(1 - x)(1 - y)(1 - (xy)^{b-1})(1 - xy)^{-1}\beta(x, y)] \\ \beta^{\text{tr}}(x^{-1}, y^{-1}) & \mathbf{A}(x, y) \end{bmatrix}$$

where $\beta(x, y)$ is a row matrix, $\mathbf{A}(x, y)$ is a square matrix with entries in Λ_2 , satisfying $\mathbf{A}(x, y) = \mathbf{A}^{\text{tr}}(x^{-1}, y^{-1})$ and $\mathbf{A}(1, 1) = \text{diag}(\pm 1, \dots, \pm 1)$. Furthermore, $\mathbf{A}(x, 1)$ (resp. $\mathbf{A}(1, y)$) is a presentation matrix for the first (resp. second) component of the link and b is the linking number of the components.

One corollary of Bailey's theorem is that the Alexander polynomial of a two-component link has the form

$$(3.3) \quad \begin{aligned} D(x, y) &= (1 - (xy)^b)(1 - xy)^{-1}A(x, y) \\ &\quad - (1 - x)(1 - y)(1 - xy)^{-1}(1 - (xy)^{b-1})B(x, y), \end{aligned}$$

where $A(x, y) = A(x^{-1}, y^{-1})$, $B(x, y) = B(x^{-1}, y^{-1})$, and $A(x, 1)$ and $A(1, y)$ are knot polynomials.

For instance, one may take

$$A(x, y) = \det \mathbf{A}(x, y), \quad B(x, y) = \det \begin{bmatrix} 0 & \beta(x, y) \\ \beta^{\text{tr}}(x^{-1}, y^{-1}) & \mathbf{A}(x, y) \end{bmatrix}.$$

Moreover, Bailey showed that a polynomial in Λ_2 has this form if and only if it satisfies (3.1).

Using Bailey’s result, Hillman has proven the following theorem.

(3.4) THEOREM (HILLMAN). *Let L be a two-component link with linking number $b > 1$ and with Alexander polynomial $\Delta(x, y)$. If the knot polynomial $(1 - x^b) \cdot (1 - x)^{-1} \Delta(x, 1)$ is (up to units) the d -cyclotomic polynomial, $\Phi_d(x)$, for some $d > 1$ dividing b and if ω is a primitive d th root of unity, then the $\mathbf{Z}[\omega]$ -ideal generated by $B(\omega, 1)$ is of the form $J\bar{J}$ for some J .*

The hypothesis of the theorem is vacuous unless d is divisible by at least two distinct primes. Hillman’s theorem suggests two questions:

(3.5) QUESTION 1. Do counterexamples to (3.1) exist whenever b is a nonprime power number?

(3.6) QUESTION 2. Do counterexamples to (3.1) exist if b is a power of a prime?

B. To answer (3.5), suppose d is a nonprime power number, $\Phi_d(x)$ is the d -cyclotomic polynomial and a is an integer. Let

$$D(x, y) = (1 - (xy)^d)(1 - xy)^{-1} \Phi_d(x) - (1 - x)(1 - y)(1 - (xy)^{d-1})(1 - xy)^{-1}(a).$$

By direct computation, one finds that $\Phi_d(x)$ is a knot polynomial; hence $D(x, y)$ satisfies (3.1). If ω is a primitive d th root of unity for $d \neq p^\alpha$, one may ask, in view of Hillman’s theorem, if there is an integer, a , such that the ideal generated by a does not factor as $J\bar{J}$ in $\mathbf{Z}[\omega]$?

Suppose that q is a prime, $q \nmid d$, and Q is a prime of $\mathbf{Z}[\omega]$ lying over q . The prime q is unramified since the only ramified primes are those dividing d . The Galois group, $\text{Gal}(\mathbf{Q}[\omega]/\mathbf{Q})$ is isomorphic to $(\mathbf{Z}/d)^\times$. The decomposition group of q , $D(Q|q)$, is the (cyclic) subgroup of $\text{Gal}(\mathbf{Q}[\omega]/\mathbf{Q})$ generated by $\omega \rightarrow \omega^q$, which corresponds to the subgroup of $(\mathbf{Z}/d)^\times$ generated by q .

Suppose that complex conjugation, σ , is an element of the decomposition group, in other words, that $Q = \bar{Q}$. This will happen, for instance, if $q \equiv -1 \pmod{d}$, and by Dirichlet’s density theorem there are infinitely many such primes. Now, any such prime q factors in $\mathbf{Z}[\omega]$ as $\prod Q_i$ with each Q_i distinct and $Q_i = \bar{Q}_i$. In particular, $(q) \neq J\bar{J}$ for any ideal J of $\mathbf{Z}[\omega]$. One may then take $a = q$ and

$$D(x, y) = (1 - (xy)^d)(1 - xy)^{-1} \Phi_d(x) - (1 - x)(1 - y)(1 - (xy)^{d-1})(1 - xy)^{-1}(q).$$

C. In order to answer (3.6) one uses an argument similar to that in [H]. Suppose the linking number is a prime power, say $b = p^\alpha$. Suppose further that in (3.3) $A(x, 1) = \lambda(x) = a(x + x^{-1}) + (1 - 2a)$ and $\lambda(-1) = 1 - 4a$ is square-free. Then $\lambda(x)$ is a knot polynomial and $R = \Lambda_2/(\lambda(x), y - 1)$ is a Dedekind domain [L-2]. (Note that $R = \mathbf{Z}[\alpha, \alpha^{-1}]$, where α is a root of $\lambda(x)$ and that the image of $A(x, 1)$ in

R is 0.) Let q be a prime ideal of R such that $q = \bar{q}$ and consider the localizations

$$\begin{array}{ccccc}
 & & \xrightarrow{\quad f \quad} & & \\
 \Lambda_2 & \longrightarrow & \Lambda_2/(1-y) & \longrightarrow & R \\
 \downarrow & & \downarrow & & \downarrow \\
 (\Lambda_2)_{f^{-1}(q)} & \xrightarrow{\quad \rho \quad} & (\Lambda_2)_{f^{-1}(q)}/(1-y) & \longrightarrow & R_q \\
 & & \xrightarrow{\quad f_q \quad} & &
 \end{array}$$

Since $f(A) = f(\det A) = 0$, $f_q(A) = 0$. R_q is a Euclidean domain, so the rows of $f_q(A)$ are linearly dependent. Hence, the first row of $f_q(A)$ can be reduced to zero by elementary row operations. By performing the conjugate column operations, the first column of $f_q(A)$ can be reduced to zero as well. An elementary f_q -matrix can be lifted to an elementary $(\Lambda_2)_{f^{-1}(q)}$ -matrix, so there is an elementary $(\Lambda_2)_{f^{-1}(q)}$ -matrix, \mathbf{P} , such that $f_q(\mathbf{P}\bar{\mathbf{A}}^{\text{tr}})$ has first row and column zero (here, bar denotes $x \rightarrow x^{-1}$). Let $\mathbf{Q} = 1 \oplus \mathbf{P}$. Then $\mathbf{Q}\bar{\mathbf{B}}\bar{\mathbf{Q}}$ has the form

$$\begin{bmatrix} 0 & \beta_1 & \gamma \\ \bar{\beta}_1 & a\lambda(x) + (y-1)b & \lambda(x)\mu + \nu(y-1) \\ \bar{\gamma}^{\text{tr}} & \lambda(x^{-1})\bar{\mu}^{\text{tr}} + (y^{-1}-1)\bar{\nu}^{\text{tr}} & \mathbf{C} \end{bmatrix}$$

where γ, μ, ν are row matrices with entries in $(\Lambda_2)_{f^{-1}(q)}$, $\mathbf{C} = \bar{\mathbf{C}}^{\text{tr}}$ is a square matrix with entries in $(\Lambda_2)_{f^{-1}(q)}$, a, b and β_1 are elements of $(\Lambda_2)_{f^{-1}(q)}$, and $\ker f_q = (y-1, \lambda(x))$.

Since $A(x, 1) = \lambda(x)$, $\ker f_q = (y-1, \lambda(x))$ and the matrix $\mathbf{P}\bar{\mathbf{A}}^{\text{tr}}$ in $(\Lambda_2)_{f^{-1}(q)}/(y-1)$ has the form

$$\begin{bmatrix} a\lambda(x) & \lambda(x)\mu \\ \lambda(x)\bar{\mu}^{\text{tr}} & \mathbf{C} \end{bmatrix}.$$

Hence,

$$\lambda(x) = \rho(\det A) = \rho(\det \mathbf{P}\bar{\mathbf{A}}^{\text{tr}}) = a\lambda(x)\rho(\det \mathbf{C}) \pmod{\lambda^2}.$$

That is,

$$1 = a\rho(\det \mathbf{C}) \pmod{\lambda},$$

so $f_q(\det \mathbf{C})$ is a unit in R_q . Therefore, the ideal $(f_q(\det \mathbf{B})) = (f_q(\beta_1))\overline{(f_q(\beta_1))}$. Since R_q is a discrete valuation ring, let $v_q(I)$ be defined by $I_q = q^{v_q(I)}$ for each ideal I of R . Thus, if $q = \bar{q}$, $v_q(f_q(\det \mathbf{B})) = 2v_q(f_q(\beta_1)) = 2w_q$. If $q \neq \bar{q}$, $v_q(f_q(\det \mathbf{B})) = v_{\bar{q}}(f_{\bar{q}}(\det \mathbf{B}))$ since $f_q(\det \mathbf{B}) = f_{\bar{q}}(\det \mathbf{B})$ for all q . Let $z_q = v_q(f_q(\det \mathbf{B}))$ in this case. Let $S = \{q \neq \bar{q} \mid z_q > 0\}$ and let $T \subset S$ contain exactly one representative of each conjugate pair. Let

$$J = \prod_{r \in T} r^{z_r} \prod_{q = \bar{q}} q^{w_q}.$$

Then $v_q(J\bar{J}) = v_q(f_q(\det \mathbf{B}))$ for all primes q of R (i.e., $(J\bar{J})_q = f_q(\det \mathbf{B}) \forall q$). Thus, $f(\det \mathbf{B}) = J\bar{J}[S]$.

Now let ω be a primitive p^β th root of unity, $\beta \leq \alpha$ ($d = p^\beta$ where d divides b). Consider $R \xrightarrow{g} \mathbf{Z}[\omega]/\lambda(\omega)$, where g is defined by evaluation. The ideal generated by the image of B in $\mathbf{Z}[\omega]/\lambda(x)$ is of the form $J\bar{J}$ for some ideal J since the involution in Λ_2 is compatible with complex conjugation in $\mathbf{Z}[\omega]$. Thus one has

(3.7) THEOREM. *Let L be a two-component link with linking number, p^α , where p is a prime. Let $\lambda(x) = a(x + x^{-1}) + (1 - 2a)$, where $\lambda(-1)$ is square-free and let $\Delta(x, y)$ be the Alexander polynomial of L . If the knot polynomial*

$$(1 - x)^{-1}(1 - x^{p^\alpha})\Delta(x, 1) = \lambda(x)$$

and if ω is a primitive p^β th root of unity for some $\beta \leq \alpha$, then the $\mathbf{Z}[\omega]/\lambda(\omega)$ -ideal generated by $\mathbf{B}(\omega, 1) \pmod{\lambda(\omega)}$ is of the form $J\bar{J}$ for some ideal J .

Question 2 can now be specialized as follows.

(3.8) QUESTION 2'. Let

$$D(x, y) = (1 - (xy)^{p^n})(1 - xy)^{-1}(a(x + x^{-1}) + 1 - 2a) - (1 - x)(1 - y)(1 - (xy)^{p^n - 1})(1 - xy)^{-1}(c).$$

Is it possible to choose a and c so that

- (i) $4a - 1$ is square-free,
- (ii) c does not generate an ideal of the form $J\bar{J}$ in $\mathbf{Z}[\omega]/\lambda(\omega)$?

The answer to the question is yes, provided $p \neq 2$.

D. Let ω be a primitive p th root of unity and let $\theta = \omega + \omega^{-1} - 2$. Then $\lambda(\omega) = 1 + a\theta$.

Consider the diagram

$$\begin{array}{ccc} L = \mathbf{Q}[\omega] & \supset & \mathbf{Z}[\omega] \\ \text{degree } 2 \downarrow & & \downarrow \\ K = \mathbf{Q}[\theta] & \supset & \mathbf{Z}[\theta] \\ \text{degree } (p-1)/2 \downarrow & & \downarrow \\ \mathbf{Q} & \supset & \mathbf{Z} \end{array}$$

The following properties are easily established [La-1, M].

- (i) $\mathbf{Z}[\omega + \omega^{-1}] = \mathbf{Z}[\theta]$.
- (ii) $f(a) = N_{K/\mathbf{Q}}(\lambda(\omega))$ splits over $\mathbf{Z}[\theta]$ into factors which are linear in a .
- (iii) $\theta = (\omega^{-1} - 1)(1 - \omega)$ and $N_{K/\mathbf{Q}}(\theta) = (-1)^{(p-1)/2}p$.
- (iv) If $q \in \mathbf{Z}$ is a prime such that $q \equiv -1 \pmod{p}$, then q splits into $r = (p - 1)/2$ distinct primes in $\mathbf{Z}[\theta]$. Furthermore, the decomposition group of q is $D = \langle \sigma \rangle$ where σ is complex conjugation, so $Q_i = \bar{Q}_i$ for each Q_i dividing q in $\mathbf{Z}[\omega]$.

Now fix $q \equiv -1 \pmod{p}$ and let Q be a prime dividing q .

(3.9) LEMMA. *If $1 + a\theta \in Q$, then for any $Q' (\neq Q)$ dividing q , $1 - a\theta \notin Q'$. Hence, if $1 + a\theta \in Q^n$, then $1 + a\theta \notin (Q')^n$.*

PROOF. If $p = 3$, $\mathbf{Q}[\theta] = \mathbf{Q}$, and there is nothing to prove. If $p > 3$, let $(q) = Q_1 \cdots Q_{(p-1)/2}$ be the splitting of q in $\mathbf{Z}[\theta]$. WLOG $Q = Q_1$ and $Q' = Q_2$. Suppose $1 + a\theta \in Q_1$ and $1 + a\theta \in Q_2$. There is $\tau \in \text{Gal}(\mathbf{Q}[\theta]: \mathbf{Q})$ such that $\tau(Q_1) = Q_2$.

Now $\tau(\theta) = \theta'$ for some $\theta \neq \theta'$ since θ is primitive. Hence,

$$\tau(1 + a\theta) = 1 + a\theta'.$$

Therefore

$$(1 + a\theta) - (1 + a\theta') = a(\theta - \theta') \in Q_2.$$

However, $a \notin Q_2$ since $a \in Q_2$ implies $1 \in Q_2$ and $\theta - \theta' \notin Q_2$ since $\theta - \theta'$ is only divisible by primes lying over p . This cannot happen since Q_2 is a prime ideal and $q \neq p$.

(3.10) LEMMA. *There is an $a \in \mathbf{Z}$ such that $1 + a\theta \in Q^2$. Hence, $f(a) \equiv 0 \pmod{q^2}$ has an integral solution, and these conditions on a are equivalent.*

PROOF. $\mathbf{Z}[\theta]/Q \simeq \mathbf{Z}/q$. Let $g(a) = 1 + a\theta$. There is a solution to $g(a) = 0$ in $\mathbf{Z}[\theta]/Q$ since $\mathbf{Z}[\theta]/Q$ is a field and θ is nonzero in $\mathbf{Z}[\theta]/Q$. For a , one takes the corresponding element of \mathbf{Z}/q . For this choice of a , $f(a) \equiv 0 \pmod{q}$. If $f(a) \not\equiv 0 \pmod{q^2}$, then a can be modified \pmod{q} so that $f(a) \equiv 0 \pmod{q^2}$. This follows because

$$\begin{aligned} f(a + kq) &\equiv f(a) + kqf'(a) \pmod{q^2} \\ &\equiv qr + kqf'(a) \pmod{q^2} \quad (\text{since } f(a) \equiv 0 \pmod{q}) \\ &\equiv q(r + kf'(a)) \pmod{q^2}. \end{aligned}$$

f and f' are relatively prime, so $f'(a) \not\equiv 0 \pmod{q}$. Hence, one seeks k such that

$$f(a + kq) \equiv q(r + kf'(a)) \equiv 0 \pmod{q^2}.$$

Equivalently,

$$r + kf'(a) \equiv 0 \pmod{q}.$$

But $h(k) = r + f'(a)k$ is a linear polynomial in k ; hence it has a root in \mathbf{Z}/q . Let $a' = a + kq$. Then $g(a') \equiv 0 \pmod{Q^2}$ and $f(a') \equiv 0 \pmod{q^2}$. Similarly,

(3.11) LEMMA. $1 - 4a \not\equiv 0 \pmod{q}$.

Finally, if $1 - 4a$ is not a prime, Dirichlet's density theorem allows modification of $a \pmod{q^2}$ so that $1 - 4a$ is a prime. That is, a can be chosen within a given residue class $\pmod{q^2}$ so that $1 - 4a$ is prime.

In summary, for $\lambda(x) = a(x + x^{-1}) + (1 - 2a)$ there is a prime $q = -1 \pmod{p}$ and an integer a such that $1 - 4a$ is prime and such that $\lambda(\omega) \in Q^2$, where Q is a prime in K dividing q . All that remains is to show that q does not factor as $J\bar{J}$ in $\mathbf{Z}[\omega]/\lambda(\omega)$.

The condition $(q) \neq J\bar{J}$ in $\mathbf{Z}[\omega]/\lambda(\omega)$ is equivalent to

$$(q) + (\lambda(\omega)) \neq J\bar{J} + (\lambda(\omega)) \quad \text{in } \mathbf{Z}[\omega].$$

Suppose $\lambda(\omega) = \prod Q_i^{e_i} \prod P_j^{f_j}$, $P_j \neq Q_i$, is a factorization of $\lambda(\omega)$ in $\mathbf{Z}[\omega]$. (It is possible that some of the e_i s are zero.) Then

$$(q) + (\lambda(\omega)) = \prod_{i=1}^{(p-1)/2} Q_i^{\min(1, e_i)},$$

$$J\bar{J} + (\lambda(\omega)) = \prod_{i=1}^{(p-1)/2} Q_i^{h_i} \prod P_j^{g_j},$$

where $h_i = e_i$ or some even integer less than e_i . (If Q_i^b is the Q_i -factor of J , then Q_i^{2b} is the Q_i -factor of $J\bar{J}$.) Thus, for inequality, it is sufficient that some e_i be at least 2. By Lemmas 3.9 and 3.10 the first time this situation occurs is when $f(a) \equiv 0 \pmod{q^2}$.

Thus, to realize a counterexamples to (3.1) for a given linking number b , one needs to fix a prime p dividing b and consider $f(a)$, the norm of $1 + a\theta$. If $f(a) \equiv 0 \pmod{q}$ with $q = -1 \pmod{p}$, then a can be modified \pmod{q} so that $f(a) \equiv 0 \pmod{q^2}$. If $1 - 4a$ is not square-free, then $1 - 4a$ can be modified $\pmod{q^2}$ so that $1 - 4a$ is square-free (in fact, $1 - 4a$ can be modified $\pmod{q^2}$ so that it is a prime). Then

$$D(x, y) = (1 - (xy)^b)(1 - xy)^{-1}(a(x + x^{-1}) + (1 - 2a))$$

$$- (1 - x)(1 - y)(1 - (xy)^{b-1})(1 - xy)^{-1}(q)$$

satisfies (3.1) but is not the Alexander polynomial for any two-component link with linking number b .

E. *Comment on the case $p^\alpha = 2^\alpha$.* If $\alpha = 1$, then condition (i) requires that $\lambda(-1)$ be square-free. Also, $\omega = -1$ in this case. Hence, $\lambda(\omega) = \lambda(-1)$. Condition (ii) then requires $\lambda(-1)$ to have a square factor. Hence, this technique will not yield a counterexample to (3.1) because (i) and (ii) cannot be satisfied simultaneously.

If $b = 4$, then $\mathbf{Z}[\omega] = \mathbf{Z}[i]$ and Question 2' (3.8) can be specialized in the following easily realizable conditions.

- (i) $1 - 4a$ is square free.
- (ii) -1 is not a square \pmod{q} and $\lambda(i) = 1 - 2a \equiv 0 \pmod{q^2}$.

Condition (ii) assures that q is a prime in $\mathbf{Z}[i]$. If $1 - 2a \equiv 0 \pmod{q^2}$, then the argument of the previous section shows that $(q) \neq J\bar{J}$ in $\mathbf{Z}[i]/\lambda(i)$. Hence, in (3.8), one takes $c = q$ and

$$D(x, y) = (1 - (xy)^4)(1 - xy)^{-1}(a(x + x^{-1}) + 1 - 2a)$$

$$- (1 - x)(1 - y)(1 - (xy)^3)(1 - xy)^{-1} \cdot q.$$

Similar counterexamples exist for any $b = 2^\alpha$ with $\alpha > 1$.

F. *Examples.* The case $p = 3$.

In this case $\mathbf{Q}[\theta] = \mathbf{Q}$, $\mathbf{Q}[\omega] = \mathbf{Q}[\sqrt{-3}]$, $\lambda(\omega) = 1 - 3a$ and $q \equiv -1 \pmod{3}$ reduces to q remains prime in $\mathbf{Z}[\omega]$ (since $r = 1$). The conditions can be reformulated as follows.

- (i) $1 - 4a$ is square-free.
- (ii) -3 is not a square \pmod{q} ($q = 2$ is excluded) and $1 - 3a \equiv 0 \pmod{q^2}$. For instance, one may take $q = 5$ and $a = 17$.

For $p = 5, q = 19$ and $a = 9 - 19^2$. Then, $f(a) = 19^2 \cdot 1721$ and $1 - 4a = 1409$, so the conditions are met.

4. A generalization for m -component links. Let $B = (b_{ij})$ be a matrix of linking number [L-1]. The entry in row i and column j, b_{ij} , is $lk(L_i, L_j)$. The diagonal entries are undefined. If B is an $(m \times m)$ matrix, a splitting, S , of B is a proper subset of the integers $\{1, \dots, m\}$ such that $b_{ij} = 0$ whenever $i \in S$ and $j \notin S$. If there is a splitting of B, B is said to be splittable. Levine [L-2] has shown that any link with an unsplittable linking matrix has nonzero Alexander polynomial and that the zero polynomial is allowed if and only if B is splittable. Let B_i denote the matrix obtained from B by deleting the i th row and column.

For convenience, let $b_{ii} = 0$ and consider

$$B = \begin{bmatrix} 0 & b & 0 & & 0 \\ b & 0 & 1 & & \\ 0 & 1 & 0 & & \\ & & & 1 & \\ & & & & 1 & 0 \end{bmatrix}$$

Consider, also, the polynomial

$$f_m(t_1, \dots, t_m) = (t_2 - 1) \cdots (t_{m-1} - 1) \Delta(t_1, t_2)$$

where $\Delta(t_1, t_2)$ is one of the examples from 3B or 3C, and hence cannot be the Alexander polynomial of a two-component link with linking number b . (Thus, $b \neq 0, 1, 2$.) In Theorem 4.3 it will be shown that f_m cannot be the Alexander polynomial of any m -component link with linking matrix B .

For $m > 2$, the Torres conditions are

T1 $\Delta(t_1, \dots, t_m) = (-1)^m t_1^{a_1} \cdots t_m^{a_m} \Delta(t_1^{-1}, \dots, t_m^{-1})$ for some a_i .

T2 $\Delta(t_1, \dots, 1, \dots, t_m) = (t_1^{b_{1i}} \cdots t_i \cdots t_m^{b_{mi}} - 1) \Gamma_i$

where Γ_i satisfies Torres' conditions for $m - 1$ variables.

(4.1) LEMMA. *The polynomial $f_3(t_1, t_2, t_3)$ determines the linking matrix B .*

PROOF. Suppose $B = (b_{ij})$. We use Torres' second conditions on $f(t_1, t_2, t_3)$ to determine the b_{ij} .

$$f_3(t_1, t_2, t_3) = (t_2 - 1) \left[\frac{1 - (t_1 t_2)^b}{t - t_1 t_2} \lambda(t_1) - (1 - t_1)(1 - t_2) \left(\frac{1 - (t_2 t_2)^{b-1}}{1 - t_1 t_2} \right) m \right],$$

where $\lambda(t_1) = \lambda(t_1^{-1}), \lambda(1) = 1$ and $m \in \mathbf{Z}$. Then

(i) by substituting $t_1 = 1$, we get

$$t_2^b - 1 = (t_2^{b_{12}} t_3^{b_{13}} - 1) \Gamma_1,$$

for Γ_1 satisfying Torres' conditions for two variables. Clearly $b_{13} = 0$, and $b_{12} \neq 0$ since $b \neq 0$. There are two choices for b_{12} which are considered below.

(ii) by substituting $t_2 = 1$, we get $f(t_1, 1, t_3) = 0$. Hence $\Gamma_2 = 0$, or $b_{12} = b_{23} = 0$, which is impossible.

(iii) by substituting $t_3 = 1$, we get

$$(t_2 - 1) \left[\frac{1 - (t_1 t_2)^b}{1 - t_1 t_2} \lambda(t_1) - (1 - t_1)(1 - t_2) \left(\frac{1 - (t_1 t_2)^{b-1}}{1 - t_1 t_2} \right) m \right] = (t_2^{b_{23}} - 1) \Gamma_3$$

for some Γ_3 satisfying Torres' conditions for two variables. (Recall $b_{13} = 0$.)

Now consider the cases for b_{12} .

Case I. b_{12} is a proper divisor of b or $b_{12} = 1$. Then $\Gamma_1 = 1 + t_2^{b_{12}} + \dots + t_2^{b-b_{12}}$, so $\Gamma_1(1, 1) = b_{23} > 1$. Then in (iii)

$$(t_2^{b_{23}} - 1) \Gamma_3 = (t_2 - 1) \left[\frac{1 - (t_1 t_2)^b}{1 - t_1 t_2} \lambda(t_1) - (1 - t_1)(1 - t_2) \left(\frac{1 - (t_1 t_2)^{b-1}}{1 - t_1 t_2} \right) m \right].$$

That is,

$$\begin{aligned} & (1 + t_2 + \dots + t_2^{b_{23}-1}) \Gamma_3 \\ &= \left[\frac{1 - (t_1 t_2)^b}{1 - t_1 t_2} \lambda(t_1) - (1 - t_1)(1 - t_2) \left(\frac{1 - (t_1 t_2)^{b-1}}{1 - t_1 t_2} \right) m \right]. \end{aligned}$$

Thus, $1 + t_2 + \dots + t_2^{b_{23}-1}$ divides the right-hand side.

Let $t_2 = 1$; then b_{23} divides $1 + t_1 + \dots + t_1^{b-1}$, which is impossible since $b_{23} > 1$.

Case II. $b_{12} = b$. Then $\Gamma_1 = 1$ and $\Gamma_1(1, 1) = b_{23} = 1$. Thus $(t_2 - 1)$ divides $f(t_1, t_2, 1)$, which is clearly true, and

$$B = \begin{bmatrix} 0 & b & 0 \\ b & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

(4.2) LEMMA. *Up to a permutation of $\{3, 4, \dots, m - 1\}$, the polynomial $f_m(t_1, \dots, t_m)$ determines the linking matrix ($m \geq 3$).*

PROOF. (Induction on m . The first step is (4.1).)

(i) Substituting $t_1 = 1$ gives

$$(t_2^b - 1)(t_3 - 1) \cdots (t_{m-1} - 1) = (t_2^{b_{12}} \cdots t_m^{b_{1m}} - 1) \Gamma_1.$$

Clearly $b_{1m} = 0$. The choices for b_{1j} , $2 \leq j \leq m - 1$, are considered below.

(ii) Substituting $t_j = 0$, $2 \leq j \leq m - 1$, gives

$$f_m(t_1, 1, t_3, \dots, t_m) = \cdots = f_m(t_1, \dots, t_{m-2}, 1, t_m) = 0.$$

(iii) Substituting $t_m = 1$ gives

$$\begin{aligned} f_m(t_1, \dots, t_{m-1}, 1) &= (t_2 - 1) \cdots (t_{m-1} - 1) \Delta(t_1, t_2) \\ &= (t_2^{b_{2m}} \cdots t_{m-1}^{b_{m-1,m}} - 1) \Gamma_m. \end{aligned}$$

Now consider the choices for b_{1j} , $2 \leq j \leq m - 1$.

Case I. $b_{12} = 0$. By permuting $\{3, \dots, m - 1\}$ if necessary, we may assume $b_{1,m-1} = 1$, $b_{1j} = 0$, $3 \leq j \leq m - 2$, the first row of B is $(0 \text{ --- } 0 \ 1 \ 0)$ and

$$\Gamma_1(t_2, \dots, t_m) = (t_2^b - 1)(t_3 - 1) \cdots (t_{m-2} - 1).$$

(since $b_{jm} = 0, j < m - 1$). This is impossible (since $b/b_{12} > 1$ so $(t_{m-1}^{b/b_{12}} - 1)$ does not divide $(t_3 - 1) \cdots (t_{m-1} - 1)(1 + t_2^{b_{12}} \cdots t_2^{b-b_{12}})$.

Case III. $b_{12} = b$. Then, $b_{ij} = 0, 3 \leq j \leq m - 1$. Hence the first row of B is $(0 \quad b \quad 0 \quad \cdots \quad 0)$, and

$$\begin{aligned} \Gamma_1(t_2, \dots, t_m) &= (t_3 - 1) \cdots (t_{m-1} - 1) \\ &= (t_2^{b_{2m}} \cdots t_{m-1}^{b_{m-1,m}} - 1) \Gamma_{1m}. \end{aligned}$$

Clearly $b_{2m} = 0$. By permuting $\{3, \dots, m - 1\}$ if necessary, we may assume $b_{m-1,m} = 1$ and $b_{jm} = 0, 3 \leq j \leq m - 2$. Hence, the last column of B is $(0 \quad \cdots \quad 0 \quad 1 \quad 0)^t$, and $\Gamma_{1m} = (t_3 - 1) \cdots (t_{m-2} - 1)$. Continuing in this way we find

$$B = \begin{bmatrix} 0 & b & 0 & & & \\ b & 0 & 1 & & & \\ 0 & 1 & 0 & & & \\ & & & \searrow & & \\ & & & & 1 & \\ & & & & & \searrow \\ & & & & & & 1 \\ & & & & & & & 0 \end{bmatrix},$$

and in (iii) above $\Gamma_m = f_{m-1}(t_1, \dots, t_{m-1})$, and in (ii) $\Gamma_2 = \cdots = \Gamma_{m-1} = 0$.

(4.3) THEOREM. *With the conditions as above $f_m(t_1, \dots, t_m)$ cannot be the Alexander polynomial of an m -component link with linking matrix B .*

PROOF (INDUCTION ON m). For $m = 3, f_3(t_1, t_2, 1) = (t_2 - 1)\Delta(t_1, t_2)$. Hence $\Gamma_3 = \Delta(t_1, t_2)$, which is not allowed by Torres' second condition.

For $m > 3,$

$$f_m(t_1, \dots, t_{m-1}, 1) = (t_{m-1} - 1) \left[\prod_{i=2}^{m-2} (t_i - 1) \right] \Delta(t_1, t_2).$$

Hence

$$\Gamma_m = \prod_{i=2}^{m-2} (t_i - 1) \Delta(t_1, t_2) = f_{m-1}(t_1, \dots, t_{m-1}),$$

which is not allowed by induction.

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