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"ALMOST" IMPLIES "NEAR"

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ABSTRACT. We formulate a formal language in which it is meaningful to say that an object almost satisfies a property. We then show that any object which almost satisfies a property is near an object which exactly satisfies the property. We show how this principle can be used to prove existence theorems. We give an example showing how one may strengthen the statement to give information about the relationship between the amount by which the object fails to satisfy the property and the distance to the nearest object which satisfies the property. Examples are given concerning commuting matrices, additive sequences, Brouwer fixed points, competitive equilibria, and differential equations.

1. Introduction. There is considerable interest in pure and applied mathematics in knowing when objects which almost satisfy a property are near objects which exactly satisfy it. We list below a number of such statements. While all of them are fairly easy to prove directly, they are also somewhat surprising.

EXAMPLE 1. Almost commuting matrices are near matrices which commute. Specifically, Luxemburg and Taylor [**16**] showed that, given any compact family of matrices K and $\varepsilon > 0$, there exists $\delta > 0$ such that, if $\|AB - BA\| < \delta$ and $A, B \in K$, then there exist $A', B' \in K$ with $\|A - A'\| < \varepsilon$, $\|B - B'\| < \varepsilon$ and $A'B' = B'A'$. This answered a portion of a question raised by Peter Rosenthal [**20**].

EXAMPLE 2. A theorem of Polya and Szego asserts that an almost additive sequence is near an additive sequence. In his introductory notes on nonstandard analysis [**15**], Luxemburg gave a simple nonstandard proof of a generalization of their result. For simplicity, we shall stay with the Polya-Szego version: If $s_1, s_2, \ldots$ is a sequence of real numbers satisfying $|s_{n+m} - s_n - s_m| < \alpha$, then there exists β such that $|s_n - n\beta| < \alpha$. The theorem is notable for the concreteness of the bound on the distance to the nearest additive sequence. A functional equation version of this result is discussed by Ulam in *The Scottish Book* [**18**, p. 11].

EXAMPLE 3. Kuhn and MacKinnon [**14**] show that, given $\varepsilon > 0$ and a continuous $f: B \to B$, where B is the closed unit ball in Euclidean space, there exists $\delta > 0$ such that if $|f(x) - x| < \delta$, then there exists y such that $|y - x| < \varepsilon$ and $f(y) = y$. The question is of interest because algorithms exist to compute points that are almost fixed (Scarf [**21**]). The proof is simple, but the result was a surprise. Indeed, a

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number of papers, appearing both before and after Kuhn and MacKinnon, assert that points that are almost fixed need not be near fixed points.

EXAMPLE 4. Competitive equilibrium and core are notions of equilibrium in mathematical economics. It has been known for a considerable time that core allocations are near competitive allocations under special hypotheses (Edgeworth [**9**], Debreu and Scarf [**8**]). A slightly weaker assertion is that the core allocations are close to the individual demands sets, which has been demonstrated under convexity assumptions (Bewley [**4**], Hildenbrand [**12**]). In the absence of convexity, only weaker statements asserting that core allocations almost satisfied the definition of demand were known.[2] Using a particular form of "almost" result in Anderson [**1**], Anderson [**2**] shows that if one draws sequences of economies at random, then with probability one, the average distance from the core allocations to the individuals' demand sets would tend to 0. Giving a complete statement of the result would require a considerable digression; however, the "almost-near" fact which is used in [**2**] will be specified once we have defined our language. A related example in mathematical economics is given in Fudenberg and Levine [**10**].

EXAMPLE 5. The following is a sort of stability theorem for differential equations which may be new. It is weaker than the usual stability theorems because it only applies to bounded intervals of time, but it is stronger because it permits perturbation of the differential equation as well as the initial values. It strengthens the results in Coddington and Levinson [**6**, pp. 57–60] on dependence of solutions of differential equations on a parameter, since we do not require that the differential equation have a unique solution. We actually give two theorems.

(a) Suppose $f: \mathbf{R}^n \times [0,1] \to \mathbf{R}^n$ is bounded and continuous, and $y_0 \in \mathbf{R}^n$. Then given $\varepsilon > 0$, there exists $\delta > 0$ such that, if $|z(0) - y_0| < \delta$ and

$$\sup_t |z'(t) - f(z(t),t)| < \delta,$$

then there exists y, $y(0) = y_0$ and $y'(t) = f(y(t),t)$ for all t, and

$$\sup_t \{|z(t) - y(t)| + |z'(t) - y'(t)|\} < \varepsilon.$$

(b) Suppose $g: \mathbf{R}^3 \to \mathbf{R}$ is continuous. Given a compact subset K of $C^1[0,1]$ and $\varepsilon > 0$, there exists $\delta \geq 0$ such that if $z \in K$ and $\sup_t |g(t, z(t), z'(t))| < \delta$ and $|z(0) - y_0| < \delta$, then there exists $y \in K$, $y(0) = y_0$, $g(t, y(t), y'(t)) = 0$ for all t, and $\sup_t\{|z(t) - y(t)| + |z'(t) - y'(t)|\} < \varepsilon$.

It is striking that nonstandard analysis (Robinson [**19**], Luxemburg and Stroyan [**22**], Davis [**7**]) was involved in the discovery of four of the five examples above. This may, of course, indicate more about the interests and experience of the author than it does about the nature of almost-near problems. Nonetheless, there is some reason to believe that nonstandard analysis can be particularly helpful in identifying almost-near theorems because the operation of taking the standard part can be used to pick out a specific object which can be shown to satisfy the property under study.

[2] One positive theorem can be obtained by combining some results that were in the literature. Combining Hildenbrand [**12**, Proposition 4, p. 200] with Mas-Colell and Neuefeind [**17**], one obtains a strong conclusion for sequences of economies whose limits lie in a residual set; see also Trockel [**23**] for a simplification of the Mas-Colell and Neuefeind result. However, this is a weaker form of genericity than having an open dense set of economies, and much weaker than having a theorem with probability one.

In this paper, we shall give a general almost-near theorem. We shall define a formal language, specify what it means to almost satisfy a property expressible in the language, and then show that any object which almost satisfies a property is near an object which exactly satisfies it. The theorem is sufficiently general to include the above examples as special cases or easy corollaries. We anticipate that the theorem will be of help in proving other almost-near theorems. Judging by the examples above, which are easy but surprising, the theorem may be more useful in guiding the search for such results than as a tool that is formally invoked in proofs.

A fascinating discussion of certain almost-near problems is given by Ulam in [**18**]; see pp. 11–12 and Problems 97 and 110. A few of the problems Ulam mentions (for example, the functional equation analogue to Example 2) can be fitted into the framework we present, but others require substantially more elaborate arguments and some are unsolved.

The almost-near theorem can also be used as a tool in proving existence results. The theorem asserts that objects almost satisfying a property are near an object exactly satisfying the property. Thus, in particular, there must *exist* an object satisfying the property. If one can easily show the existence of almost-satisfying objects, the almost-near theorem gives an existence proof for objects exactly satisfying. Some examples are given in §3.

Finally, in §4, we give an example showing how the almost-near theorem can be used as a first step in obtaining more concrete results on the rate of approximation, i.e., the relationship between ε and δ.

2. The almost-near theorem. The language is similar in a number of ways to one introduced by Ward Henson [**11**] in studying nonstandard hulls of Banach spaces.

We have in mind a fixed metric space (X, d). We say a subset A of $X^n \times X^m$ is (n, m) *lower hemicontinuous* if, given $(c_1, \ldots, c_n, c_{n+1}, \ldots, c_{n+m}) \in A$ and $\varepsilon > 0$, there exists $\delta > 0$ so that, if $|c_i' - c_i| < \delta$ for $i = n+1, \ldots, n+m$, there exist c_i', $|c_i' - c_i| < \varepsilon$ $(i = 1, \ldots, n)$, such that $(c_1', \ldots, c_n', c_{n+1}', \ldots, c_{n+m}') \in A$. Observe that A is (n, m) lower hemicontinuous if it is open, or if $m = 0$.

We form a language according to the following rules:

(1) There are variables $v_1, v_2, \ldots$. Variables are terms.

(2) To each element of X there corresponds a constant symbol c. Constants are terms.

(3) To each continuous function from X^n to X there corresponds a function symbol f. If f is an n-ary function symbol and $T_1, \ldots, T_n$ are terms, then $f(T_1, \ldots, T_n)$ is a term.

(4) Only strings generated by rules (1)–(3) are terms.

(5) There are two disjoint sets of relation symbols, $\mathbf{F}$ and $\mathbf{U}$. To each closed relation on X^n there correspond two relation symbols, one each in $\mathbf{F}$ and $\mathbf{U}$. If $T_1, \ldots, T_n$ are terms and R is an n-ary relation symbol, then $R(T_1, \ldots, T_n)$ is a formula.

(6) If F_1 and F_2 are formulas, so are $F_1 \wedge F_2$ and $F_1 \vee F_2$.

(7) Suppose

(i) A is (n, m) lower hemicontinuous, $v_1, \ldots, v_n$ are variables, and $T_1, \ldots, T_m$ are terms, and

(ii) A is open, or $m = 0$, or $T_1, \ldots, T_m$ do not contain any occurrences of $v_1, \ldots, v_n$, and

(iii) F is a formula in which $v_1, \ldots, v_n$ and all variables appearing in $T_1, \ldots, T_m$ are free.

Then

$$(\forall(v_1, \ldots, v_n))((v_1, \ldots, v_n, T_1, \ldots, T_m) \in A \Rightarrow F(v_1, \ldots, v_n, T_1, \ldots, T_m))$$

is a formula. If $m = 0$, the formula will be abbreviated as

$$(\forall(v_1, \ldots, v_n) \in A)F(v_1, \ldots, v_n).$$

(8) To each compact subset A of X^n, there corresponds a quantifier

$$(\exists(v_1, \ldots, v_n) \in A).$$

If F is a formula in which $v_1, \ldots, v_n$ are free variables, then

$$(\exists(v_1, \ldots, v_n) \in A)F(v_1, \ldots, v_n)$$

is a formula.

(9) If F is a formula, then so is (F).

(10) Only strings generated by rules (5)–(9) are formulas.

REMARK. Rule (9) is designed simply to allow for the introduction of parentheses to indicate the order in which the truth value of formulas is to be evaluated. Rule (7) is fairly complicated since it includes several cases. However, there are two easy special cases which cover most situations.

(a) For any set $A \subset X^n$ and any formula F in which $v_1, \ldots, v_n$ are free, $(\forall(v_1, \ldots, v_n) \in A)F(v_1, \ldots, v_n)$ is a formula.

(b) For any open set $A \subset X^n \times X^m$ and any formula F in which $v_1, \ldots, v_n$ and all variables in $T_1, \ldots, T_m$ are free,

$$(\forall(v_1, \ldots, v_n))((v_1, \ldots, v_n, T_1, \ldots, T_m) \in A \Rightarrow F(v_1, \ldots, v_n, T_1, \ldots, T_m))$$

is a formula.

The formulas in the language L will be interpreted in (X, d) in the conventional way. Our notion of *almost satisfying a formula* is the following: If F is a formula and $\delta > 0$, then F_δ is the formula obtained by replacing each relation symbol R in $\mathbf{F}$ (representing the relation $A \subset X^n$) with the relation symbol which corresponds to the closed ball

$$B(A, \delta) = \{(y_1, \ldots, y_n)\text{: there exists } (x_1, \ldots, x_n) \in A \text{ with } d(x_i, y_i) \leq \delta\}.$$

Note that we do not replace the relation symbols in $\mathbf{U}$. Thus, we are able to choose which relation symbols we wish to allow to be approximately satisfied. Speaking informally, we shall say that x almost satisfies F if $F_\delta(x)$ holds for some small δ.

THE ALMOST-NEAR THEOREM. *Let K be a compact subset of X^n, and let F be a formula with n free variables. Given $\varepsilon > 0$, there exists $\delta > 0$ such that, if $(x_1, \ldots, x_n) \in K$ and $F_\delta(x_1, \ldots, x_n)$ is satisfied, then there exists $(y_1, \ldots, y_n) \in K$ with $d(y_i, x_i) < \varepsilon$ satisfying $F(y_1, \ldots, y_n)$.*

The heart of the proof is the following nonstandard lemma, which is proved by induction on complexity. Given any formula F in our language, nonstandard analysis associates a unique formula ${}^*F$ which is meaningful as a statement about ${}^*X$, the nonstandard extension of X. The transfer principle of nonstandard analysis asserts that, if F is a statement, then F is true of X if and only if ${}^*F$ is true of ${}^*X$. Given $x \in {}^*X$, ${}^\circ x$ is the unique element of X such that x is infinitely close to ${}^\circ x$.

LEMMA. *Let I be an interpretation of ${}^*F$ in ${}^*X$ such that $I(v_i)$ is nearstandard for each free variable v_i. Let ${}^\circ I$ be the interpretation of F in X defined by ${}^\circ I(v_i) = {}^\circ(I(v_i))$. If $I({}^*F_\delta)$ is true for some $\delta \simeq 0$, then ${}^\circ I(F)$ is true.*

PROOF. It is routine to show by induction on complexity of terms that if I' is any intepretation such that $I'(v_i)$ is nearstandard for every variable v_i, then ${}^\circ(I')(T) = {}^\circ(I'(T))$.

Suppose F is $R(T_1, \ldots, T_n)$. If $I({}^*F_\delta)$ holds for some $\delta \simeq 0$, then ${}^*R(y_1, \ldots, y_n)$ holds for some $y_i \simeq I(T_i)$. Note that if R is in $\mathbf{U}$, we can actually take $y_i = I(T_i)$. Since R is closed, $R({}^\circ y_1, \ldots, {}^\circ y_n)$ holds. But ${}^\circ y_i = {}^\circ(I(T_i)) = {}^\circ I(T_i)$, so $R({}^\circ I(T_1), \ldots, {}^\circ I(T_n))$ holds.

If F is $F_1 \wedge F_2$, $F_1 \vee F_2$, or (F_1), the induction step is obvious.

Suppose F is

$$(\forall(v_1, \ldots, v_n))((v_1, \ldots, v_n, T_1, \ldots, T_m) \in A \Rightarrow G(v_1, \ldots, v_n, T_1, \ldots, T_m)).$$

If $I({}^*F_\delta)$ holds for some $\delta \simeq 0$, then $I'({}^*G_\delta)$ holds for all I' that agree with I except on $v_1, \ldots, v_n$ and which have $(I'(v_1), \ldots, I'(v_n), I'(T_1), \ldots, I'(T_m)) \in {}^*A$. If, in addition, such an I' maps $v_1, \ldots, v_n$ to nearstandard values, then ${}^\circ(I')(G)$ holds by induction. Suppose I'' is a standard interpretation agreeing with ${}^\circ I$ except on $v_1, \ldots, v_n$, and satisfying $(I''(v_1), \ldots, I''(v_n), I''(T_1), \ldots, I''(T_m)) \in A$. Let $I' = I$ except on $v_1, \ldots, v_n$; by (n, m) lower hemicontinuity (or openness, in case some of the v_i's appear in some of the T_i's), choose $I'(v_1) \simeq I''(v_1), \ldots, I'(v_n) \simeq I''(v_n)$ such that $(I'(v_1), \ldots, I'(v_n), I'(T_1), \ldots, I'(T_m)) \in {}^*A$. Thus, $I'({}^*G_\delta)$ holds, and so ${}^\circ(I')(G)$ holds by induction. But ${}^\circ(I') = I''$, so ${}^\circ(I')(F)$ holds.

Suppose F is $(\exists(v_1, \ldots, v_n) \in A)G(v_1, \ldots, v_n)$, where A is a compact subset of X^n and $I({}^*F_\delta)$ holds for some $\delta \simeq 0$. Then $I'({}^*G_\delta)$ holds for some I' which agrees with I except on $v_1, \ldots, v_n$ and satisfies $(I'(v_1), \ldots, I'(v_n)) \in {}^*A$. Since A is compact, $({}^\circ(I'(v_1)), \ldots, {}^\circ(I'(v_n)))$ is defined and belongs to A. By induction, ${}^\circ(I')(G)$ holds, and so ${}^\circ I(F)$ holds. This completes the proof of the lemma.

PROOF OF THE THEOREM. If the theorem is false, there exists $\varepsilon > 0$ and a compact K such that we may find $x^1, x^2, \ldots \in K^n$ such that $F_{1/m}(x_1^m, \ldots, x_n^m)$ holds, but, for all m,

$$\inf\left\{\max_i d(x_i^m, y_i) \colon F(y_1, \ldots, y_n) \text{ holds}\right\} \geq \varepsilon.$$

But if $m \in {}^*\mathbf{N} - \mathbf{N}$, ${}^*F_{1/m}(x_1^m, \ldots, x_n^m)$ holds by transfer, and thus $F({}^\circ x_1^m, {}^\circ x_n^m)$ holds by the lemma. Hence,

$$\inf\left\{\max_i {}^*d(x_i^m, y_i) \colon {}^*F(y_1, \ldots, y_n) \text{ holds}\right\} \simeq 0,$$

a contradiction which establishes the theorem.

We now return to the six examples of the Introduction.

EXAMPLE 1. Take X to be the set of $n \times n$ matrices, with the norm $\|A\| = \max_{i,j} |a_{ij}|$, $f(A, B) = AB - BA$, and R the relation symbol in $\mathbf{F}$ corresponding to the relation $\{A \colon A = 0\}$. Letting $F(A, B)$ be $R(f(A, B))$, we see that $F_\delta(A, B)$ holds exactly when $\|AB - BA\| \leq \delta$, and $F(A, B)$ holds exactly when $AB = BA$. Thus, the Luxemburg-Taylor theorem is a special case of the almost-near theorem.

EXAMPLE 2. We prove the weaker statement: given $\varepsilon > 0$ and $K > 0$, there exists $\delta > 0$ such that if $|s_{n+m} - s_n - s_m| < \delta$ and $|s_1| \leq K$, then there exists β

such that $|s_n - n\beta| < \varepsilon$. The stronger form, with $\delta = \varepsilon$, can be obtained if one is a little more careful in estimating than our language permits. The key step in Luxemburg's proof is the same as the idea of the proof of the almost-near theorem, but he does take the extra care required to nail down the relationship between δ and ε.

Let X be the space of real sequences, with the metric

$$d(x,y) = \sup_n \min\{|x_n - y_n|, 1/n\}.$$

Define $f(x)$ to be the sequence all of whose terms are equal to

$$\sup_{n,m} \min\{|x_{m+n} - x_n - x_m|, 1/(m+n)\}.$$

$f: X \to X$ is continuous. Let $R(x)$ be the relation symbol in $\mathbf{F}$ corresponding to the relation $\{x: x = 0\}$. Then letting $F(x)$ be $R(f(x))$, we see that $F(x)$ is satisfied if and only if x is additive, while $\sup_{n,m} |x_{n+m} - x_n - x_m| \le \delta$ implies $F_\delta(x)$. If $x^1, x^2, \ldots$ is a sequence in X with $|x_1^m| \le K$ and $F_{1/m}(x^m)$ satisfied, then $x^1, x^2, \ldots$ has compact closure. Hence, the claimed result follows from the theorem.

EXAMPLE 3. Let X be B with the Euclidean metric, and $R(x,y)$ the relation symbol in $\mathbf{F}$ corresponding to the relation $\{(x,y): x = y\}$. Letting $F(x)$ be $R(x, f(x))$, we see that $F(x)$ holds if and only if x is a fixed point, while $F_\delta(x)$ holds if and only if x is moved at most δ.

EXAMPLE 4. This is the most intricate of the examples, and the only one to make use of the quantifiers. As we indicated in the introduction, we shall discuss only the portion of Anderson [**2**] which used the almost-near property most directly. We suppose we are given a preference relation $\succ$ (i.e., a binary relation on $\mathbf{R}_+^k$ such that $\{(x,y): x \succ y\}$ is relatively open in $\mathbf{R}_+^k$). Given $p \in \mathbf{R}_{++}^k$ and $I \in \mathbf{R}_{++}$, the demand set is $D(p,I) = \{x: p \cdot x \le I,\, y \succ x \Rightarrow p \cdot u > I\}$. Let $S(x)$ be the relation symbol in $\mathbf{F}$ for the relation $\{x: p \cdot x \le I\}$, $R(x,y)$ the relation symbol in $\mathbf{F}$ for the relation $\{(x,y): (y \succ x)$ does *not* hold$\}$, and $A = \{y: p \cdot y \le I\}$. Then $x \in D(p,I)$ if and only if $F(x) \equiv S(x) \wedge (\forall y \in A) R(x,y)$ holds. For suitable choice of δ', it is easy to see that

$$(*) \qquad |p \cdot x - I| < \delta' \quad \text{and} \quad \inf\{p \cdot y: y \succ x\} \ge I - \delta'$$

implies $F_\delta(x)$. But $(*)$ is exactly the "almost demand" condition shown to hold for core allocations in Anderson [**1**], and hence such allocations are near the demand set.

EXAMPLE 5. Suppose statement (a) is not true. Then there exist $\varepsilon > 0$ and a sequence $z_n \in C^1[0,1]$ with $|z(0) - y_0| < 1/n$ and $\sup_t |z_n'(t) - f(z_n(t), t)| < 1/n$, but

$$\sup_t \{|z_n(t) - y(t)| + |z_n'(t) - y'(t)|\} \ge \varepsilon$$

for all n and all solutions y. $\{z_n'\}$ is equicontinuous and bounded, and so $\{z_n\}$ has compact closure in $C^1[0,1]$. Let $X = C^1[0,1]$, $R(z)$ be the relation symbol in $\mathbf{F}$ corresponding to the relation $\{z: z(0) = y_0\}$, and $S(z)$ be the relation symbol in $\mathbf{F}$ corresponding to the relation $\{z: z'(t) = f(z,(t),t)$ for all $t\}$. Then the statement is exactly the almost-near theorem applied to $F(z) \equiv R(z) \wedge S(z)$. The proof of (b) is similar but easier.

3. Proving existence results. The almost-near theorem has the form, "If x almost satisfies a property, then x is close to some y which exactly satisfies it." In particular, if one can show the existence of x, it follows that there must *exist* y which exactly satisfies the property. Thus, the almost-near theorem provides a means for proving existence theorems. We give a few examples of this phenomenon. This section was stimulated by the observation of Donald J. Brown that the nonstandard proof of the fact that a continuous function on a compact set attains its maximum is an almost-near theorem.

EXAMPLE 6. We prove that if $f\colon [0,1] \to \mathbf{R}$ is continuous, then f achieves its maximum. Let R be the relation symbol (from $\mathbf{U}$) for the relation $\{(x,y)\colon x \geq y\}$, and S the relation symbol (from $\mathbf{F}$) for the relation $\{(x,y)\colon x = y\}$. Consider the formula

$$F(x) \equiv (\forall y \in [0,1])(\exists z \in [0,1])(R(f(x), f(z)) \wedge S(y,z)).$$

Given $\delta = 1/n$ for some $n \in \mathbf{N}$, let x_δ be chosen to maximize $f(0), f(\delta), f(2\delta), \ldots, f(1)$. Then $F_\delta(x_\delta)$ is true, since $F_\delta(x_\delta)$ simply asserts that for all y there exists z such that $|y-z| \leq \delta$ and $f(x) \geq f(z)$. Hence, there exists x such that $F(x)$ holds; i.e., for all y, $f(x) \geq f(y)$, so f assumes its maximum.

EXAMPLE 7. The Intermediate Value Theorem can be proved in much the same way. Let S be the same relation symbol as in Example 6, R the relation symbol (from $\mathbf{U}$) for the relation $\{y\colon y \geq 0\}$, and R' the relation symbol (from $\mathbf{U}$) for the relation $\{y\colon y \leq 0\}$. Let

$$F(x) \equiv R'(f(x)) \wedge (\exists y \in [0,1])(S(x,y) \wedge R(f(y))).$$

Given $\delta = 1/n$, for $n \in \mathbf{N}$, choose x_δ to be the first value in $0, \delta, 2\delta, \ldots, 1$ such that $f(x_\delta) \leq 0$, but $f(x_\delta + \delta) \geq 0$. Then $F_\delta(x_\delta)$ holds, so there must exist x such that $F(x)$ holds. But $F(x)$ asserts that $f(x) = 0$.

EXAMPLE 8. The Cauchy-Peano Existence Theorem for solutions of ordinary differential equations can be proven by constructing the usual polygonal approximations. These satisfy F_δ, where $F(z) \equiv R(z) \wedge S(z)$ is the formula used in Example 5(a). Hence, there must exist z satisfying F, and so z is a solution of the differential equation.

EXAMPLE 9. The derivation of the Brouwer Fixed Point Theorem from Sperner's Lemma is immediate by applying the almost-near theorem to the formula F of Example 3. The derivation of the Schauder Fixed Point Theorem from Brouwer's Theorem also has the same form. Roughly speaking, we approximate the compact convex infinite-dimensional set by a finite-dimensional compact convex subset, and project the function onto the subset. Brouwer's Theorem gives a fixed point of the projected function; this point is almost fixed under the original function, and hence is near a fixed point of the original function. The details are omitted.

4. Rates of convergence. One might object that the almost-near theorem does not give a concrete relationship between δ and ε. However, it may provide the first step in getting such a concrete relationship. In Example 2, we noted that using a little care would yield $\delta = \varepsilon$. In this section, we show how Sard's Theorem can be used to show that, in Example 3, δ is generically linear in ε.

Consider $f \in C^1(B,B)$. We can approximate f as closely as we like by a C^1 function $f_n\colon B \to \{x\colon |x| \leq 1 - 1/n\}$: take, for example, $f_n(x) = ((n-1)/n)f(x)$. By Sard's Theorem (Hirsch [**13**]), the set of critical values of $g_n(x) \equiv f_n(x) - x$ is of measure 0 in $\mathbf{R}^n$, so that we may find y_n, $|y_n| < 1/n$, and y_n is a regular value of g_n. Defining $h_n(x) \equiv f_n(x) - y_n$, we see that $h_n \in C^1(B,B)$, $\|f - h_n\|_{C^1} \le 2/n$, and 0 is a regular value of $h_n(x) - x$. Thus, if $\mathbf{C}$ is the set of all $f \in C^1(B,B)$ such that 0 is a regular value of $f(x) - x$, $\mathbf{C}$ is dense; moreover, $\mathbf{C}$ is clearly open.

Fix $f \in \mathbf{C}$. We claim there exists L such that for all x, there exists y such that $f(y) = y$ and $|y - x| \le L|f(x) - x|$. If not, we can find x_n such that $f(y) = y$ implies that $|y - x_n| \ge n|f(x_n) - x_n|$. Since $|y - x_n|$ is bounded above, we conclude that $|f(x_n) - x_n| = O(1/n)$, and hence by the almost-near theorem, we conclude there is a subsequence x_{n_j} such that $x_{n_j} \to y$ with $f(y) = y$. Since 0 is a regular value of $f(x) - x$,

$$f(x_{n_j}) - x_{n_j} = A(x_{n_j} - y) + O(|x_{n_j} - y|^2),$$

where A is a nonsingular matrix, and thus

$$|x_{n_j} - y| = |A^{-1}(f(x_{n_j}) - x_{n_j})| + O(|x_{n_j} - y|^2).$$

Since $|x_{n_j} - y| \to 0$,

$$|x_{n_j} - y| = O(|A^{-1}(f(x_{n_j}) - x_{n_j})|) = O(|f(x_{n_j}) - x_{n_j}|),$$

a contradiction. We have thus proven the following

THEOREM. *There is an open dense set $\mathbf{C} \subset C^1(B,B)$ such that, for all $f \in \mathbf{C}$, there exists L such that for all x there exists a fixed point y with $|y-x| \le L|f(x)-x|$.*

These same methods can be extended to obtain generic estimates on the computation speed of algorithms for computing fixed points (Anderson [**3**], Boese [**5**]).

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