

ON A CONORMAL MODULE OF SMOOTH SET THEORETIC COMPLETE INTERSECTIONS

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ABSTRACT. We prove that $V \subset \mathbf{A}_k^n$ (V -smooth) is a set-theoretic complete intersection (stci) if and only if V imbedded as a zero section of its normal bundle is a stci, we give a characterization of smooth codimension 2 stci of index ≤ 4 in terms of their conormal modules.

Introduction. This paper is another author's attempt to settle Murthy's problem of whether a smooth affine variety is a set theoretic complete intersection. The main result of the first part of our paper states that $V \subset \mathbf{A}_k^n$ (V -smooth) is a set theoretic complete intersection (stci) if and only if V embedded as a zero section of its normal bundle is a stci. It follows that the property of $V \subset \mathbf{A}_k^n$ being a stci depends only on the isomorphism class of the conormal module of V . So if one believes that Murthy's problem has a negative solution, it makes sense to look for a characterization of a smooth stci in terms of its conormal module. This line of investigation is pursued in the second part of this paper when $\text{codim } V = 2$. We were able to characterize there the conormal modules of smooth stci of index ≤ 4 .

1°. **A normal bundle of smooth stci.** In the sequel A will always denote $k[X_1, X_2, \dots, X_n]$ (k any field) and $I \subset A$ its ideal. P will stand for I/I^2 —the conormal module which has a natural structure of an A/I -module. We let $S = S(P) = \bigoplus_{i \geq 0} S^i P$ be the symmetric algebra of P and $S^+ = \bigoplus_{i > 0} S^i P$ its augmentation ideal.

In this section we prove the following:

THEOREM 1.1. *Let I be an ideal of A such that $R = A/I$ is smooth over k . Then I is a stci if and only if S^+ is a stci.*

REMARK. The above theorem is an algebraic version of the result mentioned in the Introduction.

COROLLARY 1.2. *The property of being a stci for a smooth $V \subset \mathbf{A}_k^n$ depends only on the isomorphism class of its conormal module.*

PROOF. Obvious.

COROLLARY 1.3. *The property of being a stci for a smooth V does not depend on the embedding into \mathbf{A}^n provided $n \geq 2 \dim V + 1$.*

PROOF. Let $\varphi_1: V \rightarrow \mathbf{A}^n$ and $\varphi_2: V \rightarrow \mathbf{A}^n$ be two (closed) embeddings of V into \mathbf{A}^n . Then the conormal modules relative to these embeddings are stably

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isomorphic. In view of the Bass Cancellation Theorem they are isomorphic since their ranks are $> \dim V$. One has to now invoke Corollary 1.2 to finish the proof.

COROLLARY 1.4. *Suppose $V \subset \mathbf{A}^n$ where V is a smooth stci. Then a scheme of zeros of a generic section of the normal bundle of V is either empty or a stci as a subvariety of V .*

PROOF. Let R denote the coordinate ring of V and let P be its conormal module. The sections of the normal bundle of V are in one-to-one correspondence with the elements of $\text{Hom}(P, R)$. For $\gamma \in \text{Hom}(P, R)$ the scheme of zeros of the corresponding section is $\text{Spec } R/\gamma(P)$.

By [4], for a generic $\gamma \in \text{Hom}(P, R)$, $\gamma(P)$ is a height k ideal of R , where $k = \text{rk } P$, unless $\gamma(P) = R$ if $k > \dim R$. $\gamma \in \text{Hom}(P, R)$ induces the ring homomorphism $S(P) \rightarrow R$ which maps S^+ onto $\gamma(P)$. So if $k = \text{rk } P \leq \dim R$, $\gamma(P)$ is a stci since S^+ is generated up to radicals by k elements ($k = \text{ht } S^+$), and if $k > \dim R$ the scheme of zeros of a generic section of a normal bundle of V is empty.

For the proof of Theorem 1.1 we need some preliminary results.

A descending chain of ideals $I_1 > I_2 > \dots$ of any ring T gives rise to the inverse system of rings $\{T/I_i\}_{i \geq 1}$. Its maps are the obvious ones. In particular, for any ideal $I \subset A$ one obtains the inverse systems of rings $\{A/I^s\}_{s \geq 1}$ and $\{S(P)/S(P)_{\geq s}\}_{s \geq 1}$, where $S(P)_{\geq s} = \bigoplus_{i \geq s} S^i P$.

PROPOSITION 1.5. *Let I be an ideal of A such that $R = A/I$ is smooth over k . Then $\{A/I^s\}_{s \geq 1} \simeq \{S(P)/S(P)_{\geq s}\}_{s \geq 1}$ (as inverse systems of rings).*

PROOF. Note first that $S^i P \simeq I^i/I^{i+1}$ because of smoothness. The smoothness of R implies also that we have the splitting exact sequence

$$0 \rightarrow I/I^2 \rightarrow \Omega_{A/k} \otimes A/I \rightarrow \Omega_{R/k} \rightarrow 0.$$

Let $\gamma: \Omega_{A/k} \otimes A/I \rightarrow I/I^2$ be a splitting of $I/I^2 \rightarrow \Omega_{A/k} \otimes A/I$. For $f \in A$ we put $d_1 f = \gamma(\overline{df})$, where \overline{df} denotes the image of df in $\Omega_{A/k} \otimes A/I$. It follows from the definition of $d_1: A \rightarrow I/I^2$ that d_1 is an extension of the natural homomorphism $I \rightarrow I/I^2$.

We denote by d_0 the natural homomorphism $A \rightarrow A/I$. The association $X_i \rightarrow d_0 X_i + d_1 X_i \in S(P)$ defines $d: A \rightarrow S(P)$ —the homomorphism of k -algebras. We are going to prove that for each $s \geq 1$ the composition $A \xrightarrow{d} S(P) \rightarrow S(P)/S(P)_{\geq s}$ induces the isomorphism of A/I^s with $S(P)/S(P)_{\geq s}$.

Let $d_i: A \rightarrow I^i/I^{i+1}$ denote the i th component of d . It is easy to check that d_0 and d_1 coincide with the previously defined d_0 and d_1 . We have $d_i(fg) = \sum_{j+k=i} d_j(f)d_k(g)$ since d is a homomorphism.

We claim that d_i restricted to I^i is the natural homomorphism $I^i \rightarrow I^i/I^{i+1}$ for $i \geq 0$. This is true for $i = 0$ and $i = 1$. Let $f \in I^{i-1}$ and $g \in I$. By induction $d_i(fg) = d_{i-1}(f)$,

$$\begin{aligned} d_1(g) &= (\text{image of } f \text{ in } I^{i-1}/I^i)(\text{image of } g \text{ in } I/I^2) \\ &= \text{image of } fg \text{ in } I^i/I^{i+1}. \end{aligned}$$

This proves the claim since I^i is additively generated by the elements of the form fg with $f \in I^{i-1}$ and $g \in I$. It follows that for $s \geq 1$ the composition of d with $S(P) \rightarrow S(P)/S(P)_{\geq s}$ induces $\varphi_s: A/I^s \rightarrow S(P)/S(P)_{\geq s}$, which is a monomorphism.

To prove that φ_s is onto, it suffices to show that the homogeneous elements of $\bigoplus_{i \leq s-1} S^i P$ differ from the image of d by $S(P)_{\geq s}$. Let x be such an element of $\bigoplus_{i \leq s-1} S^i P$ with $\deg x = i$ ($i \leq s - 1$). Then there exists $a \in I^i$ such that $d_i(a) = x$. We shall define inductively a sequence of elements $\{a_j\}_{i \leq j \leq s-1}$ with $a_j \in I^j$. Put $a_i = a$. Suppose the sequence a_i, a_{i+1}, \dots, a_k has been defined with $a_j \in I^j$ for $i \leq j \leq k$. Let a_{k+1} be an element of I^{k+1} such that $d_{k+1} a_{k+1} = -d_{k+1}(a_i + a_{i+1} + \dots + a_k)$. Then it is easy to check that $d(\sum_{j=i}^{s-1} a_j) - x \in S(P)_{\geq s}$. This finishes the proof since the isomorphisms φ_s are obviously compatible with the maps of both inverse systems in question.

REMARK. It follows from the proof that φ_s maps I/I^s isomorphically onto $S^+/S(P)_{\geq s}$.

LEMMA 1.6. *Let J be an ideal of A such that $I \supset J \supset I^s$, and let $\varphi_{2s}(J/I^{2s}) = K/S(P)_{\geq 2s}$. Then $J/J^2 \simeq K/K^2$ as modules over A/J and $S(P)/K$, respectively, identified by the isomorphism induced by φ_{2s} .*

PROOF. One has to prove only that $K^2 \supset S(P)_{\geq 2s}$. We have $\varphi_{2s}(I^s/I^{2s}) = S(P)_{\geq s}/S(P)_{\geq 2s}$. It follows that $K \supset S(P)_{\geq s}$, since $J \supset I^s$ and $\varphi_{2s}(J/I^{2s}) = K/S(P)_{\geq 2s}$. Hence $K^2 \supset S(P)_{\geq 2s}$.

PROPOSITION 1.7. *Let R be a smooth k -algebra, and let P be a projective R -module. Then every projective $S(P)$ -module is extended from R .*

PROOF. By [1] a projective $S(P)$ -module is extended from R if it is locally so. One has to now invoke Lindel's theorem, which says that every projective $T[X_1, X_2, \dots, X_n]$ -module is free for any local ring T which is a localization of a smooth k -algebra.

PROPOSITION 1.8. *Let T be a (connected) Cohen-Macaulay ring and K its ideal, with $\text{ht } K = n$ and K/K^2 free over T/K with $\text{rk } K/K^2 = n$. Then there exists an ideal $K' \subset K$ such that $\text{rad } K' = \text{rad } K$, K' is a homomorphic image of a projective module of rank n and, moreover, $K'/(K')^2$ is free over T/K' with $\text{rk } K'/(K')^2 = n$.*

PROOF. The existence of K' which satisfies the first two conditions has been proved in [2]. The proof actually shows that there exists $t \in T$ such that $(K', t) = T$ and K'_t is generated by n elements. It follows that K'_t is generated by a regular sequence, since $\text{ht } K'_t = \text{ht } K' = n$ and T is Cohen-Macaulay. We infer that $K'/(K')^2 = K'_t/(K'_t)^2$ is free over T/K' with $\text{rk } K'/(K')^2 = n$.

We can now prove Theorem 1.1. Suppose that I is a stci. This means that there exists an ideal $J \subset A$ and $s \geq 1$ such that J is a complete intersection and $I^s \subset J \subset I$. Put $\varphi_{2s}(J/I^{2s}) = K/S(P)_{\geq 2s}$. It follows from Lemma 1.6 that K/K^2 is free over $S(P)/K$. We have $\text{rk } K/K^2 = \text{rk } J/J^2 = \text{ht } J = \text{ht } I$ and $\text{ht } K = \text{rk } P = \text{ht } I$ since $S(P)_{\geq 2s} \subset K \subset S^+$. Therefore by Proposition 1.8 there exists an exact sequence $Q \xrightarrow{f} K' \rightarrow 0$, where Q is $S(P)$ -projective of rank equal to $\text{ht } I$, $K' \subset K$, $\text{rad } K' = \text{rad } K$ and $K'/(K')^2$ is free over $S(P)/K'$ with $\text{rk } K'/(K')^2 = \text{ht } I$. f induces an isomorphism $Q/K'Q \simeq K'/(K')^2$, since both modules are projective of the same rank. So $Q/K'Q$ is free over $S(P)/K'$. $Q/S^+Q \simeq Q/K'Q \otimes S/S^+$ since $K' \subset S^+$. So Q/S^+Q is free over $S/S^+ \simeq R$.

By Proposition 1.7 Q is free. It follows that K' is a complete intersection ideal of $S(P)$. So S^+ is a stci since $\text{rad } K' = \text{rad } K$ and $\text{rad } K = S^+$.

Suppose now that S^+ is a stci. Then there exists $s \geq 1$ and an ideal K of $S(P)$ such that K is a complete intersection and $S(P)_{\geq s} \subset K \subset S^+$. In the same way as before, we can show that there exists $J \subset A$ such that $\text{rad } J = \text{rad } I$ and J/J^2 free over A/J with $\text{rk } J/J^2 = \text{ht } I$. It follows from Proposition 1.8 and the Quillen-Suslin theorem that J is a stci. So I is a stci since $\text{rad } J = \text{rad } I$.

2°. **A conormal module of smooth stci in codimension 2 case.** For any affine Cohen-Macaulay (CM) k -algebra R ω_R will denote its canonical module. We refer to [7] for its basic properties. Here we note only that in case R is smooth over k $\omega_R \simeq \bigwedge^{\dim R} \Omega_{R/k}$. So, in particular, ω_R is projective of rank 1.

DEFINITION. Let R be an affine CM k -algebra. R is called Gorenstein if $\omega_R \simeq R$.

PROPOSITION 2.1 (SERRE, QUILLEN, SUSLIN). *Let I be an ideal of A with $\text{ht } I = 2$. If $R = A/I$ is Gorenstein, then I is a complete intersection.*

REMARK. It follows from Propositions 2.1 and 1.5 that to prove that I is a stci (A/I -smooth and $\text{ht } I = 2$) one has to show that there exist $t \geq 1$ and an ideal K of $S(P)_{\leq t} = S(P) / \bigoplus_{i \geq t+1} S^i(P)$ such that $K \subset S_{\leq t}^+ = \bigoplus_{0 < i \leq t} S^i P$ and the ring $S(P)_{\leq t}/K$ is Gorenstein.

Let $t \geq 0$ and let $\varphi: S(P)_{\leq t} \rightarrow \omega$ ($\omega = \omega_R$) be an R -linear map. We define $\varphi': S(P)_{\leq t} \rightarrow \text{Hom}(S(P)_{\leq t}, \omega)$ by $\varphi'(x)(y) = \varphi(xy)$ with $x, y \in S(P)_{\leq t}$.

PROPOSITION 2.2. *Let R be an affine smooth k -algebra, and let P be a projective R -module. For $t \geq 0$ the following conditions are equivalent:*

1°. *There exists an ideal $K \subset S(P)_{\leq t}$ such that $K \subset S_{\leq t}^+$ and $S(P)_{\leq t}/K$ is Gorenstein,*

2°. *There exists $0 \neq \varphi: S(P)_{\leq t} \rightarrow \omega$ such that φ' is of constant rank (i.e., the rank of $\varphi' \otimes k(x): S(P)_{\leq t} \otimes k(x) \rightarrow \text{Hom}(S(P)_{\leq t}, \omega) \otimes k(x)$ does not depend on $x \in \text{Spec } R$).*

3°. *There exists $0 \neq \varphi: S(P)_{\leq t} \rightarrow \omega$ such that $\text{coker } \varphi'$ is projective.*

REMARK. It was proved in [3] that the existence of a homogeneous ideal K which satisfies the conditions of 1° is equivalent to the existence of $0 \leq j \leq t$ and $0 \neq \varphi \in \text{Hom}(S(P)_{\leq t}, \omega)$ such that $\varphi|_{j \neq i \leq t} S^i P = 0$ and φ' satisfies the condition of 2°.

Moreover, there is an example of a smooth surface in \mathbf{A}^4 with a coordinate ring R and conormal module P for which such a ‘homogeneous’ φ does not exist for any value of t [3].

For the proof of Proposition 2.2 we shall need the following lemma, which can be easily deduced from the results of [7].

LEMMA 2.3. *Let $f: R \rightarrow S$ be a homomorphism of affine CM k -algebras which makes S into a finitely generated R -module. If $\dim R = \dim S$, then $\omega_S \simeq \text{Hom}_R(S, \omega_R)$. Moreover, the S -module structure of ω_S is induced by that of S .*

PROOF OF PROPOSITION 2.2. Let $\gamma: M \rightarrow N$ be a homomorphism of projective R -modules (R any noetherian ring). It is well known (and easy to prove) that $\text{coker } \gamma$ is projective if and only if γ has a constant rank in $\text{Spec } R$. This shows the equivalence 2° and 3°.

Suppose now that there exists $0 \neq \varphi \in \text{Hom}(S(P)_{\leq t}, \omega)$ such that $\text{coker } \varphi'$ is projective. Put $K = \ker \varphi'$. One can easily check that K is an ideal of $S(P)_{\leq t}$. Let $x = \sum_{i=0}^t x_i \in K$ with $x_0 \neq 0$, and let s be the largest number i ($i \leq t$) with the property that $\varphi|S^i(P) \neq 0$. We have

$$0 = \varphi'(x)(S^s(P)) = \varphi(xS^s(P)) = \varphi(x_0S^s(P)) = x_0\varphi(S^s(P)).$$

It follows that $\varphi(S^s(P)) = 0$ since ω is torsion free. This contradicts the way s was defined. So $x_0 = 0$ and $K \subset S_{\leq t}^+$. We claim that $S(P)_{\leq t}/K = \text{Im } \varphi'$ is Gorenstein. $S(P)_{\leq t}/K$ is R -projective since the exact sequence $0 \rightarrow \text{Im } \varphi' \rightarrow \text{Hom}(S(P)_{\leq t}, \omega) \rightarrow \text{coker } \varphi' \rightarrow 0$ splits. It follows that $S(P)_{\leq t}/K$ is CM, since $S(P)_{\leq t}/K$ is a finitely generated projective R -module and R is smooth. Applying Lemma 2.3 with $S = S(P)_{\leq t}/K$, we get

$$\omega_B \simeq \{f \in \text{Hom}(S(P)_{\leq t}, \omega) \mid Kf = 0\} = \text{Anih}_K \text{Hom}_R(S(P)_{\leq t}, \omega),$$

where $B = S(P)_{\leq t}/K$. (Note that $\dim B = \dim R$.) It follows from the way K was defined that $K = \{x \in S(P)_{\leq t} \mid x\varphi = 0\}$. We claim $\text{Anih}_K \text{Hom}_R(S(P)_{\leq t}, \omega)$ is (freely) generated as a B -module by φ . This will prove that $\omega_B \simeq B$ and B is Gorenstein. Let $g \in \text{Anih}_K \text{Hom}(S(P)_{\leq t}, \omega)$. Then g induces $g': \text{Im } \varphi' \rightarrow \omega$. $\text{Im } \varphi'$ is a direct summand of $\text{Hom}(S(P)_{\leq t}, \omega)$ since $\text{coker } \varphi'$ is projective. Therefore g' can be extended to a homomorphism $\text{Hom}(S(P)_{\leq t}, \omega) \rightarrow \omega$. Every such homomorphism is given by the evaluation on some $r \in S(P)_{\leq t}$ since ω is projective of rank 1. So for $x \in S(P)_{\leq t}$ we have

$$g(x) = g'\varphi'(x) = \varphi'(x)(r) = \varphi(xr) = (r\varphi)(x).$$

Hence $g = r\varphi$ and $\text{Anih}_K \text{Hom}(S(P)_{\leq t}, \omega)$ is generated by φ as claimed.

Let now K be an ideal of $S(P)_{\leq t}$ such that $K \subset S_{\leq t}^+$ and $B = S(P)_{\leq t}/K$ is Gorenstein. Applying Lemma 2.3 again, we get $B \simeq \text{Hom}_R(B, \omega)$. Let $\varphi: S(P)_{\leq t} \rightarrow \omega$ be a composition of the natural homomorphism $S(P)_{\leq t} \rightarrow B$ and an element of $\text{Hom}_R(B, \omega)$ which corresponds to $1 \in B$. Obviously $\varphi \neq 0$. We claim that $\text{coker } \varphi'$ is projective. B is a projective R -module since B is CM, and the splitting exact sequence $0 \rightarrow K \rightarrow S(P)_{\leq t} \rightarrow B \rightarrow 0$ induces the splitting exact sequence

$$0 \rightarrow \text{Hom}(B, \omega) \rightarrow \text{Hom}(S(P)_{\leq t}, \omega) \rightarrow \text{Hom}(K, \omega) \rightarrow 0.$$

One can check that $\text{coker } \varphi' = \text{Hom}(K, \omega)$. So $\text{coker } \varphi'$ is projective as claimed, since K is projective. We shall need later the following two lemmas:

LEMMA 2.4. *Let $0 \neq \varphi \in \text{Hom}(S(P)_{\leq t}, \omega)$ satisfy the condition 2° (or 3°) of Proposition 2.2. Then $\varphi|S_{\leq t}^+ : S_{\leq t}^+ \rightarrow \omega$ is onto unless $\ker \varphi' \supset S_{\leq t}^+$.*

PROOF. If $\varphi|S_{\leq t}^+$ is not onto, then there exists $x \in \text{Spec } R$ such that $\varphi \otimes k(x) | S_{\leq t}^+ \otimes k(x)$ is zero. It follows that $\text{rank } \varphi' \otimes k(x) \leq 1$. So $\text{rank } \varphi' = 1$ since $\varphi \neq 0$.

From the local matrix presentation of φ' we get that $\varphi|S_{\leq t}^+ = 0$ and $\ker \varphi' \supset S_{\leq t}^+$.

Let $\varphi \in \text{Hom}(S(P)_{\leq t}, \omega)$. We define $\varphi'' : \bigoplus_{0 < i < t} S^i P \rightarrow \text{Hom}(\bigoplus_{0 < i < t} S^i P, \omega)$ as follows: $\varphi''(x)(y) = \varphi(xy)$, where $x, y \in \bigoplus_{0 < i < t} S^i(P)$. In the sequel R and P will have the same meaning as in Proposition 2.2 unless explicitly stated otherwise.

LEMMA 2.5. *Suppose $0 \neq \varphi \in \text{Hom}(S(P)_{\leq t}, \omega)$ with $t \geq 2$ has the property that φ'' is of constant rank. Then $\varphi \mid S_{\leq t}^+$ is onto unless $\ker \varphi' \supset \bigoplus_{2 \leq i \leq t} S^i P$.*

PROOF. If $\varphi \mid S_{\leq t}^+$ is not onto, then similarly, as before, there exists $x \in \text{Spec } R$ such that $\varphi \otimes k(x) \mid S_{\leq t}^+ \otimes k(x)$ is zero. From the way φ'' was defined, we infer that $\varphi'' \otimes k(x)$ is zero. So $\varphi'' = 0$, since φ'' is of constant rank. It follows easily that $\ker \varphi' \supset \bigoplus_{2 \leq i \leq t} S^i P$, which was to be proved.

From the above two lemmas and the local matrix presentation of φ' (in the suitable bases) one can easily deduce the following:

PROPOSITION 2.6. *Let $0 \neq \varphi \in \text{Hom}(S(P)_{\leq t}, \omega)$ with $\text{rank } P = 2$ and $t \geq 2$. Suppose that $\ker \varphi' \not\supset \bigoplus_{2 \leq i < t} S^i P$. Then φ' is of constant rank if and only if φ'' is of constant rank. (In fact $\text{rank}(\varphi' \otimes k(x)) = \text{rank}(\varphi'' \otimes k(x)) + 2$ for $x \in \text{Spec } R$.)*

THEOREM 2.7. *Let $\text{rank } P = 2$. There exists an ideal $K \subset S(P)_{\leq t}$ such that $K \subset S_{\leq t}^+$ and $S(P)_{\leq t}/K$ is Gorenstein for $t = 0, 1$ or 2 if and only if one of the following conditions holds:*

- (a) $\omega \simeq R$.
- (b) *There exists $f: P \rightarrow \omega$ which is onto.*
- (c) *There exists $0 \neq f: S^2 P \rightarrow \omega$ such that the induced map $P \rightarrow \text{Hom}(P, \omega)$ is of constant rank.*

PROOF. There is no problem with (a). The necessity of (b) and (c) follow from Lemma 2.4 and Proposition 2.6 respectively.

Suppose now that $f: P \rightarrow \omega$ is onto. Put $\varphi = (0, f): R \oplus P \rightarrow \omega$. One checks easily that φ' is of constant rank. Also $\varphi = (0, 0, f): R \oplus P \oplus S^2 P \rightarrow \omega$ does the job for (c). This finishes the proof.

The next result shows that the case $t = 3$ is more complicated.

THEOREM 2.8. *Suppose $\text{rank } P = 2$ and $\frac{1}{2} \in R$. There exists an ideal $K \subset S(P)_{\leq 3}$ such that $K \subset S_{\leq 3}^+$, $K \not\supset S^3 P$ and $S(P)_{\leq 3}/K$ is Gorenstein if and only if one of the following conditions holds:*

- (a) *There exists $0 \neq f: S^3 P \rightarrow \omega$ such that the induced map $P \rightarrow \text{Hom}(S^2 P, \omega)$ is of constant rank.*
- (b) *There exists a nontrivial decomposition $P = P_1 \oplus P_2$ and $f: (P_1 \otimes P_2) \oplus P_1^{\otimes 3} \rightarrow \omega$ which is onto and has the property that $f \mid P_1^{\otimes 3} \neq 0$.*

For the proof of Theorem 2.8 we shall need some preliminary notions and results.

DEFINITION. Let Q be a projective R -module (R any noetherian ring) and ω a rank 1 projective R -module. Q is called a quadratic ω -valued space if there exists a nonsingular symmetric bilinear pairing $(\ , \) : Q \times Q \rightarrow \omega$. (Nonsingular means that the induced map $Q \rightarrow \text{Hom}(Q, \omega)$ is an isomorphism.)

LEMMA 2.9. *Let P be a projective R -module (R a domain), and let ω be a projective R -module of rank 1. Suppose $f: P \rightarrow \text{Hom}(P, \omega)$ is of constant rank and has the property that $f(x)(y) = f(y)(x)$ for $x, y \in P$. Then*

- 1°. $(x, y) = f(x)(y)$ for $x, y \in P$ defines a symmetric pairing on P .
- 2°. $\text{Im } f$ with the pairing $(f(x), f(y)) = f(x)(y)$ is a quadratic ω -valued space.
- 3°. f is compatible with the pairings on Q and $\text{Im } f$.

PROOF. Easy exercise.

For any projective R -module U with a symmetric bilinear ω -valued pairing $\beta: U \times U \rightarrow \omega$, one can define a quadratic ω -valued space $M(U, \beta)$ in the following way: Put $M(U, \beta) = U \oplus \text{Hom}(U, \omega)$ and $(u + f, v + g) = \beta(u, v) + f(v) + g(u)$.

DEFINITION. A quadratic ω -valued space is called split metabolic if it is isometric with some $M(U, \beta)$. A quadratic ω -valued space isometric with $M(U, 0)$ is called hyperbolic.

LEMMA 2.10. *Suppose $\frac{1}{2} \in R$. Then every split metabolic quadratic ω -valued space is hyperbolic.*

PROOF. This result was proved in [7] in case $\omega = R$. It is easy to see that the same proof carries over to the case of arbitrary rank 1 projective ω .

PROPOSITION 2.11. *Let R be a regular domain such that $\frac{1}{2} \in R$. Suppose Q is a rank 2 ω -valued quadratic space which contains a nonzero isotropic element. Then Q is hyperbolic.*

PROOF. Let $0 \neq x \in Q$ such that $(x, x) = 0$. Put $S = \{s \in Q \mid \exists 0 \neq r \in R \text{ such that } rs \in Rx\}$. We obviously have $(S, S) = 0$. We claim that $S = S^\perp = \{a \in Q \mid (a, S) = 0\}$. Suppose $S \not\subseteq S^\perp$. Then $\text{rank } S^\perp = 2$ since Q/S is torsion free. If $\text{rank } S^\perp = 2$, then Q/S^\perp is torsion and $(Q, S) = 0$, a contradiction. So $S = S^\perp$ is claimed.

We claim now that Q/S is projective. We can assume that $\dim R \geq 2$ and R is local. If Q/S is not projective, then $Q/S \simeq I$, where I is an ideal of R with $\text{ht } I \geq 2$. So we have the exact sequence $0 \rightarrow S \rightarrow Q \xrightarrow{\gamma} I \rightarrow 0$. Let $\varphi: Q \rightarrow \text{Hom}(Q, \omega)$ denote the homomorphism induced by $(,)$ on Q , and let us put $\varphi_s = (s,) : Q \rightarrow \omega$ for $s \in S$. Every φ_s factors through γ , and each homomorphism $I \rightarrow \omega \simeq R$ is given by multiplication, since $\text{depth } I \geq 2$. So for $a \in Q$, $\varphi_s(a) = \gamma(a)\lambda(s)$ with $\lambda(s) \in \omega$ and $S = S^\perp \simeq \text{Hom}(Q/S, \omega) \simeq \text{Hom}(I, \omega) \simeq \omega$. Moreover, the obtained isomorphism of S with ω is given by λ . R is local, so $Q \simeq R^2$ and I is a ht 2 complete intersection. Let $\gamma(e_1) = x$, $\gamma(e_2) = y$, where e_1, e_2 is a canonical basis of R^2 and x, y form a regular sequence. $S = R(-y, x)$ and $\lambda(-y, x) = r \in R \simeq \omega$ with r invertible. For $s = (-y, x)$ we have $(s, e_1) = \varphi_s(e_1) = \gamma(e_1)\lambda(s) = rx$, and similarly $(s, e_2) = ry$. Put $a = (e_1, e_1)$, $b = (e_1, e_2)$ and $c = (e_2, e_2)$. We obtain the equalities

- (1) $-ya + xb = rx$,
- (2) $-yb + xc = ry$.

It follows from (1) that $b = r + \alpha y$ for some $\alpha \in R$ since x, y form a regular sequence. If we plug this in (2), we get $xc = y(2r + \alpha y)$. For the same reason as above, we conclude that $2r + \alpha y = \beta x$ for some $\beta \in R$. This is a contradiction, since $2r$ is invertible. We infer that Q/S is projective. So $Q = U \oplus S$ with $U \simeq Q/S$. The association $s \rightarrow \varphi_s \mid U$ defines a homomorphism $S \rightarrow \text{Hom}(U, \omega)$, which is an isomorphism, since $S = S^\perp$. It follows that Q is isometric with $M(U, \beta)$, where β denotes the restriction of the pairing on Q to U . So by Lemma 2.10 Q is hyperbolic, which was to be proved.

PROOF OF THEOREM 2.8. Suppose K is an ideal of $S(P)_{\leq 3}$ which has all the required properties. Then by Proposition 2.2 there exists $0 \neq \varphi \in \text{Hom}(S(P)_{\leq 3}, \omega)$ such that $\varphi': S(P)_{\leq 3} \rightarrow \text{Hom}(S(P)_{\leq 3}, \omega)$ is of constant rank. We can apply Proposition 2.6 since $K = \ker \varphi' \not\supseteq S^3P$. Thus φ'' is of constant rank. Let

$\varphi = (\varphi_0, \varphi_1, \varphi_2, \varphi_3): R \oplus P \oplus S^2P \oplus S^3P \rightarrow \omega$, and let $x \in \text{Spec } R$. If e_1, e_2 form a base of $P \otimes k(x)$, then we take $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_2$ as a base of $S^2P \otimes k(x)$ and $e_1 \otimes e_1 \otimes e_1, e_1 \otimes e_1 \otimes e_2, e_1 \otimes e_2 \otimes e_2, e_2 \otimes e_2 \otimes e_2$ as a base of $S^3P \otimes k(x)$. Let $\varphi_2 \otimes k(x): S^2(P) \otimes k(x) \rightarrow \omega \otimes k(x)$ and $\varphi_3 \otimes k(x): S^3(P) \otimes k(x) \rightarrow \omega \otimes k(x)$ have a presentation $[b_1, b_2, b_3]$ and $[c_1, c_2, c_3, c_4]$, respectively, in the above chosen bases $(b_i, c_j \in k(x))$. Then

$$\begin{pmatrix} b_1 & b_2 & c_1 & c_2 & c_3 \\ b_2 & b_3 & c_2 & c_3 & c_4 \\ c_1 & c_2 & 0 & 0 & 0 \\ c_2 & c_3 & 0 & 0 & 0 \\ c_3 & c_4 & 0 & 0 & 0 \end{pmatrix}$$

is a matrix of $\varphi'' \otimes k(x)$ whose determinant is zero. An easy calculation of the minors shows that $\text{rank } \varphi'' \otimes k(x) = 4$ if and only if

$$\text{rank} \begin{pmatrix} c_1, c_2, c_3 \\ c_2, c_3, c_4 \end{pmatrix} = 2$$

and $\text{rank } \varphi'' \otimes k(x) = 3$ implies that

$$\text{rank} \begin{pmatrix} c_1, c_2, c_3 \\ c_2, c_3, c_4 \end{pmatrix} = 1.$$

We conclude that if $\text{rank } \varphi'' = 4$ or 3 , then $\varphi_3 \neq 0$ and the map $P \rightarrow \text{Hom}(S^2P, \omega)$ is of constant rank. Suppose now that $\text{rank } \varphi'' = 2$ and put $Q = \text{Im } \varphi''$. Then by Lemma 2.9 Q is a rank 2 quadratic ω -valued space. $\varphi''(S^2P) \neq 0$, since otherwise we would have $\varphi_3 = 0$ and $K = \ker \varphi' \supset S^3P$. Moreover, $(\varphi''(S^2P), \varphi''(S^2P)) = (S^2P, S^2P) = 0$. So it follows from Proposition 2.11 that Q is hyperbolic. Let

$$\varphi'' = \begin{pmatrix} f_1, f_2 \\ f_3, f_4 \end{pmatrix} : P \oplus S^2P \xrightarrow{\text{onto}} U \oplus \text{Hom}(U, \omega),$$

where U is a rank 1 projective,

$$\begin{aligned} f_1 &\in \text{Hom}(P, U), & f_2 &\in \text{Hom}(S^2P, U), \\ f_3 &\in \text{Hom}(P, \text{Hom}(U, \omega)), & f_4 &\in \text{Hom}(S^2P, \text{Hom}(U, \omega)). \end{aligned}$$

Every isotropic element of $U \oplus \text{Hom}(U, \omega)$ is either contained in U or in $\text{Hom}(U, \omega)$, since U is of rank 1 and $\frac{1}{2} \in R$. It follows that $S^2P = \ker f_2 \cup \ker f_4$. For symmetry reasons we can assume that $S^2P = \ker f_4$ and $f_4 = 0$ (since $\text{Hom}(\text{Hom}(U, \omega), \omega) \simeq U$). It follows that $f_3: P \rightarrow \text{Hom}(U, \omega)$ is onto. So $P \simeq P_1 \oplus P_2$, with $P_1 = \text{Hom}(U, \omega)$ and $P_2 = \ker f_3$. Using the surjectivity of φ'' again, we infer that $(f_1, f_2): P_2 \oplus S^2P \rightarrow U$ is onto. We claim that $f_2 \mid (P_1 \otimes P_2) \oplus P_2^{\otimes 2} = 0$. We have

$$\varphi_3(P_2 \otimes S^2P) = (P_2, S^2P) = (\varphi''(P_2), \varphi''(S^2P)) = ([f_1(P_2), 0], [f_2(S^2P), 0]) = 0.$$

$f_3 \neq 0$, since otherwise φ'' would not be onto. Let $x \in P$ be such that $f_3(x) \neq 0$. Then

$$\begin{aligned} 0 &= \varphi_3(x \otimes P_2 \otimes P) = (x, P_2 \otimes P) = (\varphi''(x), \varphi''(P_2 \otimes P)) \\ &= ([f_1(x), f_3(x)], [f_2(P_2 \otimes P), 0]) = f_3(x)(f_2(P_2 \otimes P)). \end{aligned}$$

So $f_2(P_2 \otimes P) = 0$ as claimed, since U is of rank 1. We infer that $(f_1, f_2 \mid P_1^{\otimes 2})$: $P_2 \oplus P_1^{\otimes 2} \rightarrow U$ is onto and $f_2 \mid P_1^{\otimes 2} \neq 0$, because otherwise we would get $\varphi_3 = 0$ and $K = \ker \varphi' \supset S^3P$. $U \simeq \text{Hom}(\text{Hom}(U, \omega), \omega) = \text{Hom}(P_1, \omega) \simeq P_1^{\otimes -1} \otimes \omega$. Tensoring the map $P_2 \oplus P_1^{\otimes 2} \rightarrow U$ by P_1 , we get a surjection $(P_1 \otimes P_2) \oplus P_1^{\otimes 3} \rightarrow \omega$ such that the induced map $P_1^{\otimes 3} \rightarrow \omega$ is nonzero.

If $\text{rank } \varphi'' \leq 1$, then a local matrix presentation of φ'' shows that $\varphi_3 = 0$, which implies that $K = \ker \varphi' \supset S^3P$. So the $\text{rank } \varphi'' \leq 1$ case is impossible.

The sufficiency of (a) can be dealt with in the same way as the sufficiencies of (b) and (c) of Theorem 2.7. To prove the sufficiency of (b), we define

$$\varphi = (\varphi_0, \varphi_1, \varphi_2, \varphi_3): R \oplus P \oplus S^2P \oplus S^3P \rightarrow \omega$$

in the following way:

$$\varphi_0 = \varphi_1 = 0, \quad \varphi_2 \mid P_1 \otimes P_2 = f \mid P_1 \otimes P_2, \quad \varphi_3 \mid P_1^{\otimes 3} = f \mid P_1^{\otimes 3}$$

and

$$\varphi_2 \mid (P_1^{\otimes 2} \oplus P_2^{\otimes 2}) = \varphi_3 \mid P_2 \otimes S^2P = 0.$$

It is easy to check, using the local matrix presentation of φ'' (for φ defined as above), that $\text{rank } \varphi'' \otimes k(x) = 2$ for all $x \in \text{Spec } R$. This finishes the proof of Theorem 2.8.

REMARK. Unfortunately we have not been able to settle (completely) the problem of the existence of Gorenstein factor rings of $S(P)_{\leq t}$ in case $t = 4$.

Combining Proposition 1.5, the remark following Proposition 2.1, Theorem 2.7 and Theorem 2.8 together, one obtains the following:

COROLLARY 2.12. *Let I be an ideal of $A = k[X_1, X_2, \dots, X_n]$, ($\text{char } k \neq 2$) with $\text{ht } I = 2$ such that $R = A/I$ is smooth over k . Then I is a stci of index ≤ 4 , i.e., there exist a complete intersection ideal J such that $I \supset J \supset I^s$ for $s = 1, 2, 3$ or 4 if and only if one of the following conditions holds:*

- (a) $\omega \simeq R$.
- (b) There exists $0 \neq f: P \rightarrow \omega$ which is onto.
- (c) There exists $0 \neq f: S^2P \rightarrow \omega$ such that the induced map $P \rightarrow \text{Hom}(P, \omega)$ is of constant rank.
- (d) There exists $0 \neq f: S^3P \rightarrow \omega$ such that the induced map $P \rightarrow \text{Hom}(S^2P, \omega)$ is of constant rank.
- (e) There exists a nontrivial decomposition $P = P_1 \oplus P_2$ and $0 \neq f: (P_1 \otimes P_2) \oplus P_1^{\otimes 3} \rightarrow \omega$ which is onto, where $P = I/I^2$ and $\omega = (\Lambda^2(I/I^2))^{\otimes -1}$.

REMARK. It follows from [3] that each of the conditions (a)–(d) implies that $c_1^2(P)$ is torsion. Unfortunately it does not seem possible to conclude $c_1^2(P)$ -torsion from (e). So the original belief we had that for a smooth variety the property of being a stci might impose some nontrivial condition on the Chern classes of its conormal bundle got considerably shattered.

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