THE DUAL OF THE BERGMAN SPACE $A^1$ IN SYMMETRIC
SIEGEL DOMAINS OF TYPE II

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Abstract. An affirmative answer is given to the following conjecture of R. Coifman
and R. Rochberg: in any symmetric Siegel domain of type II, the dual of the
Bergman space $A^1$ coincides with the Bloch space of holomorphic functions and can
be realized as the Bergman projection of $L^\infty$.

Let $D$ be a symmetric Siegel domain of type II; let $v$ denote the Lebesgue
measure in $D$ and $H(D)$ the space of holomorphic functions in $D$. The Bergman
space $A^p(D)$, $0 < p \leq \infty$, is defined by $A^p(D) = H(D) \cap L^p(dv)$.

In the one-dimensional case, $D = \Omega^+ = \{ z \in \mathbb{C} : \text{Im } z > 0 \}$, R. Coifman and R.
Rochberg [4] gave a proof of the following fact: the dual of the Bergman space
$A^1(\Omega^+)$ coincides with the Bloch space of holomorphic functions and can be realized
as the Bergman projection of $L^\infty$. Then, these authors conjectured that this char-
acterization of the dual of $A^1$ should hold in any symmetric Siegel domain of type II.

In [1] (resp. [2 and 3]), an affirmative answer to this conjecture was given in the
particular case of the Cayley transform of the unit ball in $\mathbb{C}^n$, $n > 1$, defined by
$\{ (z_1, z') \in \mathbb{C} \times \mathbb{C}^{n-1} : \text{Im } z_1 > |z'|^2 \}$ (resp. in the tube $\mathbb{R}^{n+1} + i \Gamma$, $n \geq 1$, over the
spherical cone $\Gamma$ defined in $\mathbb{R}^{n+1}$ by

$\Gamma = \{ (y_0, y_1, y_2, \ldots, y_n) \in \mathbb{R}^{n+1} : y_0 y_1 - y_2^2 - \cdots - y_n^2 > 0, y_0 > 0 \}$).

The purpose of this paper is to prove the Coifman-Rochberg conjecture in the
general case of any symmetric Siegel domain of type II.

We shall denote by $B(\xi, z)$ the Bergman kernel of such a domain $D$. When $r$ is a
strictly positive integer, S. G. Gindikin [5] defined a differential operator $D_r$ in $D$
that satisfies the property

$(D_r) B(\xi, z) = C_r B_1(\xi, z), \quad \xi, z \in D$.

The Bloch space $B_r$ corresponding to the integer $r$ is defined in terms of that
operator $D_r$. A holomorphic function $g$ in $D$ is said to be a Bloch function when it
satisfies the estimate

$\sup_{z \in D} \{ |D_r g(z) | B_r(\xi, z) \} < +\infty$.

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Liouville differential operator.
Denote by \( \mathcal{N} \) the subspace of \( H(D) \) consisting of those functions satisfying \( \mathcal{D}_r g = 0 \); the Bloch space \( \mathcal{B}_r \) is defined as the quotient space \( \{ \text{Bloch functions} \} / \mathcal{N} \).

In view of the proof of the Coifman-Rochberg conjecture, let us recall at once that, unlike the bounded domains, the Bergman kernel \( B(\xi, z) \) of \( D \) is not integrable with respect to \( z \) because of its bad behaviour when \( z \) tends to infinity.

In the first chapter of this paper, we use S. G. Gindikin's general theory [5] to characterize those integers \( r \) for which the function \( z \rightarrow B^{1+r}(\xi, z) \) belongs to \( L^1(D) \), \( \xi \in D \); also enclosed in this chapter are some results that will be useful later.

In the second chapter, we prove that when \( r \) is sufficiently large, the dual of \( A^1 \) coincides with the Bloch space \( \mathcal{B}_r \). More precisely, let \( L \) be a bounded linear functional on \( A^1(D) \); by the Hahn-Banach theorem, there exists a bounded function \( b \) in \( D \) such that for any \( f \) in \( A^1(D) \), the following equality holds: \( L(f) = \int_D b f \, dv \).

We associate with \( b \) the holomorphic function \( G \) defined in \( D \) by

\[
G(\xi) = C_r \int_D B^{1+r}(\xi, z) b(z) \, dv(z);
\]

this function \( G \) possesses the following two properties:

1° \( \sup_{\xi \in D} \{|G(\xi)|B^{-r}(\xi, \xi)\} < +\infty \);

2° \( L(f) = \int_D G(\xi) \tilde{f}(\xi) B^{-r}(\xi, \xi) \, dv(\xi) \)

for any \( f \) in \( A^1(D) \).

Since it is well known (cf. [7]) that the differential equation \( \mathcal{D}_r \tilde{g} = G \) in \( D \) possesses holomorphic solutions, we then conclude that \( L \) can be represented by the element \( g \) of \( \mathcal{B}_r \), consisting of all the holomorphic solutions to this equation.

In the third chapter, we denote by \( P \) the operator that assigns to a bounded function \( b \) the above-defined element \( g \) of the Bloch space \( \mathcal{B}_r \), and we call it "Bergman projection" of \( L^\infty \) into \( \mathcal{B}_r \), for the following reason: although \( P \) is not the integral operator \( \mathcal{P} \) associated with the Bergman kernel \( B(\xi, z) \) of \( D \), which has no meaning on \( L^\infty \), we shall prove that for any function \( b \) in \( (L^2 \cap L^\infty)(D) \), the element \( Pb \) of \( \mathcal{B}_r \), can be represented by the holomorphic function \( \mathcal{P}b \). Moreover, we prove that \( P \) defines a bounded operator from \( L^\infty \) onto \( \mathcal{B}_r \), and, consequently, the dual of \( A^1(D) \) (which coincides with the Bloch space \( \mathcal{B}_r \)) can be realized as the Bergman projection of \( L^\infty \).

Finally, as usual, the same letter \( C \) will denote constants that may be different from each other.

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I. Preliminary results about the Bergman kernel

1. The canonical decomposition of the domain. Let \( V \) denote an affine-homogeneous cone in \( \mathbb{R}^n \), which contains no straight line; we first recall the canonical decomposition of \( V \) as settled in [5].
Notations. (i) At the jth step, \( j = 1, 2, \ldots \), the real line \( \mathbb{R} \) will be denoted by \( \mathbb{R}^l_j \), and at the kth step, \( k = 2, 3, \ldots \), \( \mathbb{R}^n_k \) will denote the \( n_k \)-dimensional euclidean space \( \mathbb{R}^n_k \).

(ii) Let \( P \) denote the real Siegel domain

\[
P = \{ (y, t) \in \mathbb{R}^n \times \mathbb{R}^l : y - \varphi(t, t) \in \Gamma \}
\]

where \( \Gamma \subset \mathbb{R}^n \) is an affine-homogeneous cone which contains no straight line and \( \varphi \) is a \( \Gamma \)-bilinear symmetric form defined in \( \mathbb{R}^n \); we shall denote by \( V(P) \) the cone

\[
V(P) = \{ (y, t, r) \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R} : r > 0 \text{ and } (yr, t) \in P \}.
\]

In order to obtain the canonical decomposition of the cone \( V \), we consider at the first step the cone \( V^{(1)} = (0, \infty) \subset \mathbb{R}^l_1 \).

At the second step, we associate with the cone \( V^{(1)} \) and with a \( V^{(1)} \)-bilinear symmetric form \( F^{(2)} \) defined in \( \mathbb{R}^n_2 \) a real Siegel domain \( P^{(2)} \) contained in \( \mathbb{R}^l_1 \times \mathbb{R}^n_2 \) and the cone

\[
V^{(2)} = V(P^{(2)}) \subset \mathbb{R}^l_1 \times \mathbb{R}^n_2 \times \mathbb{R}^l_2.
\]

At the third step, we associate with the cone \( V^{(2)} \) and with a \( V^{(2)} \)-bilinear symmetric form \( F^{(3)} \) defined in \( \mathbb{R}^n_3 \) a real Siegel domain \( P^{(3)} \) contained in \( \mathbb{R}^l_1 \times \mathbb{R}^n_2 \times \mathbb{R}^l_2 \times \mathbb{R}^n_3 \times \mathbb{R}^l_3 \) and the cone

\[
V^{(3)} = V(P^{(3)}) \subset \mathbb{R}^l_1 \times \mathbb{R}^n_2 \times \mathbb{R}^l_2 \times \mathbb{R}^n_3 \times \mathbb{R}^l_3.
\]

And so on, at the \( i \)th step, we associate with the cone \( V^{(i-1)} \) and with a \( V^{(i-1)} \)-bilinear symmetric form \( F^{(i)} \) defined in \( \mathbb{R}^n_i \) a real Siegel domain \( P^{(i)} \) and the cone

\[
V^{(i)} = V(P^{(i)}) \subset \mathbb{R}^l_1 \times \cdots \times \mathbb{R}^n_i \times \cdots \times \mathbb{R}^n_i.
\]

It follows from results in [5] that every affine-homogeneous cone \( V \), which contains no straight line, can be decomposed in the form \( V = V^{(i)} \) (up to an affine isomorphism). The required number of steps to obtain \( V \) in this form is called the rank \( l \) of the cone \( V \) (\( V = V^{(i)} \)). This yields the following decomposition of the space \( \mathbb{R}^n \) containing \( V \):

\[
\mathbb{R}^n = \mathbb{R}^l_1 \times \cdots \times \mathbb{R}^l_i \times \mathbb{R}^n_2 \times \cdots \times \mathbb{R}^n_i, \quad n = l + \sum_{j=2}^{i} n_j.
\]

Now, let \( F_{ij}^{(j)} \), \( 1 \leq i < j \leq l \), denote the projection of the \( V^{(j-1)} \)-bilinear symmetric form \( F^{(j)} \) on the real line \( \mathbb{R}^l_i \), and let \( \mathbb{R}^n_{ij} \) denote the \( n_{ij} \)-dimensional subspace of \( \mathbb{R}^n_j \) where the form \( F_{ij}^{(j)} \) is positive definite; then the decomposition \( \mathbb{R}^n_j = \prod_{j=1}^{l} \mathbb{R}^n_{ij} \) holds.

We recall next that the cone \( V \) is self-conjugate if and only if the integers \( n_{ij} \), \( 1 \leq i < j \leq l \), are equal among themselves. In that case, when \( x_{ij} \) denotes the projection of \( x \in \mathbb{R}^n \) on \( \mathbb{R}^n_{ij} \), the cone \( V \) is self-conjugate with respect to the scalar product

\[
\langle x, x' \rangle = \sum_{j=1}^{l} \frac{(x_{ij}, x'_{jj})}{2} + \sum_{1 \leq i < j \leq l} x_{ij}x'_{ij}, \quad x, x' \in \mathbb{R}^n.
\]
Thus, in order that the cone \( V \subset \mathbb{R}^n \) of rank \( l \) be self-conjugate, we shall assume that \( n - l \) is a multiple in \( \mathbb{N} \) of \( (l(l - 1))/2 \) and that all the \( n_{ij} \)'s, \( 1 \leq i < j \leq l \), are equal to \( p = 2(n - l)/(l - 1) \).

We shall also assume that the vectors in the subspaces \( \mathbb{R}^{p}_{ij} \) are represented by their coordinates with respect to bases satisfying the following property: for any \( \lambda_j \) in \( \mathbb{R}^p \), one has the equality \( F^{(ij)}(\lambda_j, \lambda_j) = ||\lambda_j||^2 \), where \( || \cdot || \) denotes the euclidean norm in \( \mathbb{R}^p \).

Furthermore, let \( F^{(ij)}_k \), \( 1 \leq i < j < k \leq l \), denote the projection of the \( V^{(k-1)} \)-bilinear symmetric form \( F^{(k)} \) on \( \mathbb{R}^{p}_{ij} \); the homogeneity assumption on the cone \( V \) implies that the form \( F^{(k)}_{ij} \) is concentrated on \( \mathbb{R}^{p}_{ik} \times \mathbb{R}^{p}_{jk} \).

Now, let \( D \subset \mathbb{C}^n \times \mathbb{C}^N \) denote a symmetric Siegel domain of type II, associated to the cone \( V \) and to the \( V \)-Hermitian form \( F \). We next describe the canonical decomposition of the space \( \mathbb{C}^n \times \mathbb{C}^N \) containing \( D \).

First, the decomposition

\[
\mathbb{R}^n = \prod_{j=1}^{l} \mathbb{R}^{p}_{ij} \times \prod_{1 \leq j < k \leq l} \mathbb{R}^{p}_{jk}
\]

given above yields in a natural way the decomposition

\[
\mathbb{C}^n = \prod_{j=1}^{l} \mathbb{C}^{q}_{ij} \times \prod_{1 \leq j < k \leq l} \mathbb{C}^{q}_{jk}
\]

Secondly, let \( F^{(i)}_{jj} \) denote the projection of the form \( F \) on the complex plane \( \mathbb{C}^{q}_{ij} \), and let \( \mathbb{C}^{q}_{jj} \) denote the \( q \)-dimensional subspace of \( \mathbb{C}^{N} \) where the form \( F_{ij} \) is positive definite; the decomposition \( \mathbb{C}^{N} = \prod_{i=1}^{l} \mathbb{C}^{q}_{ij} \) holds, and the canonical decomposition of the space \( \mathbb{C}^n \times \mathbb{C}^N \) containing \( D \) is given by

\[
\mathbb{C}^n \times \mathbb{C}^N = \prod_{j=1}^{l} \mathbb{C}^{q}_{ij} \times \prod_{1 \leq j < k \leq l} \mathbb{C}^{q}_{jk} \times \prod_{j=1}^{l} \mathbb{C}^{q}_{jj}
\]

Moreover, with respect to its canonical form, the cone \( V \) can be described in the following quantitative manner. Let \( \lambda \) be a point of \( V \) and let \( \lambda_j \), \( j = 2, \ldots, l \), denote the projection of \( \lambda \) on \( \mathbb{R}^p \); there exists an automorphism (in the transitive group of \( V \) given in [5]) that assigns to \( \lambda \) a point \( \lambda_j \) of \( V \) satisfying \( \lambda_j = 0 \), for every \( j = 2, \ldots, l \). Such an automorphism can be built in \( l - 1 \) stages: a first automorphism maps \( \lambda \) to \( \lambda^{(1)} \in V \) satisfying \( \lambda^{(1)} = 0 \), a second one takes \( \lambda^{(1)} \) to \( \lambda^{(2)} \in V \) satisfying \( \lambda^{(2)}_j = 0 \), and finally, the \( (l - 1) \)th automorphism assigns to \( \lambda^{(l-2)} \) satisfying \( \lambda^{(l-2)}_j = 0 \) the point \( \lambda^{(l-1)} = \lambda \). More explicitly, the points \( \lambda^{(m)} \), \( m = 0, 1, \ldots, l - 1 \), of \( V \) are given by the following recurrence formulas (formulas (1.26), (1.27) in [5]):

\[
\lambda^{(m)}_i = \lambda^{(m)}_{i-1} - \frac{\sum_{j=1}^{m} \lambda^{(m)}_{i-j} \lambda^{(m)}_{i-j}}{\sum_{j=1}^{m} \lambda^{(m)}_{i-j} \lambda^{(m)}_{i-j}}, \quad 1 \leq i < j < l - m.
\]

Now, set \( \chi_j(\lambda) = \lambda^{(j-1)}_j \); the cone \( V \) is defined by the \( l \) inequalities \( \chi_j(\lambda) > 0 \), \( j = 1, 2, \ldots, l \) (formula (1.25) in [5]).
Furthermore, let us recall that the functions \( \chi_j(\lambda) \) can also be obtained in the following way. The transitive group \( G(V) \) of the cone \( V \), defined in [5], is represented by \( l \times l \) triangular matrices, and the automorphism \( T \in G(V) \) that assigns \( \lambda \) to \( \lambda \) is a triangular matrix of \( G(V) \) whose diagonal elements are all equal to 1. On the other hand, let \( e \) denote the point of \( V \subset \prod_{j=1}^{l} \mathbb{R}_{+}^{1} \times \prod_{j=2}^{l} \mathbb{R}_{+}^{m} \) whose coordinates are \( e_{jj} = 1 \) and \( e_{k} = 0 \) for \( j = 1, \ldots, l \) and \( k = 2, \ldots, l \); then for any \( \lambda \) in \( V \), there exists in \( G(V) \) a unique matrix \( g \) such that \( g(e) = \lambda \) and \( g^{-1} \) is the product of \( T \) by a diagonal matrix. Hence, the matrix \( g \) can be written as the product \( g = td \) of a diagonal matrix \( d \) of \( G(V) \) by a triangular matrix \( t \) of \( G(V) \) whose diagonal elements are all equal to 1; the point \( \hat{\lambda} \) and the functions \( \chi_j(\hat{\lambda}) \) are then given by the formulas \( \hat{\lambda} = d(e) \) and \( \chi_j(\hat{\lambda}) = d_j \), where \( d_j \) is the \( j \)th diagonal element of \( d \).

Finally, let us recall that the self-conjugate cone \( V \) can also be defined by the \( l \) functions \( \chi_j^*(\lambda) \) that generally define the dual cone \( V^* \) of \( V \). Those functions \( \chi_j^*(\lambda) \) are given as follows.

Let \( G^*(V) \) denote the adjoint group of \( G(V) \) with respect to the scalar product \( \langle , \rangle \); the group \( G^*(V) \) acts transitively on \( V^* = \overline{V} \) and for any \( \lambda \) in \( V^* = \overline{V} \), there exists a unique automorphism \( g^* \) in \( G^*(V) \) satisfying \( g^*(e) = \lambda \). The group \( G^*(V) \) can also be represented by triangular matrices, and the matrix \( g^* \) can be uniquely written in \( G^*(V) \) as the product \( g^* = t^*d^* \) of a diagonal matrix \( d^* \) by a triangular matrix \( t^* \) whose diagonal elements are all equal to 1. The functions \( \chi_j^*(\lambda) \) are then given by \( \chi_j^*(\lambda) = d_j^* \), where \( d_j^* \) is the \( j \)th diagonal element of \( d^* \).

We shall carry the following notations: for any vector \( \rho = (\rho_1, \ldots, \rho_l) \) in \( R^l \), we set
\[
\lambda^\rho = \prod_{j=1}^{l} (\chi_j(\lambda))^\rho_j \quad \text{and} \quad (\lambda^*)^\rho = \prod_{j=1}^{l} (\chi_j^*(\lambda))^\rho_j.
\]

2. The Bergman kernel. Let \( n \) and \( l \) be two integers satisfying \( 1 \leq l \leq n, n > 1 \), and such that \( l(l - 1)/2 \) divides \( n - l \) in \( N \). By \( V \) we shall denote an affine-homogeneous, self-conjugate cone of rank \( l \) in \( \mathbb{R}^n \), and we shall assume that the cone \( V \) is irreducible: this implies that \( 1 < l < n \).

We therefore exclude the case \( n = l = 1 \), where the cone \( V \) is the positive real half-line and its associated Siegel domains, that are Cayley transforms of balls, were studied in [1].

Let \( D \) denote a symmetric Siegel domain of type II contained in \( \mathbb{C}^n \times \mathbb{C}^N \), associated to the cone \( V \) and to the \( V \)-Hermitian form \( F \) defined in \( \mathbb{C}^N \); the space \( \mathbb{C}^n \times \mathbb{C}^N \) containing \( D \) will be considered in its canonical form
\[
\mathbb{C}^n = \prod_{j=1}^{l} \mathbb{C}_{jj}^{1} \times \prod_{k=2}^{l} \mathbb{C}_{kk}^{n_k},
\]
\[
\mathbb{C}_{kk}^{n_k} = \prod_{j<k} \mathbb{C}_{jk}^{q_{jk}}, \quad p = \frac{2(n-l)}{l(l-1)},
\]
\[
\mathbb{C}^N = \prod_{j=1}^{l} \mathbb{C}_{jj}^{q_j}.
\]
We shall denote by \( q \) the vector of \( \mathbb{N}^l \) whose coordinates are \( q_j, \ j = 1, \ldots, l \), and we shall denote by \( d \) and \( m \) the vectors of \( \mathbb{R}^l \) whose respective coordinates are \( d_j = -(1 + p(l - 1)/2), \ m_j = p(j - 1) \).

It then follows from results in [5] that the Bergman kernel \( B(\xi, z) \) of \( D \) has the following two expressions:

**Proposition 1.2.1.** The Bergman kernel \( B(\xi, z) \) of \( D \) is given by the formulas

\[
B(\xi, z) = c\left( \frac{x - x}{2l} + \frac{\eta + y}{2} - F(\xi, z_2) \right)^{2l - q}
\]

where \( \xi = (\xi + i\eta, \xi_2) \) and \( z = (x + iy, z_2) \) are two points of \( D \subset \mathbb{C}^n \times \mathbb{C}^N \).

Subsequently, we shall carry the following notations:

(i) let \( \rho = (\rho_1, \ldots, \rho_l) \) and \( \rho' = (\rho'_1, \ldots, \rho'_l) \) be two vectors in \( \mathbb{R}^l \); we shall write \( \rho > \rho' \) whenever \( \rho_j > \rho'_j \) for every \( j = 1, \ldots, l \); we shall also set

\[
\rho \rho' = (\rho_1 \rho'_1, \ldots, \rho_l \rho'_l) \quad \text{and} \quad \rho/\rho' = (\rho_1/\rho'_1, \ldots, \rho_l/\rho'_l),
\]

when none of the \( \rho'_j \)'s, \( j = 1, \ldots, l \), is zero;

(ii) for any two points \( \xi = (\xi + i\eta, \xi_2) \) and \( z = (x + iy, z_2) \) in \( D \subset \mathbb{C}^n \times \mathbb{C}^N \), we let \( B^\rho(\xi, z) \) and \( B^{1+\rho}(\xi, z) \) denote the expressions

\[
B^\rho(\xi, z) = c^\rho \left( y - F(z_2, z_2) \right)^{2l - q}.
\]

We will use the following lemma, due to S. G. Gindikin [5]:

**Lemma 1.2.1.** A holomorphic function \( G \) in \( D \) belongs to \( A^2(D) \) if and only if it can be expressed in the form

\[
G(z) = \int_V g(\lambda, z_2) \exp(i\langle \lambda, z_1 \rangle) \, d\lambda,
\]

for any \( z = (z_1, z_2) \) in \( D \subset \mathbb{C}^n \times \mathbb{C}^N \), where the function \( g(\lambda, z_2) \) satisfies the following properties:

(i) the function \( z_2 \mapsto g(\lambda, z_2) \) is entire in \( \mathbb{C}^N \);

(ii) the integral

\[
\int_V \int_{\mathbb{C}^N} |g(\lambda, z_2)|^2 \exp(-2\langle \lambda, F(z_2, z_2) \rangle)(\lambda^*)^d \, dv(z_2) \, d\lambda
\]

converges and is equal to \( c\int_D |G(z)|^2 \, dv(z) \).

We state another lemma of Gindikin [5]:

**Lemma 1.2.2.** For any vector \( \rho \) in \( \mathbb{R}^l \) such that \( \rho > m/2 \) and for any \( z \) in \( \mathbb{C}^n \) such that \( \Re z \in V \), we have

\[
\int_V \exp(-\langle z, \lambda \rangle)(\lambda^*)^\rho \, d\lambda = c_\rho(z)^{-\rho}.
\]
Our next purpose is to prove the following estimate for the Bergman kernel:

**Lemma 1.2.3.** For any $\xi$ in $D$, the kernel $B^{1+\alpha}(\xi, z)$, $\alpha \in \mathbb{R}$, belongs to $L^1(dv(z))$ if and only if $(-2d + q)\alpha > m/2$; in that case,

$$\int_D |B^{1+\alpha}(\xi, z)| dv(z) = c_\alpha B^\alpha(\xi, \xi).$$

**Proof.** Because of the homogeneity of the domain $D$, it suffices to prove the lemma for $f = (ie, 0) \in D \subset C^n \times C^N$, where $e$ is the point of $V \subset \prod_{j=1}^r \mathbb{R}^j \times \prod_{j < k < l} \mathbb{R}^j \mathbb{R}^k$ whose components are $e_{jj} = 1$ for $j = 1, \ldots, l$ and $e_{jk} = 0$ when $j < k$.

Take then $f = (ie, 0)$. The assumption $(-2d + q)\alpha > m/2$ yields the inequality $-\frac{2d - q}{2} > m/2$; now, in view of Proposition 1.2.1 and Lemma 1.2.2, $B^{(1+\alpha)/2}(\xi, z)$ can be written.

$$B^{(1+\alpha)/2}(\xi, z) = c_\alpha \left( -\frac{x}{2i} + \frac{e + y}{2} \right)^{(2d-q+(2d-q)\alpha)/2}$$

$$= c_\alpha \int_V \exp \left( -\left( \lambda, -\frac{x}{2i} + \frac{e + y}{2} \right) \right) (\lambda^*)^{q+(-2d+q)\alpha/2} d\lambda.$$

Next, the Plancherel formula given in Lemma 1.2.1 yields

$$\int_D |B^{1+\alpha}(\xi, z)| dv(z) = c_\alpha \int_V \exp(-\langle \lambda, e \rangle) (\lambda^*)^{d+q-(-2d+q)\alpha}$$

$$\cdot \left( \int_{C^n} \exp(-2\langle \lambda, F(z_2, z_2) \rangle) dv(z_2) \right) d\lambda.$$

Let us show that the right-hand side of this last equality converges if $(-2d + q)\alpha > m/2$. We first integrate with respect to $z_2$; the following lemma is proved in [5]:

**Lemma 1.2.4.** The integral

$$\int_{C^n} \exp(-\langle e, F(u, u) \rangle) dv(u)$$

is convergent.

In view of lemma 1.2.4 and of the homogeneity of the cone $V$, we get

$$\int_{C^n} \exp(-2\langle \lambda, F(z_2, z_2) \rangle) dv(z_2) = c(\lambda^*)^{-q};$$

hence,

$$\int_D |B^{1+\alpha}((ie, 0), z)| dv(z) = c_\alpha \int_V \exp(-\langle \lambda, e \rangle) (\lambda^*)^{d+(-2d+q)\alpha} d\lambda.$$

It then follows from Lemma 1.2.2 that $B^{1+\alpha}((ie, 0), z)$ belongs to $L^1(dv(z))$ when $(-2d + q)\alpha > m/2$.

Conversely, S. G. Gindikin [5] has proved that the right-hand side of (3) converges only if this condition on $\alpha$ is satisfied. The proof of the lemma is complete.
Remark. The necessary and sufficient condition on $a$ given in Lemma 1.2.3 contradicts a statement of R. Coifman and R. Rochberg (Lemma 2.2 of [4]). Even in the particular case of the tube over the spherical cone of $\mathbb{R}^n$, $n > 3$ (cone of rank two), the condition $a > 0$ given by those authors was proved before to be insufficient (cf. [2 and 3]).

We shall also use Lemma 2.3 of [4]:

**Lemma 1.2.5.** $d$ denotes the Bergman distance in $D$. There exists a constant $C_D$ such that for any two points $\xi$ and $\xi_0$ in $D$ satisfying $d(\xi, \xi_0) < 1$ and for any $z$ in $D$, the following inequality holds:

$$|B(\xi, z)/B(\xi_0, z) - 1| \leq C_D d(\xi, \xi_0).$$

**II. The dual of $A^1$ and Bloch functions**

In the following, $\rho$ denotes a vector of $\mathbb{N}^l$ whose coordinates $\rho_j$, $j = 1, \ldots, l$, satisfy two conditions:

(i) $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_{l-1} \geq \rho_l = 1$;

(ii) $(\lambda^*)^\rho = \prod_{j=1}^l \lambda_j^{\rho_j}$ is a polynomial of $\lambda \in \mathbb{R}^n$;

according to the terminology in [5], a vector $\rho$ satisfying property (ii) is said to be $V$-integral, and for any cone $V$ of rank $l$ an example of a $V$-integral vector is $\rho = (2^{-2}, 2^{-3}, \ldots, 2, 1, 1)$.

We shall denote by $\mathcal{D}$ the differential polynomial in $\mathbb{C}^n$ that possesses the property

$$\mathcal{D}_{z_1} \exp(\langle \lambda, z_1 \rangle) = (\lambda^*)^\rho \exp(\langle \lambda, z_1 \rangle), \quad z_1 \in \mathbb{C}^n.$$

Referring to [5], $\mathcal{D}$ is a Riemann-Liouville differential operator of $D \subset \mathbb{C}^n \times \mathbb{C}^n$.

Let $r$ be a vector of $\mathbb{N}^l$ satisfying the following two conditions (the notations are those introduced in section 1.2):

(iii) $(\lambda^*)^r$ is a polynomial of $\lambda \in \mathbb{R}^n$; i.e., $r \rho$ is also a $V$-integral vector;

(iv) $r \rho > m/2$.

Let $\mathcal{D} = \mathcal{D}^r$ denote the differential polynomial in $\mathbb{C}^n$ that possesses the following property:

$$\mathcal{D}_{z_1} \exp(\langle \lambda, z_1 \rangle) = (\lambda^*)^r \exp(\langle \lambda, z_1 \rangle), \quad z_1 \in \mathbb{C}^n.$$

In other words, $\mathcal{D}$ is the Riemann-Liouville differential operator of $D$ obtained by iterating $\mathcal{D}$ $r$ times.

Now, fix two such vectors $\rho$ and $r$; the corresponding Bloch space $\mathcal{B} = \mathcal{B}_{\rho, r}$ of $D$ is defined as follows. A holomorphic function $g$ in $D$ is said to be a Bloch function if it satisfies

$$\|g\|_* = \sup_{(z_1, z_2) \in D \subset \mathbb{C}^n \times \mathbb{C}^n} \left\{ |\mathcal{D}_{z_1} g(z) |B^{-r \rho/(-2d+\delta)}(z, z) \right\} < +\infty.$$

Let $\mathcal{N} = \{ g \in H(D): \mathcal{D}_{z_1} g(z) = 0 \}$; the Bloch space $\mathcal{B} = \mathcal{B}_{\rho, r}$ of $D$ is the quotient space $\{\text{Bloch functions}\}/\mathcal{N}$.

The Bloch space $\mathcal{B} = \mathcal{B}_{\rho, r}$ possesses the following property:

**Proposition II.1.** Under the quotient norm induced by $\|\|_*$, the Bloch space $\mathcal{B} = \mathcal{B}_{\rho, r}$ of $D$ is a Banach space.
The proof of this proposition essentially relies on the following lemma [7, p. 477]:

**Lemma II.1.** For any holomorphic function \( h(z_1, z_2) \) in \( D \subset C^n \subset C^N \), there exists a holomorphic function \( f(z_1, z_2) \) such that \( \partial_z f = h \).

We next prove the following theorem:

**Theorem II.1.** For any vectors \( \rho \) and \( r \) in \( N^I \) satisfying conditions (i)–(iv), the dual of the Bergman space \( A^1(D) \) coincides with the Bloch space \( B = B_{\rho, r} \) of \( D \).

**Proof.** It is easy to prove that the Bloch space \( B_{\rho, r} \) of \( D \) coincides with a subspace of the dual of \( A^1(D) \). This is done as follows: any element \( g \) of \( B_{\rho, r} \) defines an element \( L \) of the dual of \( A^1(D) \) by

\[
L(f) = \int_D \partial_z g(z) \bar{f}(z) B^{-r_\rho/(-2d+q)}(z, z) \, dv(z), \quad f \in A^1(D)
\]

and the conclusion follows from the easily proved inequality \( \|L\| \leq \|g\|_* \).

Conversely, let us prove that any element \( L \) of the dual of \( A^1(D) \) can be represented by an element \( g \) of \( B_{\rho, r} \). First, by the Hahn-Banach theorem, there exists a bounded function \( b \) in \( D \) such that for any \( f \) in \( A^1(D) \),

\[
L(f) = \int_D b(z) f(z) B^{r_\rho/(-2d+q)}(z, z) \, dv(z).
\]

Secondly, in view of the condition \( r_\rho > m/2 \) and of Lemma I.2.3, we can assign to this function \( b \) the holomorphic function \( G \) defined in \( D \) by

\[
G(z) = \int_D B^{1 + r_\rho/(-2d+q)}(z, z) b(z) \, dv(z);
\]

\( G \) satisfies the estimate

\[
\sup_{\xi \in D} \{ |G(\xi)| B^{-r_\rho/(-2d+q)}(\xi, \xi) \} \leq C \|b\|_*
\]

and yields an element \( L' \) of the dual of \( A^1(D) \) defined by

\[
L'(f) = C \int_D G(\xi) \bar{f}(\xi) B^{-r_\rho/(-2d+q)}(\xi, \xi) \, dv(\xi).
\]

Now, it follows from the Fubini theorem that

\[
L'(f) = C \int_D \left( \int_D B^{1 + r_\rho/(-2d+q)}(\xi, z) \bar{f}(\xi) B^{-r_\rho/(-2d+q)}(\xi, \xi) \, dv(\xi) \right) b(z) \, dv(z).
\]

We then conclude, in view of the reproducing formula

\[
\int_D B^{1 + r_\rho/(-2d+q)}(\xi, z) \bar{f}(\xi) B^{-r_\rho/(-2d+q)}(\xi, \xi) \, dv(\xi) = C^{-1} \bar{f}(z),
\]

that the linear functionals \( L \) and \( L' \) coincide on \( A^1(D) \).

Finally, the linear functional \( L \) on \( A^1(D) \) can be represented by the element \( g \) of the Bloch space \( B = B_{\rho, r} \) defined as follows: by Lemma II.1 and using the estimate (\( \ast \)) for \( G \), take \( g \) to be the equivalence class of all holomorphic solutions to the equation

\[
\partial_z g(z) = CG(z), \quad z = (z_1, z_2) \in D \subset C^n \times C^N.
\]

Also, \( \|g\|_* \leq C \|b\|_* \). The proof of Theorem II.1 is complete.
Remark. Bloch spaces can be defined exactly in the same way on homogeneous but nonsymmetric Siegel domains of type II associated with a self-conjugate cone. Furthermore, the argument in the proof of Theorem II.1 leads to the same conclusion in such a domain because all results we need from the first chapter are still valid. Concerning the existence of such domains, let us recall that the “historical” example of a homogeneous but nonsymmetric Siegel domain of type II, due to Pyateckii-Shapiro (in his solution to E. Cartan’s problem about the existence of bounded homogeneous but nonsymmetric domains; cf. [8, p. 269]) is associated with the (self-conjugate) spherical cone of $\mathbb{R}^3$.

III. The Bergman projection of $L^\infty$.

In this chapter, $D$ again denotes a symmetric Siegel domain of type II contained in $\mathbb{C}^n \times \mathbb{C}^N$. The Bergman projection $\mathcal{P}$ of $D$ is the orthogonal projection of $L^2(dv)$ onto the subspace $A^2(D)$ of $L^2(dv)$ consisting of holomorphic functions; for any $\varphi$ in $L^2(dv)$, $\mathcal{P}\varphi$ is given by the integral formula

$$ (5) \quad \mathcal{P}\varphi(\xi) = \int_D B(\xi, z) \varphi(z) \, dv(z), \quad \xi \in D, $$

where $B$ denotes the Bergman kernel of $D$, explicitly given in (1) and (2). However, since by Lemma 1.2.3, the kernel $B(\xi, z)$ does not belong to $L^1(dv(z))$, $\xi \in D$, expression (5) does not make sense when $\varphi$ is any bounded function in $D$.

In the following, we fix two vectors $\rho$ and $\tau$ in $\mathbb{N}$, satisfying conditions (i)--(iv) introduced at the beginning of Chapter II and $\mathcal{B} = \mathcal{B}_{\rho, \tau}$ denotes the corresponding Bloch space.

Our next concern is to define an operator $P$ from $L^\infty(D)$ into the Bloch space $\mathcal{B}$ that satisfies the following two properties:

1° The image $Pb$ in $\mathcal{B}$ of a function $\varphi \in (L^\infty \cap L^2)(D)$ can be represented by the function $\mathcal{P}\varphi$ defined in (5): for this reason, we shall also call this new operator $P$ the “Bergman projection”;

2° The “Bergman projection” $P$ is a bounded operator from $L^\infty$ onto $\mathcal{B}$, and, consequently, the dual of the Bergman space $A^1$ (the Bloch space $\mathcal{B}$) coincides with the “Bergman projection” $PL^\infty$ of $L^\infty$.

Let $b$ be a bounded function in $D$. In the proof of Theorem II.1, we assigned to $b$ a holomorphic function $G$ defined in $D$ by

$$ G(\xi) = C \int_D B^{1+\rho_2/2} (\xi, z) b(z) \, dv(z), \quad \xi \in D, $$

and the equivalence class $g$ in the Bloch space $\mathcal{B}$, consisting of all holomorphic solutions $\tilde{g}$ of the differential equation

$$ \mathcal{D}_{\xi} \tilde{g}(\xi) = G(\xi), \quad G = (\xi_1, \xi_2) \in D \subset \mathbb{C}^n \times \mathbb{C}^N. $$

The operator $P$ from $L^\infty$ into $\mathcal{B}$, which assigns $g$ to $b$, will be called the “Bergman projection” in view of the following lemma:

Lemma III.1. For any $\varphi$ in $(L^\infty \cap L^2)(D)$, the element $P\varphi$ of the Bloch space $\mathcal{B}$ can be represented by the function $\mathcal{P}\varphi$ defined in (5).
Proof. This follows immediately from the equality
\[ \mathcal{D}_{\mathcal{A}} B(\xi, z) = CB^{1+r/(-2d+q)}(\xi, z), \quad \xi = (\xi_1, \xi_2) \in D \subset \mathbb{C}^n \times \mathbb{C}^N. \]

It was already shown that $P$ is a bounded operator from $L^\infty$ into $\mathcal{A}$; we now prove that $P$ is onto. More precisely, we show that any element $g$ of $\mathcal{A}$ is the Bergman projection $Pb$ of a bounded function $b$ defined by
\[ b(z) = \mathcal{D}_{\mathcal{A}} g(z) B^{-r/(-2d+q)}(z, z), \quad z = (z_1, z_2) \in D \subset \mathbb{C}^n \times \mathbb{C}^N. \]

This is an immediate consequence of the following proposition:

**Proposition III.1.** For any holomorphic function $G$ in $D$ satisfying the estimate
\[ (*) \quad \sup_{z \in D} \{ |G(z)| B^{-r/(-2d+q)}(z, z) \} < +\infty, \]
the following equality holds:
\[ (7) \quad G(\xi) = C \int_D G(z) B^{-r/(-2d+q)}(z, z) B^{1+r/(-2d+q)}(\xi, z) \, dv(z). \]

Proof. This proposition will be proved in an equivalent form on a bounded realization of $D$. We need the following lemma:

**Lemma III.2.** Every symmetric Siegel domain of type II contained in $\mathbb{C}^M$ is biholomorphic to a bounded domain $\Omega$ which possesses the following property: if $z = (z_1, \ldots, z_M)$ is a point of $\Omega$, then for any complex numbers $\alpha_j$ satisfying $|\alpha_j| \leq 1$, $j = 1, \ldots, M$, $(\alpha_1 z_1, \ldots, \alpha_M z_M)$ is also a point of $\Omega$ ($\Omega$ is said to be a bounded "circular" domain).

Furthermore, let $B_\Omega(\xi, z)$ denote the Bergman kernel of $\Omega$; there exists a nonzero constant $c_\Omega$ such that $B_\Omega(0, z) = C_\Omega$.

The first part of the lemma was proved by A. Koranyi and J. A. Wolff in [6]. The second part is contained in the proof of Lemma 2.3 of [4] (Lemma I.2.5 above); that proof, due to Koranyi, uses all of Lemma III.2.

We can now begin the proof of Proposition III.1. Let $\Phi$ denote a biholomorphic transformation of a bounded circular domain $\Omega$, associated with $D$ by Lemma III.2, onto $D$, which assigns to the origin 0 the point $e = (e_1, e_2)$ of $D \subset \mathbb{C}^n \times \mathbb{C}^N$, whose components are $(e_1)_j = 1$, $(e_1)_{jk} = 0$ when $j < k$ and $e_2 = 0$.

This transformation $\Phi$ yields the following relation between the respective Bergman kernels $B_D$ and $B_\Omega$ of the domains $D$ and $\Omega$:
\[ B_\Omega(\xi', z') = B_D(\Phi(\xi'), \Phi(z')) \left[ J\Phi(\xi') \right] \left[ J\Phi(z') \right], \]
where $J\Phi(w)$ denotes the jacobian of $\Phi$ at the point $w$ of $\Omega$. When one puts $z' = 0$ in this relation, it follows from the second part of Lemma III.2 that $J\Phi(\xi') = CB_D^{-1}(\Phi(\xi'), e)$; furthermore, according to the notations introduced in section I.2, we set
\[ [J\Phi(\xi')]^a = C_a B_D^a(\Phi(\xi'), e) \]
and
\[ B_\Omega^a(\xi', z') = B_D^a(\Phi(\xi'), \Phi(z')) [J\Phi(\xi')] \left[ J\Phi(z') \right]^a. \]
By means of the biholomorphic transformation $\Psi = \Phi^{-1}$, it is then easy to check that Proposition III.1 is equivalent to the following result:

**PROPOSITION III.2.** For any holomorphic function $\tilde{G}$ in $\Omega$ satisfying the estimate

$$\sup_{z' \in \Omega} \left\{ |\tilde{G}(z')| B_{\Omega}^{-\rho p / (-2d + q)}(z', z') |J\Phi(z')|^{1 + \rho p / (-2d + q)} \right\} < +\infty,$$

the following equality holds:

$$\tilde{G}(z') [J\Phi(z')]^{1 + \rho p / (-2d + q)} = C \int_{\Omega} \tilde{G}(z') [J\Phi(z')]^{1 + \rho p / (-2d + q)} B_{\Omega}^{1 + \rho p / (-2d + q)}(\xi, z') B_{\Omega}^{-\rho p / (-2d + q)}(z', z') \, dv(z'), \quad \xi' \in \Omega.$$

**PROOF OF PROPOSITION III.2.** With respect to the measure $B_{\Omega}^{-\rho p / (-2d + q)}(z, z) \, dv(z)$ the Bergman kernel of $D$ is $CB_{\Omega}^{1 + \rho p / (-2d + q)}(\xi, z)$. Carrying ourselves onto $\Omega$ by using the biholomorphic transformation $\Psi = \Phi^{-1}$, we easily deduce that with respect to the measure $B_{\Omega}^{-\rho p / (-2d + q)}(z', z') \, dv(z')$, the Bergman kernel of $\Omega$ is $CB_{\Omega}^{1 + \rho p / (-2d + q)}(\xi', z').$

It now suffices to prove that the function $\tilde{G}(z') [J\Phi(z')]^{1 + \rho p / (-2d + q)}$ is integrable in $D$ with respect to the measure $B_{\Omega}^{-\rho p / (-2d + q)}(z', z') \, dv(z').$

Since we assume that the function $\tilde{G}(z') [J\Phi(z')]^{2 \rho p / (-2d + q)} B_{\Omega}^{-\rho p / (-2d + q)}(z', z')$ is bounded in $\tilde{\Omega}$, it is enough to prove the following lemma:

**LEMMA III.3.** $\int_{\tilde{\Omega}} |J\Phi(z')|^{1 - \rho p / (-2d + q)} \, dv(z') < +\infty.$

**PROOF OF LEMMA III.3.** Put $\Psi = \Phi^{-1}$; the assertion in the lemma is equivalent in $D$ to the estimate

$$\int_{D} |J\Psi(z)|^{1 + \rho p / (-2d + q)} \, dv(z) < +\infty.$$

The desired conclusion then follows from Lemma I.2.3 because $J\Psi(z) = CB_{D}(e, z)$ and $\rho p > m/2$. The proof of Lemma III.3 is complete, and Propositions III.2 and III.1 are thus entirely proved.

**REMARK.** In view of Proposition III.1, under the measure $B^{-\rho p / (-2d + q)}(z, z) \, dv(z)$ the Bergman kernel of $D$ does not only reproduce functions in Bergman spaces $A_{p}$, but also those holomorphic functions in $D$, satisfying the uniform estimate ($\ast$). This property is trivial on bounded domains and seems to be new concerning unbounded domains; however, in this case, unlike the bounded domains, the meaning of the right-hand side of (7) requires the presence of a weight.

We have just proved that the Bergman projection $P$ of $D$ is bounded from $L^{\infty}$ onto the Bloch space $B$. Furthermore, if $R$ denotes the operator defined in $B$ with values in $L^{\infty}$ that assigns to an element $g$ of $B$ the bounded function

$$Rg(z) = B_{z} g(z) B^{-\rho p / (-2d + q)}(z, z), \quad z = (z_{1}, z_{2}) \in D \subset C^{n} \times C^{N},$$

$R$ is a "continuous right inverse" of $P$, i.e., $PR = Id_{B}$.
The following theorem comes as a summary of this chapter:

**Theorem III.1.** Let $D$ be a symmetric Siegel domain of type II.

1° The Bergman projection $P$ of $D$ is a bounded operator from $L^\infty$ onto the Bloch space $\mathcal{B}$; furthermore, $P$ possesses a continuous right inverse $R: \mathcal{B} \to L^\infty$; 

2° Consequently, the dual of the Bergman space $A^1$ (the Bloch space $\mathcal{B}$) can be realized as the Bergman projection of $L^\infty$.

**Remarks.** 1° At the end of the second chapter, we noticed that the dual of $A^1$ still coincides with the Bloch space in any homogeneous, but nonsymmetric Siegel domain of type II, associated with a self-conjugate cone. In such a domain, the Bergman projection can be defined exactly in the same way as an operator from $L^\infty$ into $\mathcal{B}$. The question therefore arises whether Theorem III.1 can be extended to this type of domain; a relevant problem would be to find a substitute for Lemma III.2.

2° Unlike the particular cases of the Cayley transform of the unit ball and of the tube over the spherical cone, respectively studied in [6 and 7], we have not defined here the Bergman projection $P$ of $L^\infty$ as an integral operator associated with a kernel. The problem of determining a defining kernel for $P$ will be considered in a forthcoming paper.

**References**


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