

## WEIGHTED INEQUALITIES FOR THE ONE-SIDED HARDY-LITTLEWOOD MAXIMAL FUNCTIONS

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**ABSTRACT.** Let  $M^+f(x) = \sup_{h>0} (1/h) \int_x^{x+h} |f(t)| dt$  denote the one-sided maximal function of Hardy and Littlewood. For  $w(x) \geq 0$  on  $R$  and  $1 < p < \infty$ , we show that  $M^+$  is bounded on  $L^p(w)$  if and only if  $w$  satisfies the one-sided  $A_p$  condition:

$$(A_p^+) \quad \left[ \frac{1}{h} \int_{a-h}^a w(x) dx \right] \left[ \frac{1}{h} \int_a^{a+h} w(x)^{-1/(p-1)} dx \right]^{p-1} \leq C$$

for all real  $a$  and positive  $h$ . If in addition  $v(x) \geq 0$  and  $\sigma = v^{-1/(p-1)}$ , then  $M^+$  is bounded from  $L^p(v)$  to  $L^p(w)$  if and only if

$$\int_I [M^+(x_I \sigma)]^p w \leq C \int_I \sigma < \infty$$

for all intervals  $I = (a, b)$  such that  $\int_{-\infty}^a w > 0$ . The corresponding weak type inequality is also characterized. Further properties of  $A_p^+$  weights, such as  $A_{p-\epsilon}^+ \Rightarrow A_p^+$  and  $A_p^+ = (A_1^+)(A_1^-)^{1-p}$ , are established.

**1. Introduction.** For  $f$  locally integrable on the real line  $R$ , define the maximal function  $Mf$  at  $x$  by

$$Mf(x) = \sup_{a < x < b} \frac{1}{b-a} \int_a^b |f(t)| dt.$$

In [9], B. Muckenhoupt characterized, for  $1 < p < \infty$ , the nonnegative functions, or weights,  $w(x)$  on  $R$  satisfying the weighted norm inequality

$$(N_p) \quad \int_{-\infty}^{\infty} [Mf(x)]^p w(x) dx \leq C \int_{-\infty}^{\infty} |f(x)|^p w(x) dx, \quad \text{for all } f,$$

as those weights  $w$  satisfying the  $A_p$  condition

$$(A_p) \quad \left[ \frac{1}{h} \int_a^{a+h} w(x) dx \right] \left[ \frac{1}{h} \int_a^{a+h} w(x)^{-1/(p-1)} dx \right]^{p-1} \leq C', \quad a \text{ in } R, h > 0.$$

This leaves open, however, the characterization of the corresponding norm inequalities for the original maximal function of Hardy and Littlewood [5],

$$M^-f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^x |f(t)| dt,$$

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and its counterpart

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(t)| dt.$$

( $M^+$  arises in the ergodic maximal function discussed below.) Denote by  $(N_p^+)$  and  $(N_p^-)$  the weighted inequalities corresponding to  $(N_p)$  with  $M$  replaced by  $M^+$  and  $M^-$  respectively. Observe that  $A_p$  is not a necessary condition for either  $N_p^+$  or  $N_p^-$ . In fact, the product of any nondecreasing function with an  $A_p$  weight satisfies  $N_p^+$ . More generally, if  $\{w_\alpha(x)\}_{\alpha \in R}$  is a family of weights uniformly in  $A_p$ , and if  $\mu$  is a positive measure on  $R$ , then  $w(x) = \int \chi_{(\alpha, \infty)}(x) w_\alpha(x) d\mu(\alpha)$  satisfies  $N_p^+$ .

**THEOREM 1.** *Suppose  $w(x) \geq 0$  on  $R$  and  $1 < p < \infty$ . Then the weighted inequality  $(N_p^+)$  holds if and only if  $w$  satisfies the one-sided  $A_p$  condition  $(A_p^+)$*

$$\left[ \frac{1}{h} \int_{a-h}^a w(x) dx \right] \left[ \frac{1}{h} \int_a^{a+h} w(x)^{-1/(p-1)} dx \right]^{p-1} \leq C' \quad \text{for all real } a, \text{ and } h > 0.$$

Similarly,  $(N_p^-)$  holds if and only if

$(A_p^-)$

$$\left[ \frac{1}{h} \int_a^{a+h} w(x) dx \right] \left[ \frac{1}{h} \int_{a-h}^a w(x)^{-1/(p-1)} dx \right]^{p-1} \leq C' \quad \text{for all real } a, \text{ and } h > 0.$$

**REMARKS.** (A) The following ‘‘duality’’ relationship holds:  $w$  satisfies  $A_p^+$  if and only if  $w^{1-p'}$  satisfies  $A_{p'}^-$  where  $1/p + 1/p' = 1$ .

(B) If  $M^- w_1 \leq C w_1$  and  $M^+ w_2 \leq C w_2$ , it is trivially verified that  $w = w_1(w_2)^{1-p}$  satisfies  $A_p^+$ . Theorem 1 together with Remark A and the argument of Coifman, Jones and Rubio in [4] yields the converse: If  $w$  satisfies  $A_p^+$ , then there are  $w_1, w_2$  such that  $M^- w_1 \leq C w_1$ ,  $M^+ w_2 \leq C w_2$  and  $w = w_1(w_2)^{1-p}$ . In future we say that  $w$  satisfies the  $A_1^+(A_1^-)$  condition if  $M^- w(M^+ w) \leq C w$ .

(C) A modification of B. Jawerth’s proof of the reverse Hölder inequality for  $A_1$  weights [7] shows that if  $w$  satisfies  $A_1^+$ , then  $w^{1+\delta}$  also satisfies  $A_1^+$  for some  $\delta > 0$ . (Note however that  $w$  cannot in general satisfy a reverse Hölder inequality.) Combining this with the factorization of  $A_p^+$  weights discussed in the previous remark, we obtain the implication  $A_p^+ \Rightarrow A_{p-\epsilon}^+$ : More precisely, if  $w$  satisfies  $A_p^+$  for a given  $p, 1 < p < \infty$ , then  $w$  satisfies  $A_{p-\epsilon}^+$  for some  $\epsilon > 0$ . Details can be found in §3.

(D) Suppose  $T$  is a measure preserving (not necessarily invertible) ergodic transformation on a probability space  $(\Omega, \mathcal{M}, \mu)$ . Let

$$f^*(x) = \sup_{k \geq 0} \frac{1}{k+1} \sum_{j=0}^k |f(T^j x)|$$

denote the ergodic maximal function of  $f$ . It can be shown (see Atencia and De La Torre [3]) that

$$\int_{\Omega} |f^*(x)|^p w(x) d\mu(x) \leq C \int_{\Omega} |f(x)|^p w(x) d\mu(x)$$

for all  $f$  if and only if for  $\mu$ -almost every  $x$  in  $\Omega$ ,

$$(1.1) \quad \sum_{n=0}^{\infty} \left[ \sup_{k \geq 0} \frac{1}{k+1} \sum_{j=0}^k |f(T^{n+j}x)| \right]^p w(T^n x) \leq C \sum_{n=0}^{\infty} |f(T^n x)|^p w(T^n x)$$

for all  $f$  defined on  $\{T^n x\}_{n=0}^{\infty}$ . (The ergodic property is needed only for the necessity of (1.1).) A discrete analogue of Theorem 1 together with elementary measure theory shows that (1.1) holds if and only if there is  $C'$  such that for  $\mu$ -almost every  $x$  in  $\Omega$  and every  $k \geq 0$

$$\left[ \frac{1}{k+1} \sum_{j=0}^k w(T^j x) \right] \left[ \frac{1}{k+1} \sum_{j=k}^{2k} w(T^j x)^{-1/(p-1)} \right]^{p-1} \leq C'.$$

A similar characterization for the two-sided ergodic maximal function corresponding to an invertible measure preserving ergodic transformation  $T$  was obtained in [3].

We turn now to the two-weight norm inequality for  $M^+$ :

$$(1.2) \quad \int_{-\infty}^{\infty} |M^+ f|^p w \leq C \int_{-\infty}^{\infty} |f|^p v \quad \text{for all } f.$$

It is convenient to set  $\sigma = v^{1-p'} = v^{-1/(p-1)}$  and replace  $f$  with  $f\sigma$  in (1.2). The resulting equivalent inequality reads

$$(1.3) \quad \int_{-\infty}^{\infty} |M^+(f\sigma)|^p w \leq C \int_{-\infty}^{\infty} |f|^p \sigma \quad \text{for all } f.$$

The corresponding weak type inequality for  $M^+$  is

$$(1.4) \quad |\{M^+(f\sigma) > \lambda\}|_w \leq \frac{C}{\lambda^p} \int_{-\infty}^{\infty} |f|^p \sigma \quad \text{for all } f,$$

where the notation  $|E|_w$  stands for  $\int_E w(x) dx$ . In analogy with results in [11 and 9], we have

**THEOREM 2.** *Suppose  $w(x), \sigma(x)$  are nonnegative measurable functions on  $R$  and  $1 < p < \infty$ . Then the strong type inequality, (1.3), holds if and only if*

$$(1.5) \quad \int_I |M^+(\chi_I \sigma)|^p w \leq B \int_I \sigma < \infty$$

for all intervals  $I = (a, b)$  such that  $\int_{-\infty}^a w > 0$ . If  $C$  and  $B$  are the best constants in (1.3) and (1.5) then their ratio is bounded between two positive constants depending only on  $p$ .

The weak type inequality, (1.4), holds if and only if<sup>2</sup>

$$(1.6) \quad \left[ \frac{1}{h} \int_{a-h}^a w \right] \left[ \frac{1}{h} \int_a^{a+h} \sigma \right]^{p-1} \leq A \quad \text{for all } a \in R, h > 0.$$

Again, the best constants in (1.4) and (1.6) are comparable. Corresponding results hold for  $M^-$ .

<sup>2</sup>This characterization of the weak type inequality is due to K. Andersen.

Theorem 1 follows easily from Theorem 2 using a clever argument of Hunt, Kurtz and Neugebauer [6] as follows. First,  $N_p^+$  implies  $A_p^+$  by a standard argument in [9] that involves testing  $N_p^+$  with  $f = \chi_{(a, a+h)} w^{1-p'}$ . For the converse, it suffices to show that  $A_p^+$  implies (1.5) with  $\sigma = w^{1-p'}$ . Fix an interval  $I = (a, b)$  with  $\int_{-\infty}^a w > 0$  and a point  $x$  in  $I$ . Choose  $h > 0$  so that  $x + h$  is in  $I$  and  $h^{-1} \int_x^{x+h} \sigma \geq \frac{3}{4} M^+(\chi_I \sigma)(x)$ . Since  $h^{-1} \int_x^{x+h/2} \sigma \leq \frac{1}{2} M^+(\chi_I \sigma)(x)$  by definition, we conclude

$$\begin{aligned} M^+(\chi_I \sigma)(x) &\leq \frac{4}{h} \int_{x+h/2}^{x+h} \sigma \leq C \left[ \frac{h}{2} / \int_x^{x+h/2} w \right]^{p'-1} \text{ by } A_p^+ \\ &= C \left[ \int_x^{x+h/2} w^{-1} w / \int_x^{x+h/2} w \right]^{p'-1} \leq C M_w^+(\chi_I w^{-1})(x)^{p'-1} \end{aligned}$$

where

$$M_w^+ f(x) = \sup_{h>0} \left[ \int_x^{x+h} |f| w / \int_x^{x+h} w \right].$$

Now  $M_w^+$  is bounded on  $L^q(w)$  for any  $w \geq 0$  on  $R$  and  $1 < q < \infty$ , and thus

$$\int_I |M^+(\chi_I \sigma)|^p w \leq C^p \int_I |M_w^+(\chi_I w^{-1})|^{p'} w \leq C \int_I \sigma.$$

Finally, we consider reverse weighted inequalities for  $M^+$ . For  $v, w$  nonnegative and locally integrable on  $R$  and  $1 < p < \infty$ , the reverse weighted strong type inequality

$$(1.7) \quad \int_{-\infty}^{\infty} |f|^p v \leq C \int_{-\infty}^{\infty} |M^+ f|^p w \quad \text{for all } f$$

holds only in two trivial cases: either  $\int_{-\infty}^{-1} w(x)/|x|^p dx = \infty$  or  $v(x) \leq C'w(x)$  for a.e.  $x$ . The reverse weighted weak type (1, 1) inequality,

$$(1.8) \quad |\{M^+ f > \lambda\}|_w \geq \frac{C}{\lambda} \int_{\{f>\lambda\}} f v \quad \text{for all } f \geq 0,$$

holds if and only if for almost every  $a$  in  $R$ ,

$$(1.9) \quad \inf_{h>0} \frac{1}{h} \int_{a-h}^a w(x) dx \geq C'v(a).$$

Proofs of these assertions can be found in §3. See [2 and 10] concerning the reverse weighted weak-type (1, 1) inequality for  $M$  and its applications.

Throughout this paper the letter  $C$  will denote a positive constant that may vary from line to line but will remain independent of the relevant quantities.

**2. Proof of Theorem 2.** In proving the analogue of Theorem 2 for the two-sided maximal function, the following key property of  $M$  is used:  $Mf(x) \geq (1/|I|) \int_I |f|$  for  $x$  in  $I$ . This fails for both  $M^+$  and  $M^-$  and accounts for the bulk of difficulty in dealing with one-sided maximal operators. We circumvent this obstacle with the aid of the next lemma and some known results on Hardy operators.

**LEMMA 2.1.** *Suppose  $g \geq 0$  is integrable with compact support on  $R$ . If  $I = (a, b)$  is a component interval of the open set  $\{M^+ g > \lambda\}$ ,  $\lambda > 0$ , then*

$$(2.1) \quad \frac{1}{b-x} \int_x^b g \geq \lambda \quad \text{for } a \leq x < b.$$

To prove the lemma, fix  $a < x < b$  and let  $r$  be the largest number such that  $(r - x)^{-1} \int_x^r g \geq \lambda$ . If  $r < b$ , then  $(s - r)^{-1} \int_r^s g > \lambda$  for some  $s > r$  by definition and this yields  $(s - x)^{-1} \int_x^s g \geq \lambda$ , contradicting the definition of  $r$ . Thus  $r \geq b$ . If  $r > b$ , then  $(r - b)^{-1} \int_b^r g \leq \lambda$  since  $b$  is not in  $\{M^+g > \lambda\}$ . Together with  $(r - x)^{-1} \int_x^r g \geq \lambda$ , we obtain  $(b - x)^{-1} \int_x^b g \geq \lambda$  as required.

To deal with the weak type inequality (1.4) we need the following ([12]: see Theorem 4 and the subsequent note; see also [1]).

**LEMMA 2.2.** *Let  $\sigma, w$  be nonnegative weights on  $(0, \infty)$ ,  $1 < p < \infty$  and  $T_1g(x) = x^{-1} \int_0^x g(t) dt$  for locally integrable  $g$ . Then*

$$|\{T_1(f\sigma) > \lambda\}|_w \leq C \left[ \sup_{0 < x \leq s < \infty} s^{-p} \left( \int_x^s w \right) \left( \int_0^x \sigma \right)^{p-1} \right] \frac{1}{\lambda^p} \int_0^\infty |f|^p \sigma.$$

We now prove the equivalence of (1.4) and (1.6). First, (1.4) implies (1.6) by a standard argument (see e.g. [9]) that involves testing (1.4) with  $f = \chi_I$  and  $\lambda = \frac{1}{2} \int_I \sigma$ . Conversely, suppose (1.6) holds. It suffices to prove (1.4) for functions  $f \geq 0$  such that  $f\sigma$  is bounded with compact support. So fix such an  $f$  and a  $\lambda > 0$ . Let  $\{I_j\}_j$  be the component intervals of  $\{M^+(f\sigma) > \lambda\}$ . Applying Lemmas 2.1 and 2.2 (with a linear change of variable) to a fixed interval  $I_j = (a, b)$ , we obtain

$$\begin{aligned} |I_j|_w &\leq \left| \left\{ x: \frac{1}{b-x} \int_x^b \chi_{I_j} f \sigma \geq \lambda \right\} \right|_w \\ &\leq C \left[ \sup_{a \leq s \leq x < b} (b-s)^{-p} \left( \int_s^x w \right) \left( \int_x^b \sigma \right)^{p-1} \right] \frac{1}{\lambda^p} \int_{I_j} |f|^p \sigma \\ &\leq \frac{CA}{\lambda^p} \int_{I_j} |f|^p \sigma \quad \text{by (1.6) with } a = x \text{ and } h = b - s. \end{aligned}$$

Summing over  $j$  yields (1.4).

To deal with the strong type inequality (1.3) we need an apparent strengthening of the usual weighted inequality for the adjoint Hardy operator.

**LEMMA 2.3.** *Suppose  $\sigma, u$  are nonnegative weights on  $R$  and  $1 < p < \infty$ . Then for all  $f$ ,*

$$(2.2) \quad \int_{-\infty}^\infty \left| \int_x^\infty f \sigma \right|^p u(x) dx \leq C_p \int_{-\infty}^\infty \left[ \sup_{r \leq x} \left( \int_{-\infty}^r u \right) \left( \int_r^\infty \sigma \right)^{p-1} \right] |f(x)|^p \sigma(x) dx.$$

To see this, rewrite (2.2) as  $\int_{-\infty}^\infty |f(x)|^p g(x) u(x) dx \leq C_p \int_{-\infty}^\infty |g|^p \mu v$  where  $v = \sigma^{1-p}$ ,  $g = f\sigma$  and  $\mu(x)$  denotes the supremum in square brackets on the right side of (2.2). This latter inequality holds since  $(\int_{-\infty}^r u)(\int_r^\infty (\mu v)^{1-p})^{p-1} \leq 1$  for all  $r > 0$  (see [8]).

We will also need

**LEMMA 2.4.** *For  $1 < p < \infty$  and  $\sigma, w \geq 0$  on  $R$ , condition (1.5) implies*

$$(2.3) \quad \left[ \int_{-\infty}^r \frac{w(x)}{(b-x)^p} dx \right] \left[ \int_r^b \sigma \right]^{p-1} \leq CB \quad \text{for all } -\infty < r \leq b < \infty.$$

To prove this, let  $x_0 = r > x_1 > x_2 \cdots$  satisfy  $\int_{x_k}^b \sigma = 2^k \int_r^b \sigma$  for  $k = 0, 1, 2, \dots$ . Then

$$\begin{aligned} \left[ \int_{-\infty}^r \frac{w(x)}{(b-x)^p} dx \right] \left[ \int_r^b \sigma \right]^p &= \sum_{k=0}^{\infty} \left[ \int_{x_{k+1}}^{x_k} \frac{w(x)}{(b-x)^p} dx \right] 2^{-kp} \left[ \int_{x_k}^b \sigma \right]^p \\ &\leq \sum_{k=0}^{\infty} 2^{-kp} \int_{x_{k+1}}^b |M^+(\chi_{(x_{k+1}, b)} \sigma)|^p w \\ &\leq \sum_{k=0}^{\infty} 2^{-kp} B \int_{x_{k+1}}^b \sigma = B \sum_{k=0}^{\infty} 2^{-kp+k+1} \int_r^b \sigma = CB \int_r^b \sigma \end{aligned}$$

by (1.5) which yields (2.3).

We now prove the equivalence of (1.3) and (1.5). Once again, a standard argument (see e.g. [11]) shows that (1.3) implies (1.5). Conversely, suppose (1.5) holds. It suffices to prove (1.3) for functions  $f \geq 0$  such that  $f\sigma$  is bounded with compact support. So fix such an  $f$  and for  $k$  in  $\mathbb{Z}$ , let  $I_j^k = (a_j^k, b_j^k)$ ,  $j$  an integer, be the component intervals of the open set  $\Omega_k = \{M^+ f \sigma > 2^k\}$ . With  $E_j^k = I_j^k - \Omega_{k+1}$  we have

$$(2.4) \quad \int_{-\infty}^{\infty} |M^+(f\sigma)|^p w \leq 2^p \sum_k 2^{kp} |\Omega_k - \Omega_{k+1}|_w \leq C \sum_{k,j} 2^{kp} |E_j^k|_w.$$

For future reference, let

$$\mu_j^k(x) = \sup_{a_j^k \leq r \leq x} \left[ \int_{a_j^k}^r \frac{\chi_{E_j^k} w(t) dt}{(b_j^k - t)^p} \right] \left[ \int_r^{b_j^k} \sigma \right]^{p-1} \quad \text{for } x \text{ in } I_j^k.$$

We now fix  $k, j$  momentarily and estimate  $2^{kp} |E_j^k|_w$ . For convenience in writing let  $I = (a, b) = I_j^k$ ,  $E = E_j^k$ ,  $\mu = \mu_j^k$  and for those  $I_i^{k+1}$  contained in  $I_j^k$ , let  $J_i = I_i^{k+1}$ . Define  $g = \chi_E f$  and  $h = \sum_i (|J_i|_{\sigma}^{-1} \int_{J_i} f \sigma) \chi_{J_i}$ . For  $x$  in  $E$  we have

$$2^k \leq \frac{1}{b-x} \int_x^b f \sigma = \frac{1}{b-x} \int_x^b (g+h) \sigma$$

by Lemma 2.1 and so

$$\begin{aligned} (2.5) \quad 2^{kp} |E|_w &\leq \int_E \frac{w(x)}{(b-x)^p} \left( \int_x^b (g+h) \sigma \right)^p dx \\ &\leq C \int \mu(x) [g(x)^p + h(x)^p] \sigma(x) dx \quad \text{by Lemma 2.3} \\ &\leq CB \int g(x)^p \sigma(x) dx + C \int \mu(x) h(x)^p \sigma(x) dx \end{aligned}$$

by Lemma 2.4. Reverting to our previous notation (2.5) becomes

$$(2.6) \quad 2^{kp} |E_j^k|_w \leq CB \int_{E_j^k} f^p \sigma + C \sum_{I_i^{k+1} \subset I_j^k} \mu_j^k(a_i^{k+1}) |I_i^{k+1}|_{\sigma} \left( \frac{1}{|I_i^{k+1}|_{\sigma}} \int_{I_i^{k+1}} f \sigma \right)^p.$$

Plugging (2.6) into (2.4) we obtain

$$(2.7) \quad \int_{-\infty}^{\infty} |M^+(f\sigma)|^p w \leq CB \int f^p \sigma + C \sum_{k,j} \sum_{I_i^{k+1} \subset I_j^k} \mu_j^k(a_i^{k+1}) |I_i^{k+1}|_{\sigma} \left( \frac{1}{|I_i^{k+1}|_{\sigma}} \int_{I_i^{k+1}} f \sigma \right)^p$$

since the  $E_j^k$  are pairwise disjoint.

Since the  $I_j^k$  are nested ( $k < l \Rightarrow$  either  $I_i^l \subset I_j^k$  or  $I_i^l \cap I_j^k = \emptyset$  for all  $i, j$ ), a standard interpolation argument (see [11]) shows that the second term on the right side of (2.7) is dominated by  $CB \int f^p \sigma$  provided the following Carleson condition holds:

$$(2.8) \quad \sum_{I_j^k \subset I_s^l} \gamma_j^k |I_j^k|_{\sigma} \leq CB |I_s^l|_{\sigma}, \quad \text{for all } t, s$$

where  $\gamma_j^k = \mu_i^{k-1}(a_j^k)$  and where  $l$  is such that  $I_j^k \subset I_i^{l-1}$ . It will be convenient to denote this ‘‘predecessor’’ of  $I_j^k$ , namely  $I_i^{l-1}$ , by  $(c_j^k, d_j^k)$ . Since  $\gamma_s^t \leq CB |I_s^t|_{\sigma}$  by Lemma 2.4, we need only estimate the sum in (2.8) over intervals  $I_j^k$  properly contained in  $I_s^t$ . For each  $k, j$  let  $r_j^k$  satisfy  $c_j^k \leq r_j^k \leq a_j^k$  and

$$\left[ \int_{c_j^k}^{r_j^k} \frac{\chi_{E_i^{k-1} w}(t)}{(d_j^k - t)^p} dt \right] \left[ \int_{r_j^k}^{d_j^k} \sigma \right]^{p-1} \geq \frac{\gamma_j^k}{2}.$$

Then, for fixed  $I_i^{k-1}$  contained in  $I_s^t$  we have

$$(2.9) \quad \begin{aligned} \sum_{I_j^k \subset I_i^{k-1}} \gamma_j^k |I_j^k|_{\sigma} &\leq \sum_{I_j^k \subset I_i^{k-1}} \int \frac{\chi_{E_i^{k-1} w}(t)}{(d_j^k - t)^p} \left[ \int_{r_j^k}^{d_j^k} \sigma \right]^{p-1} \chi_{(c_j^k, r_j^k)}(t) |I_j^k|_{\sigma} dt \\ &= \int_{E_i^{k-1}} \frac{w(t)}{(b_i^{k-1} - t)^p} \left\{ \sum_{I_j^k \subset I_i^{k-1}} \chi_{(c_j^k, r_j^k)}(t) \left( \int_{r_j^k}^{b_i^{k-1}} \sigma \right)^{p-1} |I_j^k|_{\sigma} \right\} dt \\ &\leq \int_{E_i^{k-1}} \frac{w(t)}{(b_i^{k-1} - t)^p} \left( \int_t^{b_i^{k-1}} \sigma \right)^p dt \\ &\leq \int_{E_i^{k-1}} |M^+(\chi_{I_s^t} \sigma)|^p w. \end{aligned}$$

Summing (2.9) over all  $I_i^{k-1}$  contained in a fixed  $I_s^t$ , we obtain

$$\begin{aligned} \sum_{I_j^k \not\subseteq I_s^t} \gamma_j^k |I_j^k|_{\sigma} &\leq \sum_{I_i^{k-1} \subset I_s^t} \int_{E_i^{k-1}} |M^+(\chi_{I_s^t} \sigma)|^p w \\ &\leq \int_{I_s^t} |M^+(\chi_{I_s^t} \sigma)|^p w \leq B |I_s^t|_{\sigma} \end{aligned}$$

by (1.5). This establishes (2.8) and completes the proof of Theorem 2.

**3. Appendix.** We now complete the proof of Remark (C). Suppose  $w$  satisfies  $A_1^+$ , i.e.  $M^-w \leq Cw$ . Fix an interval  $I = (a, b)$ . If  $\lambda > M^-w(b)$ , then  $\Omega_\lambda = \{M^-(\chi_I w) > \lambda\}$  is contained in  $I$ . If  $\{I_j\}_j$  are the component intervals of  $\Omega_\lambda$ , then  $(1/|I_j|)\int_{I_j} w \geq \lambda$  for all  $j$  by Lemma 2.1. But  $(1/|I_j|)\int_{I_j} w \leq \lambda$  since the right endpoint of  $I_j$  is not in  $\Omega_\lambda$ . Thus we have

$$(3.1) \quad |\Omega_\lambda|_w = \sum_{j \rightarrow I_j} \int_{I_j} w = \lambda \sum_j |I_j| = \lambda |\Omega_\lambda|$$

$$\leq \lambda |I \cap \{w > \lambda/C\}| \quad \text{since } M^-w \leq Cw.$$

The argument of B. Jawerth in §5 of [7] now applies as follows.

$$\int_{I \cap \{w > M^-w(b)\}} w^{1+\delta} = |I \cap \{w > M^-w(b)\}| (M^-w(b))^{1+\delta}$$

$$+ \delta \int_{M^-w(b)}^\infty \lambda^{\delta-1} |I \cap \{w > \lambda\}|_w d\lambda$$

$$\leq |I| (M^-w(b))^{1+\delta} + \delta \int_{M^-w(b)}^\infty \lambda^\delta \left| I \cap \left\{ w > \frac{\lambda}{C} \right\} \right| d\lambda$$

(using  $|I \cap \{w > \lambda\}|_w \leq |\Omega_\lambda|_w$  and then (3.1))

$$\leq |I| (M^-w(b))^{1+\delta} + C \frac{\delta}{1+\delta} \int_{I \cap \{w > M^-w(b)\}} w^{1+\delta}.$$

Choosing  $\delta > 0$  sufficiently small we get

$$\int_I w^{1+\delta} \leq C |I| (M^-w(b))^{1+\delta} \leq C |I| w^{1+\delta}(b)$$

since  $M^-w \leq Cw$ , and this shows that  $M^-(w^{1+\delta}) \leq Cw^{1+\delta}$  as required.

We now prove the assertions made in the introduction concerning reverse weighted inequalities for  $M^+$ . Suppose  $1 < p < \infty$  and  $v, w$  are nonnegative locally integrable weights satisfying the reverse weighted inequality (1.7). Suppose further that  $\int_{-\infty}^{-1} w(x)/|x|^p dx < \infty$ . We must show that  $v(x) \leq C'w(x)$  for a.e.  $x$ . Fix  $x$ , a Lebesgue point of both  $v$  and  $w$ , and let  $\epsilon > 0$  be given. Choose  $R > 0$  so that  $r^{-1}\int_{x-r}^x w \leq w(x) + \epsilon$  for  $0 < r \leq R$ . For  $k \geq 1$ , set  $r_k = 2^{-k}R$ . With  $f = \chi_{(x-r_k, x)}$  in (1.7) we obtain

$$\frac{1}{r_k} \int_{x-r_k}^x v \leq C \sum_{j=0}^k \left(\frac{1}{2^j}\right)^{p-1} \left(\frac{1}{2^j r_k} \int_{x-2^j r_k}^x w\right) + Cr_k^{p-1} \int_{-\infty}^{x-R} \frac{w(y)}{(r_k + |y|)^p} dy$$

$$\leq C_p (w(x) + \epsilon) + C \left(\frac{R}{2^k}\right)^{p-1} \int_{-\infty}^{x-R} \frac{w(y)}{|x-y|^p} dy.$$

The integral in the second term on the right above is finite since  $\int_{-\infty}^{-1} w(x)/|x|^p dx < \infty$ . Thus, if  $k \rightarrow \infty$ , we get  $v(x) \leq C_p(w(x) + \epsilon)$  and since  $\epsilon > 0$  is arbitrary,  $v(x) \leq C_p w(x)$ .

We now prove the equivalence of the reverse weighted weak type (1, 1) inequality, (1.8), and condition (1.9). Fix  $a$ , a Lebesgue point of  $v$ , and  $h > 0$ . For  $0 < \epsilon < h$ ,

let  $f_\epsilon = \epsilon^{-1}\chi_{(a-\epsilon,a)}$ . Then  $\{M^+f_\epsilon > 1/h\} = (a-h, a)$  and (1.8) yields  $\int_{a-h}^a w \geq Ch\epsilon^{-1}\int_{a-\epsilon}^a v$ . Letting  $\epsilon \rightarrow 0$  we obtain (1.9) with  $C' = C$ . Conversely, fix  $f \geq 0$  bounded with compact support and  $\lambda > 0$ . Let  $(I_j)_j$  be the component intervals of the open set  $\Omega_\lambda = \{M^+f > \lambda\}$ . We claim

$$(3.2) \quad |I_j|_w \geq \frac{C}{\lambda} \int_{I_j} fv \quad \text{for all } j.$$

To see (3.2), suppose (for convenience) that  $I_j = (0, 1)$ . Then

$$\frac{1}{t} \int_t^{2t} f \leq 2 \frac{1}{2t} \int_0^{2t} f \leq 2M^+f(0) \leq 2\lambda$$

for  $0 < t < 1$ . Thus

$$\begin{aligned} \int_0^1 w &\geq \frac{1}{2\lambda} \int_0^1 \left[ \frac{1}{t} \int_t^{2t} f(x) dx \right] w(t) dt \\ &\geq \frac{1}{2\lambda} \int_0^1 f(x) \left[ \int_{x/2}^x \frac{w(t)}{t} dt \right] dx \\ &\geq \frac{C'}{4\lambda} \int_0^1 f(x)v(x) dx \end{aligned}$$

by (1.9) as required. Summing (3.2) over  $j$  yields

$$|\Omega_\lambda|_w = \sum_j |I_j|_w \geq \frac{C}{\lambda} \int_{\{M^+f > \lambda\}} fv \geq \frac{C}{\lambda} \int_{\{f > \lambda\}} fv.$$

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