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ISOMETRIES ON $L_{p,1}$

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ABSTRACT. The extreme points of the sphere of the Lorentz function space $L_{p,1}[0,1]$ are computed. As an application, the linear isometries from $L_{p,1}$ into itself are completely described.

1. Introduction. Since its introduction in 1950 by G. G. Lorentz, the Lorentz function space $L_{p,1} = L_{p,1}[0,1]$ has found frequent application to problems in interpolation theory and weighted-norm inequalities. In recent years the isomorphic structure of $L_{p,1}$ (as a Banach space) has received increasing attention and is now reasonably well understood while very little has been written on the isometric structure of $L_{p,1}$. In §2, we develop several interesting isometric tools; in particular, we compute the extreme points of the closed unit ball of $L_{p,1}$ (Theorem 1). As an application of these results, we give, in §3, a complete description of the linear isometries from $L_{p,1}$ into itself (Theorem 2).

Recall that Lamperti's proof of Banach's classical theorem on the linear isometries $T\colon L_p \to L_p$ [**18**, p. 333] proceeds in two major steps. In the first it is shown that T must preserve disjointness (this via an observation concerning the L_p-norm). The second step is quite general: T now induces a homomorphism of the measure algebra, and this homomorphism is necessarily induced by an automorphism, τ, of the underlying measure space. It now follows easily that T may be written: $Tf = h \cdot (f \circ \tau)$, where $h = T1$. Moreover, the converse is also true; that is, if h is norm-one in L_p and τ is an automorphism of $[0,1]$ with $\tau^{-1}[0,1] =$ support of h, then $Tf = h \cdot (f \circ \tau)$ defines an isometry on L_p.

By modifying the first step of this argument, Bru and Heinich [**2**] are able to show that the positive (onto) isometries on a large class of Banach lattices (which includes $L_{p,1}$) are likewise induced by automorphisms of the underlying measure space and so may also be written: $Tf = h \cdot (f \circ \tau)$, where $h = T1$. However, even in the case of $L_{p,1}$, it is not clear whether the converse holds. Rather than deduce information for $L_{p,1}$ from this lattice result, we opt for a somewhat more analytic approach to the question. We treat the isometries on $L_{p,1}$ as an application of specific geometric and "distributional" tools intrinsic to $L_{p,1}$. In particular, by using a stronger version of

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the first step outlined above (see Lemma 5, below), we shall see that there are actually fewer isometries on $L_{p,1}$ than might be anticipated from the results in [**2**]. Specifically, not every norm-one h in $L_{p,1}$ can be written as $T1$ for some isometry T.

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Our notation is, for the most part, standard and follows that of Lindenstrauss and Tzafriri [**15**]. We write $\mu(A)$ for the Lebesgue measure of subset of A of $\mathbf{R}$ and, given a measurable function f: $[0,1] \to \mathbf{R}$, we define

$$\operatorname{dist}(f;t) = d_f(t) = \mu(\{s\colon |f(s)| > t\}), \quad f^*(t) = \inf\{s\colon d_f(s) \leqslant t\},$$

$$F(t) = \int_0^t f^*(s)\,ds, \quad \text{and} \quad \operatorname{supp} f = \{s\colon f(s) \neq 0\}.$$

Notice that d_f is actually the (probability) distribution of $|f|$. Also we apologize in advance for all the usual abuses (and omissions) of the phrase "almost everywhere." For example, we shall sometimes write $f \geqslant 0$ when we mean $f \geqslant 0$ a.e., and $A \subset B$ instead of $\mu(B \setminus A) = 0$, etc.

Now, for $1 < p < \infty$, the Lorentz function space $L_{p,1} = L_{p,1}[0,1]$ is defined to be the collection of all (equivalence classes of) measurable functions f: $[0,1] \to \mathbf{R}$ for which $\|f\|_{p,1} < \infty$ where

$$\|f\|_{p,1} = \int_0^1 f^*(t)\,d(t^{1/p}). \tag{1}$$

Simple change-of-variable and integration-by-parts arguments show that (1) can be written in a variety of guises:

$$\begin{aligned} \|f\|_{p,1} &= \int_0^\infty d_f(t)^{1/p}\,dt = \int_0^1 \left(\frac{1}{p}\right) t^{1/p-1}\,dF(t) \\ &= \int_0^1 F(t)\,d\left(\left(\frac{-1}{p}\right) t^{1/p-1}\right) + \|f\|_1. \end{aligned} \tag{2}$$

(Notice that if $f \in L_{p,1}$, then $t^{1/p-1}F(t) \to 0$ as $t \to 0^+$.) We shall also use the usual L_p-spaces with norm

$$\|f\|_p = \left(\int_0^1 |f(t)|^p\,dt\right)^{1/p},$$

and also the well-known duality: $(L_{p,1})^* = L_{p',\infty}$, $1/p + 1/p' = 1$, where

$$\|f\|_{p',\infty} = \sup_{0<t<1} t^{-1/p}\int_0^t f^*(s)\,ds. \tag{3}$$

(See [**16** or **13**].)

It is also well known that $L_{p,1}$ is a separable dual space [**10**] with an unconditional basis [**15**, p. 156]. Moreover, $L_{p,1}$ is known to satisfy a lower p-estimate for disjoint elements [7, **1**]; that is, if $f_1, \ldots, f_n \in L_{p,1}$ are disjointly supported, then

$$\left\|\sum_{i=1}^n f_i\right\|_{p,1}^p \geqslant \sum_{i=1}^n \|f_i\|_{p,1}^p. \tag{4}$$

In particular, (4) implies that $\|f\|_p \leqslant \|f\|_{p,1}$. (Also see [**13** or **15**, Proposition 2.6.9].)

2. Extreme points. Our first two lemmas (which are essentially known) examine the triangle inequality in $L_{p,1}$.

LEMMA 1. *If $f,\ g \in L_{p,1}$ with $\|f+g\|_{p,1} = \|f\|_{p,1} + \|g\|_{p,1}$, then $(f+g)^* = f^* + g^*$.*

PROOF. First notice that since

$$\|f\|_{p,1} + \|g\|_{p,1} = \|f+g\|_{p,1} \leqslant \|\,|f| + |g|\,\|_{p,1} \leqslant \|f\|_{p,1} + \|g\|_{p,1},$$

we must have $|f+g| = |f| + |g|$ a.e. Thus $f \cdot g \geqslant 0$ a.e. and $\|f+g\|_1 = \|f\|_1 + \|g\|_1$. Now set

$$F_1(x) = \int_0^x (f+g)^*(t)\,dt \quad \text{and} \quad F_2(x) = \int_0^x [f^*(t) + g^*(t)]\,dt$$

for $0 \leqslant x \leqslant 1$. Then $F_1 \leqslant F_2$ and we need to show that $F_1 = F_2$. (See [**15**, p. 125].) But

$$0 = \|f\|_{p,1} + \|g\|_{p,1} - \|f+g\|_{p,1} = \int_0^1 [F_2(t) - F_1(t)]\,d\left(-\left(\frac{1}{p}\right)t^{1/p-1}\right),$$

and $-(1/p)t^{1/p-1}$ is increasing; thus $F_2 - F_1 \equiv 0$. Consequently, $(f+g)^* = f^* + g^*$. $\square$

REMARK. Note that $(f+g)^* = f^* + g^*$ implies that $f \cdot g \geqslant 0$ and that

$$\operatorname{supp}(f+g)^* = \operatorname{supp} f^* \cup \operatorname{supp} g^*.$$

Thus

$$\mu(\operatorname{supp} f \cup \operatorname{supp} g) = \max\{\mu(\operatorname{supp} f),\ \mu(\operatorname{supp} g)\};$$

that is, we either have $\operatorname{supp} f \subset \operatorname{supp} g$ or else $\operatorname{supp} g \subset \operatorname{supp} f$.

LEMMA 2. *For $f \in L_{p,1}$ and $a > 0$, let $f^a = |f| \vee a - a$ and $f_a = |f| \wedge a$. Then $\|f\|_{p,1} = \|f^a\|_{p,1} + \|f_a\|_{p,1}$. In particular, if $\|f\|_{p,1} = 1$ and $0 < a < 1$, then $\|f^a\|_{p,1} > 0$ and $\|f_a\|_{p,1} > 0$, and hence $|f|$ is a convex combination of $f^a/\|f^a\|_{p,1}$ and $f_a/\|f_a\|_{p,1}$.*

PROOF. A straightforward computation shows that $f^* = (f^*)^a + (f^*)_a = (f^a)^* + (f_a)^*$ and so $\|f\|_{p,1} = \|f^a\|_{p,1} + \|f_a\|_{p,1}$. Now, for $0 < a < 1$, it is easy to see that $0 < a \cdot [d_f(a)]^{1/p} \leqslant \|f_a\|_{p,1} \leqslant a < 1$, and hence also $\|f^a\|_{p,1} \geqslant 1 - a > 0$. $\square$

We are now in a position to give a simple description of the extreme points of the closed unit ball of $L_{p,1}$.

THEOREM 1. *Let $1 < p < \infty$ and let $f \in L_{p,1}$ with $\|f\|_{p,1} = 1$. Then the following are equivalent*:

(i) $\|f\|_p = 1$.

(ii) *f is an extreme point of the closed unit ball of $L_{p,1}$.*

(iii) $|f| = \mu(E)^{-1/p}\chi_E$ *for some* $E \subset [0,1]$.

(iv) $\|\,|f|^{p-1}\|_{p',\infty} = 1$, *where* $1/p + 1/p' = 1$.

PROOF. Suppose (i) holds. If $f = (g+h)/2$ with $\|g\|_{p,1} = \|h\|_{p,1} = 1$, then $\|g\|_p = \|h\|_p = 1$. The strict convexity of L_p then implies $f = g = h$.

Now suppose (ii) holds. Then, for any $0 < a < 1$, Lemma 2 implies that $f_a = \lambda|f|$ and $f^a = (1-\lambda)|f|$ where $\lambda = \|f_a\|_{p,1}$. But this easily implies that $|f| = \mu(E)^{-1/p}\chi_E$ for some $E \subset [0,1]$.

That (iii) implies (iv) is obvious; so finally suppose (iv) holds. Then $\|f\|_p^{p-1} = \||f|^{p-1}\|_{p'} \geqslant \||f|^{p-1}\|_{p',\infty} = 1$ and so $\|f\|_p = 1$. □

REMARK. The analogue of Theorem 1 for the Lorentz sequence space $l_{p,1}$ is well known and is due to W. J. Davis (cf. e.g. [**3**]).

Since $L_{p,1}$ is a separable dual space, its closed unit ball should have a wealth of strongly exposed points. (See [**8** or **9**].) As it happens, each extreme point of the closed unit ball of $L_{p,1}$ is also strongly exposed. To see this, suppose $f \in L_{p,1}$ is an extreme point of the closed unit ball and let $g = (\operatorname{sgn} f) \cdot |f|^{p-1}$. Then $\|f\|_p = \|g\|_{p'} = 1 = \langle f, g\rangle$ and f (considered as an element of L_p) is strongly exposed by g (considered as an element of $L_{p'}$). Thus, if $\|f_n\|_{p,1} \leqslant 1$ and $\langle f_n, g\rangle \to \langle f, g\rangle$, then $f_n \to f$ in L_p and hence $\|f_n\|_p \to \|f\|_p = 1$. But this implies that $\|f_n\|_{p,1} \to \|f\|_{p,1} = 1$. That these conditions are sufficient to imply the convergence of the sequence (f_n) to f in $L_{p,1}$ is given as

LEMMA 3. *Let (f_n) be a sequence in $L_{p,1}$ such that (f_n) converges to f in L_p and $(\|f_n\|_{p,1})$ converges to $\|f\|_{p,1}$. Then (f_n) converges to f in $L_{p,1}$.*

PROOF. First notice that it is enough to show that every subsequence of (f_n) has a further subsequence converging to f in $L_{p,1}$. Consequently, we may assume that (f_n) converges to f a.e. Also, for convenience, we shall take $\|f\|_{p,1} = 1$. Now let $\varepsilon > 0$ and choose $\delta > 0$ so that $\|f\chi_{A^c}\|_{p,1} < \varepsilon$ whenever $\mu(A) < \delta$. Next, use Egorov's theorem to choose A so that $\mu(A) < \delta$ and so that f_n converges uniformly to f on A^C. Finally, choose n sufficiently large so that the following hold:

(i) $\|(f - f_n)\chi_{A^c}\|_{p,1} < \varepsilon$,

(ii) $\|f_n\|_{p,1} < 1 + \varepsilon$, and

(iii) $\|f_n\|_{p,1} - \|f_n\chi_{A^c}\|_{p,1} < \varepsilon$.

Then

$$\begin{aligned}\|f - f_n\|_{p,1} &\leqslant \|f\chi_A\|_{p,1} + \|f_n\chi_A\|_{p,1} + \|(f - f_n)\chi_{A^c}\|_{p,1} \\ &\leqslant 2\varepsilon + \left(\|f_n\|_{p,1}^p - \|f_n\chi_{A^c}\|_{p,1}^p\right)^{1/p} \leqslant 2\varepsilon + \left(p(1+\varepsilon)^{p-1}\varepsilon\right)^{1/p} < 2p\varepsilon^{1/p}.\end{aligned}$$

Thus (f_n) converges to f in $L_{p,1}$. □

Before we can entertain any discussion of isometries on $L_{p,1}$, we shall need some condition stated in terms of the norm in $L_{p,1}$ which will guarantee that two functions are disjointly supported. The next two lemmas provide such conditions; the first is quite general (and really just a minor variation of Lemma 7.2 in [**14**]), while the second examines the case of equality in (4).

LEMMA 4. *Given f, $g \in L_{p,1}$, let $f \oplus g$ denote the sum of disjoint copies of f and g; that is, $d_{f\oplus g} = d_f + d_g$. (Of course, we may need to take $f \oplus g \in L_{p,1}[0,2]$.) If $f \cdot g \geqslant 0$, then $\|f + g\|_{p,1} \geqslant \|f \oplus g\|_{p,1}$ and equality occurs only when $f \cdot g = 0$.*

PROOF. The first conclusion is a general fact in any rearrangement invariant space. Indeed, just as in the proof of Lemma 1, we need only to observe that if

$$H_1(x) = \int_0^x (f \oplus g)^*(t)\,dt \quad \text{and} \quad H_2(x) = \int_0^x (f+g)^*(t)\,dt,$$

then $H_1 \leqslant H_2$. But, since $f \cdot g \geqslant 0$,

$$\begin{aligned} H_1(x) &= \sup_{\mu E = X} \int_E |f \oplus g|(s)\,ds \leqslant \sup_{\mu E = X} \left\{ \int_E |f(s)|\,ds + \int_E |g(s)|\,ds \right\} \\ &= \sup_{\mu E = X} \int_E |f+g|(s)\,ds = H_2(x). \end{aligned}$$

Again, as in Lemma 1, the case $\|f+g\|_{p,1} = \|f \oplus g\|_{p,1}$ would imply that $H_1 = H_2$; that is $d_{f+g} = d_f \oplus g = d_f + d_g$. Then $f \cdot g \geqslant 0$ implies

$$\mu(\operatorname{supp} f \cup \operatorname{supp} g) = d_{f+g}(0) = d_f(0) + d_g(0) = \mu(\operatorname{supp} f) + \mu(\operatorname{supp} g).$$

Thus $f \cdot g = 0$. □

LEMMA 5. *Let $f, g \in L_{p,1}$ with $f \cdot g \geqslant 0$. If $\|f+g\|_{p,1}^p = \|f\|_{p,1}^p + \|g\|_{p,1}^p$, then $f \cdot g = 0$ and, moreover, d_f and d_g are proportional.*

PROOF. The first conclusion is immediate from (4) and Lemma 4. Indeed,

$$\|f\|_{p,1}^p + \|g\|_{p,1}^p = \|f+g\|_{p,1}^p \geqslant \|f \oplus g\|_{p,1}^p \geqslant \|f\|_{p,1}^p + \|g\|_{p,1}^p$$

and thus $f \cdot g = 0$. But now

$$\begin{aligned} \left\{ \int_0^\infty \left(d_f(t) + d_g(t) \right)^{1/p} dt \right\}^p &= \|f+g\|_{p,1}^p = \|f\|_{p,1}^p + \|g\|_{p,1}^p \\ &= \left\{ \int_0^\infty d_f(t)^{1/p}\,dt \right\}^p + \left\{ \int_0^\infty d_g(t)^{1/p}\,dt \right\}^p. \end{aligned}$$

That is, we have equality in the triangle inequality in $L_{1/p}[0,\infty)$. Hence d_f and d_g are proportional. (In particular, $\|f\|_{p,1} = \|g\|_{p,1}$ would imply that $d_f = d_g$.) □

REMARK. The observation made in [2] is that $L_{p,1}$ is "order convex"; that is, if $f \cdot g \geqslant 0$ and if $\|f-g\|_{p,1} = \|f+g\|_{p,1}$, then $f \cdot g = 0$. Notice that $f \cdot g \geqslant 0$ and $\|f+g\|_{p,1}^p = \|f\|_{p,1}^p + \|g\|_{p,1}^p$ imply that $\|f+g\|_{p,1} = \|f-g\|_{p,1}$.

3. Isometries. Finally we are ready to describe the linear isometries from $L_{p,1}$ into itself. What might not be expected here is that the only linear isometries are the obvious ones: changes of sign, rearrangements, and dilations. That is, if $T: L_{p,1} \to L_{p,1}$ is an isometry and $\lambda = \mu(\operatorname{supp} T1)$, then

$$(Tf)^*(t) = \lambda^{-1/p} f^*(t/\lambda) \tag{5}$$

for every $f \in L_{p,1}$ and $0 \leqslant t \leqslant 1$. (We take $f^*(s) = 0$ for $s > 1$.)

Before we set a plan of attack for proving (5), let us first reduce to the case of positive isometries. In what follows, $T: L_{p,1} \to L_{p,1}$ is any linear isometry (not necessarily onto or positive) and, for each $n = 1, 2, \ldots$ and $i = 1, 2, \ldots, n$, $z_{n,i}$ will denote the characteristic function of the interval $[(i-1)/n, i/n)$.

LEMMA 6. *Let $T\colon L_{p,1} \to L_{p,1}$ be a linear isometry. For every $f \in L_{p,1}$,*

$$\operatorname{supp} Tf \subset \operatorname{supp} T1. \tag{6}$$

PROOF. It suffices to show that (6) holds in the case $f = z_{n,i}$ for any $n > 2^{p-1}$ and $i = 1, 2, \ldots, n$. To do this, it suffices to show that $Tz_{n,i}$ and $T(1 - z_{n,i})$ are disjointly supported for all $n > 2^{p-1}$ and all $i = 1, 2, \ldots, n$. Fix $n > 2^{p-1}$ and $1 \leqslant i \leqslant n$, and set $f = Tz_{n,i}$, $g = T(1 - z_{n,i})$, $h = f + g = T1$, $k = f - g = T(2z_{n,i} - 1)$. Then, since T is an isometry, it is easy to see that

$$\|f\|_{p,1} = n^{-1/p}, \quad \|g\|_{p,1} = (1 - n^{-1})^{1/p}, \quad \|h\|_{p,1} = \|k\|_{p,1} = 1, \tag{7}$$

and

$$\begin{aligned} &\|f + h\|_{p,1} = \|f\|_{p,1} + \|h\|_{p,1}, \quad \|f + k\|_{p,1} = \|f\|_{p,1} + \|k\|_{p,1}, \\ &\|g + h\|_{p,1} = \|g\|_{p,1} + \|h\|_{p,1}, \quad \|g - k\|_{p,1} = \|g\|_{p,1} + \|k\|_{p,1}. \end{aligned} \tag{8}$$

The equations in (8) follow from the fact that $\|1 + x\|_{p,1} = 1 + \|x\|_{p,1}$ for any $x \in L_{p,1}$, $x \geqslant 0$.

The equations in (8) combine with the remark following Lemma 1 to yield several consequences:

$$f \cdot h \geqslant 0, \quad f \cdot k \geqslant 0, \quad g \cdot h \geqslant 0, \quad (-g) \cdot k \geqslant 0 \tag{9}$$

and

$$\begin{aligned} &\operatorname{supp} f \subset \operatorname{supp} h \quad \text{or} \quad \operatorname{supp} h \subset \operatorname{supp} f, \\ &\operatorname{supp} f \subset \operatorname{supp} k \quad \text{or} \quad \operatorname{supp} k \subset \operatorname{supp} f, \\ &\operatorname{supp} g \subset \operatorname{supp} h \quad \text{or} \quad \operatorname{supp} h \subset \operatorname{supp} g, \\ &\operatorname{supp} g \subset \operatorname{supp} k \quad \text{or} \quad \operatorname{supp} k \subset \operatorname{supp} g. \end{aligned} \tag{10}$$

The inequalities in (9) imply that at any point for which $f \cdot g \neq 0$ we have $h \cdot k = 0$ and, moreover, that $f \cdot g \geqslant 0$ on $\operatorname{supp} h$ and $f \cdot g \leqslant 0$ on $\operatorname{supp} k$. Consequently, if we set:

$$\begin{aligned} A &= \operatorname{supp} f \setminus \operatorname{supp} g, \\ B &= \operatorname{supp} g \setminus \operatorname{supp} f, \\ C &= \operatorname{supp} h \setminus (A \cup B) = \{\operatorname{sgn} f = \operatorname{sgn} g\}, \\ D &= \operatorname{supp} k \setminus (A \cup B) = \{\operatorname{sgn} f = -\operatorname{sgn} g\}, \end{aligned} \tag{11}$$

then A, B, C, D are pairwise disjoint and

$$\begin{aligned} f &= f\chi_A \phantom{{}+g\chi_B} + \tfrac{1}{2}h\chi_C + \tfrac{1}{2}k\chi_D, \\ g &= \phantom{f\chi_A+{}} g\chi_B + \tfrac{1}{2}h\chi_C - \tfrac{1}{2}k\chi_D. \end{aligned} \tag{12}$$

But now (10) implies several conditions on A, B, C, and D. In fact, it is not hard to see that either $A = B = \varnothing$ or else $C = D = \varnothing$ (that is, either $h \cdot k = 0$ or $f \cdot g = 0$). Indeed, suppose for instance that $A \neq \varnothing$ and $C \neq \varnothing$. Then $\operatorname{supp} g = B \cup C \cup D$ and $\operatorname{supp} k = A \cup B \cup D$ cannot satisfy either of the containments $\operatorname{supp} g \subset \operatorname{supp} k$ or $\operatorname{supp} k \subset \operatorname{supp} g$. Thus we need only point out that $A = B = \varnothing$ (i.e., $h \cdot k = 0$) is impossible. But $\|h + k\|_{p,1} = 2\|f\|_{p,1} = 2n^{-1/p}$ while from (4), $h \cdot k = 0$ would imply $\|h + k\|_{p,1} \geqslant 2^{1/p}$; our choice of $n > 2^{p-1}$ makes this impossible. Thus $f \cdot g = 0$ as desired. □

LEMMA 7. *Let T: $L_{p,1} \to L_{p,1}$ be a linear isometry. The map S: $L_{p,1} \to L_{p,1}$ defined by $Sf = (\operatorname{sgn} T1) \cdot Tf$ is a positive isometry. In particular, if $f \cdot g \geqslant 0$, then $Tf \cdot Tg \geqslant 0$.*

PROOF. As mentioned above, for any $f \geqslant 0$, we have $\|T1 + Tf\|_{p,1} = \|T1\|_{p,1} + \|Tf\|_{p,1}$ and so, from Lemma 1, $T1 \cdot Tf \geqslant 0$. Since Lemma 6 states that $\operatorname{supp} Tf \subset \operatorname{supp} T1$, we can conclude that $(\operatorname{sgn} T1) \cdot Tf = |Tf|$. $\square$

Our next goal will be to provide a more tractable (i.e., linear) replacement for (5). To this end, it may be helpful to think of T as an isometry from $L_{p,1}$ into $L_{p,1}([0,1]^2)$. The reasons for this are essentially cosmetic: if we define $f \otimes g$ by $(f \otimes g)(s,t) = f(s)g(t)$, then it is easy to see that the map $f \to f \otimes g$ defines an isometry satisfying (5) whenever $|g| = \mu(E)^{-1/p}\chi_E$. Indeed, the distribution of $f \otimes \chi_E$ is $\mu(E) \cdot d_f$ and so, in this case,

$$(f \otimes g)^*(t) = \mu(E)^{-1/p} f^*(t/\mu(E)). \tag{13}$$

Now it is also known that

$$\|f\|_{p,1} \cdot \|g\|_p \leqslant \|f \otimes g\|_{p,1} \leqslant \|f\|_{p,1} \cdot \|g\|_{p,1} \tag{14}$$

for any f, $g \in L_{p,1}$ [**4**, **5** and **17**, Theorem 7.4] and so our program is easy to outline. We shall first show that Tf must have same distribution as $f \otimes g$ where $g = T1$. We shall then show that $g = T1$ forces equality in (14). This implies that $\|g\|_p = \|g\|_{p,1} = 1$ or equivalently, by Theorem 1, that $g = \mu(E)^{-1/p}\chi_E$ for some measurable set E.

LEMMA 8. *Let T: $L_{p,1} \to L_{p,1}$ be a linear isometry and let $g = T1$. Then Tf has the same distribution as $f \otimes g$ for every f in $L_{p,1}$.*

PROOF. Since T is linear and continuous and since step functions of the form $\sum_{i=1}^n a_i z_{n,i}$ are dense in $L_{p,1}$, it is enough to show that, for each n, the functions $Tz_{n,i}$, $i = 1,2,\ldots,n$, are disjointly supported and have the same distribution. Fix n and $1 \leqslant i \neq j \leqslant n$. Since T is a linear isometry, we have $\|Tz_{n,i}\|_{p,1} = \|Tz_{n,j}\|_{p,1}$ and $\|Tz_{n,i} + Tz_{n,j}\|_{p,1}^p = \|Tz_{n,i}\|_{p,1}^p + \|Tz_{n,j}\|_{p,1}^p$. But now Lemma 5 and Lemma 7 give us that $Tz_{n,i}$ and $Tz_{n,j}$ are disjointly supported and have the same distribution. Necessarily $Tz_{n,i}$ has the same distribution as $z_{n,i} \otimes g$ and linearity implies that $T(\sum_{i=1}^n a_i z_{n,i})$ has the same distribution as $(\sum_{i=1}^n a_i z_{n,i}) \otimes g$. $\square$

Next we consider the case of (near) equality on the left-hand side of inequality (14). The following lemma is suggested by the proof of Lemma 8.8 of [**14**]. (Also see [**15**, Theorem 2.7.2 and **6**].)

LEMMA 9. *Given a positive integer k and $\alpha > 1$, define $f_{k,\alpha} \in L_{p,1}$ by $f_{k,\alpha}(t) = k^{\alpha/p} \wedge t^{-1/p}$. Then, for any $h = \sum_{i=1}^k a_i z_{k,i}$,*

$$\|f_{k,\alpha} \otimes h\|_{p,1} \leqslant (1 + \alpha^{-1}) \cdot \|h\|_p \cdot \|f_{k,\alpha}\|_{p,1}. \tag{15}$$

PROOF. We shall show that if $\|h\|_p \leqslant 1$, then $(f_{k,\alpha} \otimes h)^* \leqslant f_{k,\alpha+1}$; (15) then follows from a simple calculation:

$$\begin{aligned}\|f_{k,\alpha+1}\|_{p,1} &= 1 + \frac{\alpha+1}{p}\log k \leqslant (1 + \alpha^{-1}) \cdot \left(1 + \frac{\alpha}{p}\log k\right) \\ &= (1 + \alpha^{-1}) \cdot \|f_{k,\alpha}\|_{p,1}.\end{aligned}$$

Assume that $h = \sum_{i=1}^{k} a_i z_{k,i}$ satisfies $h \geqslant 0$ and $\|h\|_p^p = k^{-1}\sum_{i=1}^{k} a_i^p \leqslant 1$; we wish to estimate $(f_{k,\alpha} \otimes h)^*$. But $f_{k,\alpha} \otimes h = \sum_{i=1}^{k} a_i(f_{k,\alpha} \otimes z_{k,i})$ and the functions $f_{k,\alpha} \otimes z_{k,i}$ are disjointly supported and have distribution k^{-1} dist$(f_{k,\alpha}; t)$. Thus

$$\operatorname{dist}(f_{k,\alpha} \otimes h; t) = k^{-1} \sum_{i=1}^{k} \operatorname{dist}\left(f_{k,\alpha}; \frac{t}{a_i}\right).$$

Now dist$(f_{k,\alpha}; t) = t^{-p} \wedge \chi_{[0,\, k^{\alpha/p})}(t)$, and it is easy to check that, for $0 < a \leqslant k^{1/p}$, we have

$$\operatorname{dist}(f_{k,\alpha}; t/a) = a^p t^{-p} \wedge \chi_{[0,\, k^{\alpha/p})}(t/a) \leqslant \chi_{[0,\,1)}(t) + a^p t^{-p} \chi_{[1,\, k^{(\alpha+1)/p})}(t).$$

Consequently,

$$\begin{aligned}
\operatorname{dist}(f_{k,\alpha} \otimes h; t) &= k^{-1} \sum_{i=1}^{k} \operatorname{dist}\left(f_{k,\alpha}; \frac{t}{a_i}\right) \\
&\leqslant \chi_{[0,\,1)}(t) + \left(k^{-1} \sum_{i=1}^{k} a_i^p\right) \cdot t^{-p} \cdot \chi_{[1,\, k^{(\alpha+1)/p})}(t) \\
&\leqslant t^{-p} \wedge \chi_{[0,\, k^{(\alpha+1)/p})}(t) = \operatorname{dist}(f_{k,\alpha+1}; t).
\end{aligned}$$

Thus, $(f_{k,\alpha} \otimes h)^* \leqslant f_{k,\alpha+1}$. □

We are finally ready to combine all of the preceding observations to give a simple proof of our main result.

THEOREM 2. *Let T: $L_{p,1} \to L_{p,1}$ be a linear isometry and let $\lambda = \mu(\operatorname{supp} T1)$. Then T satisfies*

$$(Tf)^*(t) = \lambda^{-1/p} f^*(t/\lambda) \tag{5}$$

for every f in $L_{p,1}$ and $0 \leqslant t \leqslant 1$.

PROOF. Let $g = T1$. Lemma 8 shows that $(Tf)^* = (f \otimes g)^*$ for every $f \in L_{p,1}$. Thus, by (13) and Theorem 1, we need only show that $\|g\|_p = 1$.

Let $0 < \varepsilon < 1$ and let $h = \sum_{i=1}^{k} a_i z_{k,i}$ be a step function such that $\|g - h\|_{p,1} \leqslant \varepsilon\|g\|_p$. Next let $f = f_{k,1/\varepsilon}$ be the function given in Lemma 9 (for $\alpha = 1/\varepsilon$). Then

$$\begin{aligned}
\|f\|_{p,1} = \|Tf\|_{p,1} = \|f \otimes g\|_{p,1} &\leqslant \|f \otimes h\|_{p,1} + \|f \otimes (g - h)\|_{p,1} \\
&\leqslant (1 + \varepsilon)\|h\|_p \cdot \|f\|_{p,1} + \|(g - h)\|_{p,1} \cdot \|f\|_{p,1} \\
&\leqslant \left[(1 + \varepsilon)^2 + \varepsilon\right] \cdot \|g\|_p \cdot \|f\|_{p,1}
\end{aligned}$$

(where the second inequality follows from (14) and (15)). Letting ε tend to 0 yields $\|g\|_p = 1$ as promised. □

REMARK. Notice that every linear isometry on $L_{p,1}$ turns out to be an isometry on L_p and, in fact, a multiple of an isometry on every L_r. Thus a linear isometry T on $L_{p,1}$ maps extreme points to extreme points and is onto exactly when $|T1| = 1$.

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