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ATTRACTING ORBITS IN NEWTON'S METHOD

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ABSTRACT. It is well known that the dynamical system generated by Newton's Method applied to a real polynomial with all of its roots real has no periodic attractors other than the fixed points at the roots of the polynomial. This paper studies the effect on Newton's Method of roots of a polynomial "going complex". More generally, we consider Newton's Method for smooth real-valued functions of the form $f_\mu(x) = g(x) + \mu$, μ a parameter. If μ_0 is a point of discontinuity of the map $\mu \to$ (the number of roots of f_μ), then, in the presence of certain nondegeneracy conditions, we show that there are values of μ near μ_0 for which the Newton function of f_μ has nontrivial periodic attractors.

Recently there have been several studies of Newton's root-finding algorithm for a function $f\colon R \to R$

$$x_{n+1} = Nf(x_n) = x_n - \left[f(x_n)/f'(x_n)\right] \tag{1.1}$$

as a dynamical system **[Sm, CGS, CoM, SaU, HM, W]**. Among these, **[CoM, SaU, HM, W]** contain proofs of results due to Barna **[B1-4]** concerning the existence of chaotic dynamics in Newton's Method for polynomials. (A brief description of these complicated dynamics is given below, at the end of §2.) Let $\mathrm{NC}(f)$ denote the set of nonconvergent points for Nf,

$$\mathrm{NC}(f) = \left\{ x \mid (Nf)^j(x) \text{ does not converge to a root of } f \text{ as } j \to \infty \right\}. \tag{1.2}$$

Both the size of this set and the dynamical complexity of the restriction of Nf to $\mathrm{NC}(f)$ are significant in determining how well Newton's Method works as a root-finding algorithm. A much-studied case is that of a polynomial, $p(x)$, all of whose roots are real. Here Barna has shown that $\mathrm{NC}(p)$ is a closed set of Lebesgue measure zero. A classical result of Fatou applies in this case to show that $\mathrm{NC}(p)$ is uniformly repelling: there is a neighborhood U of $\mathrm{NC}(p)$ and constants $A > 0$, $r > 1$ satisfying, for every $x \notin \mathrm{NC}(p)$

$$\operatorname{dist}\left(Np^j(x), \mathrm{NC}(p)\right) > A \cdot r^j \cdot \operatorname{dist}(x, \mathrm{NC}(p))$$

as long as the first $j-1$ iterates of x by Np stay in the neighborhood U. (Fatou's result is discussed in §8 of **[Bla]**; also see **[Sul, MSS]**; a proof of this inequality that does not use Fatou's theorem can be found in **[B4]** and in **[CoM]**.) When one considers this result along with the inherent inaccuracies that occur whenever one

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does numerical work on a computer, it is not surprising that Newton's Method should work reasonably well to find the roots of a polynomial that has only real roots.

On the other hand, there are known examples of functions whose nonconvergent set has positive Lebesgue measure. In addition to the trivial examples (where $f(x)$ has no real roots), there are Newton functions Nf that have attracting periodic orbits. A periodic orbit is called *attracting* if there is a compact neighborhood of the orbit that is mapped into itself by Nf and such that the iterates of any point in the neighborhood approach the orbit as the number of iterations grows. The set of points that approach the periodic orbit under iteration is called the *basin* of the periodic attractor. If p is a periodic point of period k with $|(Nf^k)'(p)| < 1$ then the orbit of p is necessarily attracting. In the special case where $(Nf^k)'(p) = 0$ we will say that the periodic orbit is *superattracting*. Examples of maps whose Newton functions have periodic attractors are known [**B3, HM**]:

$$f(x) = 3x^5 - 10x^3 + 23x, \tag{1.3(a)}$$

$$f(x) = 11x^6 - 34x^4 + 39x^2 \tag{1.3(b)}$$

(each of these maps has $\{-1, 1\}$ as a superattracting period 2 orbit). The basic question we wish to address in this paper is how the disappearance of real roots in a one-parameter family of polynomials leads to the appearance of attracting periodic orbits in the Newton functions of these polynomials. In fact, we will study the somewhat more general question of how the dynamics of Newton functions Nf_μ change as a parameter μ is varied in such a way that the number of roots of the maps f_μ changes. We give rough statements of our main results below; more precise statements of these theorems will be given in §3.

THEOREM A. *Suppose that g is C^3, that g has a repeated root of multiplicity two, that g' has no repeated roots, and that g'' has a real root. Let $f_\mu(x) = g(x) + \mu$. For all sufficiently large integers k there is an open interval I_k on the μ-axis such that for each μ in I_k, Nf_μ has an attracting periodic orbit of period k, and the intervals I_k converge to* 0 *as k goes to infinity.*

THEOREM B. *Suppose g is C^3, satisfies the hypotheses of Theorem* A, *as well as certain other technical hypotheses. For all sufficiently large integers k there is an open interval M_k on the μ-axis such that for each μ in M_k, Nf_μ has an attracting periodic orbit of period k. For sufficiently large k the intervals M_k are disjoint from any bounded set, and each M_k is disjoint from the interval I_k of Theorem* A.

Relevant to these results is the recent thesis of C. McMullen [**Mc**], which shows that there is no generally convergent rational algorithm (in the sense of Smale [**Sm2**]) for finding the roots of complex polynomials of degree at least 4. In fact, McMullen shows that for these polynomials there is no rational root-finding algorithm with a uniform bound on the periods of the attracting orbits. McMullen also gives an example of a generally convergent algorithm for cubics (Newton's method is generally convergent for quadratics).

We make the following notational conventions.

(1.4) Nf^j should be read as $(Nf)^j$ as opposed to $N(f^j)$; similarly, Nf' will denote $(Nf)'$, not $N(f')$.

(1.5) By "periodic orbit" we will mean an orbit $\{Nf^j(x) | j = 0, 1, 2, \dots\}$ that is finite and has cardinality at least 2, so a fixed point will not be considered a periodic orbit.

2. Basic facts about Newton functions. Suppose f is a smooth (at least C^3) map from R to R. We will always assume that f satisfies the nondegeneracy condition

$$f'(x) = 0 \quad \text{implies} \quad f''(x) \neq 0. \tag{2.1}$$

The Newton function of f, Nf, is defined by

$$Nf(x) = \begin{cases} x - [f(x)/f'(x)] & \text{if } f'(x) \neq 0, \\ x & \text{if } f(x) = f'(x) = 0, \end{cases} \tag{2.2}$$

$(Nf(x)$ is undefined in all other cases$)$.

Note that

$$Nf(x) = x \text{ if and only if } f(x) = 0 \tag{2.3}$$

and

$$(Nf)'(x) = [f(x)f''(x)]/[f'(x)^2] \quad \text{whenever } f'(x) \neq 0. \tag{2.4}$$

Consequently, if $f(q) = 0$ and $f'(q) \neq 0$ then

$$(Nf)'(q) = 0. \tag{2.5(a)}$$

More generally, if f is C^{n+1} and f as well as its first $n - 1$ derivatives all vanish at q, but its nth derivative at q is nonzero, then defining $Nf(q) = q$ makes Nf a C^1 function in a neighborhood of q, and

$$(Nf)'(q) = (n - 1)/n. \tag{2.5(b)}$$

(Thus for f satisfying the nondegeneracy condition 2.1, if $f(q) = f'(q) = 0$, then $(Nf)'(q) = 1/2$.)

(2.6) If $f'(c) = 0$ and $f(c) \neq 0$, then Nf has a vertical asymptote at c, and the sign of $Nf(x)$ changes as x is increased past c.

If the second derivative of f is bounded away from 0 as $|x|$ goes to infinity, then l'Hôpital's rule shows

$$Nf(x) \to \infty \text{ as } x \to \infty \quad \text{and} \quad Nf(x) \to -\infty \text{ as } x \to -\infty. \tag{2.7}$$

(This condition will be useful later.)

As a consequence of (2.1)–(2.7), the graph of Nf will typically be like the graph in Figure 1.

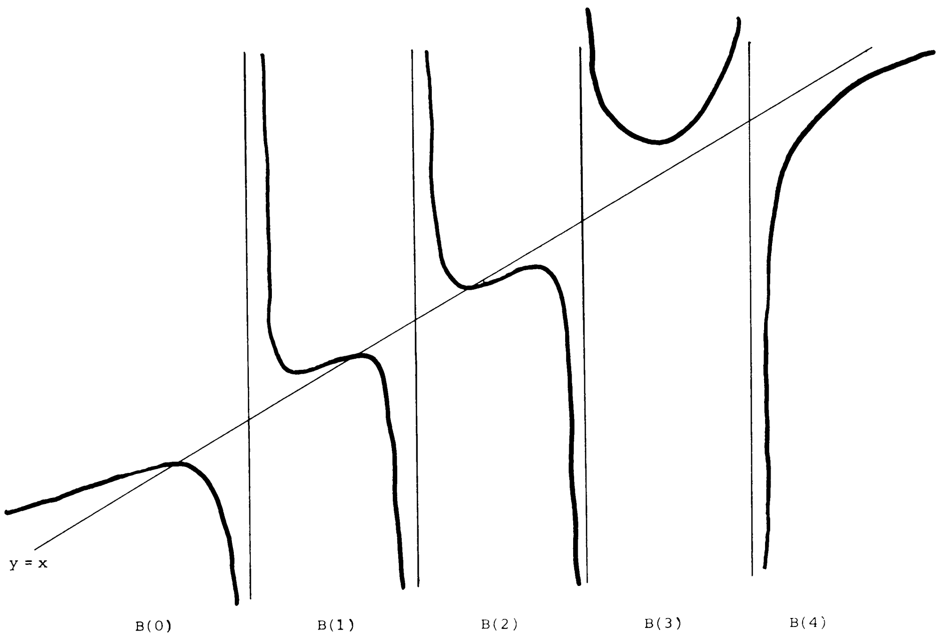

FIGURE 1. Graph of a typical Newton function

A maximal open interval on which Nf is continuous is called a *band* for Nf. In Figure 1 the bands are labelled $B(i)$, $0 \leqslant i \leqslant 4$. Note that there are several pairs i, j satisfying $B(j) \subset Nf(B(i))$, for instance

$$\begin{aligned} &B(1) \cup B(2) \cup B(3) \cup B(4) \subset Nf(B(1)), \\ &B(1) \cup B(2) \cup B(3) \cup B(4) \subset Nf(B(2)), \\ &B(4) \subset Nf(B(3)), \\ &B(1) \cup B(2) \cup B(3) \subset Nf(B(4)). \end{aligned} \tag{2.8}$$

Moreover, one can choose compact subintervals $K(i) \subset B(i)$ for each i such that all of the inclusions listed in (2.8) will continue to hold when each $B(i)$ is replaced by $K(i)$. In this situation standard arguments show that Nf has chaotic dynamics: periodic points of infinitely many different periods, uncountably many points that are neither periodic nor asymptotically periodic, positive topological entropy, etc. This phenomenon is quite general; for instance, it occurs whenever f has at least four real roots. For a more complete discussion see [**B1-4, SaU, HM, W, CoM**].

3. Newton's method and quadratic bifurcations. In what follows we will study the change in the dynamics of Newton's Method when the underlying map loses a root by undergoing a quadratic tangency with the x-axis. Specifically, let g denote a smooth (C^3) map satisfying the nondegeneracy condition (2.1). For convenience we will also assume that

$$g(0) = 0, \qquad g'(0) = 0, \qquad g''(0) > 0 \tag{3.1}$$

so the graph of $g(x)$ has a quadratic tangency with the x-axis at $x = 0$. We will be considering the Newton functions Nf_μ for the maps f_μ defined by

$$f_\mu(x) = g(x) + \mu \tag{3.2}$$

depending upon the parameter μ. Note that (3.1) implies that the number of roots of f_μ in a neighborhood of 0 changes from 2 to 1 to 0 as μ increases past $\mu = 0$. The effect of this on the graph of Nf_μ is depicted in Figure 2.

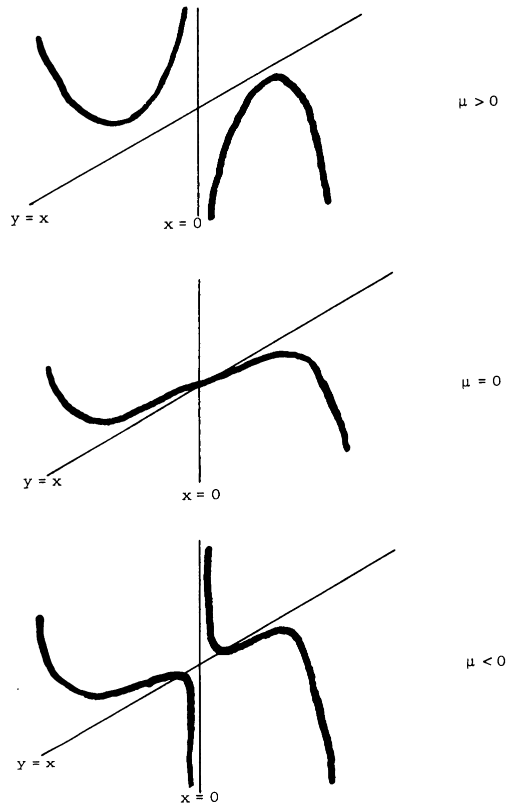

FIGURE 2. Graphs of $y = Nf_\mu(x)$ near $(x, \mu) = (0, 0)$

We list below several easily verified facts.

$$Nf_\mu(x) = Ng(x) - [\mu/g'(x)], \tag{3.3}$$

$$(Nf_\mu)'(x) = (Ng)' + [\mu g''/(g')^2]. \tag{3.4}$$

If x is a point with $g'(x) \neq 0$, then the partial derivative of Nf_μ with respect to μ at x is

$$D_\mu(Nf_\mu) = -1/g'(x). \tag{3.5}$$

It is important that this derivative is independent of μ and is nonzero whenever defined. There are some exceptional points (x_0, μ_0) where $D_\mu Nf_\mu$ does not exist, as can be seen in Figure 2 at $(x, \mu) = (0,0)$. These points are the places where $g'(x_0) = 0$ and $\mu_0 = -g(x_0)$, so that for $\mu = \mu_0$ both f_μ and f_μ' are 0 at $x = x_0$.

Note also that the bands for Nf_μ are essentially independent of μ and are the same as those for g. (As in the previous paragraph, the only exceptions are at the values of μ where f_μ has a root that is also a critical point.) Additionally, if the second derivative of g is bounded away from 0 as $|x|$ goes to infinity, then the same will be true of f_μ, so that the limits (2.7) will hold for Nf_μ. We list for reference the following trivial lemma.

LEMMA 1. *If I is a compact interval on which g' is nonzero, then Nf_μ converges uniformly to Ng and Nf_μ' converges uniformly to Ng' on I as μ goes to* 0.

PROOF. By (3.3) and (3.4) we have $Nf_\mu(x) - Ng(x) = -\mu/g'(x)$ and $Nf_\mu' - Ng' = \mu g''/(g')^2$. □

In looking for periodic attractors in Newton's Method it is useful to follow the orbits of the critical points of the Newton function. There are at least two reasons for this. First, if a critical point of Nf_μ is periodic of period k, then the orbit is necessarily superattracting. Second, if Nf_μ is a rational function, then there is a strong connection between the attracting periodic orbits of Nf_μ and the orbits of the critical points of Nf_μ.

THEOREM (JULIA, **[J, Bla]**). *If h is a meromorphic function defined on the complex plane, then every attracting periodic orbit of h contains a critical point of h in its basin.*

In what follows we will study the orbits of critical points of Nf_μ as the graph of $f_\mu(x)$ undergoes a quadratic tangency with the x-axis at $(x, \mu) = (0, 0)$. We assume that such a critical point exists; that is, we assume that there are points $p < 0 < q$ satisfying

$$g''(p) = 0 \quad \text{and} \quad g''(x) > 0 \qquad \text{on } (p, q). \tag{3.6}$$

(From (3.6), (3.1), and the mean value theorem it follows that the only point in (p, q) where either g or g' vanishes is at $x = 0$, and that $g(x) \geqslant 0$ on (p, q), $g'(x) > 0$ on $(0, q)$, $g'(x) < 0$ on $(p, 0)$.) By 2.5(b), $Ng'(0) = 1/2$, so that $Ng'(x) > 0$ for $p < x < q$. Together with (3.5) and (3.6), this shows that for x in $(p, q) - \{0\}$ and $\mu > 0$,

$$(Nf_\mu)'(x) > Ng'(x) > 0. \tag{3.7}$$

Another application of (3.5) shows that

$$D_\mu(Nf_\mu) > 0 \quad \text{on } (p, 0) \qquad \text{and} \qquad D_\mu(Nf_\mu) < 0 \quad \text{on } (0, q). \tag{3.8}$$

LEMMA 2. *Suppose $\mu > 0$. If $(Nf_\mu)^i(p) < 0$ for $0 \leqslant i < k$, then $D_\mu(Nf_\mu^k)(p) > 0$.*

PROOF. Combine (3.7), (3.8), and the chain rule. □

The hypotheses of Lemma 2 are satisfied for small values of μ:

LEMMA 3. *Given any $k > 0$ there is a constant $\mu_0 > 0$ such that if $|\mu| < \mu_0$ then*

$$p < Nf_\mu(p) < Nf_\mu^2(p) < \cdots < Nf_\mu^k(p) < 0.$$

PROOF. Ng is monotonically increasing on $[p, q]$, and for any x in $[p, q]$, $Ng(x)$ is between x and 0, so for any such x the sequence $(Ng)^i(x)$ converges monotonically to $x = 0$. In addition, for any positive constant β, Nf_μ converges C^1 uniformly to Ng on $[p, -\beta] \cup [\beta, q]$ as μ goes to 0, so if β is chosen to be less than both of $Ng^k(q)$ and $|Ng^k(p)|$, one can find μ_0 as asserted in the lemma. □

LEMMA 4. *Given any $k > 0$ there is a constant $\mu_1 > 0$ such that if $0 < \mu < \mu_1$ then*
(1) $p < Nf_\mu(p) < \cdots < Nf_\mu^{k-1}(p) < 0$;
(2) $Nf_\mu^k(p) > Nf_\mu^{k-1}(p)$;
(3) $Nf_\mu^{k-1}(p) \to 0$ *as μ increases to μ_1, so that $Nf_\mu^k(p) \to \infty$ as μ increases to μ_1.*

PROOF. Combine (3.5) and (3.7) with Lemmas 2 and 3. The idea is that as long as conclusion (1) holds, the set $\{Nf_\mu^i(p)\}$ is monotonic in i $(0 \leqslant i \leqslant k)$ and in μ $(\mu > 0)$. □

By Lemmas 3 and 4, for each $k \geqslant 1$ there is a μ-interval $[a_k, b_k]$ such that if μ is contained in $[a_k, b_k]$ then

$$(3.9) \qquad \begin{aligned} &(Nf_\mu)^i(p) \in (p, 0), \quad 1 \leqslant i < k, \qquad (Nf_\mu)^k(p) \in [0, q], \\ &(Nf_{a_k})^k(p) = 0, \qquad (Nf_{b_k})^k(p) = q. \end{aligned}$$

In addition, for each k, $a_k > b_{k+1} > 0$, and a_k tends to 0 as k goes to infinity. Define $j_k\colon [a_k, b_k] \to [0, q]$ by $j_k(\mu) = (Nf_\mu)^k(p)$. Each map j_k is monotone increasing by Lemma 2 and maps onto $[0, q]$ by (3.9).

For small positive μ, Nf_μ maps the interval $(0, q)$ over the interval $[p, 0]$ so the set $(Nf_\mu)^{-1}(p) \cap (0, q)$ is nonempty. In fact, (3.7) and (3.8) show that this set is the graph of an increasing function $x = Z(\mu)$. The curves Z and j_k are sketched in Figure 3. For all large k there is a point (μ_k, x_k) satisfying $x_k = Z(\mu_k) = j_k(\mu_k)$. Consequently, for $\mu = \mu_k$ the critical point p lies on a superattracting periodic orbit of period $k + 1$ for Nf_μ.

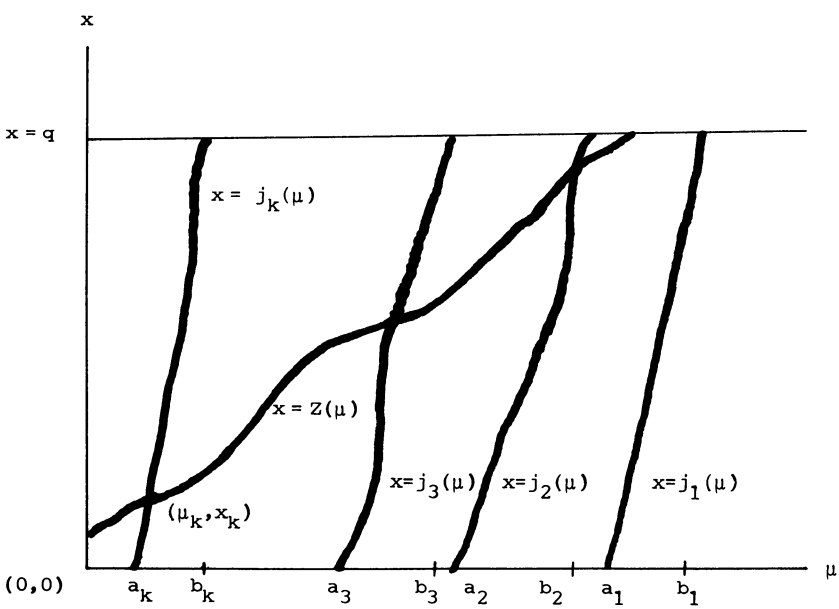

FIGURE 3

THEOREM A. *Suppose g is C^3, satisfies* (2.1), *and* $g(0) = 0$, $g'(0) = 0$, $g''(0) \neq 0$. *Let $f_\mu(x) = g(x) + \mu$. If there is a point p where g'' vanishes then for all sufficiently large values of k there will be an open interval I_k on the μ-axis such that*

(1) *for each μ in I_k, Nf_μ has an attracting periodic orbit of period k*;

(2) *each I_k contains at least one point μ_k such that p is a superattracting periodic point of Nf_μ when $\mu = \mu_k$*;

(3) *if $I_k = (L_k, R_k)$, then both L_k and R_k converge to* 0 *as k goes to infinity.*

PROOF. The previous discussion deals with the case $p < 0$ and $g''(0) > 0$. In that situation we have demonstrated the existence of the superattracting periodic orbits at values $\mu = \mu_k$. The implicit function theorem shows that this attracting periodic orbit will continue to exist for all values of μ sufficiently close to μ_k. Let I_k be an open interval containing μ_k such that for each μ in I_k Nf_μ has an attracting periodic orbit of period k that contains p in its basin. If $p > 0$ the argument is essentially the same. To deal with the case when $g''(0) < 0$, note that the Newton function for $g(x) + \mu$ is exactly the same as that for $-g(x) - \mu$, to which the previous case applies. □

Using the same notation as above, there are two possibilities concerning the band for Ng that contains q: either this band is bounded or it is not. In either case, additional arguments (and some additional hypotheses in the case when the band is unbounded) can be made to show the existence of other families of attracting periodic orbits for $\{Nf_\mu\}$.

COROLLARY A. *In addition to the hypotheses of Theorem* A *suppose that there are two points, y and z, where g' vanishes, with $y < 0 < z$, so that there are points $p < 0 < q$ with $g''(p) = g''(q) = 0$ and with $g'' \neq 0$ on (p, q). Then for each sufficiently large k there are at least two values of μ for which p is a superattracting periodic orbit of period k, and at least two values of μ for which q is a superattractng periodic orbit of period k.*

PROOF. The mean value theorem applied to g' and the assumptions on g ensure the existence of the points p, q. The assumption concerning the vanishing of g' implies that Nf_μ will have vertical asymptotes to the left of $x = p$ and to the right of $x = q$, which we may assume are located at $x = y$ and $x = z$ respectively. In the language of §2, the intervals $[y, 0]$ and $[0, z]$ are bands for Nf_μ, neither of which contains a fixed point of Nf_μ. (The graph of Nf_μ on the band $[y, 0]$ looks like the graph in band $B(3)$ in Figure 1; the graph of Nf_μ on $[0, z]$ is similarly shaped but upside down. Consequently, if μ is sufficiently close to 0 then $(Nf_\mu)^{-1}(p)$ will contain at least two points in $[0, z]$, so that for all large k this set in (μ, x) space will intersect each of the graphs j_k in at least two points; see Figure 4. As before, each of these intersections corresponds to a value of μ for which p is superattracting of period k. The existence of superattracting periodic orbits through q is demonstrated similarly. □

In the next theorem we assume that 0 is the largest critical point of g, and we make assumptions concerning the behavior of $Ng(x)$ as x goes to infinity. Specifically, we assume that $g''(x)$ is bounded away from 0 as $|x| \to \infty$ (so that $Ng(x)$

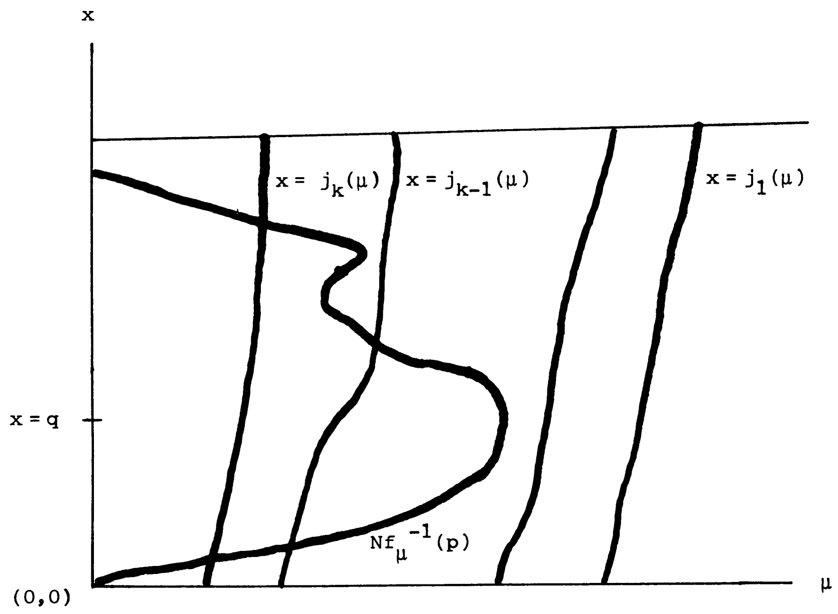

FIGURE 4

goes to infinity with x; see (2.7)) and that $(Ng)'(x)$ is both bounded and bounded away from 0 as x goes to infinity. This second condition is satisfied for all polynomials as well as for rational functions and exponentials that satisfy the first condition.

THEOREM B. *Suppose g is* C^3 *and satisfies* (2.1) *and* (3.1). *In addition, assume that*

(a) $g''(x)$ *is bounded away from* 0 *as* $|x|$ *goes to infinity*;

(b) $(Ng)'$ *is both bounded and bounded away from* 0 *as* $|x| \to \infty$;

(c) *either* (i) $g'(x) \neq 0$ *for all* $x > 0$ *and* $g''(p) = 0$ *for some* $p < 0$, *or* (ii) $g'(x) \neq 0$ *for all* $x < 0$ *and* $g''(p) = 0$ *for some* $p > 0$.

Then for each $k = 2, 3, \ldots$ *there is an open interval* M_k *on the* μ*-axis such that*

(1) *if* μ *lies in* M_k *then* Nf_μ *has an attracting periodic point of period* k;

(2) *in each interval* M_k *there is at least one value of* μ *for which this periodic orbit is superattracting*;

(3) *if* $k > 2$ *then* M_k *is disjoint from the interval* I_k *described in Theorem* A, *so the attracting periodic orbits described in* (2) *are distinct from those of Theorem* A;

(4) *either all of the* M_k *lie on the positive* μ*-axis, or else they all lie on the negative* μ*-axis*;

(5) *given any bounded subset B of the* μ*-axis,* M_k *is disjoint from B for all large k.*

PROOF. For definiteness assume that (i) holds in assumption (c). We will study the orbit of the largest negative value where g'' vanishes; for convenience we assume that this value is p. For $x, \mu > 0$,

$$x > Nf_\mu(x) = x - \left[f_\mu(x)/g'(x)\right] \tag{3.10}$$

(both terms in the quotient in brackets are positive), so by (2.6) and (2.7), for each $\mu > 0$ Nf_μ maps $(0, \infty)$ onto R. (The graph of Nf_μ on $(0, \infty)$ looks like the graph in band $B(4)$ of Figure 1.) As before, we are interested in finding values of μ for which p is an attracting periodic point of Nf_μ. Define

$$U = \left\{ (x, \mu) | x > 0,\ \mu \geqslant 0,\ Nf_\mu(x) = p \right\}.$$

The previous remarks show that the projection of U onto its second coordinate contains all $\mu > 0$.

It is useful to note that (3.1) and (c) imply that $g'(x) > 0$ for all $x > 0$, so by (a) $g''(x) > 0$ for all large x. Let W denote the open half-space

$$W = \left\{ (x, \mu) | x > 0,\ \mu \in R \right\}$$

and define $S\colon W \to R^2$ by

$$S(x, \mu) = (S_1(x, \mu), \mu) \quad \text{where } S_1(x, \mu) = Nf_\mu(x). \tag{3.11}$$

The fact that $D_\mu Nf_\mu = -1/g'(x) \neq 0$ for all $x > 0$ means that p is a regular value of S_1, so by the regular value theorem U consists of a collection of smooth curves in W. In fact, the implicit function theorem shows that each curve in U is the graph of some smooth function $\mu = h(x)$.

Claim 1. There is one such curve $\mu = H(x)$ with H defined on an interval (A, ∞), $A \geqslant 0$.

Proof. Differentiate the equality $Nf_{h(x)}(x) = p$ to obtain

$$h'(x) = \left[f_{h(x)} \cdot g''(x) \right] / g'(x) = \left[(g + h) \cdot g'' \right] / g'. \tag{3.12}$$

Each of the maps h is positive by definition, and g'' is positive and bounded away from 0 for all large x, so there is a constant T with the property that if h is any of these maps, then

$$h'(x) > 0 \quad \text{whenever } x > T. \tag{3.13}$$

Each of the curves $(x, h(x))$ is a closed submanifold of $(0, \infty) \times [0, \infty)$ so if one of the maps h is defined at some point $x_0 > T$ then this same map is defined for all $x > x_0$. To find such a curve, pick $x_0 > T$ large enough that $Ng(x_0) > p$. Since $D_\mu Nf_\mu(x_0) = -1/g'(x_0) < 0$, there is some $\mu_0 > 0$ with (x_0, μ_0) contained in U. Let $(x, H(x))$ be the specific curve in U satisfying $H(x_0) = \mu_0$. Then H is defined and positive on some maximal open subinterval $(A, \infty) \subset (0, \infty)$ as claimed.

Claim 2. $H'(x) \to \infty$ as $x \to \infty$.

Proof. Since $g'(x) \neq 0$ for $x > 0$, (3.12) gives a well-defined linear differential equation whose solution is

$$H(x) = (x + c) g'(x) - g(x)$$

for some constant c, so that

$$H'(x) = (x + c) g''(x). \tag{3.14}$$

The claim follows from (3.14) and (a).

CLAIM 3. $H(x)$ tends to 0 as x decreases to A.

PROOF. If x_0 is as in the definition of H, then $x < x_0$ implies that $H(x) < \mu_0$ (use (3.13)). In particular $H(x)$ is bounded for x between x_0 and A so there is a sequence $\{x_n\}$ decreasing to A with $H(x_n)$ converging to some value μ_*. If $A > 0$ then the map (3.11) is continuous at (A, μ_*), so $Nf_{\mu_*}(A) = p$, and since $A > 0$ the implicit function theorem can be applied to extend H to a larger domain of definition. This would contradict the definition of A unless $\mu_* = 0$. Now consider the case $A = 0$. By (3.1) and (3.12) there is a constant $Q > 0$ such that $H'(x) > 0$ if $0 < x < Q$, so near $x = 0$ $H(x)$ decreases monotonically to μ_* as x decreases to 0. In addition, for $\mu > 0$ Nf_μ tends to $-\infty$ as x decreases to 0. If $\mu_* > 0$, choose $z > 0$ such that $Nf_{\mu_*}(x) < p$ for $0 < x < z$. Since $D_\mu Nf_\mu(x) < 0$ if $x > 0$, it follows that $Nf_{H(x)}(x) < Nf_{\mu_*}(x) < p$ for $0 < x < z$. This is absurd, since $Nf_{H(x)}(x) = p$ by definition.

As a consequence of the three claims we know that the range of H contains all $\mu > 0$ (the range also contains $\mu = 0$ if $A > 0$, but not if $A = 0$). A typical curve $\mu = H(x)$ is sketched in Figure 5. Also in Figure 5 is the graph of the function $\mu \to Nf_\mu(p)$. Since $g'(p) < 0$, (3.5) shows that this function is a straight line with positive slope that crosses the upper half of the μ-axis. Claim 2 ensures that this line intersects the curve $\mu = H(x)$ as indicated. This intersection represents a value of μ for which p is a superattracting period two orbit.

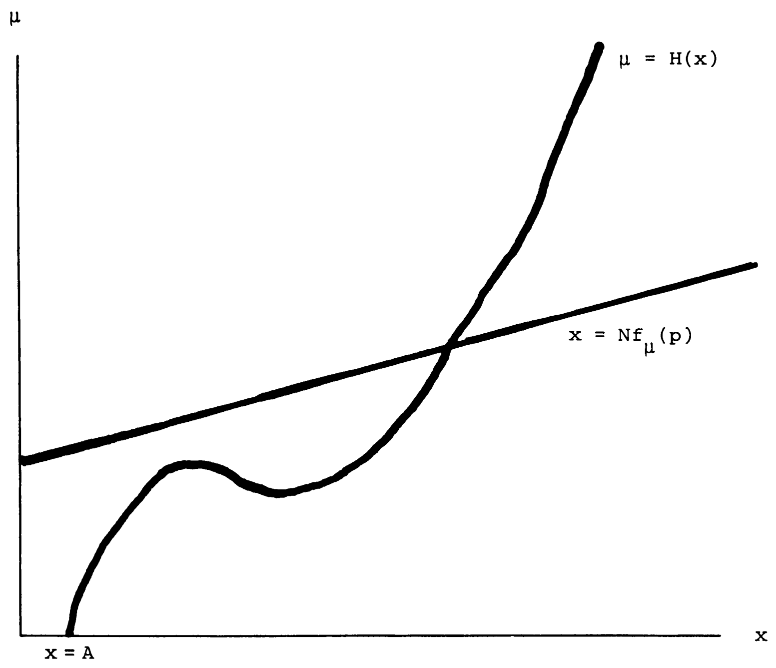

FIGURE 5

To find superattracting orbits of higher periods, we proceed as follows. Let V_2 be the line segment

$$V_2 = \{ S(p, \mu) | \mu \geqslant 0 \} \cap W$$

where S is the map defined in (3.11). V_2 is the segment depicted in Figure 5. For $k > 2$ define

$$V_k = S(V_{k-1}) \cap W.$$

If (x, μ) is in V_k then $(Nf_\mu)^k(p) = x$, so if such a point is also on the graph of H, (i.e., if $\mu = H(x)$) then p is a superattracting periodic point of period $k + 1$ for $Nf_{H(x)}$.

CLAIM 4. Each curve V_k intersects the graph of H.

PROOF. The derivative of S is given by

$$DS_{(x,\mu)} = \begin{pmatrix} (Nf_\mu)'(x) & -1/g'(x) \\ 0 & 1 \end{pmatrix}.$$

This map takes a tangent line of slope m at (x, μ) to one of slope $m/[Nf_\mu'(x) - (m/g'(x))]$ at $S(x, \mu)$. If there are bounds on m of the form $B > m > b > 0$ then assumptions (a) and (b) show that as long as x is large the slope of the tangent line at $S(x, \mu)$ is also bounded and bounded away from 0. Since V_2 is a straight line, it follows that if (x, μ) lies on V_2 and x is sufficiently large, then there are constants $T_2 > T_1 > 0$ such that the slope of the tangent line to $V_3 = S(V_2)$ is contained in (T_1, T_2). Note also that S moves points in the first quadrant of the (x, μ) plane to the left ($S(x, \mu) = (y, \mu)$ with $y < x$) so the projection of V_3 on the x-axis is equal to $(0, \infty)$. Consequently, for sufficiently large x V_3 is a graph of the form $\mu = L_3(x)$ with L_3' bounded and bounded away from 0. Now the preceding argument can be applied inductively with V_{k+1} in place of V_k to show

(i) $\{x |$ for some $\mu > 0$ $(x, \mu) \in V_k\} = (0, \infty)$;

(ii) $\{\mu |$ for some $x > 0$ $(x, \mu) \in V_k\} = (m_k, \infty)$ with $m_k > 0$;

(iii) for large x, V_k is a graph of the form $\mu = L_k(x)$ with L_k' bounded and bounded away from 0.

Because $H'(x)$ goes to infinity with x and $H(x)$ goes to 0 as x decreases to $A \geqslant 0$, (i)–(iii) suffice to establish the claim. See Figure 6.

PROOF OF THE THEOREM. For each $k > 1$ choose a parameter value μ_k such that $\mu_k = H(x_k)$ and $(x_k, \mu_k) \in V_k$. Let M_k be an open interval containing μ_k such that for each μ in M_k Nf_μ has an attracting periodic orbit of period k containing p in its basin. For μ in M_k there is only one point on this attracting periodic orbit that is on the same side of the vertical asymptote at $x = 0$ as p, so assertion (3) holds. Assertion (4) follows from conclusion (ii) in the proof of Claim 4.

To establish (5) it is enough to show that μ_k goes to infinity with k. Recall that S moves points in the first quadrant of the (x, μ) plane to the left, so for $k > 2$, V_k lies above the line V_2 in the first quadrant. Consequently the interval M_1 lies between 0

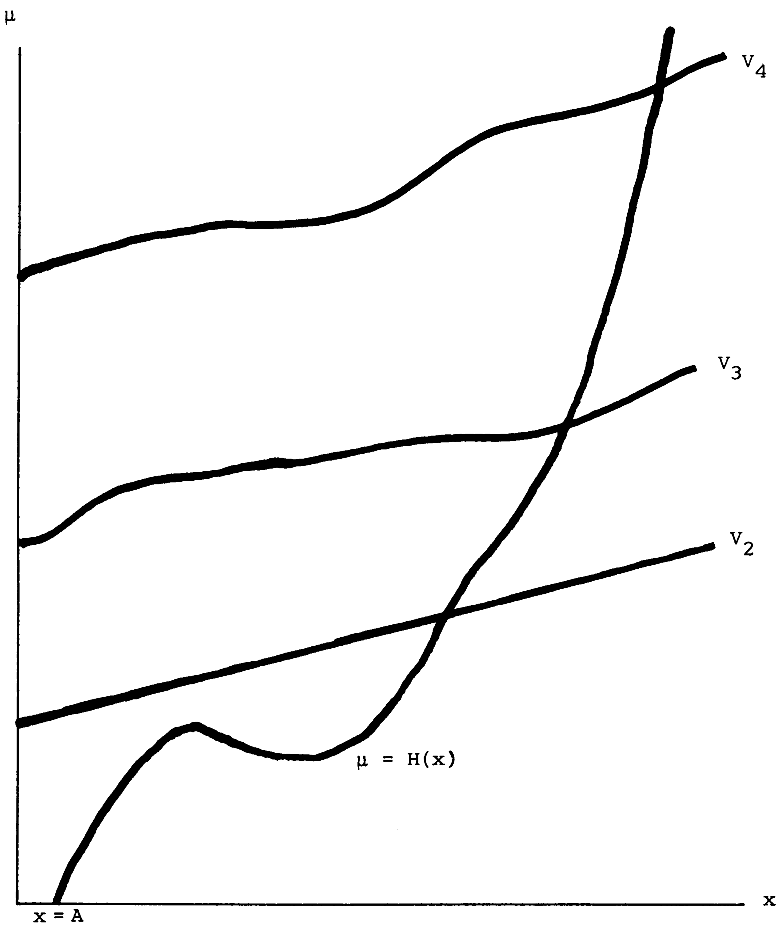

FIGURE 6

and M_k on the μ-axis, and so the collection $\{\mu_k\}$ is bounded away from 0. As described above, for $\mu = \mu_k$ the first $k - 1$ iterates of p by Nf_μ are all bigger than 0. Since $\{\mu_k\}$ is bounded away from 0, this means that the sequence $Nf_{\mu_k}(p)$ must go to infinity. But $Nf_{\mu_k}(p) = Ng(p) - [\mu_k/g'(p)]$ so the sequence μ_k must also go to infinity. □

The two theorems describe the "simplest" attracting orbits. Similar techniques can be used to find additional families of attracting periodic orbits. For instance, one can combine the proofs of Theorems A and B to obtain the following.

COROLLARY B. *Keep the assumptions and notation of Theorem* B. *For any pair of positive integers* m, n *there is a value* $\mu = \mu(m, n)$ *such that*

(1) *p is a periodic point of Nf_μ of period $m + n$;*

(2) *$(Nf_\mu)^j(p)$ and p have the same sign for $0 \leqslant j < m$;*

(3) *$(Nf_\mu)^j(p)$ and p have the opposite sign for $m \leqslant j < n$.*

The parameter values $\mu(m, n)$ can be chosen to satisfy

(4) *$\mu(m_1, n_1)$ is between 0 and $\mu(m_2, n_2)$ if either $m_1 > m_2$ or both $m_1 = m_2$ and $n_2 > n_1$.*

REMARK. If one has more information about the behavior of Ng on each of its bands, then one can use the standard techniques of the theory of continuous maps of the real line ("kneading theory", the understanding of the phenomenon of period-doubling, etc.) to find other families of periodic attractors in $\{Nf_\mu\}$. A good reference for this theory is **[CoE]**.

4. Numerical results. The results of the previous section establish the existence of parameter values $\mu = \mu_k$ for which Nf_μ has an attracting periodic orbit of period k. In some cases these values μ_k appear to scale geometrically and so can be found fairly easily. We will describe the results of a numerical study of a certain family of cubic polynomials below. A previous study of Newton's Method for cubics can be found in **[CuGS]**, which considers Newton's Method for the complex polynomials $p_A(z) = z^3 + (A - 1)z - A$, where A is a complex parameter. Among the things that are observed for this family is the existence of various periodic attractors. Since p_A depends on A in a more complicated way than f_μ depends on μ, the results of the current paper do not apply directly to the family p_A, even if we consider only real values of A and z. Nonetheless, there is apparently a connection between some of the attracting orbits found in **[CuGS]** and those predicted by Theorems A and B. From the point of view of the current paper, the interesting parameter values in the family p_A are $A = 1/4$ and $A = 1$. (For the rest of the discussion we consider only real values of A so that p_A can be thought of as a real mapping.) The number of real roots of p_A changes from 3 to 2 to 1 as A increases past $1/4$ (p_A has a double root at $A = 1/2$), so the attractors predicted by Theorem A seem to be related to those found by **[CuGS]** near $A = 1/4$. On the other hand, as A increases past 1, the two real critical points of p_A move together and then go complex. In terms of the results of this paper, it seems that the essential features of Theorem B are satisfied for p_A with A increasing towards 1, so again it would appear that the orbits described in Theorem B correspond to some of these bound numerically near $A = 1$ in **[CuGS]**. (Because of the more complicated parameter dependence in the family $\{p_A\}$, the parameter interval $1/4 < A < 1$ corresponds to the parameter interval $0 < \mu < \infty$.) We strongly urge the reader to consult **[CuGS]**, especially its Figure 3.2.

Next we describe a numerical study of the attracting orbits that arise as predicted by Theorem B for a one parameter family of cubics with simpler parameter dependence than in the family $\{p_A\}$.

EXAMPLE. $f_\mu(x) = x^3 - 3x + \mu$.

This family of maps has a quadratic tangency with the x-axis at $x = 1$ when $\mu = 2$. The only zero of f_μ'' is at $x = 0$, so Julia's Theorem implies that if Nf_μ has an attracting periodic orbit of period greater than 1, then the iterates of 0 must tend towards that orbit. By following the iterates of 0 on a computer we have found the periodic orbits described in Theorem B up to periods of nearly 40. Some of the data are sumarized in Table 1. As usual, we let μ_k denote the parameter value for which 0 is periodic of period k. Numerically μ_{k+1}/μ_k appears to tend to a limit of $1.8371\ldots$ which to several decimal place accuracy is equal to $(1.5)^{1.5}$. The following argument, although not rigorous, seems to explain why this constant occurs.

TABLE 1. Approximate parameter values (μ) for which the Newton function Nf_μ of the map $f_\mu(x) = x^3 - 3x + \mu$ has a superattractnig periodic orbit as described in Theorem B.

Period (k)	Parameter value (μ_k)	μ_k/μ_{k-1}
2	3.674234614	----
3	7.751340820	2.10964775
4	15.556213393	2.00690612
5	30.155532349	1.93848795
6	57.264691352	1.8989779
7	107.416273052	1.87578541
16	26585.780130	1.83799350
17	48856.706076	1.83770067
18	89774.484656	1.83750587
19	164949.501860	1.83737620
20	303060.043608	1.83728983
21	556791.700373	1.83723229
30	132732626.08139	1.83712029
31	243845669.13925	1.83711193
32	447973423.03963	1.83711186
33	822980125.40498	1.83711181
34	1511911517.77782	1.83711178
35	2777559411.32909	1.83711177
36	5102703195.02650	1.837111756
37	9374265245.36477	1.837111748
38	17221626017.00249	1.837111742

For any fixed value of μ the quantity $|Nf_\mu(x) - 2x/3|$ tends to 0 as x goes to infinity. Trivial calculations show that $Nf_\mu(0) = \mu/3$ and $(Nf_\mu)^{-1}(0) = (\mu/2)^{1/3}$, so if 0 is a periodic point of period k for Nf_μ then all of the other iterates of 0 lie between these two values, and

(4.1) $$Nf_\mu^{k-2}(\mu/3) = (\mu/2)^{1/3}.$$

By assuming that Nf_μ can be well approximated by $2x/3$ on this interval one obtains an approximate version of (4.1):

(4.2) $$2^{k-2}\mu/3^{k-1} = (\mu/2)^{1/3}.$$

Solving (4.2), one finds $\mu = C \cdot (1.5)^{1.5k}$ where C is a constant independent of k. Using this value as an approximation of μ_k, one obtains the approximate value $(1.5)^{1.5}$ for the ratio μ_{k+1}/μ_k.

The problem with this argument is that μ is growing with k, and $|Nf_{\mu_k}(x) - 2x/3|$ does not tend to 0 as k grow for values of x on the orbit of 0 and between $\mu/3$ and $(\mu/2)^{1/3}$. The numerical results in Table 1 can be viewed as evidence that the approximation $2x/3$ of $Nf_\mu(x)$ is good enough at most of the relevant points along the orbit so that $(Nf_\mu)^{k-2}(x)$ is well approximated by $(2/3)^{k-2} \cdot x$.

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