TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 297, Number 1, September 1986

A REGULARITY THEOREM FOR MINIMIZING HYPERSURFACES MODULO ν

FRANK MORGAN

ABSTRACT. It is proved that an $(n-1)$-dimensional, area-minimizing flat chain modulo ν in $\mathbf{R}^n$, with smooth extremal boundary of at most $\nu/2$ components, has an interior singular set of Hausdorff dimension at most $n-8$. Similar results hold for more general integrands.

1. Introduction. This paper gives a regularity theorem for hypersurfaces that minimize area for a given boundary in the class of flat chains modulo ν. It is well known that area-minimizing flat chains modulo ν, unlike rectifiable currents, can have codimension-1 singular sets. Nevertheless, it is shown that certain global hypotheses on the boundary imply that area-minimizing flat chains modulo ν enjoy the same strong regularity as rectifiable currents.

1.1. *Area-minimizing flat chains modulo ν.* The surface of least area with a given boundary depends, of course, on the class of admissible surfaces. Consider, for example, the boundary B of Figure 1(1), consisting of two close, parallel, unit circles, both oriented counterclockwise. In the standard class of oriented surfaces called rectifiable currents (integral flat chains), the surface of least area consists of two disks as in Figure 1(2). Next consider the standard class of unoriented surfaces called flat chains modulo 2 (since $+1 \equiv -1 \pmod 2$, orientation is ignored). Here the surface of least area is the catenoid of Figure 1(3), previously disallowed by the

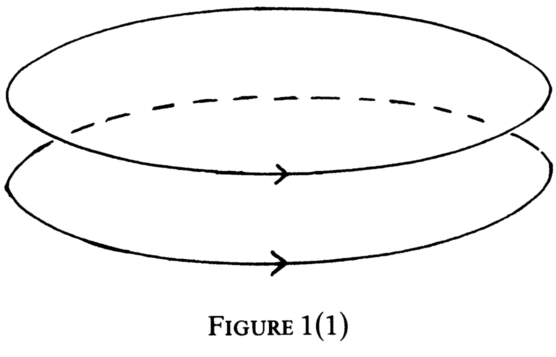

FIGURE 1(1)
A boundary *B* of 2 parallel circles

Received by the editors February 11, 1985 and, in revised form, September 27, 1985.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 49F22, 53C42.
Key words and phrases. Area-minimizing, flat chain modulo ν, regularity.

0002-9947/86 $1.00 + $.25 per page

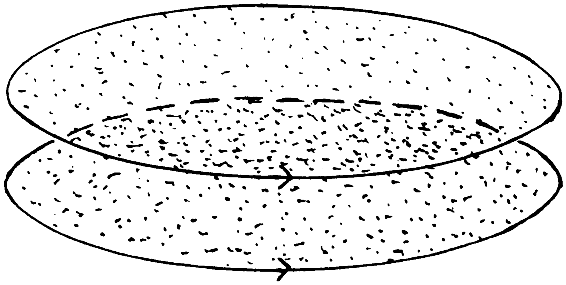

FIGURE 1(2)
The area-minimizing rectifiable current is 2 disks

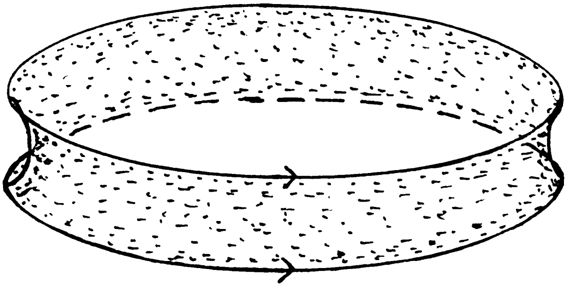

FIGURE 1(3)
The area-minimizing flat chain modulo 2 is the catenoid

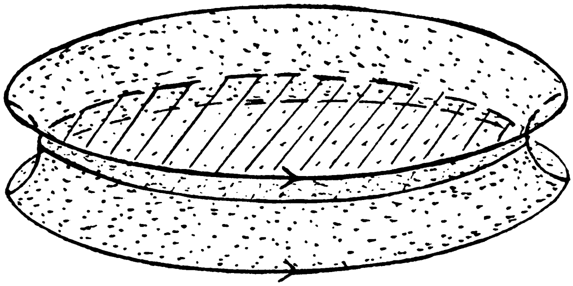

FIGURE 1(4)
The area-minimizing flat chain modulo 3

given boundary orientation. Finally, among flat chains modulo 3, the presumed surface of least area, as pictured in Figure 1(4), consists of two catenoids and a disk, meeting along an auxiliary circle. The three pieces are oriented so that the auxiliary circle occurs as boundary with multiplicity 3, and hence does not count as boundary at all modulo 3.

All three pictured surfaces occur as soap films, the surface of Figure 1(4) the most easily obtained. Moreover, beyond their own intrinsic and geometric interest, flat chains modulo ν have proved useful in regularity arguments for standard rectifiable currents [**W1**, **M3**, proof of Proposition 4.1].

1.2. DEFINITIONS. Flat chains modulo ν are defined as congruence classes of integral flat chains, where $T \equiv 0 \pmod{\nu}$ means either that T equals νS, for some integral flat chain S, or more generally that T is the limit of a sequence νS_j in the flat norm. In a natural way one defines for any flat chain modulo ν T a boundary ∂T, an area or mass $\mathbf{M}^\nu(T)$, and a support $\operatorname{spt}^\nu T$. A full discussion of flat chains modulo ν appears in [**F1**, 4.2.26].

1.3. *Regularity.* (Cf. *Open problems in geometric measure theory*, #4.20, [**B**].) Let T be an $(n-1)$-dimensional, area-minimizing flat chain modulo ν in $\mathbf{R}^n$. Let E be the interior singular set, i.e., the points where $\operatorname{spt}^\nu T - \operatorname{spt}^\nu \partial T$ fails to be a smooth, embedded manifold. Examples such as pictured in Figure 1(4) show that for $\nu \geqslant 3$, E can have Hausdorff dimension $n-2$. A few positive results are known. For $\nu = 2$, the Hausdorff dimension of E is at most $n-8$ ([**F2**], e.g. with [**M2**, 4.2]). For $\nu = n = 3$, J. Taylor [**T**] proved that E consists of smooth curves along which 3 minimal surfaces meet at equal angles. For $\nu = 4$, B. White [**W3**] proved that the Hausdorff dimension of E is at most $n-2$; indeed, that $\operatorname{spt}^\nu T - \operatorname{spt}^\nu \partial T$ is an immersed manifold except for a set of Hausdorff dimension at most $n-8$. Finally, for ν odd, White [**W2**] has recently proved that the Hausdorff dimension of E is at most $n-2$.

Our Regularity Theorem 2.4 shows that for all ν, if ∂T is a compact, extremal C^1 submanifold of $\mathbf{R}^n$ of at most $\nu/2$ components, then the Hausdorff dimension of E is at most $n-8$. Thus, a certain, nonlocal hypothesis ensures that area-minimizing flat chains modulo ν share the strong regularity of rectifiable currents. Similar results (see 2.4) hold for more general integrands. There are also consequences (2.8) on finiteness for the number of area-minimizing flat chains modulo ν with a fixed boundary.

1.4. *The theorem and proof.* Here we state a corollary of our main result and sketch the proof.

THEOREM (COROLLARY 2.5). *For $n \geqslant 2$, $\nu \geqslant 2$, let B be an $(n-2)$-dimensional, compact, oriented, extremal, C^1 submanifold of $\mathbf{R}^n$, of at most $\nu/2$ components. Let T be an $(n-1)$-dimensional, area-minimizing flat chain modulo ν with $\partial T \equiv B$* (mod ν). *Then* $\operatorname{spt}^\nu T - B$ *is an embedded, minimal manifold except for a singular set of Hausdorff dimension at most $n-8$.*

The proof reduces the question to the well-known regularity results for rectifiable currents. First, a decomposition theorem (2.2), following another of B. White, shows that T has a decomposition $T \equiv \sum_{j=1}^\nu T_j$ into rectifiable currents T_j such that $\mathbf{M}^\nu(T) = \sum_{j=1}^\nu \mathbf{M}(T_j)$ and $\partial T_j = B_j + Q$, where the B_j are (possibly empty or disconnected) components of $B = \sum B_j$. Since B has at most $\nu/2$ components, we may assume that $B_j = 0$ for $j > \nu/2$. Consider the rectifiable current

$$S = \sum_{j=1}^{[\nu/2]} T_j - \sum_{j=[\nu/2]+1}^{2[\nu/2]} T_j.$$

Now

$$\partial S = \sum B_j + \left[\frac{\nu}{2}\right] Q - \left[\frac{\nu}{2}\right] Q = B, \quad \text{and} \quad \mathbf{M}(S) \leqslant \sum \mathbf{M}(T_j) = \mathbf{M}^\nu(T).$$

Since T is area-minimizing, therefore equality must hold, S as a sum involves no cancellation, $\operatorname{spt} S = \operatorname{spt}^\nu T$, and S is an area-minimizing rectifiable current. Therefore by the standard regularity theory [**F2**], $\operatorname{spt}^\nu T - B = \operatorname{spt} S - \operatorname{spt} \partial S$ is an embedded, minimal manifold except for a set of Hausdorff dimension at most $n-8$. A fortiori, T has a representative R modulo ν such that $\operatorname{spt}\partial R = \operatorname{spt}^\nu \partial T$.

ACKNOWLEDGMENTS. The author appreciates some helpful early conversations with Noga Alon and Gil Kalai. He wants to thank the referee for a correction and improvement of the main theorem 2.4. This work was partially supported by a National Science Foundation grant.

2. The regularity theorem. The main result of this paper, Regularity Theorem 2.4, depends on Decomposition Theorem 2.2, in turn based on a similar result of B. White [**W2**, §2]. It is convenient first to state a formulation of the constancy theorem [**F1**, 4.1.31] in the context of flat chains modulo ν. The proof is essentially the same as for rectifiable currents (cf. [**F1**, pp. 431–432]).

2.1. CONSTANCY THEOREM. *Let C be a connected, m-dimensional, C^1 embedded manifold in $\mathbf{R}^n$. For $\nu \geqslant 2$, let T be an m-dimensional flat chain modulo ν of finite mass with*

$$T = T \llcorner C, \quad (\operatorname{spt}^\nu \partial T) \cap C = \varnothing.$$

If C is nonorientable, then ν is even and T is the associated flat chain modulo ν of multiplicity $\nu/2$. Otherwise we may assume C is an oriented manifold and also denote by C the associated current; then $T \equiv lC \pmod{\nu}$ for some $-\nu/2 < l \leqslant \nu/2$.

Temporarily it will be convenient to distinguish between an oriented, embedded manifold M of compact closure and finite area and the associated rectifiable current, to be denoted by $\mathbf{t}(M)$.

2.2. DECOMPOSITION THEOREM. *For $n \geqslant 2$, $\nu \geqslant 2$, let T be an $(n-1)$-dimensional rectifiable current in $\mathbf{R}^n$. Suppose there is an $(n-1)$-dimensional, oriented, embedded, C^1 manifold M of compact closure and finite area such that $\partial\mathbf{t}(M) \equiv \partial T \pmod{\nu}$ and $M \cap \operatorname{spt} T = \varnothing$. Then there are rectifiable currents $T_1, T_2, \ldots, T_\nu$, a decomposition of M into (perhaps empty or disconnected) components $M_1, M_2, \ldots, M_\nu$, and an integral flat chain Q such that*

$$T = \sum T_i, \quad \|T\| = \sum \|T_i\|, \quad \partial T_i = \partial\mathbf{t}(M_i) + Q \qquad (1 \leqslant i \leqslant \nu).$$

PROOF. Since $\partial(T - \mathbf{t}(M)) \equiv 0 \pmod{\nu}$, by a decomposition theorem of B. White [**W2**, §2], there are rectifiable currents $S_1, S_2, \ldots, S_\nu$ such that $T - \mathbf{t}(M) = \Sigma_{i=1}^{\nu} S_i$, $\|T - \mathbf{t}(M)\| = \Sigma_{i=1}^{\nu}\|S_i\|$, and $\partial S_i = \nu^{-1}(\partial T - \partial\mathbf{t}(M))$. Let $Q = \nu^{-1}(\partial T - \partial\mathbf{t}(M)) = \partial S_i$. Let $T_i = S_i \llcorner \operatorname{spt} T$. Note that $T = \Sigma T_i$. Also,

$$\sum \mathbf{M}(T_i) = \left(\sum \|S_i\|\right)(\operatorname{spt} T) = \|T - \mathbf{t}(M)\|(\operatorname{spt} T) = \mathbf{M}(T),$$

because $M \cap \operatorname{spt} T = \varnothing$. Hence $\Sigma\|T_i\| = \|T\|$.

The description of ∂T_i remains to be proved. First,

$$\sum (T_i - S_i) = \sum T_i - \sum S_i = T - (T - \mathbf{t}(M)) = \mathbf{t}(M).$$

Second, since $\Sigma\|S_i\| = \|\Sigma S_i\|$,

$$\sum \|T_i - S_i\| = \left\| \sum (T_i - S_i) \right\| = \|\mathbf{t}(M)\|.$$

Third, since $\operatorname{spt}\partial(T_i - S_i) \subset \operatorname{spt} T$,

$$\operatorname{spt}\partial(T_i - S_i) \cap M = \varnothing.$$

It follows from these three facts and the Constancy Theorem 2.1 that M is the union of ν disjoint submanifolds $M_1, M_2, \ldots, M_\nu$, each open and closed in M, such that $T_i - S_i = \mathbf{t}(M_i)$. Hence

$$\partial T_i = \partial \mathbf{t}(M_i) + Q,$$

as asserted.

2.3 Lemma. *Let Φ be an $(n-1)$-dimensional parametric integrand on $\mathbf{R}^n$. Let T_i $(1 \leqslant i \leqslant N)$ be $(n-1)$-dimensional rectifiable currents in $\mathbf{R}^n$. Suppose that $T = \Sigma T_i$ is Φ-minimizing and $\|T\| = \Sigma \|T_i\|$. Then for any nonnegative integers r_i, the current $\Sigma r_i T_i$ is Φ-minimizing.*

Proof. Let $r = \max r_i$. Since any portion of a Φ-minimizing current is Φ-minimizing, it suffices to prove that rT is Φ-minimizing.

Let S be any rectifiable current with $\partial S = \partial(rT)$. By White's decomposition theorem [**W2**, §2], there are rectifiable currents S_i $(1 \leqslant i \leqslant r)$ such that $S = \Sigma S_i$, $\|S\| = \Sigma \|S_i\|$, and $\partial S_i = T$. Hence $\Phi(S) = \Sigma \Phi(S_i) \geqslant \Sigma \Phi(T) = \Phi(rT)$. Therefore rT is Φ-minimizing.

2.4. Regularity Theorem. *For $n \geqslant 2$, $\nu \geqslant 2$, let Φ be a C^3, elliptic, parametric, even, $(n-1)$-dimensional integrand in $\mathbf{R}^n$. Let T be an $(n-1)$-dimensional Φ-minimizing flat chain modulo ν. Suppose that there is an $(n-1)$-dimensional, oriented, embedded, C^1 manifold M of compact closure and finite area, with current boundary B, such that $\partial T \equiv B \pmod{\nu}$ and $M \cap \operatorname{spt}^\nu T = \varnothing$. Suppose that the number k of connected components of M satisfies $k \leqslant \nu/2$. Then $\operatorname{spt}^\nu T - \operatorname{spt} B$ is a C^2, embedded manifold except for a singular set of $\mathscr{H}^{n-3}$ measure 0. If Φ is the area integrand, then the singular set has Hausdorff dimension at most $n-8$.*

In fact, T has a representative R modulo ν such that $\operatorname{spt} \partial R = \operatorname{spt} B$. If $k < \nu/2$, then $\partial R = B$. If $k = \nu/2 = 1$, then R may be chosen such that $\partial R = B$. If $k = \nu/2 > 1$ and $\operatorname{spt} B$ is a C^1 submanifold, then R may be chosen such that $\|\partial R\| = \|B\|$.

Example 1. Let B be the three meridian semicircles on the unit sphere at longitudes 0°, 120°, and 240°, the first oriented downward with multiplicity 2, the others oriented upward with multiplicity 1. See Figure 2.4(3). The truncated cone Y

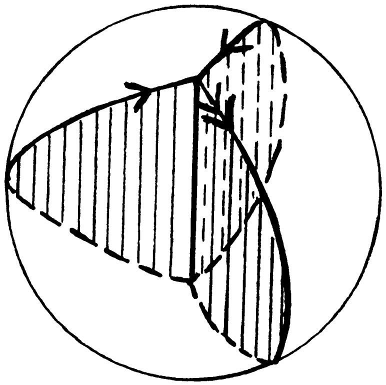

Figure 2.4(3)
The truncated cone Y over B

over this boundary is area-minimizing modulo 3. Note that Y has a 1-dimensional singular set, a segment along the polar axis. To apply the theorem, note that B bounds a suitable manifold M in $\mathbf{R}^3 - \mathbf{B}^3(0,1)$ of two components. Let T be an area-minimizing flat chain modulo ν with $\partial T \equiv B \pmod{\nu}$, for $\nu \geqslant 4$. Since $\operatorname{spt}^\nu T \subset \mathbf{B}^3(0,1)$, $M \cap \operatorname{spt}^\nu T = \varnothing$. Therefore $\operatorname{spt}^\nu T - B$ has no singularities at all. Presumably it looks like Figure 2.4(4). Finally, mod 2, B consists of two semicircles and bounds a connected manifold M' outside the unit ball. Again the theorem implies that any area-minimizing flat chain mod 2 is regular. See Figure 2.4(5). Only for modulo 3 are there interior singularities.

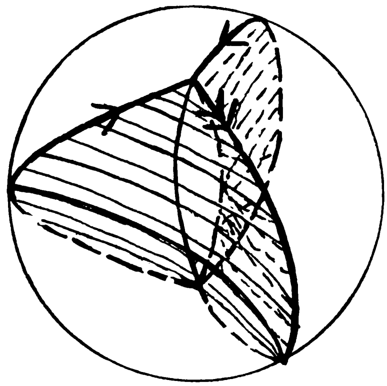

FIGURE 2.4(4)
A minimal surface with boundary B

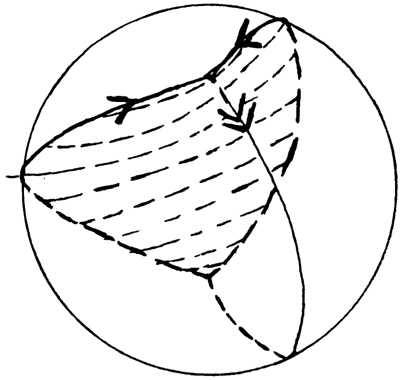

FIGURE 2.4(5)
A minimal surface with boundary B (mod 2)

EXAMPLE 2. Let Γ be a system of oriented, C^1 Jordan curves on the unit sphere in $\mathbf{R}^3$. Let l be the number of pairs of adjacent, similarly-oriented components of Γ. See Figure 2.4(6). Then there is a C^1 manifold M with boundary Γ of $l + 1$ components such that M intersects the unit ball only in Γ. (Obviously there is such a manifold M with at most as many components of Γ. But if two adjacent components of Γ have opposite orientation, the associated components of M may be connected by a tube that respects orientation.) In particular, no area-minimizing surface bounded by Γ intersects the interior of M. Consequently, by the Regularity Theorem 2.4, for $\nu \geqslant 2l+2$, if T is an area-minimizing flat chain mod ν bounded by Γ, then $\operatorname{spt}^\nu T - \Gamma$ is an embedded, minimal manifold.

REMARK 1. The inequality on the number of components of M, $k \leqslant \nu/2$, is sharp for $\nu \geqslant 3$, even for minimizing length in $\mathbf{R}^2$. Indeed, given $\nu \geqslant 3$, let $k = [\nu/2]+1$, $\varepsilon = 1/k^2$. Let B be $2k$ oriented points

$$B = \sum_{j=1}^{k} \left([(1, j\varepsilon)] - [(-1, j\varepsilon)]\right).$$

It can be shown that a length-minimizing flat chain modulo ν with $\partial T \equiv B \pmod{\nu}$ must have an interior singularity. See Figure 2.4(7).

For $\nu = 2$, the hypothesis $k \leqslant \nu/2$ is superfluous in proving that T has a representative R modulo 2 such that $\operatorname{spt} \partial R = \operatorname{spt} B$ and consequently in proving interior regularity for T [**M2**, Lemma 4.2].

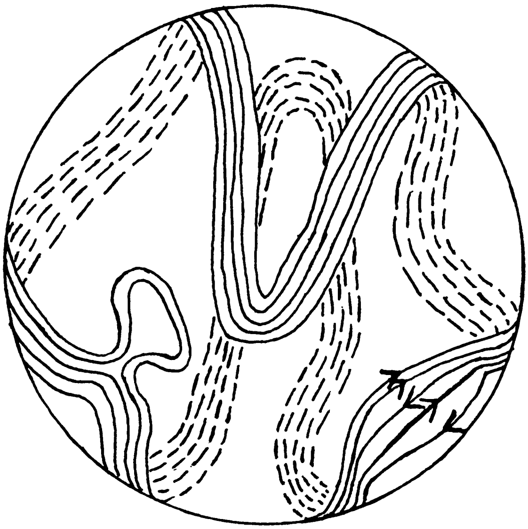

FIGURE 2.4(6)
A system Γ of curves with $l = 1$.
Regularity holds for $\nu \geqslant 2l + 2 = 4$

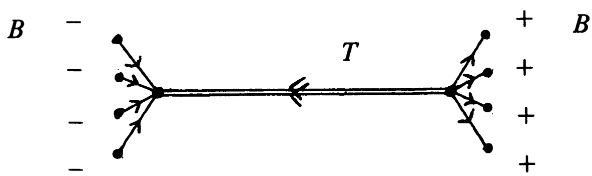

FIGURE 2.4(7)
A singular curve, presumably length minimizing mod 6

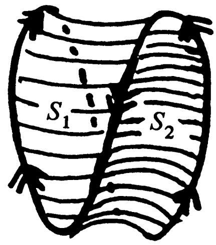

FIGURE 2.4(8)
$T = -S_1 - S_2$ has no representative
R mod 4 with $\partial R = B$

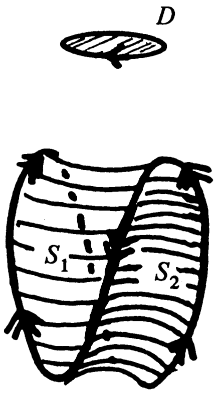

FIGURE 2.4(9)
$T = -S_1 - S_2 + D$ has no representative
R mod 4 with $\partial R = \pm B$

REMARK 2 (with thanks to the referee). For $k = \nu/2$, it is not always possible to choose a representative R for T with $\partial R = B$. For example, for $k = \nu/2 = 2$, let B_0 be an oriented curve in the sphere in $\mathbf{R}^3$ which bounds two distinct area-minimizing rectifiable currents, S_1 and S_2. Let $B = 2B_0$, $R = -S_1 - S_2$. Then the equivalence class T of R modulo 4 is area-minimizing, $\partial R = -2B_0 \equiv B \pmod 4$, and R is the unique representative for T modulo 4. See Figure 2.4(8).

This example may be modified to yield $\partial R \neq \pm B$ as follows. Let D be a small disc a distance above, let $B = 2B_0 + \partial D$, and let $R = -S_1 - S_2 + D$. Note $\partial R = -2B_0 + D \equiv B \pmod 4$, $\partial R \neq \pm B$, but $\|R\| = \|B\|$. See Figure 2.4(9).

REMARK 3. Suppose that the hypothesis that T be Φ-minimizing is dropped. For $\nu = 2$, T still has a representative R modulo ν with $\partial R = B$ (cf. [**M2**, 4.2]). For $\nu \geqslant 3$, T need not even have a representative R with $\operatorname{spt} \partial R \subset \operatorname{spt} B$, as Figure 2.4(10) indicates.

PROOF OF THEOREM. Since T is Φ-minimizing, $\mathbf{M}^\nu(T) < \infty$ and T is rectifiable. Applying the Decomposition Theorem 2.2 to a representative R mod ν for T yields rectifiable currents $T_1, T_2, \ldots, T_\nu$, a decompositon of M into components M_1, $M_2, \ldots, M_\nu$, and an integral flat chain Q such that $R = \Sigma T_i$, $\|T\|^\nu = \|R\| = \Sigma\|T_i\|$,

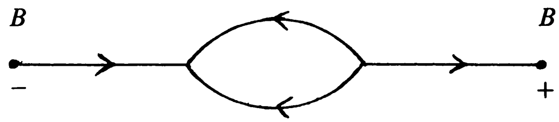

FIGURE 2.4(10)
T has no representative R mod 3
with spt $\partial R \subset$ spt B

and $\partial T_i = \partial \mathbf{t}(M_i) + Q$. Since M has k components, we may assume that when $i > k$, $M_i = \varnothing$. Let $S = \sum_{i=1}^{k} T_i - \sum_{i=k+1}^{2k} T_i$. Then $\partial S = \partial \mathbf{t}(M) = B$ and $\Phi(S) \leqslant \sum_{i=1}^{\nu} \Phi(T_i) = \Phi(T)$. Since T is Φ-minimizing, therefore equality holds, $\operatorname{spt} S = \operatorname{spt}^{\nu} T$, and S is a Φ-minimizing rectifiable current. Therefore by the standard regularity theory [**SSA**, Theorem II.7, **F2**], $\operatorname{spt}^{\nu} T - \operatorname{spt} B = \operatorname{spt} S - \operatorname{spt} \partial S$ is a C^2 orientable, embedded manifold except for a singular set of the size asserted.

In particular, by the Constancy Theorem 2.1, we may assume that $\operatorname{spt} \partial R \subset \operatorname{spt} B$. Suppose $k < \nu/2$. Since

$$\sum_{i=1}^{2k} \Phi(T_i) = \Phi(S) = \Phi(T) = \sum_{i=1}^{\nu} \Phi(T_i),$$

then $T_\nu = 0$, $Q = 0$, and $\partial R = B$.

If $k = \nu/2 = 1$, then S is a representative for T modulo 2 with boundary B.

Finally, suppose $k = \nu/2 > 1$. Let X be the oriented manifold on which $T_{k+1}, \ldots, T_{2k}$ all coincide, each with multiplicity 1. By subtracting X from all $2k$ T_i's (which does not change the congruence class of $R = \sum T_i$ modulo ν) and subtracting ∂X from Q, we may henceforth assume that $X = \varnothing$.

Let

$$T' = \sum_{i=1}^{k} T_i = \frac{R+S}{2} \quad \text{and} \quad T'' = \sum_{i=k+1}^{2k} T_i = \frac{R-S}{2}.$$

Under the hypothesis that $\operatorname{spt} B$ is a C^1 submanifold, it follows from the Constancy Theorem 2.1 that ∂R, ∂S, and hence $\partial T'$, $\partial T''$ are just sums of oriented components C of $\operatorname{spt} B$. We will proceed by cases to prove that $\partial R \llcorner C = \pm B \llcorner C$, which implies that $\|\partial R\| = \|B\|$.

Case 1: $C \not\subset \operatorname{spt} \partial T'$. In this case $\partial R \llcorner C = -\partial S \llcorner C = -B \llcorner C$.

Case 2: $C \not\subset \operatorname{spt} \partial T''$. In this case $\partial R \llcorner C = \partial S \llcorner C = B \llcorner C$.

Case 3: $C \subset \operatorname{spt} \partial T' \cap \operatorname{spt} \partial T''$. In this case $\partial T' \llcorner C = lC$ and $\partial T'' \llcorner C = mC$ for nonzero integers l, m. Since both $R = T' + T''$ and $S = T' - T''$ are Φ-minimizing, by Lemma 2.3, $U = mT' - lT''$ is Φ-minimizing. Since $(\operatorname{spt} \partial U) \cap C = \varnothing$, regularity theory implies that, on C, $\operatorname{spt} R = \operatorname{spt} U$ is a C^2 embedded manifold except for a set of H^{n-3} measure 0. Since $\|T' + T''\| = \|T' - T''\| = \|T'\| + \|T''\|$, therefore $T_{k+1}, \ldots, T_{2k}$ must all lie on the same side of C and their supports coincide locally. Since the multiplicity of R is at most $k = \nu/2$, each T_i $(i \geqslant k+1)$ must occur locally with multiplicity 1. By our assumption above that $X = \varnothing$, this case does not occur.

Thus in all cases, for each component C of $\operatorname{spt} B$, $\partial R \llcorner C = \pm B \llcorner C$. Therefore $\|\partial R\| = \|B\|$, as desired.

2.5. COROLLARY. *For $n \geqslant 2$, $\nu \geqslant 3$, $\alpha > 0$, let B be an $(n-2)$-dimensional, compact, oriented, extremal, $C^{1,\alpha}$ submanifold of $\mathbf{R}^n$, of at most $\nu/2$ connected components. Let T minimize area among all rectifiable currents with $\partial T \equiv B \pmod{\nu}$. Then $\partial T = B$. Consequently, $\operatorname{spt} T$ is a $C^{1,\alpha}$ submanifold with boundary B, except for a compact singular set disjoint from B of Hausdorff dimension at most $n-8$.*

PROOF. *B extremal* means that B lies on the boundary of its convex hull. Let M_0 be a compact, oriented, C^1 manifold with boundary B, of at most $\nu/2$ connected components, which intersects the convex hull C of B only in B. Let $M = M_0 - B$. Since $\operatorname{spt} T \subset C$ (cf. [**AS**, Appendix A]), $M \cap \operatorname{spt} T = \varnothing$. Hence Regularity Theorem 2.4 provides in the equivalence class of T mod ν a representative R with $\|\partial R\| = \|B\|$. Since $\partial R = B \pmod{\nu}$, B has multiplicity 1, and $\nu \geqslant 3$, therefore $\partial R = B$. Since R is area-minimizing, the standard regularity theory [**F2**, **HS**] says that $\operatorname{spt} R$ is a $C^{1,\alpha}$ submanifold with boundary B, except for a compact singular set disjoint from B of Hausdorff dimension at most $n-8$. It follows that R has multiplicity 1 almost everywhere. Since $\nu \geqslant 3$, the equivalence class of R mod ν has a unique representative. Of course, the minimizing property of T ensures that T is also a representative mod ν. Therefore, $T = R$ and the corollary is proved.

REMARK. For $\nu = 2$, the same proof produces a representative R mod 2 congruent to T mod 2 with $\partial R = B$. However, for $\nu = 2$, such a representative is not unique and T may well have additional boundary.

2.6. COROLLARY. *For $n \geqslant 2$, $\nu \geqslant 2$, $\alpha > 0$, let B be an $(n-2)$-dimensional, compact, oriented, extremal, $C^{1,\alpha}$ submanifold of $\mathbf{R}^n$, of at most $\nu/2$ connected components. Let S be an area-minimizing rectifiable current with boundary B. Then S is area-minimizing modulo ν.*

PROOF. Let T minimize area among rectifiable currents with $\partial T \equiv B \pmod{\nu}$. For $\nu \geqslant 3$, by Corollary 2.5, $\partial T = B$ and therefore area $S \leqslant$ area T. For $\nu = 2$, by the remark following 2.5, there is a representative R mod 2 congruent to T with $\partial R = B$. Hence area $S \leqslant$ area $R =$ area T.

EXAMPLE. For $\nu \geqslant 2$, the cone over $\mathbf{S}^3 \times \mathbf{S}^3$ in $\mathbf{R}^8$ is area-minimizing modulo ν. Cf. [**M2**, Theorem 4].

2.7. COROLLARY. *Let K be a compact, convex domain in $\mathbf{R}^n$ and let $f\colon K \to \mathbf{R}$ be a C^2 solution to the minimal surface equation. Then the graph of f is area-minimizing as a flat chain modulo ν for all $\nu \geqslant 2$.*

PROOF. This result is well known for rectifiable currents (cf. [**F1**, 5.4.18]). By Corollary 2.6, it follows for flat chains mod ν.

2.8. FINITENESS THEOREM. *For $n \leqslant 7$, $\alpha > 0$, $\nu \geqslant 2$, let B be an $(n-2)$-dimensional, compact, oriented, $C^{2,\alpha}$, extremal submanifold of $\mathbf{R}^n$, with a positive-integer multiplicity assigned to each component. Suppose that the number of components of B, counting multiplicities, does not exceed $\nu/2$. Then B bounds only finitely many area-minimizing flat chains modulo ν.*

REMARK. The hypothesis on the number of components of B may be replaced by the weaker hypothesis that there exist an $(n-1)$-dimensional, oriented, embedded, C^1 manifold M of compact closure and finite area, of multiplicity 1, of at most $\nu/2$ components, with current boundary B, disjoint from the convex hull of B.

PROOF. In light of the preceding remark, the Regularity Theorem 2.4 applies, and provides for any area-minimizing flat chain mod ν T with $\partial T \equiv B \pmod{\nu}$ a representative R with $\|\partial R\| = \|B\|$. This admits only finitely many possibilities for ∂R. Since R is area-minimizing, the theorem follows from the corresponding finiteness theorem [**M4**, Theorem 4.3] for area-minimizing rectifiable currents.

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DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139