COUNTABLE DIMENSIONAL UNIVERSAL SETS

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Abstract. The main results of this paper are a construction of a countable union of zero dimensional sets in the Hilbert cube whose complement does not contain any subset of finite dimension $n \geq 1$ (Theorem 2.1, Corollary 2.3) and a construction of universal sets for the transfinite extension of the Menger-Urysohn inductive dimension (Theorem 2.2, Corollary 2.4).

1. Terminology and notation. All spaces considered in this paper are metrizable and separable. Our terminology follows Kuratowski [Ku] and Nagata [Na 2].

1.1. Notation. We denote by $I$ the interval $[-1,1]$, $I^\infty$ is the countable product of $I$, i.e. the Hilbert cube, $p_i: I^\infty \to I$ is the projection onto the $i$th coordinate, $P$ is the set of the irrationals from $I$ and $\omega$ is the set of natural numbers. Given a point $t \in I$, we let $Q_t = \{(x_1, x_2, \ldots) \in I^\infty: x_1 = t\}$.

1.2. Partitions. A partition in a space $X$ between a pair of disjoint sets $A$ and $B$ is a closed set $L$ such that $X \setminus L = U \cup V$, where $U$ and $V$ are disjoint open sets with $A \subset U$ and $B \subset V$.

1.3. Countable dimensional spaces and the transfinite inductive dimension $\text{ind}$. A space $X$ is countable dimensional if it is a countable union $X = \bigcup_{i=1}^{\infty} X_i$ of zero dimensional sets $X_i$ [Hu].

The transfinite dimension $\text{ind}$ is the extension by transfinite induction of the classical Menger-Urysohn inductive dimension: $\text{ind} X = -1$ means $X = \emptyset$, $\text{ind} X \leq \alpha$ if and only if each point $x$ in $X$ can be separated in $X$ from any closed set not containing $x$ by a partition $L$ with $\text{ind} L < \alpha$, $\alpha$ being an ordinal, we let $\text{ind} X$ be the smallest ordinal $\alpha$ with $\text{ind} X \leq \alpha$ if such an ordinal exists, and we put $\text{ind} X = \infty$ otherwise. If $\text{ind} X \neq \infty$, then $\text{ind} X$ is a countable ordinal, $X$ having a countable base.

The transfinite dimension $\text{ind}$ was first discussed by Hurewicz [Hu, §5], [H-W, p. 50] (although the idea goes back to Urysohn’s memoir [Ur, p. 66]). A comprehensive survey of the topic is given by Engelking [En 2].

Hurewicz [Hu, En 2, 4.1, 4.15] proved that for a complete space $X$, $\text{ind} X \neq \infty$ if and only if $X$ is countable dimensional and that each space $X$ with $\text{ind} X \neq \infty$ has a countable dimensional compactification.
1.4. **Hereditarily infinite dimensional spaces.** We say that an infinite dimensional space \( X \) is hereditarily infinite dimensional (hereditarily uncountable dimensional) if each nonempty subset of \( X \) is either zero dimensional or infinite dimensional (uncountable dimensional), cf. [G-S, Wa, Tu].

2. **Introduction.** The following two theorems are main results of this paper.

2.1. **Theorem.** There exists a countable dimensional set \( C \) in the Hilbert cube \( I^\infty \) such that for each countable dimensional subset \( A \) of \( I^\infty \) the difference \( A \setminus C \) is at most zero dimensional.

2.2. **Theorem.** For each countable ordinal \( \alpha \) there exists a \( G_\delta \)-set \( E_\alpha \) in \( I^\infty \) with transfinite inductive dimension \( \text{ind} E_\alpha = \alpha \) such that for every \( G_\delta \)-set \( G \) in \( I^\infty \) with \( \text{ind} G \leq \alpha \) there is an irrational \( t \in I \) for which \( G \cap Q_t = E_\alpha \cap Q_t \), where \( Q_t = \{(x_1,x_2,\ldots) \in I^\infty: x_1 = t\} \).

Since each separable metrizable space embeds in \( I^\infty \), Theorem 2.1 yields the following corollary.

2.3. **Corollary.** Each uncountable dimensional separable metrizable space contains a countable dimensional subset with hereditarily uncountable dimensional complement.

Each subset of \( I^\infty \) can be enlarged to a \( G_\delta \)-set in \( I^\infty \) with the same transfinite dimension \( \text{ind} \) [En 2, 5.5] and hence the sets \( E_\alpha \) in Theorem 2.2 have the following property:

2.4. **Corollary.** For each countable ordinal \( \alpha \), every separable metrizable space \( X \) with \( \text{ind} X \leq \alpha \) can be embedded homeomorphically into the space \( E_\alpha \).

Therefore, \( E_\alpha \) is a universal space in the class of separable metrizable spaces with transfinite dimension \( \text{ind} \leq \alpha \). The question about the existence of such universal spaces for \( \alpha \geq \omega_0 \) was asked by Engelking [En 2, Problem 5.11], cf. also Luxemburg [Lu 2, Problem 8.4].

Using the zero dimensional set \( E_0 \) described in Theorem 2.2 one obtains the following fact (cf. §4.2):

2.5. **Corollary.** There exists a countable dimensional \( G_{\delta_0} \)-set \( E_\infty \) in \( I^\infty \) such that for every countable dimensional \( G_{\delta_0} \)-set \( G \) in \( I^\infty \) there is an irrational \( t \in I \) with \( G \cap Q_t = E_\infty \cap Q_t \).

The results of this paper are based implicitly on a notion of “universal functions” for a given collection of sets and some related diagonal arguments; these ideas go back to the origins of descriptive set theory, cf. Moschovakis [Mo, Remark 15 on p. 63], Kuratowski [Ku, §30, XIII]. Certain “universal functions” for the collections of compacta in \( I^\infty \) with transfinite dimension \( \text{ind} \leq \alpha \) have been considered in [Po 2, §4] and a significant part of the present paper is a modification and an extension of the methods from [Po 2, §4].

Theorem 2.1 can be proved by making use of the zero dimensional universal set \( E_0 \) described in Theorem 2.2 (cf. Remark 4.2.1). We give, however, an independent direct proof of this theorem (though based on the same ideas as the construction of
the sets $E_\alpha$ in Theorem 2.2). This proof also provides a simple construction of sets with properties only slightly weaker than those of $E_0, E_1, \ldots$ (for finite $\alpha$) and the set $E_\infty$ in Corollary 2.5.

The paper is organized as follows.

In §3.1 we construct a “universal sequence” of sets for zero dimensional sets in $I^\infty$ and in §3.2 we intensify certain singular properties of these sets, following an idea from Walsh [Wa], to obtain the countable dimensional set $C$ described in Theorem 2.1.

In §4.1 we construct “universal functions” $M_\alpha \subset P \times I^\infty$, in the product of the irrationals and the Hilbert cube, for the collection of $G_\delta$-sets in $I^\infty$ with transfinite dimension $\text{ind} \leq \alpha$ and in §4.2 we apply a standard diagonal construction to get from these $M_\alpha$’s the sets $E_\alpha$ described in Theorem 2.2.

§5 is a slight departure from the main subject of this paper (and it is formally independent of the other sections). We define here, by a method similar to that in §3.1, a “universal sequence” of partitions between the opposite faces in $I^\infty$, and we use these partitions along a path outlined by Walsh [Wa, §§3 and 7] to obtain a rather unexpectedly simple construction of hereditarily infinite dimensional compacta.

In §6 we collect some comments related to the subject of this paper.

I would like to thank Henryk Toruńczyk for pointing out a direct argument used in the proofs of property (I) in §3.1 and Lemma 4.1.3(h), which simplified my original proofs.

3. A countable union of 0-dimensional sets in $I^\infty$ whose complement has no subsets of dimension $n \geq 1$. In this section we give a proof of Theorem 2.1.

3.1. A universal sequence $N_1, N_2, \ldots$ for 0-dimensional sets in $I^\infty$. Let $T$ be an arbitrary set in $I$ homeomorphic to the irrationals $P$, let $\Gamma$ be the space of all homeomorphic embeddings $h: I^\infty \to I^\infty$ of the Hilbert cube into itself endowed with the topology of uniform convergence and let

$$u = (u_1, u_2, \ldots): T \to \Gamma \times \Gamma \times \cdots$$

be a continuous map of the set $T$ onto the countable product of the completely metrizable separable space $\Gamma$, cf. [Ku, §36, II].

For each $i = 1, 2, \ldots$ we let

$$N_i = \{(x_1, x_2, \ldots) \in I^\infty: x_1 \in T \text{ and } u_i(x_1)(x_1, x_2, \ldots) \in P \times P \times \cdots\}.$$  

We shall verify that the sets $N_i$ have the following two properties, where for each $t \in I$,

$$Q_t = \{(x_1, x_2, \ldots) \in I^\infty: x_1 = t\}$$

(cf. §4.2(I) and (II)):  

(I) The sets $N_i$ are zero dimensional.

(II) Given an arbitrary sequence $G_1, G_2, \ldots$ of zero dimensional sets in $I^\infty$ there exists $t \in T$ such that $G_i \cap Q_t \subset N_i \cap Q_t$ for each $i = 1, 2, \ldots$.

**Proof of (I).** Let $Q_T = \{(x_1, x_2, \ldots) \in I^\infty: x_1 \in T\}$ and let, for each $i \in \omega$, a continuous map $f_i: Q_T \to T \times I^\infty$ be defined by

$$f_i(x_1, x_2, \ldots) = (x_1, u_i(x_1)(x_1, x_2, \ldots)).$$
The map $f_i$ is closed (the projection $Q_T \to T$ being parallel to a compact factor, see [Bo, Chapter I, §§10, 1 and 2]) and injective (the maps $u_i(t)$ being embeddings), and hence $f_i$ embeds $Q_T$ homeomorphically into the product $T \times I^\infty$. Property (I) follows now from the fact that $f_i$ embeds $N_i$ into the zero dimensional space $T \times (P \times P \times \cdots)$.

**Proof of (II).** We shall use the following universal property of the product of the irrationals $P \times P \times \cdots$ established by Nagata [Na 1, Na 2, VI.2.A] (a simple proof is given in §6.4):

3.1.1. **Lemma (Nagata).** For each zero dimensional set $G$ in a metrizable separable space $X$ there exists a homeomorphic embedding $h: X \to I^\infty$ such that $h(G) \subset P \times P \times \cdots$.

Let $G_1, G_2, \ldots$ be an arbitrary sequence of zero dimensional sets in $I^\infty$ and, for each $i \in \omega$, let $h_i: I^\infty \to I^\infty$ be an embedding such that

$$h_i(G_i) \subset P \times P \times \cdots$$

Let us choose a $t \in T$ such that

$$u(t) = (u_1(t), u_2(t), \ldots) = (h_1, h_2, \ldots).$$

Then, for each $i \in \omega$, we have (see (2), (3), (1))

$$G_i \cap Q_i \subset \{(t, x_2, x_3, \ldots): h_i(t, x_2, x_3, \ldots) \in P \times P \times \cdots\}$$

$$= \{(t, x_2, x_3, \ldots): u_i(t)(t, x_2, x_3, \ldots) \in P \times P \times \cdots\} = N_i \cap Q_i,$$

which proves property (II).

We close this section with an observation that property (II) yields, by a simple diagonal argument, the following property of the union

$$N_\infty = \bigcup_{i=1}^{\infty} N_i.$$

(III) If $A$ is a subset of $I^\infty$ disjoint from $N_\infty$ whose projection onto the first coordinate contains the set $T$, then $A$ is uncountable dimensional.

Assume on the contrary that $A = \bigcup_{i=1}^{\infty} G_i$, where the sets $G_i$ are zero dimensional. By property (II) there exists a $t \in T$ such that $G_i \cap Q_i \subset N_i \cap Q_i$ for all $i \in \omega$ and hence $\emptyset \neq A \cap Q_i = \bigcup_{i=1}^{\infty} G_i \cap Q_i \subset N_\infty \cap Q_i$, contradicting the fact that $A$ was disjoint from $N_\infty$.

3.2. **Proof of Theorem 2.1.** Reasoning in this section follows some ideas of Walsh [Wa].

Let $T_1, T_2, \ldots$ be a sequence of topological copies of the irrationals in $I$, with pairwise disjoint closures in $I$, such that each nondegenerate interval in $I$ contains some $T_i$.

For each $i \in \omega$, the construction described in §3.1 with $T = T_i$ yields a countable dimensional set $N_\infty^{(i)}$ in $I^\infty$ such that (see property (III)) each subset in $I^\infty$ disjoint from $N_\infty^{(i)}$ whose projection onto the first coordinate contains $T_i$ is uncountable.
dimensional. Therefore, the countable dimensional set

\[ C' = N^{(1)}_\infty \cup N^{(2)}_\infty \cup \ldots \]

has the following property:

(IV) If \( A \) is a countable dimensional set in \( I^\infty \) disjoint from \( C' \) then the projection of \( A \) onto the first coordinate does not contain any nondegenerate interval in \( I \), i.e. the projection is zero dimensional.

Let \( \pi_i : I^\infty \to I^\infty \) be the permutation of the coordinates interchanging the first coordinate with the \( i \)th one and let \( C_i = \pi_i(C') \) for \( i \in \omega \). Each countable dimensional set \( C_i \) has then the property analogous to that of \( C' \) stated in (IV), where the first coordinate is replaced by the \( i \)th one. The countable dimensional set \( C = C_1 \cup C_2 \cup \cdots \) satisfies the assertion of Theorem 2.1: if \( A \) is a nonempty countable dimensional set in \( I^\infty \) disjoint from \( C \) then, \( A \) being disjoint from each \( C_i \), for every \( i \in \omega \) the projection \( p_i(A) \) of \( A \) onto the \( i \)th coordinate is zero dimensional and so is the set \( A \subset \prod_{i=1}^{\infty} p_i(A) \), cf. Walsh [Wa, §3].

4. Universal sets for the transfinite extension of the inductive Menger-Urysohn dimension. In this section we construct the sets \( E_\alpha \) described in Theorem 2.2 and we prove Corollary 2.5.

4.1. Universal functions \( M_\alpha \). Given a set \( S \) in the product \( P \times I^\infty \) of the irrationals \( P \) and the Hilbert cube \( I^\infty \), for each \( t \in P \), we let

\[ S(t) = \{ x \in I^\infty : (t, x) \in S \} . \]

4.1.1. Proposition. For each countable ordinal \( \alpha \) there exists a \( G_\delta \)-set \( M_\alpha \) in \( P \times I^\infty \) such that

(i) \( \text{ind } M_\alpha = \alpha \),

(ii) for each \( G_\delta \)-set \( G \) in \( I^\infty \) with \( \text{ind } G \leq \alpha \) there is a \( t \in P \) with \( M_\alpha(t) = G \).

**Proof.** Let \( p_i : I^\infty \to I \) be the projection onto the \( i \)th coordinate and let

\[ C_i = p_i^{-1}(-1), \quad D_i = p_i^{-1}(1), \quad H_i = p_i^{-1}(0), \]

i.e. \( H_i \) is a partition between the pair \( C_i \) and \( D_i \) of the \( i \)th opposite faces in \( I^\infty \). Let

\[ Z = I^\infty \setminus \bigcup \{ C_{2i-1} \cup D_{2i-1} : i = 1, 2, \ldots \} . \]

We shall construct the sets \( M_\alpha \) by transfinite induction. Let \( M_{-1} = \emptyset \) and let us assume that for some ordinal \( \alpha \) the sets \( M_\beta \) with \( \beta < \alpha \) have been already constructed. We shall define the set \( M_\alpha \).

(I) Let us split the set of even natural numbers into disjoint infinite sets \( \Sigma_{-1}, \Sigma_0, \Sigma_1, \ldots, \Sigma_\beta, \ldots, \beta < \alpha \), and let \( \Gamma \) be the space of all homeomorphic embeddings \( h : I^\infty \to I^\infty \) satisfying the following two conditions

(*) if \( x \in h^{-1}(Z) \) and \( F \) is a closed set in \( I^\infty \) not containing \( x \), then \( h(x) \in C_{2i} \) and \( h(F) \subset D_{2i} \) for some \( i \in \omega \);

(**) if \( j \in \Sigma_\beta \) then \( \text{ind } h^{-1}(Z \cap H_j) \leq \beta \).

We shall consider \( \Gamma \) with the topology of uniform convergence.
4.1.2. Lemma. For each $G_\delta$-set $G$ in $I^\infty$ with $\text{ind} G \leq \alpha$ there exists an embedding $h \in \Gamma$ such that $G = h^{-1}(Z)$.

**Proof.** Let $\mathscr{B}$ be a countable base in $I^\infty$. Let us consider the collection of all pairs $(A, B)$ of disjoint closed sets in $I^\infty$, each of which being a finite sum of the closures of the elements of $\mathscr{B}$ such that there exists a partition $L$ in $I^\infty$ between $A$ and $B$ with $\text{ind}(L \cap G) = \gamma < \alpha$, let $\gamma(A, B)$ be the minimal such ordinal $\gamma$ for the pair $(A, B)$, and finally, let us arrange this collection of pairs into a sequence $(A_1, B_1), (A_2, B_2), \ldots$ letting $\gamma(i) = \gamma(A_i, B_i)$. Choose an injection $\tau: \omega \to \omega$ such that $\tau(i) \in \Sigma_{\gamma(i)}$ and let $f_i: I^\infty \to I$ be continuous maps with

$$f_i^{-1}(-1) = A_i, \quad f_i^{-1}(1) = B_i,$$

$$\text{ind}(G \cap f_i^{-1}(0)) = \gamma(i).$$

Let $I^\infty \setminus G = X_1 \cup X_2 \cup \cdots$, where $X_i$ are compact sets and let $g_i: I^\infty \to [0,1]$ be continuous maps with $g_i^{-1}(1) = X_i$. Let us finally split the odd natural numbers into two disjoint infinite sets $2'$ and $2''$ and let us choose bijections $\nu: 2' \to \nu$ and $\mu: 2'' \to \mu$.

An embedding $h: I^\infty \to I^\infty$ with required properties can be defined now by $h(x_1, x_2, \ldots) = (y_1, y_2, \ldots)$ where

$$y_j = \begin{cases} g_{\nu(j)}(x_1, x_2, \ldots), & \text{if } j \in \Sigma', \\ \frac{1}{2}x_{\mu(j)}, & \text{if } j \in \Sigma'', \\ f_i(x_1, x_2, \ldots), & \text{if } j = \tau(i), \\ 1 & \text{if } j \not\in \Sigma' \cup \Sigma'' \cup \tau(\omega). \end{cases}$$

The first two formulas in (6) guarantee that $h$ is an embedding and $G = h^{-1}(Z)$. Given an $x \in G$ and a closed set $F$ in $I^\infty$ not containing $x$, the assumption $\text{ind} G \leq \alpha$ yields the existence of a pair $(A_i, B_i)$ such that $x \in A_i$ and $F \subset B_i$ (cf. [En 1, Lemma 1.2.9]). Then (see (2), (4), (5) and (6)), $x \in h^{-1}(C_{\tau(i)}), F \subset h^{-1}(D_{\tau(i)})$ and $\text{ind} h^{-1}(Z \cap H_{\tau(i)}) = \text{ind}(G \cap f_i^{-1}(0)) = \gamma(i)$. Therefore $h$ satisfies condition (*) and condition (**) holds for all $j \in \tau(\omega)$ (recall that $\tau(i) \in \Sigma_{\gamma(i)}$). But if $j$ is an even number not belonging to $\tau(\omega)$, the last formula in (6) shows that $h(I^\infty) \cap H_j = \emptyset$ and so (**) is satisfied also in that case.

(II) For each even number $2i$ let $\beta(i)$ be the ordinal such that $2i \in \Sigma_{\beta(i)}$ (see (I)). Let us consider an embedding $h \in \Gamma$, where $\Gamma$ is the space defined in (I). For every even number $2i$ property (**) and universality of the set $M_{\beta(i)}$ yield the existence of an irrational $t_i \in P$ such that

$$h^{-1}(Z \cap H_{2i}) = M_{\beta(i)}(t_i).$$

Let $\Lambda \subset \Gamma \times P \times P \times \cdots$ be the space of all sequences $(h, t_1, t_2, \ldots)$ such that for each $i \in \omega$ the pair $(h, t_i)$ satisfies condition (7). The space $\Lambda$ being metrizable and separable, there exists a subset $S$ of the irrationals in $I$ and a continuous map $u = (u_0, u_1, u_2, \ldots): S \to \Lambda$ onto $\Lambda$ (cf. [Ku, §36, III]). Let us define a continuous map $k: S \times I^{\infty} \to I^{\infty}$ by $k(s, x) = u_0(s)(x)$ and let

$$M = \{(s, x) \in S \times I^{\infty}: u_0(s)(x) \in Z\} = k^{-1}(Z),$$

$$L_i = \{(s, x) \in S \times I^{\infty}: u_0(s)(x) \in H_{2i}\} = k^{-1}(H_{2i}).$$
4.1.3. Lemma. The set $M$ and the sets $L_i$ have the following properties:

(i) for each $G_x$-set $G$ in $I^\infty$ with $\text{ind} G \leq \alpha$ there exists an $s \in S$ such that $M(s) = G$,
(ii) $\text{ind}(L_i \cap M) \leq \beta(i)$, $i \in \omega$,
(iii) $\text{ind} M = \alpha$.

Proof. (i) Let $G$ be a $G_x$-set in $I^\infty$ with $\text{ind} G \leq \alpha$. By Lemma 4.1.2 there exists an embedding $h \in \Gamma$ such that $G = h^{-1}(Z)$ and since $u_0(S) = \Gamma$, $h = u_0(s)$ for some $s \in S$. Then (see (8)) $M(s) = u_0(s)^{-1}(Z) = G$.

(ii) By (7), for each $s \in S$ and $i \in \omega$, we have

$$u_0(s)^{-1}(Z \cap H_{2i}) = M_{\beta(i)}(u_i(s)),$$

i.e. (see (8) and (9))

$$M \cap L_i(s) = M_{\beta(i)}(u_i(s)).$$

For each $i \in \omega$ define a continuous map $g_i: S \times I^\infty \to S \times P \times I^\infty$ by

$$g_i(s, x) = (s, u_i(s), x).$$

The map $g_i$ homeomorphically embeds the set $M \cap L_i$ into the product $S \times M_{\beta(i)}$ (see (10)) and therefore $\text{ind}(M \cap L_i) \leq \text{ind} M_{\beta(i)} = \beta(i)$, the space $S$ being zero dimensional.

(iii) Let $(s, x) \in M$ and let $F$ be a closed set in $S \times I^\infty$ not containing $(s, x)$. Since $u_0(s) \in \Gamma$, property $(*)$ in (I) and (8) yield the existence of an $i \in \omega$ such that $u_0(s)(x) \in C_{2i}$ and $u_0(s)(F(s)) \subset D_{2i}$. Therefore the set $L_i$ separates in $S \times I^\infty$ the point $(s, x)$ from the closed set $F \cap (\{s\} \times I^\infty)$ (see (9)). The projection $S \times I^\infty \to S$ parallel to the compact factor being closed, there exists an open and closed neighborhood $W$ of $s$ in $S$ such that $L_i \cap (W \times I^\infty)$ is a partition in $S \times I^\infty$ between the point $(s, x)$ and the set $F$. Since, by (ii), $\text{ind}(M \cap L_i) < \alpha$, this shows that $\text{ind} M \leq \alpha$ and it completes the proof, as $\text{ind} M \geq \alpha$, by (i).

(III) It remains to modify slightly the set $M$ constructed in (II) to obtain a $G_x$-set in $P \times I^\infty$ satisfying conditions (i) and (ii) in Lemma 4.1.3.

By (8), $M$ is a $G_x$-set in $S \times I^\infty \subset P \times I^\infty$ and therefore there exists a $G_x$-set $M^*$ in $P \times I^\infty$ such that $\text{ind} M^* = \text{ind} M$ and $M^* \cap (S \times I^\infty) = M$, cf. [En 2, 5.5]. Let $S^*$ be the projection of the set $M^*$ onto the $P$-coordinate, let $w: P \to S^*$ be a continuous map onto $S^*$, cf. [Ku, §37, 1], and let

$$M_\alpha = \{(t, x) \in P \times I^\infty: (w(t), x) \in M^*\}.$$

The set $M_\alpha$ is a $G_x$-set in $P \times I^\infty$ (being the preimage of the set $M^*$ under the map $(t, x) \to (w(t), x)$) and for each $G_x$-set $G$ in $I^\infty$ with $\text{ind} G \leq \alpha$, there exists an irrational $t \in P$ with $M_\alpha(t) = G$ (as $G = M(s)$ for some $s \in S$ and $s = w(t)$ for some $t \in P$). Finally, $\text{ind} M_\alpha \leq \alpha$, since the map $(t, x) \to (t, w(t), x)$ embeds homeomorphically the set $M_\alpha$ into the product $P \times M^*$ (cf. the proof of Lemma 4.1.3(ii)).

This completes the inductive proof of Proposition 4.1.1.
4.2. Diagonal constructions related to the universal functions $M_{\alpha}$. For each countable ordinal $\alpha$, let $M_{\alpha} \subseteq P \times I^\infty$ be the universal function constructed in §4.1 and let

$$E_{\alpha} = \left\{ (x_1, x_2, \ldots) \in I^\infty : (x_1, (x_1, x_2, \ldots)) \in M_{\alpha} \right\}.$$  

Given a point $t \in I$, we put

$$Q_t = \left\{ (x_1, x_2, \ldots) \in I^\infty : x_1 = t \right\}.$$

Let $G$ be an arbitrary $G_\delta$-set in $I^\infty$ with $\text{ind } G \leq \alpha$. By Proposition 4.1.1, there exists an irrational $t \in P$ such that $M_{\alpha}(t) = G$ and hence

$$E_{\alpha} \cap Q_t = \left\{ (x_1, x_2, \ldots) : x_1 = t \text{ and } (x_1, (x_1, x_2, \ldots)) \in M_{\alpha} \right\},$$

$$= \left\{ (x_1, x_2, \ldots) : x_1 = t \text{ and } (x_1, x_2, \ldots) \in M_{\alpha}(t) \right\},$$

$$= \left\{ (x_1, x_2, \ldots) : x_1 = t \text{ and } (x_1, x_2, \ldots) \in G \right\},$$

$$= G \cap Q_t.$$

Moreover, Proposition 4.1.1 shows also that $E_{\alpha}$ is a $G_\delta$-set in $I^\infty$ with $\text{ind } E_{\alpha} = \alpha$. Therefore the sets $E_{\alpha}$ satisfy the assertions of Theorem 2.2.

Let us construct now the set $E_{\infty}$ described in Corollary 2.5.

Let $u = (u_1, u_2, \ldots) : P \to P \times P \times \cdots$ be a continuous map of the irrationals onto its countable product and let

$$N_{i}^* = \left\{ (x_1, x_2, \ldots) \in I^\infty : x_1 \notin P \text{ and } (u_i(x_1), (x_1, x_2, \ldots)) \in M_0 \right\},$$

where $M_0 \subseteq P \times I^\infty$ is the universal function for zero dimensional sets in $I^\infty$. Let us verify that the sets $N_{i}^*$ have the following two properties (cf. §3.1(I) and (II)):

(I) Each $N_{i}^*$ is a zero dimensional $G_\delta$-set in $I^\infty$.

(II) Given an arbitrary sequence $G_1, G_2, \ldots$ of zero dimensional $G_\delta$-sets in $I^\infty$ there exists an irrational $t \in P$ such that $G_i \cap Q_t = N_{i}^* \cap Q_t$ for each $i = 1, 2, \ldots$.

Property (I) follows from the fact that the map $(x_1, x_2, \ldots) \to (x_1, u_i(x_1), (x_1, x_2, \ldots))$ homeomorphically embeds the set $N_{i}^*$ into the product $P \times M_0$.

Let $G_1, G_2, \ldots$ be a sequence of zero dimensional $G_\delta$-sets in $I^\infty$, let $t_1, t_2, \ldots$ be irrationals such that $M_0(t_i) = G_i$, for $i = 1, 2, \ldots$ and let $t$ be an irrational such that $u(t) = (u_1(t), u_2(t), \ldots) = (t_1, t_2, \ldots)$. Then, for each $i \in \omega$, we obtain (see 12)

$$G_i \cap Q_t = \left\{ (x_1, x_2, \ldots) : x_1 = t \text{ and } (t_i, (x_1, x_2, \ldots)) \in M_0 \right\},$$

$$= \left\{ (x_1, x_2, \ldots) : x_1 = t \text{ and } (u_i(t), (x_1, x_2, \ldots)) \in M_0 \right\},$$

$$= N_{i}^* \cap Q_t.$$  

Let us put now (cf. §3.1(4))

$$E_{\infty} = \bigcup_{i=1}^{\infty} N_{i}^*,$$

and let us verify that the countable dimensional $G_{\delta \alpha}$-set $E_{\infty}$ satisfies the assertions of Corollary 2.5. Let $G$ be an arbitrary countable dimensional $G_{\delta \alpha}$-set in $I^\infty$; one can find zero dimensional $G_\delta$ sets $G_1, G_2, \ldots$ in $I^\infty$ such that $G = G_1 \cup G_2 \cup \cdots$. By property (II) there exists an irrational $t \in P$ such that $G_i \cap Q_t = N_{i}^* \cap Q_t$ for each $i \in \omega$, and therefore $G \cap Q_t = E_{\infty} \cap Q_t$. 
4.2.1. Remark. The sequence $N_1^*, N_2^*, \ldots$ has properties slightly stronger than those of the sequence $N_1, N_2, \ldots$ described in §3.1 and therefore one can obtain the results in §3 using the sets $N_i^*$ instead of $N_i$. Let us notice that in the case of the universal set $M_0$, used to define the sets $N_i^*$, the construction given in §4.1 simplifies essentially: the sets $\Sigma_\beta$ in (I) do not appear and "ind $< \alpha^*$ means just "empty". Still, however, the proof of Theorem 2.1 given in §3 seems more direct than an alternative one based on the construction of the space $M_0$. Let us observe finally that if $N$ is a zero dimensional $G_\delta$-set in $I^\infty$ containing the set $N_1$ defined in §3.1, $G \subset P \times I^\infty$ is a $G_\delta$-set universal for $G_\delta$-sets in $I^\infty$ [Ku, §30, XIII], and $w = (w_1, w_2)$ maps continuously $P$ onto $P \times P$, then the set

$$M = \{(t, x_1, x_2, \ldots) \in P \times I^\infty: (w_1(t), w_2(t), x_1, x_2, \ldots) \in (P \times N) \cap G\}$$

is a $G_\delta$-set in $S \times I^\infty$, $S$ being the projection of $M$ onto $P$-coordinate, and each zero dimensional $G_\delta$-set in $I^\infty$ is of the form $M(t)$ for some $t \in S$, cf. Jayne and Rogers [J-R, proof of Theorem 9.1], and therefore the space $M_0$ can be easily obtained from $M$ by the method described in §4.1(III).

5. A universal sequence of partitions between the opposite faces in $I^\infty$. In this section we show that the method of parametrizing function spaces applied in §3.1 (cf. also the proof of Proposition 4.1.1(ii)) can be used to define a sequence of partitions between the opposite faces in $I^\infty$ with some "universal" properties. This, combined with ideas of Walsh [Wa] and Rubin [Ru 1] provides a quite simple construction of hereditarily strongly infinite dimensional compacta.

Recall that $p_i: I^\infty \to I$ is the projection onto the $i$th coordinate and let

$$C_i = p_i^{-1}(-1), \quad D_i = p_i^{-1}(1), \quad H_i = p_i^{-1}(0),$$

i.e. $C_i$ and $D_i$ is the pair of the $i$th opposite faces in $I^\infty$ and $H_i$ is a partition between them.

Let $0 < a_i < 1$ and let

$$C_i^* = p_i^{-1}[-1, -a_i], \quad D_i^* = p_i^{-1}[a_i, 1].$$

Finally, let $T$ be any subset of $I$ homeomorphic to the irrationals and, for each $t \in I$, let

$$Q_t = \{(x_1, x_2, \ldots) \in I^\infty: x_1 = t\}.$$
Proof. Let $\Lambda$ be the space of all continuous maps $f: I^\infty \to I^\infty$ such that

$$f(C_i^*) \subset C_i \quad \text{and} \quad f(D_i^*) \subset D_i,$$

$\Lambda$ being endowed with the topology of uniform convergence, and let $u: T \to \Lambda$ be a continuous map onto the completely metrizable separable space $\Lambda$, cf. [Ku, §36, II]. Let $Q_T = \{(x_1, x_2, \ldots) \in I^\infty: x_1 \in T\}$ and let $F: Q_T \to I^\infty$ be a continuous map defined by (cf. §3.1(1))

$$F(x_1, x_2, \ldots) = u(x_1)(x_1, x_2, \ldots).$$

By (1), the set $F^{-1}(H_i)$ is a partition in $Q_T$ between $C_i^* \cap Q_T$ and $D_i^* \cap Q_T$ and since $C_i$ and $D_i$ are in the interior of $C_i^*$ and $D_i^*$ respectively, there exists a partition $L_i$ in $I^\infty$ between $C_i$ and $D_i$ extending $F^{-1}(H_i)$, cf. Engelking [En 2, Lemma 1.2.9], i.e.

$$L_i \cap Q_T = F^{-1}(H_i), \quad i = 1, 2, \ldots.$$  

We shall verify that the sequence $L_1, L_2, \ldots$ has the required property. Given partitions $S_i$ in $I^\infty$ between $C_i^*$ and $D_i^*$ ($i \in \omega$), let $f_i: I^\infty \to I$ be continuous functions such that $C_i^* = f_i^{-1}(-1)$, $D_i^* = f_i^{-1}(1)$ and $S_i = f_i^{-1}(0)$. The diagonal map $f = (f_1, f_2, \ldots): I^\infty \to I^\infty$ belongs to $\Lambda$ and therefore $f = u(t)$ for some $t \in T$. Since $S_i = f^{-1}(H_i)$, for $i \in \omega$, we obtain (see (3) and (2))

$$L_i \cap Q_t = F^{-1}(H_i) \cap Q_t = u(t)^{-1}(H_i) \cap Q_t = f^{-1}(H_i) \cap Q_t = S_i \cap Q_t.$$  

5.2. Corollary. Let $L_1, L_2, \ldots$ be the sequence of partitions between the opposite faces in $I^\infty$ described in Proposition 5.1. For each $\sigma \subset \omega \setminus \{1\}$ and for each set $M \subset \cap\{L_i: i \in \sigma\}$ whose projection onto the first coordinate contains $T$, we have

(i) if $\sigma$ is a $k$-element set, then $M$ is at least $k$-dimensional;
(ii) if $\sigma$ is infinite, then $M$ is strongly infinite dimensional.

Proof. The reasoning in both cases is the same. Assume that the assertion is not true. Then (using again a simple lemma on extension of partitions [En 2, Lemma 1.2.9]) one can find partitions $S_i$ between $C_i$ and $D_i$ in $I^\infty$, where $i \in \sigma$, such that $M \cap \cap\{S_i: i \in \sigma\} = \emptyset$, cf. [Ku, §27, II; R-S-W, §3]. But, on the other hand, there exists a $t \in T$ such that $S_i \cap Q_t = L_i \cap Q_t$ for each $i$, and this would yield a contradiction, $\emptyset \neq M \cap Q_t \subset M \cap \cap\{S_i: i \in \sigma\} = \emptyset$.

5.3. Hereditarily strongly infinite dimensional compacta. We shall repeat in this section some arguments due to Walsh [Wa, §§3, 7] and Rubin [Ru 1, §6] to derive from Corollary 5.2 a construction of hereditarily strongly infinite dimensional spaces.

Choose in $I$ a collection $T_1, T_2, \ldots$ of homeomorphic copies of the irrationals such that each nondegenerate interval in $I$ contains some $T_i$ and let $\sigma_{ik} \subset \omega \setminus \{1, i\}$, where $i, k = 1, 2, \ldots$, be pairwise disjoint infinite sets.

Let us fix a pair of natural numbers $i, k$. Changing the $i$th coordinate with the first one and letting $T = T_k$, we obtain from Corollary 5.2 partitions $L_i$ between the $j$th opposite faces in $I^\infty$, where $j \in \sigma_{ik}$, such that each subset of the intersection $L_{ik} = \cap\{L_j: i \in \sigma_{ik}\}$ whose projection onto the $j$th coordinate contains $T_k$ is
strongly infinite dimensional. Since \( 1 \not\in \bigcup_{i,k} \sigma_{ik} \), each partition in \( I^\infty \) between the opposite faces \( C_1 \) and \( D_1 \) hits the intersection \( L = \bigcap \{ L_j : j \in \bigcup_{i,k} \sigma_{ik} \} = \bigcap_{i,k} L^{i,k} \), cf. [Ku, §28, IV], and therefore \( L \) is a compactum of positive dimension. If \( M \) is a nonempty set in \( L \), then either for some \( i \) and \( k \), \( T_k \subset p_i(M) \) and then \( M \) is strongly infinite dimensional, or else no projection \( p_i(M) \) contains a nondegenerate interval, and then \( M \) is zero dimensional, being a subset of the product \( \prod_{i=1}^\infty p_i(M) \) of zero dimensional sets.

5.4. Remark. For other constructions of hereditarily infinite dimensional compacta we refer the reader to the papers by Walsh [Wa], Rubin [Ru 1, Ru 3] and Krasinkiewicz [Kr]; an illuminating account of the topic is given by Garity and Schori [G-S, §2], cf. also Nagata [Na 2, p. 125].

Separators of certain special type between the opposite faces in \( I^\infty \) with properties similar to (i) and (ii) in Corollary 5.2 were constructed by Walsh [Wa, §4], cf. also [R-S-W and S-W]. A sequence of partitions between the opposite faces in \( I^\infty \) satisfying condition (ii) in Corollary 5.2 was constructed by Rubin [Ru 1] (a simplified, but still rather involved, exposition of this construction was given in [Ru 3]). Rubin [Ru 2] has shown that the existence of such partitions yields a result that each strongly infinite dimensional space contains a closed hereditarily strongly infinite dimensional subspace.

An important element in the constructions in [R-S-W, Wa, Ru 1 and Kr] is a continuous parametrization of some collections of compacta and forming a “diagonal compactum” for that collection, and this element is also hidden in the proof of Proposition 5.1. This idea can be traced back to Mazurkiewicz [Ma] and Knaster [Kn], cf. Lelek [Le, Example, p. 80].

6. Comments.

6.1. Tumarkin’s property. The following property of separable metrizable spaces \( X \) was considered by Tumarkin [Tu, Na, p. 125]:

\((T)\) each infinite dimensional subspace of \( X \) contains subsets of arbitrarily large finite dimension.

Corollary 2.5. and the fact that the spaces \( X \) with \( \text{ind } X \neq \infty \) have property (T) yield the following fact.

6.1.1. Proposition. Property (T) implies countable dimensionality, and in the class of completely metrizable separable spaces countable dimensionality is equivalent to property (T).

It is an open problem whether there exists a countable dimensional space which fails property (T), cf. [Tu, Wa, §7, En 2, 4.14]. In connection with this problem, let us make the following remark. One can repeat the construction in §3, starting with a continuous mapping \( u = (u_1, u_2) : T \to \Gamma \times \Gamma \) onto the square of \( \Gamma \) instead of its countable product. This yields zero dimensional sets \( N_1, N_2 \) (see 3.1(1)), a one dimensional set \( E = N_1 \cup N_2 \) (cf. 3.1(4)), and finally, it provides a one dimensional set \( D' \) in \( I^\infty \) defined analogously to the set \( C' \) described in §3.2. The set \( D' \) has the property that each one dimensional set \( S \subset I^\infty \setminus D' \) has zero dimensional projection.
onto the first coordinate (cf. §3.2 (IV)). Therefore, if we let \( D = \pi_1(D') \cup \pi_2(D') \cup \cdots \), where \( \pi_i : I^\infty \to I^\infty \) is the permutation of the coordinates changing the \( i \)th one with the first one, we obtain a countable dimensional set \( D \) intersecting each one dimensional set in \( I^\infty \) (see the reasoning at the end of §3.2) and hence, the complement \( I^\infty \setminus D \) does not contain any subset of positive finite dimension (as any such set contains a one-dimensional subset). It is still conceivable that there exists an infinite dimensional countable dimensional set \( S \) in \( I^\infty \setminus D \) (any such \( S \) would provide a solution to the problem we have formulated), however, the nature of the construction of \( D \) makes it difficult to clarify what are exactly the properties of this set.

Let us also mention that there exist uncountable dimensional compacta all of whose closed infinite dimensional subspaces contain closed subsets of arbitrarily large finite dimension—such a compactum is defined in [Po 1].

6.2. Totally disconnected complete spaces \( D_\alpha \) with \( \text{ind} \, D_\alpha = \alpha \). We shall use the spaces \( E_\alpha \) defined in Theorem 2.2 to obtain spaces \( D_\alpha \) described in the title of this section. Various constructions of totally disconnected complete spaces of arbitrarily large finite dimension can be found in [Ma, Le, p. 80; R-S-W, Kr].

Let \( \alpha = \beta + 1 \) be a non-limit-countable ordinal and let \( E_\beta \) be the universal set described in Theorem 2.2. Let \( \mathbb{P} \times I^\infty \) be the product of the irrationals in \( I \) and the Hilbert cube and let

\[
G_\beta = \{(t, (x_1, x_2, \ldots)) \in \mathbb{P} \times I^\infty : (t, x_1, x_2, \ldots) \in E_\beta \}.
\]

Let \( p : \mathbb{P} \times I^\infty \to \mathbb{P} \) be the projection. The universal properties of \( E_\beta \) yield immediately that \( G_\beta \) intersects each set \( S \) in \( \mathbb{P} \times I^\infty \) with \( \text{ind} \, S \leq \beta \) and \( p(S) = \mathbb{P} \).

Let \( K_\alpha \subset I^\infty \) be a compactum with \( \text{ind} \, K_\alpha = \alpha \), cf. [En 2, 2.2; Na 2, p. 148]. The set \( \mathbb{P} \times K_\alpha \setminus G_\beta = F_\alpha \) is an \( F_\beta \)-set in \( \mathbb{P} \times K_\alpha \) and, since \( \text{ind} \, G_\beta = \beta < \text{ind} \, K_\alpha = \alpha \), each vertical section \( F_\alpha \cap p^{-1}(t) \) is nonempty. Therefore, there exists a function \( f : \mathbb{P} \to K_\alpha \) of the first Baire class whose graph \( D_\alpha \) is contained in \( F_\alpha \); see [Ku, §43, IX].

The set \( D_\alpha \) is a totally disconnected \( G_\beta \)-set in \( \mathbb{P} \times K_\alpha \). Moreover, \( D_\alpha \) is disjoint from \( G_\beta \) and \( p(D_\alpha) = \mathbb{P} \), so \( \beta < \text{ind} \, D_\alpha \leq \text{ind}(\mathbb{P} \times K_\alpha) = \beta + 1 = \alpha \).

If \( \alpha \) is a limit ordinal it is enough to let \( D_\alpha \) be the free union of the sets \( D_{\beta+1} \) with \( \beta < \alpha \).

6.3. Compactifications of spaces with \( \text{ind} = \alpha \). Let \( M_\alpha \) be the universal space described in Proposition 4.1. By the theorem of Hurewicz quoted at the end of §1.3, there exists a countable dimensional compact extension \( M_\alpha^* \) of \( M_\alpha \) and let \( \phi(\alpha) \) be the minimal transfinite dimension of such extensions. The function \( \phi \) augmented by \( \phi(\infty) = \infty \) has the following property:

For each separable metrizable space \( X \) there exists a compactification \( X^* \) of \( X \) such that \( \text{ind} \, X \leq \text{ind} \, X^* \leq \phi(\text{ind} \, X) \).

Luxemburg [Lu 1] constructed examples which show that for any limit ordinal \( \alpha \), \( \phi(\alpha) > \alpha \). The exact nature of the function \( \phi \) is, however, unknown, see [Lu 1, Conjecture on p. 443, En 2, 5.10].

6.4. A proof of Nagata’s Lemma 3.1.1. We shall give a simple proof of Lemma 3.1.1. By a standard diagonal embedding argument it is enough to show that given a zero dimensional set \( G \) in \( X \) and a pair of disjoint closed sets \( A \) and \( B \) in \( X \), there is
a continuous map \( f: X \to J \) into the interval \( J = [-1/\sqrt{2}, 1/\sqrt{2}] \) such that \( A \subset f^{-1}(-1/\sqrt{2}) \), \( B \subset f^{-1}(1/\sqrt{2}) \) and \( f(G) \subset P \), \( P \) being the irrationals.

Nagata [Na 1, Na 2, VI.2.A] constructed such a map \( f \) by a modification of a standard proof of Urysohn’s Lemma. We shall construct this map by a simple approximation procedure.

Arrange all rational numbers from \( T \) into a sequence \( r_1, r_2, \ldots \) (without repetitions) and let \( \delta_i = \min\{|r_j - r_i|, |r_j + 1/\sqrt{2}| : k < j < i\} \), \( \epsilon_i = 2^{-(i+3)} \cdot \delta_i \), \( a_i = r_i - \epsilon_i \), \( b_i = r_i + \epsilon_i \), \( A_i = [-1/\sqrt{2}, a_i] \), \( B_i = [b_i, 1/\sqrt{2}] \), \( J_i = (a_i, b_i) \). Let us define continuous maps \( f_i: X \to J \), \( i = 0,1,2, \ldots \), inductively as follows: let \( f_0 \) be such that \( A \subset f_0^{-1}(-1/\sqrt{2}) \), \( B \subset f_0^{-1}(1/\sqrt{2}) \), assume that the map \( f_i \) has been already defined and put \( C = f_i^{-1}(A_{i+1}) \), \( D = f_i^{-1}(B_{i+1}) \). Choose an open set \( U \) in \( X \) such that \( C \subset U \), \( U \cap D = \emptyset \) and \( (U \setminus U) \cap G = \emptyset \) and let \( f_{i+1}: X \to J \) be a continuous map such that \( f_{i+1}^{-1}(A_{i+1} \cup B_{i+1}) = C \cup D \), \( f_{i+1} \) coincides with \( f_i \) on \( C \cup D \) and \( f_{i+1}^{-1}(r_i) = U \setminus U \). Since \( |f_{i+1}(x) - f_i(x)| < 2^{-i} \cdot \epsilon_{i+1} = 2^{-(i+3)} \cdot \delta_{i+1} \), the sequence \( \{f_i\}_{i=0}^{\infty} \) converges uniformly to a continuous map \( f: X \to J \). Since all \( f_i \) coincide with \( f_0 \) on \( A \cup B \), \( A \subset f^{-1}(-1/\sqrt{2}) \) and \( B \subset f^{-1}(1/\sqrt{2}) \) and it is routine to check that for every rational number \( r_i \) from \( J \), \( f^{-1}(r_i) \cap G = \emptyset \), i.e. \( f(G) \subset P \).

REFERENCES


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