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# THE RADON TRANSFORM ON $SL(2,\mathbf{R})/SO(2,\mathbf{R})$

D. I. WALLACE AND RYUJI YAMAGUCHI

ABSTRACT. Let $G$ be $SL(2,\mathbf{R})$. $G$ acts on the upper half-plane $\mathcal{H}$ by the Möbius transformation, providing $\mathcal{H}$ with the Riemannian metric structure along with the Laplacian, $\Delta$. We study the integral transform along each geodesic. $G$ acts on $\mathcal{P}$, the space of all geodesics, in a natural way, providing $\mathcal{P}$ with its invariant measure and its own Laplacian. ($\mathcal{P}$ actually is a coset space of $G$.) Therefore the above transform can be viewed as a map from a suitable function space on $\mathcal{H}$ to a suitable function space on $\mathcal{P}$. We prove a number of properties of this transform, including the intertwining properties with its Laplacians and its relation to the Fourier transforms.

**1. Introduction.** Both the Radon transform and the X-ray transform on a symmetric space arise from the problem of reconstructing a function from its integrals along certain paths. For the Euclidean case, a problem can be stated in one of two ways. Suppose all the integrals of some function $f$ along all straight lines are known. It is then possible to reconstruct $f$ from its line integrals. For $\mathbf{R}^2$, the solution to this problem was the inversion of the original Radon-John transform. The ability to invert this particular transform rested in a duality in integral geometry between the points in $\mathbf{R}^2$ and the set of lines in $\mathbf{R}^2$. Technically, the inversion depends heavily on Fourier analysis on $\mathbf{R}^2$. Alternatively, if the space in question is $\mathbf{R}^n$, one might wish to reconstruct $f$ from its integrals over hyperplanes of dimension $k$. Solutions to both of these problems can be found in Helgason [**4**, **9**]. For the Euclidean case, we shall speak of the Radon transform when we mean an integral over a $k$-plane where $k < n$, and of the X-ray transform when we mean an integral over a straight line. Some references for the Euclidean case include Helgason [**5**, **9**], Strichartz [**17**], Radon [**13**], Peters [**12**], Solmon [**15**, **16**], and Smith, Solmon and Wagner [**14**].

When one wishes to define the analogous transform on a general symmetric space, several options become available. The source of these options lies in the generalization of the notion of hyperplane. In $\mathbf{R}^2$, for example, there are two ways of thinking of an $(n-1)$-dimensional hyperplane. One is as a totally geodesic $n-1$ submanifold. The term "totally geodesic" means that a straight line tangent to a particular plane is actually contained in it. Thus, one generalization of the Radon transform which has been exploited is the non-Euclidean case of integrals over totally geodesic submanifolds of dimension $k$. A theory, including inversion formula, has been worked out for the case where $k$ is even, in Helgason [**9**]. More can be found in Gel′fand et al. [**2**], and Lax and Phillips [**11**].

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A second and equally productive way of thinking of an $n-1$ hyperplane in $\mathbf{R}^2$ is as a set of trajectories through a fixed point orthogonal to a family of parallel geodesics. In this case, the non-Euclidean analog of a hyperplane is not a geodesic at all. In fact, for a symmetric space $G/K$, it can be characterized as the orbit $gNg^{-1}$, where $g$ is a fixed element of the group and $N$ is the nilpotent part of the Iwasawa decomposition $G = ANK$. Details can be found in Helgason [**10**] or Terras [**18**]. The virtue of this point of view is that the Fourier transform decomposes into two integral transforms. The integral over the surface where the exponential function defined in [**9**] is constant is exactly the above-mentioned orbital integral. This makes the inversion formulas somewhat easier and allows one to explain somewhat the structure of some differential operators on $G/K$.

In this paper we attempt to construct the results analogous to those for $\mathbf{R}^2$ in the case of symmetric space $\mathcal{H} = SL(2,\mathbf{R})/SO(2,\mathbf{R})$, i.e., the hyperbolic upper half of the complex plane $\mathbf{C}$. The Radon transform is defined, decompositions which mirror those for $\mathbf{R}^2$ are given. The space of geodesics is characterized and the intertwining operators are given for the Laplace-Beltrami operator on $\mathcal{H}$.

**2. The space of geodesics $\mathcal{P}$.** Let $G$ be $SL(2,\mathbf{R})$, the group of $2\times 2$ matrices with real entries having determinant 1. Let $ANK$ be its Iwasawa decomposition. $A(t)$, $N(n)$ and $R(\theta)$ denote matrices

$$\begin{pmatrix} e^{+t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \quad \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

respectively. $G$ acts on the complex plane $\mathbf{C}$ by Möbius transformation; let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an element of $SL(2,\mathbf{R})$. It maps an upper half-plane $\mathcal{H} = \{x+iy\colon y>0\}$ into itself. $A(t)$ acts on $z$ as a dilation; $A(t)z = e^t z$. $N(n)$ translates $z$ along the $x$-axis; $N(n)z = x+n+iy$. $R(\theta)$ rotates $z$ along the circle centered at $(\cosh r)i$ with radius $\sinh r$ for some $r$. This $r$ is related to the Poincaré metric that is defined below. As it turns out, $r$ is the non-Euclidean distance from the point $i$ to the point $z$.

A $G$-invariant measure, in terms of $x$ and $y$ coordinates, is $dxdy/y^2$. We sometimes refer to this measure as $dz$. The $G$-invariant differential operator on $\mathcal{H}$ is the Laplacian $\Delta = y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$.

The $G$-invariant metric on $\mathcal{H}$, called the Poincaré metric, is defined as follows; let $u$ and $v$ be tangent vectors at a point $z = x+iy$, then $\langle u,v\rangle_z = u\cdot v/y^2$. The Riemannian structure endows $\mathcal{H}$ with the set of geodesics $\mathcal{P}$. The set consists of all semicircles with centers on the $x$-axis and all straight lines parallel to the $y$-axis in $\mathcal{H}$. The action of $G$ maps geodesics to geodesics, hence $G$ also acts on $\mathcal{P}$. This metric also provides us with a measure on each smooth curve $\xi$ in $\mathcal{H}$ such that it stays invariant when it is mapped into another curve in $\mathcal{H}$ by an element of $G$. Denote this measure by $d\sigma_\xi$. If $\xi$ is a semicircle with center $s$ and radius $r$ so that $\xi = \{s + r\cos\alpha + ir\sin\alpha\colon 0<\alpha<\pi\}$, then $d\sigma_\xi = d\alpha/(\sin\alpha)$. There is another way to parametrize $\mathcal{P}$ and $\xi$ in it. Each geodesic $\xi$ is $\{R(\theta)N(n)A(t)i\colon -\infty<t<\infty\}$ for some $\theta$ and $n$. Then $d\sigma_\xi = dt$. $\mathcal{P}$ has a manifold structure inherited from $G/A$. We see later that $G/A$ is a double covering of $\mathcal{P}$.

Let $f$ be a compactly supported smooth function on $\mathcal{H}$. Define the Radon transform $Tf(\xi)$ for each $\xi$ by the equation $Tf(\xi) = \int_\xi f(z)\,d\sigma_\xi(z)$.

In this section, we find a $G$-invariant measure and a $G$-invariant second-order differential operator on $\mathcal{P}$. We then compare those to the $G$-invariant measure and operator on the space of horocycles which has already been studied by Helgason (see [**5**, **6**, and **9**]).

In §3, we define a dual transform of the Radon transform which maps a function on $\mathcal{P}$ back to a function on $\mathcal{H}$. This definition agrees with Helgason [**5**]. The intertwining properties of these transforms with $G$-invariant differential operators on $\mathcal{H}$ and $\mathcal{P}$ are proved.

In §4, we discuss an inversion formula.

We work with four different parametrizations of $\mathcal{P}$. Since each geodesic is either a semicircle or a straight line parallel to the $y$-axis, we can parametrize $\mathcal{P}$ with $s$ and $r$ (or $1/r$), where $s$ is the center of the semicircle and $r$ is the radius. When a geodesic is a straight line, then we let $r = +\infty$ (or $1/r = 0$). We can also consider two endpoints $x_1$ and $x_2$ of geodesics for parameters of $\mathcal{P}$. We can always choose $x_1 < x_2$ and allow values $+\infty$ for $x_2$ and $-\infty$ for $x_1$. Thus the pair $x_1 = -\infty$ and $x_2 = a$, and the pair $x_1 = a$ and $x_2 = +\infty$ represent the same geodesic $x = a$. From $x_1$ and $x_2$, we can also choose $\theta_1$ and $\theta_2$ such that $0 \le \theta_1 < \theta_2 \le \pi$ and $\cot\theta_1 = -x_1$ and $\cot\theta_2 = -x_2$. $\theta_1$ and $\theta_2$ are then the angles that the straight lines from $i$ to $x_1$ and $x_2$ make with the $x$-axis. The manifold structure of $\mathcal{P}$ is perhaps best expressed with this parametrization. $\mathcal{P}$ is diffeomorphic to the set $\{(\theta_1,\theta_2)\colon 0 \le \theta_1 < \theta_2 \le \pi \text{ and if } \theta_1 = 0 \text{ then } \theta_2 \ne \pi\}$ with boundaries $\theta_1 = 0$ and $\theta_2 = \pi$ identified by the equation $(0,\alpha) = (\alpha,\pi)$ for an angle $\alpha$, $0 < \alpha < \pi$.

We note that the geodesic $x_1 = -\cot\theta_1$ and $x_2 = -\cot\theta_2$ is a rotation $R(\theta_1)$ of a vertical line $x = n$ for some $n$. In fact, all geodesics are rotations of straight lines.

The fourth parametrization of $\mathcal{P}$ is derived from the Iwasawa decomposition of $G$. All geodesics are of the form $\{R(\theta)N(n)A(t)i\colon -\infty < t < \infty\}$ for some $\theta$ and $n$. When $\theta = 0$, the geodesic $\xi$ is a straight line $x = n$. The rotation $R(\theta)$ maps two endpoints of this line to the endpoints of a semicircle. $x_1$ and $x_2$ can easily be computed from the Möbius transformation $R(\theta)N(n)A(t)$ applied to the point $i$ and by letting $t$ approach $+\infty$. The result is $x_1 = -\cot\theta$ and $x_2 = (nx_1+1)/(x_1-n)$ if $n < \cot\theta$; $x_1$ and $x_2$ are switched if $n > \cot\theta$. Here, $\theta$ is considered to be in the interval $[0,\pi]$. Two geodesics $\xi_1(t) = R(\theta_1)N(n_1)A(t)i$ and $\xi_2(t) = R(\theta_2)N(n_2)A(t)i$ are the same if either $\theta_1 = \theta_2$ and $n_1 = n_2$ or $\theta_2 = \theta_1 + \cot^{-1}(-n_1)$ and $n_1 = -n_1$. This shows that the coset space $G/A$ is a double covering of $\mathcal{P}$. $\mathcal{P}$ is in fact $G/MA$ where $M = \{I, -I, R(\pi/2), -R(\pi/2)\}$.

PROPOSITION 2.1. *There is a $SL(2,\mathbf{R})$-invariant measure $d\mu$ on $\mathcal{P}$.*

(1) *In terms of center $s$ and radius $r$,*

$$d\mu = \frac{ds\,dr}{r^2}.$$

(2) *In terms of endpoints $x_1$ and $x_2$,*

$$d\mu = \frac{dx_1dx_2}{2(x_2-x_1)^2}.$$

(3) *When $x_1 = \cot\theta_1$ and $x_2 = \cot\theta_2$ then*

$$d\mu = \frac{d\theta_1 d\theta_2}{2\sin^2(\theta_1-\theta_2)}.$$

(4) *When $\xi(t)$ is written as $R(\theta)N(n)A(t)i$ with $n > 0$ then $d\mu = d\theta\,dn$.*

PROOF. We have already discussed the relations between the above four different parametrizations of $\mathcal{P}$. That the four different expressions express the same measure is an easy consequence of these relations.

It remains to prove that $d\mu$ is $G$-invariant using any of the above four coordinate systems. We choose $x_1, x_2$ coordinates. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an element of $SL(2, \mathbf{R})$; then $g$ takes endpoints $x_1, x_2$ of a semicircle to

$$x_1' = \frac{ax_1 + b}{cx_1 + d} \quad \text{and} \quad x_2' = \frac{ax_2 + b}{cx_2 + d}.$$

It is easy to verify that $dx_1' = (cx_1 + d)^{-2} dx_1$ and $dx_2' = (cx_2 + d)^{-2} dx_2$. Also $(x_2' - x_1')^2 = (x_1 - x_2)^2 (cx_1 + d)^{-2} (cx_2 + d)^{-2}$. Hence $dx_1' dx_2' / (x_2' - x_1')^2 = dx_1 dx_2 / (x_2 - x_1)^2$. This proves the invariance of $d\mu$.

PROPOSITION 2.2. *There is an $SL(2, \mathbf{R})$-invariant differential operator $\Box$ on $\mathcal{P}$.*

(1) *In terms of $s$ and $r$,*

$$\Box = r^2 \left( \frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial r^2} \right).$$

(2) *In terms of $x_1$ and $x_2$,*

$$\Box = \frac{1}{4}(x_2 - x_1)^2 \frac{\partial^2}{\partial x_1 \partial x_2}.$$

(3) *In terms of $x_1 = \cot \theta_1$ and $x_2 = \cot \theta_2$,*

$$\Box = \sin^2(\theta_1 - \theta_2) \frac{\partial^2}{\partial \theta_1 \partial \theta_2}.$$

(4) *When $\xi(t)$ is written as $R(\theta)N(n)A(t)i$, then*

$$\Box = \frac{1}{1 + n^2} \left( \frac{\partial^2}{\partial \theta^2} + 2 \frac{\cos \theta}{\sin \theta} \frac{\partial}{\partial \theta} \right) + \frac{\partial^2}{\partial n \partial \theta}.$$

PROOF. A change of variables shows that all the expressions in (1)–(4) represent the same differential operator $\Box$. Now, $\Box$ is $G$-invariant if $\Box$ is invariant under each action $A(t)$, $N(n)$ and $R(\theta)$. From (2), it is easy to see that $\Box$ is invariant under both a dilation $A(t)$ and a translation $N(n)$. A computation shows that a rotation $R(\theta)$ maps a geodesic $x_1 = \cot \theta_1$ and $x_2 = \cot \theta_2$ to a geodesic $x_1' = \cot(\theta_1 + \theta)$ and $x_2' = \cot(\theta_2 + \theta)$. From (3), we see that $\Box$ stays invariant under $R(\theta)$. This proves the proposition.

**3. The dual transform and the intertwining properties.** Let $f$ be a compactly supported smooth function on $\mathcal{P}$. The Radon transform maps this function $f$ to a function $Tf$ on $\mathcal{P}$ defined by the equation $Tf(\xi) = \int_\xi f(z)\, d\sigma_\xi(z)$. Let $d\xi$ be the $G$-invariant measure defined in Proposition 2.1. Let $\phi$ be a compactly supported smooth function on $\mathcal{P}$. The inner product on $\mathcal{P}$,

$$\langle Tf, \phi \rangle = \int_{\mathcal{P}} Tf(\xi) \phi(\xi)\, d\xi,$$

is well defined. The dual transform $T^*$ is characterized by the property that $T^*$ maps a function $\phi$ on $\mathcal{P}$ to a function $T^*\phi$ on $\mathcal{H}$ such that the inner product on $\mathcal{H}$,

$$\langle f, T^*\phi\rangle = \int_{\mathcal{H}} f(z)T^*\phi(z)\,dz,$$

has the same value as $\langle Tf, \phi\rangle$. To write down $T^*\phi$, we shall first use $x, y$ coordinates for $\mathcal{H}$ and $s, r$ coordinates for $\mathcal{P}$.

PROPOSITION 3.1. *The dual transform $T^*$ of the Radon transform maps a compactly supported smooth function $\phi$ on $\mathcal{P}$ to $T^*\phi$ on $\mathcal{H}$, given by the equation*

$$T^*\phi(x,y) = \int_{\theta=0}^{\pi} \phi(x - y\cot\theta, y\csc\theta)\,d\theta.$$

PROOF. We have

$$\begin{aligned}\langle Tf, \phi\rangle &= \iint_{\mathcal{P}} Tf(s,r)\phi(s,r)\frac{ds\,dr}{r^2}\\ &= \iint_{\mathcal{P}}\int_{\theta=0}^{\pi} f(s + r\cos\theta, r\sin\theta)\phi(s,r)\frac{d\theta}{\sin\theta}\frac{ds\,dr}{r^2}.\end{aligned}$$

Letting $x = s + r\cos\theta$, $y = r\sin\theta$, and

$$\frac{ds\,dr}{r^2} = \frac{dx\,dy}{\sin\theta}\frac{1}{(y/\sin\theta)^2},$$

we can write the preceding integral as

$$\iint_{\mathcal{H}}\int_{\theta=0}^{\pi} f(x,y)\phi(x - y\cot\theta, y\csc\theta)\,d\theta\frac{dx\,dy}{y^2}.$$

Hence $T^*\phi(x,y) = \int \phi(x - y\cot\theta, y\csc\theta)\,d\theta$.

We now examine the integrand $\phi(x - y\cot\theta, y\csc\theta)$ a little more closely. Here, we are integrating geodesics $\xi_\theta$ with respect to $\theta$, where each $\xi_\theta$ has a center $s = x - y\cot\theta$ and a radius $y\csc\theta$. We note that $\xi_\theta = \left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}\sqrt{y} & 0\\ 0 & 1/\sqrt{y}\end{smallmatrix}\right)\eta_\theta$, where the matrices are elements of $SL(2,\mathbf{R})$ and act on a geodesic $\eta_\theta$ with center $s = -\cot\theta$ and radius $r = \csc\theta$. $\eta_\theta$ is independent of $x$ and $y$ and passes through the point $i$ in $\mathbf{C}$. So, in fact, we are integrating, with respect to $\theta$, all geodesics $\xi_\theta$ which pass through the point $x + iy$. In other words, $T^*\phi(x,y) = \int \phi(s,r)\,d\mu_{x,y}(s,r)$ for a distribution $d\mu_{x,y}$ supported on a subset of $\mathcal{P}$ containing all geodesics having the point $x + iy$ on them. These definitions coincide with those in Helgason [**5**] for a general pair of coset spaces.

In terms of covering $G/A$, $\eta_\theta$ has coordinates $R(\theta)A$. We can write $T^*\phi$ using the $G/K$-covering for coordinates of $\mathcal{H}$ and the $G/A$-covering for coordinates of $\mathcal{P}$:

$$\begin{aligned}&T^*\phi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{y} & 0\\ 0 & 1/\sqrt{y}\end{pmatrix}K\right)\\ &\qquad = \int_{\theta=0}^{\pi}\phi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{y} & 0\\ 0 & 1/\sqrt{y}\end{pmatrix}R(\theta)A\right)d\theta\end{aligned}$$

of equivalently

$$T^*\phi(A(t)N(n)K) = \int_{\theta=0}^{\pi} \phi(A(t)N(n)R(\theta)A)\, d\theta.$$

What we would now like to show is the intertwining property of $\Delta$ and $\Box$ with respect to $T$ and $T^*$. This can be expressed as follows:

$$\Delta(T^*\phi) = -T^*(\Box\phi), \qquad \Box(Tf) = -T(\Delta f),$$

where $\phi$ and $f$ are compactly supported smooth functions on $\mathcal{P}$ and $\mathcal{H}$ respectively.

In order to prove the above, we will rewrite the operators $\Delta$ and $\Box$ as a composition of first-order differential operators as follows:

$$\Delta = y^2\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}\right) = y\frac{\partial}{\partial y}\left(y\frac{\partial}{\partial y}\right) - y\frac{\partial}{\partial y} + y^2\left(\frac{\partial}{\partial x}\frac{\partial}{\partial x}\right),$$

$$\Box = r^2\left(\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial r^2}\right) = r^2\frac{\partial}{\partial s}\left(\frac{\partial}{\partial s}\right) - r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + r\frac{\partial}{\partial r}.$$

We write $Dy = y\,\partial/\partial y$, $Dx = \partial/\partial x$, $Dr = r\,\partial/\partial r$ and $Ds = \partial/\partial s$. Each of these operators can be written as an infinitesimal generator of a semigroup operator as follows:

$$Dy\, f = \lim_{p\to 1} \frac{f(x,py) - f(x,y)}{p-1},$$

$$Dx\, f = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h},$$

etc. This decomposition allows us to bring the generators $Dx$, $Dy$, $Dr$ and $Ds$ inside the integral in Lemmas 3.2, 3.3, 3.6 and 3.7. Also, this particular choice of variables allows us to change an action on $\mathcal{H}$ into an action on $\mathcal{P}$.

LEMMA 3.2. *$(\partial/\partial x)(T^*\phi) = T^*(\partial\phi/\partial s)$ for all compactly supported smooth functions $\phi$.*

PROOF.

$$\begin{aligned}
&\frac{\partial}{\partial x}\int_{\theta=0}^{\pi} \phi(x - y\cot\theta, y\csc\theta)\, d\theta \\
&\quad = \lim_{h\to 0}\frac{1}{h}\int_{\theta=0}^{\pi} \phi(x+h-y\cot\theta, y\csc\theta) - \phi(x-y\cot\theta, y\csc\theta)\, d\theta \\
&\quad = \int_{\theta=0}^{\pi} \lim_{h\to 0} \frac{\phi(x+h-y\cot\theta, y\csc\theta) - \phi(x-y\cot\theta, y\csc\theta)}{h}\, d\theta \\
&\quad = \int \lim_{h\to 0}\frac{\phi(s+h,r) - \phi(s,r)}{h}\, d\mu_{x,y}(s,r) \\
&\quad = \int \frac{\partial\phi}{\partial s}(s,r)\, d\mu_{x,y}(s,r) \\
&\quad = T^*\left(\frac{\partial\phi}{\partial s}\right)(x,y).
\end{aligned}$$

LEMMA 3.3.

$$y\frac{\partial}{\partial y}(T^*\phi) = T^*\left((s-x)\frac{\partial}{\partial s} + r\frac{\partial\phi}{\partial r}\right)$$

*for all compactly supported smooth functions $\phi$.*

PROOF.

$$
\begin{aligned}
y\frac{\partial}{\partial y}&(T^*\phi(x,y)) \\
&= \lim_{p\to 1}\frac{1}{p-1}\int_{\theta=0}^{\pi}\phi(x-py\cot\theta, py\csc\theta)-\phi(x-y\cot\theta, y\csc\theta)\,d\theta \\
&= \int_{\theta=0}^{\pi}\lim_{p\to 1}\frac{\phi(x-py\cot\theta, py\csc\theta)-\phi(x-y\cot\theta, y\csc\theta)}{p-1}\,d\theta \\
&= \int \lim_{p\to 1}\frac{\phi(s+(1-p)(x-s), pr)-\phi(s,r)}{p-1}\,d\mu_{x,y}(s,r) \\
&= \int\left((s-x)\frac{\partial\phi}{\partial s}+r\frac{\partial\phi}{\partial r}\right)d\mu_{x,y}(s,r) \\
&= T^*\left((s-x)\frac{\partial\phi}{\partial s}+r\frac{\partial\phi}{\partial r}\right)(x,y).
\end{aligned}
$$

LEMMA 3.4. *Assume that $\phi$ is a compactly supported smooth function; then*

$$\int\left(r^2\frac{\partial\phi}{\partial s}+r(s-x)\frac{\partial\phi}{\partial r}\right)d\mu_{x,y}(s,r)=0 \quad \textit{for all } x \textit{ and } y.$$

PROOF. Since $\phi$ is a smooth function, for all $x$ and $y$

$$\lim_{\theta\to 0}\phi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{y} & 0\\ 0 & 1/\sqrt{y}\end{pmatrix}\eta_\theta\right)=\lim_{\theta\to 0}\phi(x-y\cot\theta, y\csc\theta)$$

and

$$\lim_{\theta\to \pi}\phi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{y} & 0\\ 0 & 1/\sqrt{y}\end{pmatrix}\eta_\theta\right)=\lim_{\theta\to \pi}\phi(x-y\cot\theta, y\csc\theta)$$

are the same. Hence

$$
\begin{aligned}
0 &= \int_{\theta=0}^{\pi}\frac{\partial}{\partial\theta}(\phi(x-y\cot\theta, y\csc\theta))\,d\theta \\
&= \int_{\theta=0}^{\pi}\frac{\partial}{\partial s}\phi(x-y\cot\theta, y\csc\theta)\frac{\partial s}{\partial\theta} \\
&\quad+\frac{\partial}{\partial r}(x-y\cot\theta, y\csc\theta)\frac{\partial r}{\partial\theta}\,d\theta.
\end{aligned}
$$

But

$$\frac{\partial s}{\partial\theta}=\frac{d}{d\theta}(x-y\cot\theta)=y\csc^2\theta=\frac{r^2}{y},$$

and

$$\frac{\partial r}{\partial\theta}=\frac{d}{d\theta}(y\csc\theta)=-y\cot\theta\csc\theta=\frac{(s-x)r}{y}.$$

This gives

$$0=\int\left(\frac{r^2}{y}\frac{\partial\phi}{\partial s}+\frac{(s-x)r}{y}\frac{\partial\phi}{\partial r}\right)d\mu_{x,y}(s,r),$$

hence

$$0=\int\left(r^2\frac{\partial\phi}{\partial s}+(s-x)r\frac{\partial\phi}{\partial r}\right)d\mu_{x,y}(s,r).$$

THEOREM 3.5. *For all compactly supported smooth functions $\phi$ and $\mathcal{P}$,*

$$(\Delta T^*\phi) = -T^*(\Box\phi).$$

PROOF. Using the decomposition of $\Delta$ given previously, we have

$$\Delta(T^*\phi) = \left[\left(y\frac{\partial}{\partial y}\right)^2 - y\frac{\partial}{\partial y} + y^2\frac{\partial^2}{\partial x^2}\right](T^*\phi).$$

By Lemmas 3.2 and 3.3, this is equal to

$$\begin{aligned}T^*&\left[\left((s-x)\frac{\partial}{\partial s} + r\frac{\partial}{\partial r}\right)^2 - \left((s-x)\frac{\partial}{\partial s} + r\frac{\partial}{\partial r}\right) + y^2\left(\frac{\partial}{\partial s}\right)^2\right]\phi \\ &= T^*\left[(s-x)^2\frac{\partial^2\phi}{\partial s^2} + 2(s-x)r\frac{\partial\phi}{\partial s\partial r} + y^2\frac{\partial^2\phi}{\partial s^2}\right].\end{aligned}$$

Now by Lemma 3.4, the above is equal to

$$T^*\left[(s-x)^2\frac{\partial^2\phi}{\partial s^2} - 2r^2\frac{\partial^2\phi}{\partial s^2} + r^2\frac{\partial^2\phi}{\partial r^2}\right].$$

$T^*$ is an integral over all geodesics $(s,r)$ passing through the point $x+iy$, hence $(s-x)^2+y^2=r^2$ holds. Thus the preceding equation now becomes

$$\begin{aligned}\Delta T^* &= T^*\left[r^2\frac{\partial^2\phi}{\partial s^2} - r^2\frac{\partial^2\phi}{\partial r^2}\right] \\ &= -T^*(\Box\phi). \quad \text{Q.E.D.}\end{aligned}$$

For the reverse property, $\Box(Tf) = -T(\Delta f)$, we shall state the lemmas and leave the proofs to the reader, but we will include an abbreviated proof of the theorem. Throughout, $f$ is a compactly supported smooth function on $\mathcal{H} = \{x+iy\colon y>0\}$.

LEMMA 3.6.

$$T\left(\frac{\partial f}{\partial x}\right)(s,r) = \frac{\partial}{\partial s}(Tf(s,r)).$$

LEMMA 3.7.

$$T\left((x-s)\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right)(s,r) = r\frac{\partial}{\partial r}(Tf(s,r)).$$

LEMMA 3.8.

$$T\left((x-s)y\frac{\partial f}{\partial y} - y^2\frac{\partial f}{\partial x}\right) = 0.$$

THEOREM 3.9. $\Box(Tf) = -T(\Delta f)$.

PROOF.

$$\Box = r^2\frac{\partial^2}{\partial s^2} - r^2\frac{\partial^2}{\partial r^2} = r^2\frac{\partial^2}{\partial s^2} - r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + r\frac{\partial}{\partial r}.$$

Now by Lemmas 3.6 and 3.7 we have

$$\Box(Tf)(s,r) = T\left(r^2\frac{\partial^2 f}{\partial x^2} - (x-s)^2\frac{\partial^2 f}{\partial x^2} - 2(x-s)y\frac{\partial^2 f}{\partial x\partial y} - y^2\frac{\partial^2 f}{\partial y^2}\right)(s,r).$$

But by Lemma 3.8,

$$T\left((x-s)y\frac{\partial^2 f}{\partial x \partial y}\right) = T\left(y^2\frac{\partial^2 f}{\partial x^2}\right).$$

Hence

$$\Box(Tf)(s,r) = T\left((r^2-(x-s)^2) - 2y^2\frac{\partial^2 f}{\partial x^2} - y^2\frac{\partial^2 f}{\partial y^2}\right)(s,r).$$

Now $T$ is an integral over points $(x,y)$ belonging to a geodesic $(s,r)$, hence $(x-s)^2+y^2=r^2$ holds. Therefore

$$\begin{aligned}\Box Tf(s,r) &= T\left(-y^2\frac{\partial^2 f}{\partial x^2} - y^2\frac{\partial^2 f}{\partial y^2}\right)(s,r)\\ &= -T(\Delta f)(s,r). \quad \text{Q.E.D.}\end{aligned}$$

**4. Inversion formula.** We begin this section with a discussion of horocycle transforms, studied extensively by Helgason. A more detailed version of the background information can be found in Helgason [**9**], Terras [**18**], and Vergne [**19**]. A horocycle can be defined as a trajectory which is orthogonal to a family of geodesics. For the upper half-plane $\mathcal{H}$, a horocycle is a circle tangent to the $x$-axis or a line parallel to the $x$-axis.

EXAMPLE.

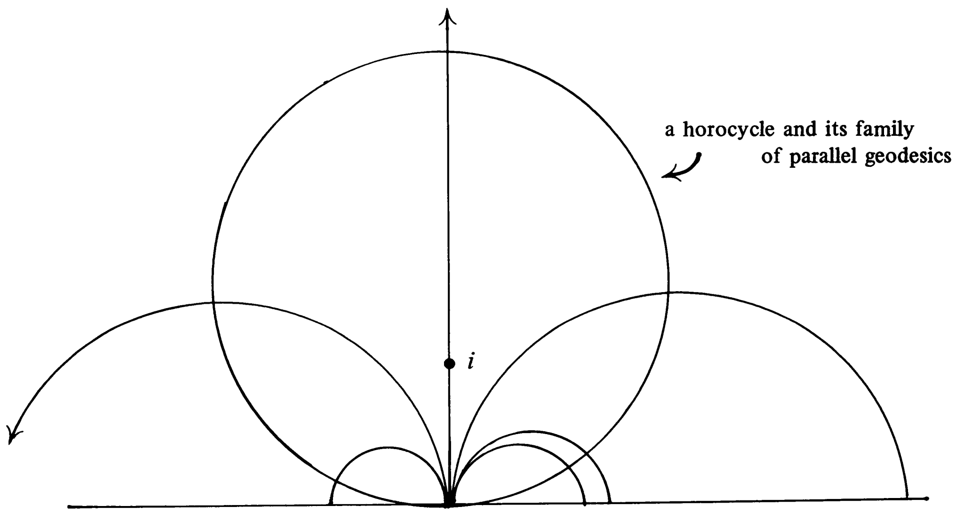


FIGURE 1

Horocycles are orbits $\{R(\theta)A(t)N(n)i\colon -\infty < n < +\infty\}$ in terms of the Iwasawa decomposition of $G$. The example shows a horocycle with $\theta = \pi/2$. The point of tangency, $b = -\cot\theta$, is called the "normal" of the horocycle. Define the bracket $\langle z, b\rangle$ to be the directed non-Euclidean distance from $i$ to the horocycle passing through $z$ and the boundary point $b$. Obviously, $\langle z,b\rangle = \langle w,b\rangle$ if $z$ and $w$ lie on the same horocycle with the boundary point $b$. Also, $\langle z,b\rangle$ is negative if the point $i$ is inside the horocycle. To relate the X-ray transform in $\mathbf{R}^2$ with Fourier analysis, the horocycles (straight lines in $\mathbf{R}^2$) are parametrized by the directed distances from the origin $(0,0)$ and an angle $\theta$. The quantity $\langle z,b\rangle$ in $\mathcal{H}$ is the natural analog of the directed distance in $\mathbf{R}^2$. The function defined by Helgason [9],

$$e_{\lambda,b}\colon z\mapsto e^{\lambda\langle z,b\rangle},\qquad \lambda\in\mathbf{C},\ z\in\mathcal{H},$$

plays the role of the usual exponential function in what is called the *Helgason transform* for the symmetric space $G/K$. (See Terras [18].) The purpose of this section is to give a decomposition of the Helgason transform in terms of the Radon transform. We begin by stating the inversion formula and properties for $e_{\lambda,b}$ found in Helgason.

DEFINITION. For a compactly supported smooth function $f$ on $\mathcal{H}$, define $\tilde{f}(\lambda,b)$ by

$$\tilde{f}(\lambda,b)=\int_{\mathcal{H}} f(z)e^{(1-i\lambda)\langle z,b\rangle}\,dz.$$

THEOREM 4.1 (HELGASON).

$$f(z)=\int_{\lambda\in\mathbf{R}}\int_{\theta=0}^{2\pi}\tilde{f}(\lambda,-\cot\theta)e^{(1-i\lambda)\langle z,b\rangle}\,d\mu,$$

*where* $d\mu(\lambda,b)=(1/4\pi^2)\tanh(\pi\lambda)\,d\lambda\,d\theta$.

PROPOSITION 4.2 (HELGASON). *Let $g$ be an element of $G=SL(2,\mathbf{R})$, and $g\cdot z$, the Möbius transform of $z$ by $g$; then*

$$\langle g\cdot z,g\cdot b\rangle=\langle z,b\rangle+\langle g\cdot i,g\cdot b\rangle.$$

PROPOSITION 4.3 (HELGASON). $|d(g\cdot b)/db|=e^{2\langle g^{-1}\cdot i,b\rangle}$.

The proofs of 4.1–4.3 are in Helgason [9].

PROPOSITION 4.4. *Let $z=x+iy$, $y>0$. Let $b$ be a boundary point in $\mathbf{R}$ of $\mathcal{H}$. Then*

$$\langle z,b\rangle=\ln\left[\frac{(1+b^2)y}{(x-b)^2+y^2}\right].$$

PROOF. $\langle z,b\rangle$ is the shortest distance from $i$ to the horocycle determined by $z$ and $b$. Since the Möbius transforms permute horocycles and preserve non-Euclidean distances, we select one which preserves the point $i$ and moves the horocycle to a horizontal line (i.e., takes $b$ to $\pm\infty$). A Möbius transformation which does this is

$$g=(1+b^2)^{-1/2}\begin{pmatrix} b & 1\\ -1 & b\end{pmatrix}.$$

If we parametrize points on the original horocycle as $w=(b+r\cos\theta)+i(r+r\sin\theta)$, then it is easy to verify that the imaginary part of $g\cdot w$ is a constant for all $\theta$. In fact, $\operatorname{Im}(g\cdot w)=(1+b^2)/2r$. In terms of $w=u+iv$, this is

$$\operatorname{Im}(g\cdot w)=\frac{(1+b^2)v}{(u-b)^2+v^2}.$$

Lastly, it is trivial to see that the non-Euclidean distance from $i$ to the horizontal horocycle (above) is

$$\ln\frac{(1+b^2)y}{(x-b)^2+y^2}$$

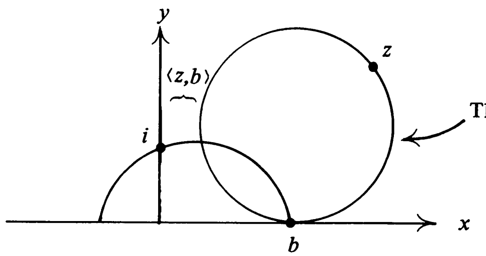


FIGURE 2

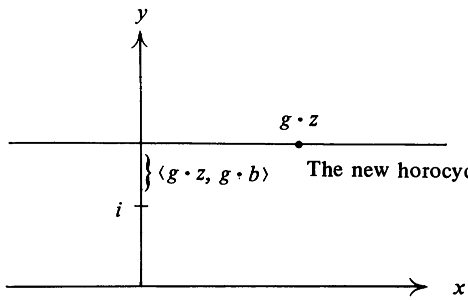


FIGURE 3

(see Figures 2 and 3). Note that this is just a special case of 4.2 because $\langle i, b\rangle = 0$ for all $b$.

REMARK. The horocycle containing $z$ and having the normal $b$ can be parametrized as $\{R(\theta)A(t)N(n)i\colon\ -\infty < n < +\infty\}$, where $\theta$ and $t$ are chosen so that $-\cot\theta = b$ and $z = R(\theta)A(t)N(n_0)$ for some $n_0$. The Möbius transformation $g$ in the proof is $R(-\theta)$. The distance from $i$ to the horocycle is $\langle z, b\rangle = t$.

Let us now define the horocycle (Radon) transform, $Lf(\eta)$, for each compactly supported smooth function on $\mathcal{H}$ and a horocycle $\eta$ by

$$Lf(\eta) = \int_\eta f(z)\, d\sigma_\eta(z)$$

where $d\sigma_\eta$ is the invariant measure on $\eta$. If we parametrize the space of horocycles by $\{R(\theta)A(t)N(n)i\colon\ -\infty < n < +\infty\}$, then $d\sigma$ on $\eta$ is $dn$. Two horocycles $\eta_{\theta_1,t_1} = \{R(\theta_1)A(t_1)N(n)i\colon\ -\infty < n < \infty\}$ and $\eta_{\theta_2,t_2} = \{R(\theta_2)A(t_2)N(n)i\colon -\infty < n < \infty\}$ are the same if and only if either $\theta_1 = \theta_2$ and $t_1 = t_2$ or $\theta_1 - \theta_2 = \pi$ and $t_1 = -t_2$.

Another parametrization of the space of horocycles is $(b, r)$ where $b$ is the normal of the horocycle and $r$ is the radius of the circle. Hence

$$\{(r\cos\alpha + b) + i(r\sin\alpha + r)\colon 0 \le \alpha < 2\pi\}$$

parametrizes the horocycle. Then $d\sigma$ is $d\alpha/(1+\sin\alpha)$, and the integral becomes

$$Lf(b,r) = \int_{\alpha=0}^{2\pi} f(b + r\cos\alpha, r\sin\alpha + r)\frac{d\alpha}{1+\sin\alpha}.$$

It is now easy to see the following result.

THEOREM 4.5. *$\tilde{f}(\lambda,b) = ((1+b^2)/2)^{1-i}M_r[Lf(b,r)](i\lambda-1)$ where $M_r$ is the Mellin transform in the variable $r$.*

PROOF.
$$\begin{aligned}\tilde{f}(\lambda,b) &= \int f(x,y)e^{(-i\lambda+1)\langle x+iy,b\rangle}\frac{dxdy}{y^2}\\ &= \int f(x,y)\left(\frac{(1+b^2)y}{(x-b)^2+y^2}\right)^{1-i\lambda}\frac{dxdy}{y^2}.\end{aligned}$$
Write $x = r\cos\alpha + b$ and $y = r\sin\alpha + r$; then its Jacobian is $r + r\sin\alpha$. Here $dxdy/y^2 = (d\alpha/(1+\sin\alpha))(dr/r)$. Making the change of variables, we have

$$\begin{aligned}\tilde{f}(\lambda,b) &= \int_{r=0}^{\infty}\int_{\alpha=0}^{2\pi} f(r\cos\alpha+b, r\sin\alpha+b)\left(\frac{1+b^2}{2r}\right)^{1-i\lambda}\frac{d\alpha}{1+\sin\alpha}\frac{dr}{r}\\ &= \left(\frac{1+b^2}{2}\right)^{1-i\lambda}\int_{r=0}^{\infty} Lf(b,r)r^{i\lambda-1}\frac{dr}{r}.\end{aligned}$$

REMARK. By the support theorems in Helgason [**9**], $L$ is one-to-one on $C_c^\infty(\mathcal{H})$. On the other hand $f \to \tilde{f}$ is also one-to-one on $C_c^\infty(\mathcal{H})$ and an inversion formula is known. See Terras [**18**]. Denote by $g \to \hat{g}$ the inverse of the Helgason transform $f \to \tilde{f}$. Write

$$M(Lf) = \left(\frac{1+b^2}{2}\right)^{1-i\lambda}\int_{r=0}^{\infty}(Lf)(b,r)r^{i\lambda-1}\frac{dr}{r};$$

then $\widehat{M(Lf)} = f$ by the above comments if $f \in C_c^\infty(\mathcal{H})$.

Before we proceed, we take another look at Theorem 4.5. In the theorem, the Helgason transform was written as a composition of the horocycle transform followed by the Mellin transform with a multiplicative factor. The multiplicative factor is not really necessary if we use the Iwasawa decomposition to parametrize the space of horocycles as will be seen below:

Let us fix an angle $\theta_0$. Write every point $z$ in $\mathcal{H}$ as $R(\theta_0)A(t)N(n)i$ for some $t$ and $n$. Then $dz = dx\,dy/y^2 = dt\,dn$. A horocycle $\eta = \{R(\theta_0)A(t)N(n)i\colon -\infty < n < +\infty\}$ is normal to $b = -\cot\theta_0$ and its distance to the point $i$ is $t$. Hence

$$\tilde{f}(\lambda,-\cot\theta_0) = \int f(R(\theta_0)A(t)N(n)i)e^{(1-\lambda_i)t}\,dt\,dn.$$

On the other hand, the integral

$$\int \tilde{f}(R(\theta_0)A(t)N(n)i)\,d\eta$$

is an integral over a horocycle $\eta$ with the measure $d\sigma_\eta = d\eta$. Hence this is exactly the horocycle transform $Lf(\eta)$. We now have the following theorem.

THEOREM 4.6. *Let the space of horocycles be parametrized by $\{R(\theta)A(t)N(n)i\colon -\infty < n < \infty\}$ where $0 < \theta < \pi$ and $-\infty < t < \infty$. Then*

$$\tilde{f}(\lambda,-\cot\theta) = \int_{-\infty}^{\infty} Lf(\theta,t)e^{(1-\lambda_i)t}\,dt,$$

*where $Lf(\theta,t)$ is the horocycle transform*

$$Lf(\theta,t)=\int_{-\infty}^{+\infty} f(R(\theta)A(t)N(n)i)\,d\eta.$$

In order to do the analogous theorem for the X-ray transform (i.e., the transform over geodesics) it is necessary to invert a Mellin-Fourier type transform which does for geodesics what the Helgason transform does for horocycles. For the Helgason transform, one regards every point on the upper half-plane as belonging to a unique element of the set of horocycles with normal $b$. If we wish to to the same for geodesics, we can look at every point on the upper half-plane as belonging to a unique geodesic with "center" $b$. We note that Gel′fand, Gindikin, and Shapiro [**2**] define this to be an "admissible family" of curves.

Now we need a notion of distance, so define $\langle\langle z,b\rangle\rangle$ to be the non-Euclidean directed distance from $i+b$ to the geodesic containing $z$ with center $b$. It is trivial to see that $\langle\langle z,b\rangle\rangle=\ln r$ for $z=x+iy$ with $(x-b)^2+y^2=r^2$. We define an exponential function $E_{\nu,b}$ by

$$E_{\nu,b}\colon z\mapsto \exp(\nu\langle\langle z,b\rangle\rangle).$$

So we have $E_{\nu,b}(z)=r^{\nu}$. We define the following transform:

$$\hat f(\nu,b)=\int_{\mathcal{H}} f(z)E_{\nu,b}(z)\,dz=\int f(z)((x-b)^2+y^2)^{\nu/2}\,\frac{dxdy}{y^2}.$$

If we assume $f$ has compact support then $\hat f(\nu,b)$ must converge. We then have the following result.

THEOREM 4.7. *If $f$ is a compactly supported smooth function and if $g(x,y)=f(x,y)y^{-1}$ then*

$$\hat f(\nu,b)=M_r[Tg(b,r)](\nu+1),$$

*where $T$ is the* X-*ray transform over the geodesic with center $b$ and radius $r$, and $M_r$ is the Mellin transform in the variable $r$.*

PROOF.

$$\hat f(\nu,b)=\int_{\mathcal{H}} f(z)((x-b)^2+y^2)^{\nu/2}\,\frac{dxdy}{y^2}.$$

Letting $x=r\cos\alpha+b$ and $y=r\sin\alpha$ we have its Jacobian $r$, so

$$\begin{aligned}\hat f(\nu,b)&=\int f(r\cos\alpha+b,r\sin\alpha)r^{\nu}\frac{r\,dr\,d\alpha}{r^2\sin^2\alpha}\\&=\int g(r\cos\alpha+b,r\sin\alpha)r^{\nu+1}\frac{dr}{r}\frac{d\alpha}{\sin\alpha}.\end{aligned}$$

Switching the order of integration will give us the desired result.

In conclusion, we would like to point out that although the Radon transform for the upper half-plane has been known for a long time, it has not been used much as a number-theoretical tool. With sufficient development it should be possible to use it as such. L. Ehrenpreis also has some results for the horocycle case which make use of a point of view different from that of this paper.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720

DEPARTMENT OF MATHEMATICAL SCIENCES, FLORIDA INTERNATIONAL UNIVERSITY, MIAMI, FLORIDA 33199