

BOUNDS FOR PRIME SOLUTIONS OF SOME DIAGONAL EQUATIONS. II

MING-CHIT LIU

ABSTRACT. Let b_j and m be certain integers. In this paper we obtain a bound for prime solutions p_j of the diagonal equations of order k , $b_1 p_1^k + \cdots + b_s p_s^k = m$. The bound obtained is $C^{(\log B)^2} + C|m|^{1/k}$ where $B = \max_j \{e, |b_j|\}$ and C are positive constants depending at most on k .

1. Introduction. Throughout p denotes a prime number and $k \geq 2$ is an integer. Let $\theta \geq 0$ be the largest integer such that p^θ divides k . We write $p^\theta || k$. Let

$$(1.1) \quad s_0 = \begin{cases} 3k - 1 & \text{if there is a } p \text{ satisfying } p|k \text{ and } k = ((p-1)/2)p^\theta, \\ 2k & \text{otherwise.} \end{cases}$$

$$(1.2) \quad s_1 = \begin{cases} 2^k + 1 & \text{if } 2 \leq k \leq 11, \\ 2k^2(2 \log k + \log \log k + 2.5) - 1 & \text{if } k \geq 12. \end{cases}$$

$$(1.3) \quad \nu = \begin{cases} \theta + 2 & \text{if } p = 2 \text{ and } 2|k, \\ \theta + 1 & \text{otherwise.} \end{cases}$$

$$(1.4) \quad K = \prod_{(p-1)|k} p^\nu.$$

In this paper we shall prove

THEOREM 1. *Let b_1, \dots, b_s be any nonzero integers which do not have the same sign. Let m be any integer satisfying*

$$(1.5) \quad \sum_{j=1}^s b_j \equiv m \pmod{K}.$$

If s is the least integer with $s \geq s_1$ and if no prime can divide more than $s - s_0$ b_j then there are constants $C_j(k)$ depending on k only such that the equation

$$(1.6) \quad \sum_{j=1}^s b_j p_j^k = m$$

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always has a solution in odd primes p_j satisfying

$$(1.7) \quad \max_{1 \leq j \leq s} p_j < C_1 |m|^{1/k} + C_2^{(\log B)^2}$$

where $B = \max\{|b_1|, \dots, |b_s|, e\}$.

Investigations on bounds for *integral* solutions of diagonal equations similar to type (1.6) were made by Cassels [3], Birch and Davenport [2], Pitman and Ridout [11], Pitman [12]. On the other hand, results on bounds for *prime* solutions of (1.6) were obtained by Baker [1] and the author [9]. In all previous works on prime solutions, bounds obtained are of the form $C(k, \delta)^{\max\{b_j\}^\delta}$ for any $\delta > 0$. So (1.7) in Theorem 1 gives an essentially better bound than the previous one [9, (1.6)] and our Theorem 1 improves Theorem 1 in [9]. The new bound, $C^{(\log B)^2}$ is obtained by using [5, Theorem 6] a zero density estimate for L -functions which, as a consequence, replaces the Siegel-Walfisz theorem on prime distribution applied in both [1, Lemma 1] and [9, Lemma 6]. By this zero density estimate we can obtain a better error estimate as shown in our Lemma 2 which enables us to treat terms belonging to category (A) in §4 below. This change causes not only an improvement on the bound but also a greatly different emphasis in methods.

By (1.1) and (1.2) we see that the divisibility condition on b_j in Theorem 1 is better than (for $k \geq 4$) the condition, $(b_j, b_l) = 1$ for $j \neq l$, which is usually assumed in additive problems involving primes. By (1.4) and (1.5) our condition on m coincides with that in the Waring-Goldbach problem [7, p. 100 and p. 108] where the case $b_j = 1$ was considered.

2. Notation. Throughout we assume that N satisfies

$$(2.1) \quad \log N \geq N_0 (\log B)^2$$

where $N_0 > 0$ is a large constant depending on k only.

$\chi \pmod q$ denotes a Dirichlet character and $\chi_0 \pmod q$ denotes the principal character. $\chi^* \pmod r$ is a primitive character, $\tilde{\chi} \pmod{\tilde{r}}$ is the exceptional primitive character and $\tilde{\beta}$ is the exceptional zero (see Lemma 1 below). Throughout the constants c_j and all implicit constants in the Vinogradov symbols \ll , the O -symbols are positive and depend at most on k . The constants A_j are positive absolute. $\phi(q)$ is the Euler function and for real α write $e(\alpha) = \exp(i2\pi\alpha)$. Let

$$(2.2) \quad P = P(N) = \exp(\sqrt{A_1 \log N} / 10), \quad Q = N^k P^{-1},$$

where A_1 is given in Lemma 1. The constant $\sqrt{A_1} / 10$ in (2.2) will be needed in the proof of Lemma 2. Let

$$W(a, \chi) = \sum_{n=1}^q \chi(n) e\left(\frac{an^k}{q}\right),$$

$$S(b\alpha) = \sum_{G < p \leq N} \log p e(b\alpha p^k), \quad S(b\alpha, \chi) = \sum_{G < p \leq N} \chi(p) \log p e(b\alpha p^k),$$

where

$$G = N(6^k s |b|)^{-1/k}.$$

For $1 \leq a \leq q \leq P$, $(a, q) = 1$ let $\mathcal{M}(q, a)$ be the major arc which is the set of real α satisfying $|\alpha - a/q| \leq \delta_q$ with

$$(2.3) \quad \delta_q = (qQ)^{-1}.$$

These major arcs are disjoint. Let \mathcal{M} be the union of all major arcs and m denote minor arcs which is the complement of \mathcal{M} with respect to the set of α satisfying $Q^{-1} \leq \alpha \leq 1 + Q^{-1}$.

For $\alpha \in \mathcal{M}(q, a)$ write $\alpha = a/q + \eta$. If $p > P$ then $(q, p) = 1$, since $q \leq P$. It follows from the orthogonal relation of characters that

$$(2.4) \quad S(b\alpha) = \phi(q)^{-1} \sum_x W(ab, \bar{\chi}) S(b\eta, \chi).$$

Note that if $p > P$ then

$$(2.5) \quad S(b\eta, \chi) = S(b\eta, \chi^*)$$

where $\chi^* \pmod r$ induces $\chi \pmod q$. Put

$$(2.6) \quad \begin{cases} I(b\eta) = \sum_{|b|G^k < n \leq |b|N^k} e(\pm \eta n) n^{-1+1/k} (k|b|^{1/k})^{-1}, \\ \tilde{I}(b\eta) = - \sum_{|b|G^k < n \leq |b|N^k} e(\pm \eta n) n^{-1+\tilde{\beta}/k} (k|b|^{\tilde{\beta}/k})^{-1} \end{cases}$$

where \pm denotes the sign of b . $\tilde{I}(b\eta)$ is defined only if there is $\tilde{\beta}$. Let

$$(2.7) \quad \Delta(b\eta, \chi) = \begin{cases} S(b\eta, \chi_0) - I(b\eta) & \text{if } \chi = \chi_0, \\ S(b\eta, \tilde{\chi}\chi_0) - \tilde{I}(b\eta) & \text{if } \chi = \tilde{\chi}\chi_0, \\ S(b\eta, \chi) & \text{if } \chi \neq \chi_0 \text{ and } \chi \neq \tilde{\chi}\chi_0. \end{cases}$$

By (2.5) we have

$$(2.8) \quad \Delta(b\eta, \chi) = \Delta(b\eta, \chi^*).$$

3. Lemmas.

LEMMA 1. *Let $z = \sigma + it$. There is A_1 such that the Dirichlet L -function $L(z, \chi^*) \neq 0$ whenever $\sigma \geq 1 - A_1/\log(P(|t| + 2))$ for all primitive characters $\chi^* \pmod r$ and $r \leq P$ with the possible exception of at most one primitive character, $\tilde{\chi} \pmod{\tilde{r}}$. If there is such an exceptional character then it is quadratic and the unique exceptional zero $\tilde{\beta}$ of $L(z, \tilde{\chi})$ is real and simple and satisfies*

$$(3.1) \quad A_2/\tilde{r}^{1/2}(\log \tilde{r})^2 \leq 1 - \tilde{\beta} \leq A_1/\log P.$$

PROOF. See [4, §14].

LEMMA 2. *For any real $\lambda \geq 1$ we have*

$$\sum_{r \leq P} \sum_{\chi^*} \left(\int_{-\delta_r}^{\delta_r} |\Delta(b\eta, \chi^*)|^\lambda d\eta \right)^{1/\lambda} \ll |b|N^{1-k/\lambda} P^{-2}$$

where the summation \sum_{χ^*} is taken over all $\chi^* \pmod r$.

PROOF. The proof is essentially the same as Theorem 7 [5]. In the proof we apply Theorem 6 [5] and put the T there to be P^7 .

LEMMA 3. Let $q = q_1 \cdots q_t$ with $(q_j, q_l) = 1$ for $j \neq l$. Let $\chi \pmod{q}$ be factorized into $\prod_{j=1}^t \chi_j \pmod{q_j}$. If $(a, q) = 1$ then there exist uniquely $a_j \pmod{q_j}$ with

$$(3.2) \quad (a_j, q_j) = 1 \quad (j = 1, \dots, t), \quad a = \sum_{j=1}^t \frac{a_j q}{q_j}$$

and

$$W(ab, \chi) = \prod_{j=1}^t W(a_j b, \chi_j).$$

PROOF. This is essentially Theorem 4.1 in [8, p. 159].

LEMMA 4. Let $h_1 = h/(h, q)$ and $q_1 = q/(h, q)$. Let $\chi^* \pmod{r}$ induce $\chi \pmod{q}$. Then

$$W(h, \chi) = \begin{cases} 0 & \text{if } r \nmid q_1, \\ \phi(q)\phi(q_1)^{-1}W(h_1, \chi_1) & \text{if } r|q_1 \text{ where } \chi_1 \pmod{q_1} \\ & \text{is induced by } \chi^* \pmod{r}. \end{cases}$$

REMARKS. Lemma 4 is parallel to the known result on the Ramanujan sum and its generalization [6, p. 450]. In fact, we can also prove that $W(h, \chi) = 0$ if $r|q_1$ and $(r, q_1/r) \nmid k$.

PROOF. Write $q_2 = q/q_1$ and $n = uq_1 + v$ with $u = 0, 1, \dots, q_2 - 1; v = 1, 2, \dots, q_1$. Then

$$(3.3) \quad \sum_{n=1}^q \chi(n) e\left(\frac{hn^k}{q}\right) = \sum_{\substack{v=1 \\ (v, q_1)=1}}^{q_1} e\left(\frac{h_1 v^k}{q_1}\right) T(v)$$

where $T(v) = \sum_{u=1}^{q_2} \chi(uq_1 + v)$.

Let $r \nmid q_1$. By the same argument as in showing $S(v) = 0$ in [4, p. 66] we can prove that $T(v) = 0$ and hence $W(h, \chi) = 0$.

Next consider $r|q_1$. Let $d = \prod_{p|q_2, p \nmid q_1} p$ and $\mathcal{J} = \{uq_1 + v: 1 \leq u \leq q_2\}$. If $(v, q_1) = 1$ then

$$(3.4) \quad \sum_{\substack{j \in \mathcal{J} \\ (j, q)=1}} 1 = \sum_{j \in \mathcal{J}} \sum_{n|(j, d)} \mu(n) = \sum_{n|d} \frac{\mu(n)q_2}{n} = q_2 \prod_{p|d} (1 - p^{-1}).$$

It follows from $\chi^*(uq_1 + v) = \chi^*(v)$ and (3.4) that if $(v, q_1) = 1$ then

$$T(v) = \chi^*(v) \sum_{\substack{u=1 \\ (uq_1 + v, q)=1}}^{q_2} 1 = \chi^*(v) \phi(q)\phi(q_1)^{-1}.$$

By (3.3) this proves Lemma 4.

LEMMA 5. (a) If $(a, p) = 1$ and p^l is the modulus of χ then $|W(a, \chi)| \leq 2kp^{l/2}$.

(b) If $(a, q) = 1$ and q is the modulus of χ then for any $\epsilon > 0$ there is a positive constant $C(\epsilon, k)$ depending at most on ϵ, k such that

$$|W(ab, \chi)| \leq C(k, \epsilon)(q, b)^{1/2} q^{1/2+\epsilon}.$$

PROOF. Part (a) follows from a similar argument as part 1 of the proof of Lemma 8.5 [7].

(b) Let $\chi^* \pmod r$ induce $\chi \pmod q$, $q' = q/(b, q)$, $b' = b/(b, q)$. Suppose that $r | q'$. Put $q' = \prod_{j=1}^t p_j^{l_j}$ and factorize $\chi' \pmod{q'}$ into $\prod_{j=1}^t \chi_j \pmod{p_j^{l_j}}$, where $\chi' \pmod{q'}$ is induced by $\chi^* \pmod r$. Then by Lemmas 4, 3, and Lemma 5(a)

$$\begin{aligned} |W(ab, \chi)| &= \phi(q)\phi(q')^{-1}|W(ab', \chi')| \leq (b, q) \prod_{j=1}^t |W(a_j b', \chi_j)| \\ &\leq (b, q)^{1/2} ((b, q)q')^{1/2} \prod_{j=1}^t 2k. \end{aligned}$$

This proves Lemma 5(b).

4. Major arcs. I. Write

$$(4.1) \quad \begin{cases} \mathscr{W}_j = \phi(q)^{-1} \sum_x W(ab_j, \bar{\chi}) \Delta(b_j \eta, \chi), \\ \mathscr{J}_j = \phi(q)^{-1} I(b_j \eta) W(ab_j, \chi_0), \\ \tilde{\mathscr{J}}_j = \phi(q)^{-1} \tilde{I}(b_j \eta) W(ab_j, \tilde{\chi} \chi_0), \end{cases}$$

where $\tilde{\mathscr{J}}_j$ is defined only when the exceptional character exists. By (2.4), (2.7) we have

$$(4.2) \quad \begin{aligned} R_1(m) &= \sum_{q \leq P} \sum'_a \int_{-\delta_q}^{\delta_q} e\left(-m\left(\frac{a}{q} + \eta\right)\right) \prod_{j=1}^s S(b_j \alpha) d\eta \\ &= \sum_{q \leq P} \sum'_a e\left(\frac{-ma}{q}\right) \int_{-\delta_q}^{\delta_q} e(-m\eta) \prod_{j=1}^s (\mathscr{W}_j + \mathscr{J}_j + \tilde{\mathscr{J}}_j) d\eta \end{aligned}$$

where the sum \sum'_a is taken over all a with $1 \leq a \leq q$ and $(a, q) = 1$.

There are two categories of terms in the last product of (4.2), namely, (A) terms having at least a factor \mathscr{W}_j ; (B) terms having no factor \mathscr{W}_j . We shall treat category (A) in this section and category (B) in §6.

Let \mathscr{J}'_j denote either \mathscr{J}_j or $\tilde{\mathscr{J}}_j$. In category (A) for each fixed $h = 1, 2, \dots, s$ we choose $\prod_{j=1}^h \mathscr{W}_j \prod_{j=h+1}^s \mathscr{J}'_j$ as the representative of those terms having exactly h factors \mathscr{W}_j . Put

$$(4.3) \quad T_h(m) = \sum_{q \leq P} \sum'_a e\left(\frac{-ma}{q}\right) \int_{-\delta_q}^{\delta_q} \prod_{j=1}^h \mathscr{W}_j \prod_{j=h+1}^s \mathscr{J}'_j e(-m\eta) d\eta \quad (h = 1, \dots, s).$$

Let

$$(4.4) \quad \chi'_j \pmod q = \chi_0 \pmod q \quad \text{or} \quad \tilde{\chi} \chi_0 \pmod q.$$

$$(4.5) \quad I'(b_j \eta) = I(b_j \eta) \quad \text{or} \quad \tilde{I}(b_j \eta).$$

Then by Schwarz's inequality and (4.1), (4.3) we have

(4.6)

$$|T_h(m)| \leq \sum_{p \leq P} \phi(q)^{-s} \sum_{\substack{\chi_j \\ j=1, \dots, h}} \left| \sum'_a e\left(\frac{-ma}{q}\right) \prod_{j=1}^h W(ab_j, \bar{\chi}_j) \prod_{j=h+1}^s W(ab_j, \chi'_j) \right| \\ \times \prod_{j=1}^h \left(\int_{-\delta_q}^{\delta_q} |\Delta(b_j \eta, \chi_j)|^{n_j} d\eta \right)^{1/n_j} \prod_{j=h+1}^s \left(\int_{-\delta_q}^{\delta_q} |I'(b_j \eta)|^{n_j} d\eta \right)^{1/n_j},$$

where $\sum_{\chi_j, j=1, \dots, h}$ denotes h summations each of which is taken over all $\chi \pmod{q}$ and $n_j \geq 1$ are integers satisfying $\sum_{j=1}^s 1/n_j = 1$. Note that each $\chi_j \pmod{q}$ is induced by a unique $\chi_j^* \pmod{r_j}$ with $r_j | q$ and that each $\chi_j^* \pmod{r_j}$ and each q with $r_j | q$ induce a unique $\chi \pmod{q}$. Then by (2.8), (4.6) we have

(4.7)

$$|T_h(m)| \leq \sum_{\substack{r_j \leq P \\ j=1, \dots, h}} \sum_{\chi_j^*} \left\{ \sum_{\substack{q=1 \\ r_j | q, j=1, \dots, h}}^{\infty} \phi(q)^{-s} \left| \sum'_a e\left(\frac{-ma}{q}\right) \prod_{j=1}^h W(ab_j, \chi_0 \bar{\chi}_j^*) \right| \right. \\ \left. \times \prod_{j=h+1}^s W(ab_j, \chi'_j) \right\} \\ \times \prod_{j=1}^h \left(\int_{-\delta_{r_j}}^{\delta_{r_j}} |\Delta(b_j \eta, \chi_j^*)|^{n_j} d\eta \right)^{1/n_j} \prod_{j=h+1}^s \left(\int_{-\delta_1}^{\delta_1} |I'(b_j \eta)|^{n_j} d\eta \right)^{1/n_j}.$$

By Lemma 5 with $\epsilon = (10s)^{-1}$, the infinite sum inside the curly brackets of (4.7) is

$$(4.8) \quad \ll \sum_{q=1}^{\infty} \phi(q)^{-s+1} \prod_{j=1}^s |b_j|^{1/2} q^{1/2+1/10s} \ll B^{s/2}$$

since by (1.2) we have $s \geq 5$ for any $k \geq 2$. Also by (2.6) we have $I'(b_j \eta) \ll N$ and then by (2.3), (2.2)

$$(4.9) \quad \left(\int_{-\delta_1}^{\delta_1} |I'(b_j \eta)|^{n_j} d\eta \right)^{1/n_j} \ll N^{1-k/n_j} P^{1/n_j}.$$

It follows from (4.7), (4.8), (4.9) and Lemma 2 that

$$(4.10) \quad T_h(m) \ll B^{s/2} \left(\prod_{j=1}^h |b_j| N^{1-k/n_j} P^{-2} \right) \left(\prod_{j=h+1}^s N^{1-k/n_j} P^{1/n_j} \right) \\ \ll B^{3s/2} N^{s-k} P^{-1} = E_1, \quad \text{say,}$$

since $\sum_{j=1}^s 1/n_j = 1$.

5. Singular series.

LEMMA 6. For a given p let $p^\theta || k$ and $p^\phi || b$. Suppose that p^t and p^j are the moduli of χ_0 and χ_1 respectively and

$$u = 2\phi + \theta + \begin{cases} 3 & \text{if } p = 2, \\ 1 & \text{if } p \geq 3. \end{cases}$$

If $1 \leq j \leq u - 2\phi - \theta$, $t \geq u + 1$ and $(a, p) = 1$ then

$$W(ab, \chi_0) = W(ab, \chi_1\chi_0) = 0.$$

PROOF. The proof is essentially the same as Lemma 1 [9].

LEMMA 7. Let $q = q_1q_2$, $(q_1, q_2) = 1$ and factorize $\chi_j \pmod{q}$ into $\prod_{j=1}^2 \chi_{j_l}$ $\pmod{q_l}$ ($j = 1, 2, \dots, s$). If

$$B(m, q) = \phi(q)^{-s} \sum'_a e\left(\frac{-ma}{q}\right) \prod_{j=1}^s W(ab_j, \chi_j),$$

then

$$B(m, q) = B(m, q_1)B(m, q_2).$$

PROOF. Apply Lemma 3.

By Lemma 1 the exceptional character $\tilde{\chi} \pmod{\tilde{r}}$ is real and primitive. Then it is known [8, p. 159] that

$$(5.1) \quad \tilde{r} = 2^l p_2 \cdots p_t$$

where p_j are distinct odd primes and $l = 0$ or 2 or 3. If $\tilde{r} | q$ write

$$(5.2) \quad q = q_1q_2, \quad (q_1, q_2) = 1 \quad \text{and} \quad q_1 = 2^{l_1} p_2^{l_2} \cdots p_t^{l_t}$$

where $l_j \geq 1$ ($j = 2, \dots, t$); $l_1 \geq l$ if $l \neq 0$ and $l_1 = 0$ if $l = 0$. Put

$$(5.3) \quad B_h(m, q) = \phi(q)^{-s} \sum'_a e\left(\frac{-ma}{q}\right) \prod_{j=1}^s W(ab_j, \chi'_j) \quad (h = 0, 1, \dots, s),$$

where χ'_j is defined in (4.4) and there are exactly h $\chi'_j = \tilde{\chi}\chi_0 \pmod{q}$ in the last product of (5.3). Define singular series ($h = 0$) and pseudosingular series ($h = 1, 2, \dots, s$) by

$$(5.4) \quad \mathcal{S}_0(m) = \sum_{q=1}^{\infty} B_0(m, q) \quad \text{and} \quad \mathcal{S}_h(m) = \sum_{\substack{q=1 \\ \tilde{r}|q}}^{\infty} B_h(m, q).$$

By Lemma 5(b) all series in (5.4) are absolutely convergent.

LEMMA 8. Let \tilde{r} and q_1 be defined as in (5.1), (5.2). If $B_h(m, q_1) \neq 0$ then $q_1 = d_k \tilde{r}$ or $2d_k \tilde{r}$ where d_k is a divisor of k .

PROOF. For each p_j ($j = 1, \dots, t$) in (5.1) with $p_1 = 2$ let $p_j^{\theta_j} || k$. Suppose that $l_1 \geq 4 + \theta_1$ or $l_j \geq 2 + \theta_j$ for some $j \geq 2$. For simplicity we only give the details for the case $j = 2$. Let

$$(5.5) \quad l_2 \geq \theta_2 + 2.$$

Since no prime can divide all b_j , we may assume that $p_2 + b_1$. Factorizing the exceptional character $\tilde{\chi}$ and the character χ'_1 in (5.3) we have

$$\tilde{\chi} \pmod{\tilde{r}} = \tilde{\chi}_1 \pmod{2^l} \prod_{j=2}^t \tilde{\chi}_j \pmod{p_j},$$

$$\chi'_1 \pmod{q_1} = \prod_{j=1}^t \chi'_{1_j} \pmod{p_j^{l_j}},$$

where each χ'_{1j} is either $\chi_0 \pmod{p_j^{l_j}}$ or $\tilde{\chi}_j \chi_0 \pmod{p_j^{l_j}}$. By (3.2) for each a with $(a, q_1) = 1$ there are a_j ($j = 1, \dots, t$) with $(a_j, p_j) = 1$ such that $W(ab_1, \chi'_1) = \prod_{j=1}^t W(a_j b_1, \chi'_{1j})$. Then by (5.5) and Lemma 6 with $\phi = 0$, for each a in Σ'_a of (5.3) we have $W(a_2 b_1, \chi'_{12}) = 0$. So by (5.3) if $B_h(m, q_1) \neq 0$ then $l \leq l_1 \leq l + 1 + \theta_1$ and $1 \leq l_j \leq 1 + \theta_j$ ($j = 2, 3, \dots, s$). This proves Lemma 8.

LEMMA 9. (a) $\mathcal{S}_0(m) \gg B^{s(1-s)}$ and (b) $\mathcal{S}_h(m) \ll \mathcal{S}_0(m)(\log N)^{-1/2}$ ($h = 1, 2, \dots, s$).

PROOF. Part (a) is Lemma 5 in [9].

We come now to prove part (b). For each q with $\tilde{r} | q$ define q_1 and q_2 as in (5.2). Since, by the hypothesis on b_j , no prime can divide more than $s - s_0$ b_j , we have

$$\prod_{j=1}^s (q_1, b_j) \leq q_1^{s-s_0}.$$

Then by (5.3) and Lemma 5 with $\epsilon = (10s)^{-1}$ we have

$$B_h(m, q_1) \ll \phi(q_1)^{-s+1} q_1^{s/2+1/10} q_1^{(s-s_0)/2} \ll q_1^{6/5-s_0/2}.$$

Then by Lemma 8 and $s_0 \geq 2k \geq 4$ (see (1.1)) we have

$$B_h(m, q_1) \ll \tilde{r}^{-4/5}.$$

So by Lemma 8 again we have

$$(5.6) \quad \sum_{q_1=1}^{\infty} B_h(m, q_1) \ll \tilde{r}^{-4/5}.$$

On the other hand, by Lemma 5(a), the divisibility hypothesis on b_j and $|W(ab_j, \chi'_j)| \leq \phi(p')$, we see that the product in $B_0(m, p')$ in (5.3) satisfies

$$\left| \prod_{j=1}^s W(ab_j, \chi'_j) \right| \leq (2k)^{s_0} p^{ts_0/2} \phi(p')^{s-s_0}.$$

So by (5.3) and $s_0 \geq 4$ we have

$$(5.7) \quad |B_0(m, p')| \leq \phi(p')^{-s_0+1} (2k)^{s_0} p^{ts_0/2} \leq (4k)^s p^{t(1-s_0/2)} < c_1 p^{-t}.$$

For each p there exists some $b_j = b_1$, say, which is not divisible by p . By Lemma 6 for each a with $(a, p) = 1$ we have $W(ab_1, \chi_0) = 0$ if $t \geq \nu + 2$ where ν is defined in (1.3) and p' is the modulus of χ_0 . So by (5.3) we have $B_0(m, p') = 0$ if $t \geq \nu + 2$. Then by Lemma 7 and $(\tilde{r}, q_2) = 1$

$$(5.8) \quad \begin{aligned} \sum_{q_2=1}^{\infty} B_0(m, q_2) &= \prod_{p|\tilde{r}} \left(1 + \sum_{t=1}^{\nu_1} B_0(m, p^t) \right) \\ &= \sum_{q=1}^{\infty} B_0(m, q) / \prod_{p|\tilde{r}} \left(1 + \sum_{t=1}^{\nu_1} B_0(m, p^t) \right) \end{aligned}$$

where $\nu_1 = \nu + 1$. Separate the last product $\prod_{p|\tilde{r}}$ into $\prod_{p|\tilde{r}, p \leq c_2}$ and $\prod_{p|\tilde{r}, p > c_2}$ where $c_2 = 4c_1$. Same as that in the proof of Lemma 5 in [9, see (4.16) and the product \prod_1 on p. 197] which depends essentially on (1.1)–(1.5) and the divisibility

condition on b_j in Theorem 1, we have that the first product $\prod_{p|\tilde{r}, p \leq c_2}$ satisfies

$$\begin{aligned} \prod_{\substack{p|\tilde{r} \\ p \leq c_2}} \left(1 + \sum_{t=1}^{\nu_1} B_0(m, p^t) \right) &\geq \prod_{\substack{p|\tilde{r} \\ p \leq c_2}} \phi(p^{\nu_1})^{-s} p^{\nu_1} \\ &\geq \prod_{p \leq c_2} p^{\nu_1(1-s)} = c_3 > 0. \end{aligned}$$

For the second product $\prod_{p|\tilde{r}, p > c_2}$, by (5.7) we have

$$\begin{aligned} (5.9) \quad \prod_{\substack{p|\tilde{r} \\ p > c_2}} \left(1 + \sum_{t=1}^{\nu_1} B_0(m, p^t) \right) &> \prod_{c_2 < p \leq \tilde{r}} \left(1 - c_1 \sum_{t=1}^{\infty} p^{-t} \right) \\ &> \prod_{c_2 < p \leq \tilde{r}} (1 - c_2/2p) \gg (\log \tilde{r})^{-c_2}. \end{aligned}$$

The last inequality is a simple modification of Theorem 9.3 in [8, p. 92]. Now by (5.8), (5.9) we have

$$(5.10) \quad \sum_{q_2=1}^{\infty} B_0(m, q_2) \ll \mathcal{S}_0(m)(\log \tilde{r})^{c_2}.$$

Finally, by (5.2) we see that $\tilde{\chi}\chi_0 \pmod{q}$ can be factorized as the product of $\tilde{\chi}\chi_0 \pmod{q_1}$ and $\chi_0 \pmod{q_2}$. Then by (5.4), Lemma 7, (5.6), (5.10) we have

$$\mathcal{S}_h(m) = \sum_{q_1=1}^{\infty} B_h(m, q_1) \sum_{q_2=1}^{\infty} B_0(m, q_2) \ll \mathcal{S}_0(m)(\log N)^{-1/2}$$

since by (3.1) we have

$$\tilde{r}^{4/5}(\log \tilde{r})^{-c_2} \gg (\log N)^{1/2}.$$

This proves Lemma 9.

6. Major arcs. II.

LEMMA 10. *We have*

$$\int_{(qQ)^{-1}}^{1/2} \left| \prod_{j=1}^s I'(b_j\eta) \right| d\eta \ll (qQ)^{s-1} N^{s(1-k)}$$

where $I'(b_j\eta)$ is defined in (4.5).

PROOF. If $0 < \eta \leq 1/2$ then for any $n \geq 1$ we have $\sum_{l=0}^n e(l\eta) \ll |\eta|^{-1}$. Let $\phi = 1/k$ or $\tilde{\beta}/k$. Then by Abel's partial summation formula and (2.6)

$$\begin{aligned} b^\phi I'(b\eta) &\ll |\eta|^{-1} \left\{ |bN^k|^\phi + \int_{|b|G^k}^{|b|N^k} \left| \frac{d}{dy} y^{\phi-1} \right| dy \right\} \\ &\ll |\eta|^{-1} (|b|G^k)^\phi \ll |\eta|^{-1} N^{1-k}. \end{aligned}$$

So the lemma follows.

Let

$$(6.1) \quad J_h(m) = \int_{-1/2}^{1/2} \prod_{j=1}^h \tilde{I}(b_j\eta) \prod_{j=h+1}^s I(b_j\eta) e(-m\eta) d\eta \quad (h = 0, 1, \dots, s).$$

LEMMA 11. (a) $|J_h(m)| \leq J_0(m)$ ($h = 1, 2, \dots, s$).

(b) If

$$(6.2) \quad |m| \leq (N/4)^k s^{-1}$$

then

$$J_0(m) \gg B^{-s/k} N^{s-k}.$$

PROOF. Part (a) follows from (6.1) and part (b) is essentially Lemma 8 [9].

We come now to treat those terms in category (B) defined in §4. In category (B) we choose $\prod_{j=1}^h \tilde{\mathcal{J}}_j \prod_{j=h+1}^s \mathcal{J}_j$ ($h = 0, 1, \dots, s$) to represent those terms $\prod_{j=1}^s \mathcal{J}'_j$ having exactly h factors $\tilde{\mathcal{J}}_j$. Put

$$(6.3) \quad \begin{aligned} T_0(m) &= \sum_{q \leq P} \sum'_a e\left(\frac{-ma}{q}\right) \int_{-\delta_q}^{\delta_q} \left(\prod_{j=1}^s \mathcal{J}_j\right) e(-m\eta) d\eta, \\ \tilde{T}_h(m) &= \sum_{\substack{q \leq P \\ \bar{\Gamma}|q}} \sum'_a e\left(\frac{-ma}{q}\right) \int_{-\delta_q}^{\delta_q} \left(\prod_{j=1}^h \tilde{\mathcal{J}}_j \prod_{j=h+1}^s \mathcal{J}_j\right) e(-m\eta) d\eta \\ &\quad (h = 1, 2, \dots, s). \end{aligned}$$

By (4.1), $s \geq 5$, Lemmas 10 and 5 with $\epsilon = (10s)^{-1}$ we have

$$(6.4) \quad \begin{aligned} \sum_{\substack{q \leq P \\ \bar{\Gamma}|q}} \sum'_a e\left(\frac{-ma}{q}\right) \int_{\delta_q}^{1/2} \left(\prod_{j=1}^h \tilde{\mathcal{J}}_j \prod_{j=h+1}^s \mathcal{J}_j\right) e(-m\eta) d\eta \\ \ll N^{s(1-k)} Q^{s-1} \sum_{q \leq P} \phi(q)^{-s+1} \left(\prod_{j=1}^s (q, b_j)^{1/2} q^{1/2+\epsilon}\right) q^{s-1} \\ \ll N^{s-k} B^{s/2} P^{-3/10} = E_2, \quad \text{say.} \end{aligned}$$

So, if we replace the integral $\int_{-\delta_q}^{\delta_q}$ in (6.3) by $\int_{-1/2}^{1/2}$ we have the error E_2 given in (6.4). Then by (6.1), (6.3), (4.1) we have

$$(6.5) \quad \begin{aligned} \tilde{T}_h(m) &= J_h(m) \left\{ \sum_{\substack{q \leq P \\ \bar{\Gamma}|q}} \phi(q)^{-s} \sum'_a e\left(\frac{-ma}{q}\right) \prod_{j=1}^h W(ab_j, \tilde{\chi}\chi_0) \prod_{j=h+1}^s W(ab_j, \chi_0) \right\} \\ &\quad + E_2. \end{aligned}$$

Similarly, by Lemma 5, if we replace the sum $\sum_{q \leq P, \bar{\Gamma}|q}$ in (6.5) by $\sum_{q=1, \bar{\Gamma}|q}^\infty$ we have an error $\ll B^{s/2} P^{-3/10}$. So by (5.4), (6.5) we have

$$(6.6) \quad \tilde{T}_h(m) = J_h(m)(\mathcal{S}_h(m) + E_3) + E_2 \quad (h = 1, 2, \dots, s),$$

where $E_3 = O(E_2 N^{k-s})$. By the same argument we have

$$(6.7) \quad T_0(m) = J_0(m)(\mathcal{S}_0(m) + E_3) + E_2.$$

Note that each representative in either category (A) or (B) defined in §4 represents at most $O(1)$ terms. It follows from (4.2), (4.10), (6.7), (6.6) that

$$(6.8) \quad \begin{aligned} R_1(m) &= J_0(m)(\mathcal{S}_0(m) + E_3) + O\left(\sum_{h=1}^s J_h(m)\{\mathcal{S}_h(m) + E_3\}\right) \\ &\quad + O(E_2 + E_1). \end{aligned}$$

By (4.10), (6.4), Lemmas 9(a) and 11(b) we see that for $j = 1, 2$

$$(6.9) \quad E_3/\mathcal{L}_0(m) \quad \text{and} \quad E_j/J_0(m)\mathcal{L}_0(m) \ll B^{2s^2}P^{-3/10}.$$

It follows from (6.8), (6.9), (2.1), Lemmas 11(a) and 9(b) that

$$(6.10) \quad R_1(m) > \frac{1}{2}J_0(m)\mathcal{L}_0(m).$$

7. Minor arcs.

LEMMA 12. *If $\alpha \in m$ then*

$$\sum_{p \leq N} e(\alpha bp^k) \ll N|b|P^{-\omega(k)}$$

where $\omega(k)^{-1} = 4^{(k+2)}(k+1)$.

PROOF. This is essentially Lemma 11 [9] (see also Lemma 5 [1]).

Let

$$R_2(m) = \int_m \prod_{j=1}^s S(b_j\alpha) e(-m\alpha) d\alpha.$$

Then by Lemma 12 and the same argument as Lemma 12 in [9] we have

$$R_2(m) \ll N^{s-k}B^sP^{-\omega(k)}(\log N)^{c_4}.$$

By (4.2), (6.10), Lemmas 9(a), 11(b) and (2.1)

$$(7.1) \quad \int_{Q^{-1}}^{1+Q^{-1}} \prod_{j=1}^s S(b_j\alpha) e(-m\alpha) d\alpha = R_1(m) + R_2(m) \\ \gg N^{s-k}B^{-s^2} \{1 - c_5B^{s(s+1)}P^{-\omega(k)}(\log N)^{c_4}\} > 0.$$

Choose the least N satisfying (2.1) and (6.2). So (7.1) implies the existence of a solution of $\sum_{j=1}^s b_j p_j^k = m$ in primes p_j and

$$\max_{1 \leq j \leq s} p_j \leq N \leq C_1(k)m^{1/k} + C_2(k)(\log B)^2.$$

This completes the proof of Theorem 1.

REMARK. Combining the Circle Method with the Sieve Method, when $k = 1$ and $s = 3$, the author [10] is able to obtain a bound for solutions of (1.6) to be B^A where $A > 0$ is an absolute constant. However, for $k \geq 2$ it seems that these two methods do not combine well to replace the $(\log B)^2$ in (1.7) by $\log B$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HONG KONG, HONG KONG