BOUND FOR PRIME SOLUTIONS
OF SOME DIAGONAL EQUATIONS. II

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ABSTRACT. Let $b_j$ and $m$ be certain integers. In this paper we obtain a bound for
prime solutions $p_t$ of the diagonal equations of order $k$, $b_1 p_t^k + \cdots + b_s p_t^k = m$.
The bound obtained is $C (\log B)^2 + C |m|^{1/k}$ where $B = \max_j \{ e, |b_j| \}$ and $C$ are
positive constants depending at most on $k$.

1. Introduction. Throughout $p$ denotes a prime number and $k \geq 2$ is an integer.
Let $\theta \geq 0$ be the largest integer such that $p^\theta$ divides $k$. We write $p^\theta \| k$. Let

$$s_0 = \begin{cases} 3k - 1 & \text{if there is a } p \text{ satisfying } p \| k \text{ and } k = \left( \frac{p - 1}{2} \right) p^\theta, \\ 2k & \text{otherwise.} \end{cases}$$

$$s_1 = \begin{cases} 2k + 1 & \text{if } 2 \leq k \leq 11, \\ 2k^2 (2 \log k + \log \log k + 2.5) - 1 & \text{if } k \geq 12. \end{cases}$$

$$\nu = \begin{cases} \theta + 2 & \text{if } p = 2 \text{ and } 2 \| k, \\ \theta + 1 & \text{otherwise.} \end{cases}$$

$$K = \prod_{(p-1) \| k} p^n.$$ 

In this paper we shall prove

THEOREM 1. Let $b_1, \ldots, b_s$ be any nonzero integers which do not have the same sign.
Let $m$ be any integer satisfying

$$\sum_{j=1}^s b_j \equiv m \pmod{K}.$$ 

If $s$ is the least integer with $s \geq s_1$ and if no prime can divide more than $s - s_0$ $b_j$
then there are constants $C_j(k)$ depending on $k$ only such that the equation

$$\sum_{j=1}^s b_j p_j^k = m$$

is
always has a solution in odd primes $p_j$ satisfying

$$
\max_{1 \leq j \leq s} p_j < C_1 |m|^{1/k} + C_2 (\log B)^2
$$

where $B = \max\{|b_1|, \ldots, |b_s|, e\}$.

Investigations on bounds for integral solutions of diagonal equations similar to type (1.6) were made by Cassels [3], Birch and Davenport [2], Pitman and Ridout [11], Pitman [12]. On the other hand, results on bounds for prime solutions of (1.6) were obtained by Baker [1] and the author [9]. In all previous works on prime solutions, bounds obtained are of the form $C(k, \delta)^{\max\{|b_j|\}^\delta}$ for any $\delta > 0$. So (1.7) in Theorem 1 gives an essentially better bound than the previous one [9, (1.6)] and our Theorem 1 improves Theorem 1 in [9]. The new bound, $C^{(\log B)^2}$ is obtained by using [5, Theorem 6] a zero density estimate for $L$-functions which, as a consequence, replaces the Siegel-Walfisz theorem on prime distribution applied in both [1, Lemma 1] and [9, Lemma 6]. By this zero density estimate we can obtain a better error estimate as shown in our Lemma 2 which enables us to treat terms belonging to category (A) in §4 below. This change causes not only an improvement on the bound but also a greatly different emphasis in methods.

By (1.1) and (1.2) we see that the divisibility condition on $b_j$ in Theorem 1 is better than (for $k \geq 4$) the condition, $(b_j, b_l) = 1$ for $j \neq l$, which is usually assumed in additive problems involving primes. By (1.4) and (1.5) our condition on $m$ coincides with that in the Waring-Goldbach problem [7, p. 100 and p. 108] where the case $b_j = 1$ was considered.

2. Notation. Throughout we assume that $N$ satisfies

$$
\log N \geq N_0 (\log B)^2
$$

where $N_0 > 0$ is a large constant depending on $k$ only.

$\chi (mod q)$ denotes a Dirichlet character and $\chi_0 (mod q)$ denotes the principal character. $\chi^* (mod r)$ is a primitive character, $\tilde{\chi} (mod \tilde{r})$ is the exceptional primitive character and $\tilde{\beta}$ is the exceptional zero (see Lemma 1 below). Throughout the constants $c_j$ and all implicit constants in the Vinogradov symbols $\ll$, the O-symbols are positive and depend at most on $k$. The constants $A_j$ are positive absolute. $\phi(q)$ is the Euler function and for real $\alpha$ write $e(\alpha) = \exp(i2\pi \alpha)$. Let

$$
P = P(N) = \exp(\sqrt{A_1 \log N / 10}), \quad Q = N^k P^{-1},
$$

where $A_1$ is given in Lemma 1. The constant $\sqrt{A_1 / 10}$ in (2.2) will be needed in the proof of Lemma 2. Let

$$
W(a, \chi) = \sum_{n=1}^{q} \chi(n) e\left(\frac{an^k}{q}\right),
$$

$$
S(ba, \chi) = \sum_{G < P \leq Q} \chi(P) \log P \ e(bap^k),
$$

where

$$
G = N(6^k s |b|)^{-1/k}.
$$
For $1 \leq a \leq q \leq P$, $(a, q) = 1$ let $M(q, a)$ be the major arc which is the set of real $\alpha$ satisfying $|\alpha - a/q| \leq \delta_q$ with

$$\delta_q = (qQ)^{-1}.$$  

These major arcs are disjoint. Let $M$ be the union of all major arcs and $m$ denote minor arcs which is the complement of $M$ with respect to the set of $\alpha$ satisfying $Q^{-1} \leq \alpha \leq 1 + Q^{-1}$.

For $\alpha \in M(q, a)$ write $\alpha = a/q + \eta$. If $p > P$ then $(q, p) = 1$, since $q \leq P$. It follows from the orthogonal relation of characters that

$$\tag{2.4} S(b\alpha) = \phi(q)^{-1} \sum_{\chi} W(ab, \bar{\chi}) S(b\eta, \chi).$$

Note that if $p > P$ then

$$\tag{2.5} S(b\eta, \chi) = S(b\eta, \chi^*)$$

where $\chi^* \pmod{r}$ induces $\chi \pmod{q}$. Put

$$\tag{2.6} \begin{cases} I(b\eta) = \sum_{|b[G] < n \leq |b|N^k} \varepsilon(\pm \eta n) n^{-1 + 1/k} (k|b|^{1/k})^{-1}, \\
\tilde{I}(b\eta) = - \sum_{|b[G] < n \leq |b|N^k} \varepsilon(\pm \eta n) n^{-1 + \tilde{\beta}/k} (k|b|^{\tilde{\beta}/k})^{-1}
\end{cases}$$

where $\pm$ denotes the sign of $b$. $\tilde{I}(b\eta)$ is defined only if there is $\tilde{\beta}$. Let

$$\tag{2.7} \Delta(b\eta, \chi) = \begin{cases} S(b\eta, \chi_0) - I(b\eta) & \text{if } \chi = \chi_0, \\
S(b\eta, \tilde{\chi} \chi_0) - \tilde{I}(b\eta) & \text{if } \chi = \tilde{\chi} \chi_0, \\
S(b\eta, \chi) & \text{if } \chi \neq \chi_0 \text{ and } \chi \neq \tilde{\chi} \chi_0.
\end{cases}$$

By (2.5) we have

$$\tag{2.8} \Delta(b\eta, \chi) = \Delta(b\eta, \chi^*).$$

3. Lemmas.

**Lemma 1.** Let $z = \sigma + it$. There is $A_1$ such that the Dirichlet $L$-function $L(z, \chi^*) \neq 0$ whenever $\sigma \geq 1 - A_1/\log(P(|t| + 2))$ for all primitive characters $\chi^* \pmod{r}$ and $r \leq P$ with the possible exception of at most one primitive character, $\tilde{\chi} \pmod{r}$. If there is such an exceptional character then it is quadratic and the unique exceptional zero $\tilde{\beta}$ of $L(z, \tilde{\chi})$ is real and simple and satisfies

$$\tag{3.1} A_2/\tilde{\beta}^{1/2} (\log \tilde{\beta})^2 \leq 1 - \tilde{\beta} \leq A_1/\log P.$$

**Proof.** See [4, §14].

**Lemma 2.** For any real $\lambda \geq 1$ we have

$$\sum_{r \leq P} \sum_{\chi^*} \left( \int_{-\delta}^{\delta} |\Delta(b\eta, \chi^*)|^\lambda \, d\eta \right)^{1/\lambda} \ll |b|N^{1-\lambda} P^{-2},$$

where the summation $\sum_{\chi^*}$ is taken over all $\chi^* \pmod{r}$. 

Proof. The proof is essentially the same as Theorem 7 [5]. In the proof we apply Theorem 6 [5] and put the $T$ there to be $P^7$.

Lemma 3. Let \( q = q_1 \cdots q_t \) with \( (q_j, q_i) = 1 \) for \( j \neq i \). Let \( \chi \) (mod q) be factorized into \( \prod_{j=1}^{t} \chi_j \) (mod \( q_j \)). If \( (a, q) = 1 \) then there exist uniquely \( a_j \) (mod \( q_j \)) with

\[
(a_j, q_j) = 1 \quad (j = 1, \ldots, t), \quad a = \sum_{j=1}^{t} \frac{a_j q_j}{q_j}
\]

and

\[
W(ab, \chi) = \prod_{j=1}^{t} W(a_j b, \chi_j).
\]

Proof. This is essentially Theorem 4.1 in [8, p. 159].

Lemma 4. Let \( h_1 = h/(h, q) \) and \( q_1 = q/(h, q) \). Let \( \chi^* \) (mod r) induce \( \chi \) (mod q). Then

\[
W(h, \chi) = \begin{cases} 
0 & \text{if } r \nmid q_1, \\
\phi(q)\phi(q_1)^{-1}W(h_1, \chi_1) & \text{if } r \mid q_1 \text{ where } \chi_1 \text{ (mod } q_1) \\
& \text{is induced by } \chi^* \text{ (mod } r). 
\end{cases}
\]

Proof. This is essentially Theorem 4.1 in [8, p. 159].

Lemma 4 is parallel to the known result on the Ramanujan sum and its generalization [6, p. 450]. In fact, we can also prove that \( W(h, \chi) = 0 \) if \( r \mid q_1 \) and \( (r, q_1/r) \nmid k \).

Proof. Write \( q_2 = q/q_1 \) and \( n = uq_1 + v \) with \( u = 0, 1, \ldots, q_2 - 1; \ v = 1, 2, \ldots, q_1 \). Then

\[
(3.3) \quad \sum_{n=1}^{q_1} \chi(n)e\left(\frac{hn^k}{q}\right) = \sum_{n=1}^{q_1} e\left(\frac{h_1 v^k}{q_1}\right)T(v)
\]

where \( T(v) = \sum_{(u, q_1)} \chi(uq_1 + v) \).

Let \( r \nmid q_1 \). By the same argument as in showing \( S(v) = 0 \) in [4, p. 66] we can prove that \( T(v) = 0 \) and hence \( W(h, \chi) = 0 \).

Next consider \( r \mid q_1 \). Let \( d = \prod_{p \mid q_2, p + q_1} p \) and \( \mathcal{J} = \{ uq_1 + v: 1 \leq u \leq q_2 \} \). If \( (v, q_1) = 1 \) then

\[
(3.4) \quad \sum_{j \in \mathcal{J}} \sum_{n \mid d} \mu(n) = \sum_{n \mid d} \frac{\mu(n)q_2}{q} = q_2 \prod_{p \mid d} (1 - p^{-1}).
\]

It follows from \( \chi^*(uq_1 + v) = \chi^*(v) \) and (3.4) that if \( (v, q_1) = 1 \) then

\[
T(v) = \chi^*(v) \sum_{u=1}^{q_2} \frac{1}{(uq_1 + v, q)} = \chi^*(v)\phi(q)\phi(q_1)^{-1}.
\]

By (3.3) this proves Lemma 4.
Lemma 5. (a) If \((a, p) = 1\) and \(p'\) is the modulus of \(\chi\) then \(|W(a, \chi)| \leq 2kp^{1/2}\).

(b) If \((a, q) = 1\) and \(q\) is the modulus of \(\chi\) then for any \(\varepsilon > 0\) there is a positive constant \(C(\varepsilon, k)\) depending at most on \(\varepsilon, k\) such that

\[
|W(ab, \chi)| \leq C(k, \varepsilon)(q, b)^{1/2}q^{1/2+\varepsilon}.
\]

Proof. Part (a) follows from a similar argument as part 2 of the proof of Lemma 8.5 [7].

(b) Let \(\chi^* \pmod{r}\) induce \(\chi \pmod{q}\), \(q' = q/(b, q)\), \(b' = b/(b, q)\). Suppose that \(r \mid q'\). Put \(q' = \prod_{j=1}^h p_j^{j'}\) and factorize \(\chi' \pmod{q'}\) into \(\prod_{j=1}^h \chi_j \pmod{p_j^{j'}}\), where \(\chi' \pmod{q'}\) is induced by \(\chi^* \pmod{r}\). Then by Lemmas 4, 3, and Lemma 5(a)

\[
|W(ab, \chi)| = \phi(q)\phi(q')|W(ab', \chi')| \leq (b, q) \prod_{j=1}^t |W(a_jb', \chi_j)|
\]

\[
\leq (b, q)^{1/2}((b, q)q')^{1/2} \sum_{j=1}^t 2k.
\]

This proves Lemma 5(b).

4. Major arcs. I. Write

\[
\begin{aligned}
\mathcal{W}_j &= \phi(q)^{-1}\sum_{\chi} W(ab_j, \bar{x}_j)\Delta(b_j\eta, \chi), \\
\mathcal{J}_j &= \phi(q)^{-1}I(b_j\eta)W(ab_j, \chi_0), \\
\mathcal{J}^*_j &= \phi(q)^{-1}I(b_j\eta)W(ab_j, \bar{x}_\chi_0),
\end{aligned}
\]

(4.1)

where \(\mathcal{J}^*_j\) is defined only when the exceptional character exists. By (2.4), (2.7) we have

\[
R_1(m) = \sum_{q \leq P} \sum_{a} \int_{-\delta_q}^{\delta_q} e\left(-m\left(\frac{a}{q} + \eta\right)\right) \prod_{j=1}^s S(b_j a) \ d\eta
\]

(4.2)

\[
= \sum_{q \leq P} \sum_{a} e\left(-\frac{ma}{q}\right) \int_{-\delta_q}^{\delta_q} e\left(-m\eta\right) \prod_{j=1}^s \mathcal{W}_j + \mathcal{J}_j + \mathcal{J}^*_j \ d\eta
\]

where the sum \(\Sigma_a\) is taken over all \(a\) with \(1 \leq a \leq q\) and \((a, q) = 1\).

There are two categories of terms in the last product of (4.2), namely, (A) terms having at least a factor \(\mathcal{W}_j\); (B) terms having no factor \(\mathcal{W}_j\). We shall treat category (A) in this section and category (B) in \(\S 6\).

Let \(\mathcal{J}'_j\) denote either \(\mathcal{J}_j\) or \(\mathcal{J}^*_j\). In category (A) for each fixed \(h = 1, 2, \ldots, s\) we choose \(\prod_{j=1}^h \mathcal{W}_j \prod_{j=h+1}^s \mathcal{J}'_j\) as the representative of those terms having exactly \(h\) factors \(\mathcal{W}_j\). Put

\[
T_h(m) = \sum_{q \leq P} \sum_{a} e\left(-\frac{ma}{q}\right) \int_{-\delta_q}^{\delta_q} \prod_{j=1}^h \mathcal{W}_j \prod_{j=h+1}^s \mathcal{J}'_j e\left(-m\eta\right) \ d\eta
\]

(4.3)

\[
(h = 1, \ldots, s).
\]

Let

\[
\chi_j \pmod{q} = \chi_0 \pmod{q} \quad \text{or} \quad \bar{x}_\chi_0 \pmod{q},
\]

(4.4)

\[
I'(b_j\eta) = I(b \eta) \quad \text{or} \quad \bar{I}(b \eta).
\]

(4.5)
Then by Schwarz's inequality and (4.1), (4.3) we have

\[ |T_h(m)| \leq \sum_{\phi(p) \leq P} \phi(q)^{-s} \sum_{j=1, \ldots, h} \left| \sum_a e\left( \frac{-ma}{q} \right) \prod_{j=1}^{h} W(ab_j, \chi_j) \prod_{j=h+1}^{s} W(ab_j, \chi_j') \right| \]

\[ \times \prod_{j=1}^{h} \left( \int_{\Delta \delta_q} \Delta(b_j \eta, \chi_j) d\eta \right)^{1/n_j} \prod_{j=h+1}^{s} \left( \int_{\Delta \delta_q} I'(b_j \eta) d\eta \right)^{1/n_j}, \]

where \( \sum_{\chi_j, j=1, \ldots, h} \) denotes \( h \) summations each of which is taken over all \( \chi \pmod{q} \) and \( n_j \geq 1 \) are integers satisfying \( \sum_{j=1}^{s} 1/n_j = 1 \). Note that each \( \chi_j \pmod{q} \) is induced by a unique \( \chi_j^* \pmod{r_j} \) with \( r_j \mid q \) and that each \( \chi_j^* \pmod{r_j} \) and each \( q \) with \( r_j \mid q \) induce a unique \( \chi \pmod{q} \). Then by (2.8), (4.6) we have

\[ |T_h(m)| \leq \sum_{\phi(p) \leq P} \sum_{\chi^*_j} \left\{ \sum_{q=1}^{\infty} \phi(q)^{-s} \left| \sum_a e\left( \frac{-ma}{q} \right) \prod_{j=1}^{h} W(ab_j, \chi_0 \chi_j^*) \right| \right. \]

\[ \times \left. \prod_{j=h+1}^{s} W(ab_j, \chi_j') \right\} \]

\[ \times \left( \int_{\Delta \delta_i} |I'(b_j \eta)|^{n_j} d\eta \right)^{1/n_j}. \]

By Lemma 5 with \( \epsilon = (10s)^{-1} \), the infinite sum inside the curly brackets of (4.7) is

\[ \ll \sum_{q=1}^{\infty} \phi(q)^{-s+1} \prod_{j=1}^{s} |b_j|^{1/2} q^{1/2 + 1/10s} \ll B^{s/2} \]

since by (1.2) we have \( s \geq 5 \) for any \( k \geq 2 \). Also by (2.6) we have \( I'(b_j \eta) \ll N \) and then by (2.3), (2.2)

\[ \left( \int_{\Delta \delta_i} |I'(b_j \eta)|^{n_j} d\eta \right)^{1/n_j} \ll N^{1-k/n_j} P^{1/n_j} \]

It follows from (4.7), (4.8), (4.9) and Lemma 2 that

\[ T_h(m) \ll B^{s/2} \left( \prod_{j=1}^{h} |b_j| N^{1-k/n_j} P^{-2} \right) \left( \prod_{j=h+1}^{s} N^{1-k/n_j} P^{1/n_j} \right) \]

\[ \ll B^{3s/2} N^{s-k} P^{-1} = E_1, \]

since \( \sum_{j=1}^{h} 1/n_j = 1 \).

5. Singular series.

**Lemma 6.** For a given \( p \) let \( p^\theta \| k \) and \( p^\theta \| b \). Suppose that \( p^i \) and \( p^j \) are the moduli of \( \chi_0 \) and \( \chi_1 \) respectively and

\[ u = 2\phi + \theta + \begin{cases} 3 & \text{if } p = 2, \\ 1 & \text{if } p \geq 3. \end{cases} \]
If \(1 \leq j \leq u - 2\phi - \theta, \ t \geq u + 1\) and \((a, p) = 1\) then
\[
W(ab, \chi_0) = W(ab, \chi_1\chi_0) = 0.
\]

**Proof.** The proof is essentially the same as Lemma 1 [9].

**Lemma 7.** Let \(q = q_1q_2, \ (q_1, q_2) = 1\) and factorize \(\chi_j \pmod{q}\) into \(\prod_{j=1}^{\phi} \chi_{j_i} \pmod{q_i}\) \((j = 1, 2, \ldots, s)\). If
\[
B(m, q) = \phi(q)^{-s} \sum_{a} e\left(\frac{ma}{q}\right) \prod_{j=1}^{s} W(ab, \chi_j),
\]
then
\[
B(m, q) = B(m, q_1) B(m, q_2).
\]

**Proof.** Apply Lemma 3.

By Lemma 1 the exceptional character \(\tilde{\chi} \pmod{\tilde{r}}\) is real and primitive. Then it is known [8, p. 159] that
\[
\tilde{r} = 2^{l_1}p_1 \cdots p_t,
\]
where \(p_j\) are distinct odd primes and \(l = 0\) or \(2\) or \(3\). If \(\tilde{r} \mid q\) write
\[
q = q_1q_2, \quad (q_1, q_2) = 1 \quad \text{and} \quad q_1 = 2^{l_1}p_2^{l_2} \cdots p_t^{l_t}
\]
where \(l_j \geq 1\) \((j = 2, \ldots, t)\); \(l_1 > l\) if \(l \neq 0\) and \(l_1 = 0\) if \(l = 0\). Put
\[
B_h(m, q) = \phi(q)^{-s} \sum_{a} e\left(\frac{ma}{q}\right) \prod_{j=1}^{s} W(ab, \chi'_{j}) \quad (h = 0, 1, \ldots, s),
\]
where \(\chi'_{j}\) is defined in (4.4) and there are exactly \(h\) \(\chi'_{j} = \tilde{\chi}\chi_0 \pmod{q}\) in the last product of (5.3). Define singular series \((h = 0)\) and pseudosingular series \((h = 1, 2, \ldots, s)\) by
\[
\mathcal{S}_0(m) = \sum_{q=1}^{\infty} B_0(m, q) \quad \text{and} \quad \mathcal{S}_h(m) = \sum_{q=1}^{\infty} B_h(m, q).
\]
By Lemma 5(b) all series in (5.4) are absolutely convergent.

**Lemma 8.** Let \(\tilde{r}\) and \(q_1\) be defined as in (5.1), (5.2). If \(B_h(m, q_1) \neq 0\) then \(q_1 = d_k\tilde{r}\) or \(2d_k\tilde{r}\) where \(d_k\) is a divisor of \(k\).

**Proof.** For each \(p_j\) \((j = 1, \ldots, t)\) in (5.1) with \(p_1 = 2\) let \(p_j^{\theta_j} \parallel k\). Suppose that \(l_1 > 4 + \theta_1\) or \(l_j > 2 + \theta_j\) for some \(j \geq 2\). For simplicity we only give the details for the case \(j = 2\). Let
\[
l_2 \geq \theta_2 + 2.
\]
Since no prime can divide all \(b_j\), we may assume that \(p_2 \nmid b_1\). Factorizing the exceptional character \(\tilde{\chi}\) and the character \(\chi_1'\) in (5.3) we have
\[
\tilde{\chi} \pmod{\tilde{r}} = \tilde{\chi}_1 \pmod{2^{l_1}} \prod_{j=2}^{t} \tilde{\chi}_j \pmod{p_j},
\]
\[
\chi_1'(\pmod{q_1}) = \prod_{j=1}^{l} \chi_{1,j}' \pmod{p_j^l},
\]
where each \( \chi'_{ij} \) is either \( \chi_0 \) (mod \( p_j \)) or \( \tilde{\chi}_i \) (mod \( p_j \)). By (3.2) for each \( a \) with \( (a, q_1) = 1 \) there are \( a_j \) \( (j = 1, \ldots, t) \) with \( (a_j, p_j) = 1 \) such that \( W(ab_1, \chi'_{1}) = \prod_{j=1}^{t-1} W(a_j b_1, \chi'_{1}) \). Then by (5.5) and Lemma 6 with \( \phi = 0 \), for each \( a \) in \( \Sigma^0_a \) of (5.3) we have \( W(a_2 b_1, \chi'_{1}) = 0 \). So by (5.3) if \( B_h(m, q_1) \neq 0 \) then \( l \leq l_1 \leq l + 1 + \theta_1 \) and \( 1 \leq l_j \leq 1 + \theta_j \) \( (j = 2, 3, \ldots, s) \). This proves Lemma 8.

Lemma 9. (a) \( \mathcal{S}_0(m) \gg B^{s(1-s)} \) and (b) \( \mathcal{S}_h(m) \ll \mathcal{S}_0(m)(\log N)^{-1/2} \) \( (h = 1, 2, \ldots, s) \).

Proof. Part (a) is Lemma 5 in [9].

We come now to prove part (b). For each \( q \) with \( \tilde{r} | q \) define \( q_1 \) and \( q_2 \) as in (5.2). Since, by the hypothesis on \( b_j \), no prime can divide more than \( s - s_0 \) \( b_j \), we have

\[
\prod_{j=1}^{s} (q_1, b_j) \leq q_1^{s-s_0}.
\]

Then by (5.3) and Lemma 5 with \( \epsilon = (10s)^{-1} \) we have

\[
B_h(m, q_1) \ll \phi(q_1)^{-s+1} q_1^{s/2 + 1/10} q_1^{(s-s_0)/2} \ll q_1^{6/5-s_0/2}.
\]

Then by Lemma 8 and \( s_0 \geq 2k \geq 4 \) (see (1.1)) we have

\[
B_h(m, q_1) \ll \tilde{r}^{-4/5}.
\]

So by Lemma 8 again we have

\[
\sum_{q_1=1}^{s} B_h(m, q_1) \ll \tilde{r}^{-4/5}.
\]

On the other hand, by Lemma 5(a), the divisibility hypothesis on \( b_j \) and \(|W(ab_j, \chi'_{j})| \leq \phi(p')\), we see that the product in \( B_0(m, p') \) in (5.3) satisfies

\[
\left| \prod_{j=1}^{s} W(ab_j, \chi'_{j}) \right| \leq (2k)^{s_0} p^{s_0/2} \phi(p')^{s-s_0}.
\]

So by (5.3) and \( s_0 \geq 4 \) we have

\[
B_0(m, p') \ll \phi(p')^{s_0-1} (2k)^{s_0} p^{s_0/2} \leq (4k)^s p^{(1-s_0/2)} < c_1 p^{-t}.
\]

For each \( p \) there exists some \( b_j = b_1 \), say, which is not divisible by \( p \). By Lemma 6 for each \( a \) with \( (a, p) = 1 \) we have \( W(ab_1, \chi_0) = 0 \) if \( t \geq v + 2 \) where \( v \) is defined in (1.3) and \( p' \) is the modulus of \( \chi_0 \). So by (5.3) we have \( B_0(m, p') = 0 \) if \( t \geq v + 2 \). Then by Lemma 7 and \( (q, q_2) = 1 \)

\[
(5.8) \quad \sum_{q_2=1}^{\infty} B_0(m, q_2) = \prod_{p \mid \tilde{r}} \left( 1 + \sum_{t=1}^{v_1} B_0(m, p') \right)
= \sum_{q=1}^{\infty} B_0(m, q) \prod_{p \mid \tilde{r}} \left( 1 + \sum_{t=1}^{v_1} B_0(m, p') \right)
\]

where \( v_1 = v + 1 \). Separate the last product \( \prod_{p \mid \tilde{r}, p \leq c_2} \) and \( \prod_{p \mid \tilde{r}, p > c_2} \)

where \( c_2 = 4c_1 \). Same as that in the proof of Lemma 5 in [9, see (4.16) and the product \( \prod_1 \) on p. 197] which depends essentially on (1.1)–(1.5) and the divisibility
condition on $b_j$ in Theorem 1, we have that the first product $\prod_{p \mid \tau, p \leq c_2} \left(1 + \sum_{t=1}^{\rho_1} B_0(m, p^t)\right)$ satisfies

$$\prod_{p \mid \tau, p \leq c_2} \left(1 + \sum_{t=1}^{\rho_1} B_0(m, p^t)\right) \geq \prod_{p \mid \tau, p \leq c_2} \phi(p)^{\tau - s} p^{\rho_1} \geq \prod_{p \leq c_2} p^{\rho_1(1-s)} = c_3 > 0.$$

For the second product $\prod_{p > c_2} \left(1 + \sum_{t=1}^{\rho_1} B_0(m, p^t)\right)$ by (5.7) we have

$$\prod_{p > c_2} \left(1 + \sum_{t=1}^{\rho_1} B_0(m, p^t)\right) \geq \prod_{c_2 < p \leq \tau} \left(1 - c_1 \sum_{t=1}^{\infty} p^{-t}\right) \geq \prod_{c_2 < p \leq \tau} (1 - c_2/2p) \gg (\log \tau)^{-c_2}. \tag{5.9}$$

The last inequality is a simple modification of Theorem 9.3 in [8, p. 92]. Now by (5.8), (5.9) we have

$$\sum_{q_2 = 1}^{\infty} B_2(m, q_2) \ll \mathcal{S}_0(m)(\log \tau)^{-c_2}. \tag{5.10}$$

Finally, by (5.2) we see that $\tilde{x} \chi_0 \pmod{q}$ can be factorized as the product of $\tilde{x} \chi_0 \pmod{q_1}$ and $\chi_0 \pmod{q_2}$. Then by (5.4), Lemma 7, (5.6), (5.10) we have

$$\mathcal{S}_h(m) = \sum_{q_1 = 1}^{\infty} B_h(m, q_1) \sum_{q_2 = 1}^{\infty} B_0(m, q_2) \ll \mathcal{S}_0(m)(\log N)^{-1/2}$$

since by (3.1) we have

$$\tau^{4/5}(\log \tau)^{-c_2} \gg (\log N)^{1/2}.$$ 

This proves Lemma 9.

6. Major arcs. II.

**Lemma 10.** We have

$$\int_{(qQ)^{-1}}^{1/2} \left| \prod_{j=1}^{s} I'(b_j \eta) \right| d\eta \ll (qQ)^{s-1} N^{s(1-k)}$$

where $I'(b_j \eta)$ is defined in (4.5).

**Proof.** If $0 < \eta < 1/2$ then for any $n \geq 1$ we have $\Sigma_{n=0}^{\infty} e(n \eta) \ll |\eta|^{-1}$. Let $\phi = 1/k$ or $\phi/k$. Then by Abel's partial summation formula and (2.6)

$$b^\phi I'(b \eta) \ll |\eta|^{-1} \left( |bN^k|^{-1} + \int_{|b|\leq G^k} \left| \frac{d}{dy} y^{\phi-1} \right| dy \right) \ll |\eta|^{-1} (|b|G^k)^{\phi-1} \ll |\eta|^{-1} N^{1-k}.$$ 

So the lemma follows.

Let

$$J_h(m) = \int_{-1/2}^{1/2} \prod_{j=1}^{h} I'(b_j \eta) \prod_{j=h+1}^{s} I(b_j \eta) e(-m \eta) d\eta \quad (h = 0, 1, \ldots, s).$$
Lemma 11. (a) $|J_h(m)| \leq J_0(m)$ $(h = 1, 2, \ldots, s)$.

(b) If

$$|m| \leq (N/4)^{s-1}$$

then

$$J_0(m) \gg B^{-s/k}N^{s-k}.$$

Proof. Part (a) follows from (6.1) and part (b) is essentially Lemma 8 [9].

We come now to treat those terms in category (B) defined in §4. In category (B) we choose $\prod_{j=1}^{h} \mathcal{J}_j \mathcal{Y}_{j=h+1}^s$ $(h = 0, 1, \ldots, s)$ to represent those terms $\prod_{j=1}^{s} \mathcal{J}_j'$ having exactly $h$ factors $\mathcal{J}_j$. Put

$$T_0(m) = \sum_{q \leq P} \sum_{\alpha} e \left( -\frac{ma}{q} \right) \int_{-\delta_q}^{\delta_q} \left( \prod_{j=1}^{h} \mathcal{J}_j \prod_{j=h+1}^{s} \mathcal{J}_j \right) e(-m\eta) \, d\eta,$$

(6.3)

$$\bar{T}_h(m) = \sum_{q \leq P} \sum_{\alpha} e \left( -\frac{ma}{q} \right) \int_{-\delta_q}^{\delta_q} \left( \prod_{j=1}^{h} \mathcal{J}_j \prod_{j=h+1}^{s} \mathcal{J}_j \right) e(-m\eta) \, d\eta$$

$(h = 1, 2, \ldots, s)$.

By (4.1), $s \geq 5$, Lemmas 10 and 5 with $\epsilon = (10s)^{-1}$ we have

$$\sum_{q \leq P} \sum_{\alpha} e \left( -\frac{ma}{q} \right) \int_{-\delta_q}^{\delta_q} \left( \prod_{j=1}^{h} \mathcal{J}_j \prod_{j=h+1}^{s} \mathcal{J}_j \right) e(-m\eta) \, d\eta$$

$$\ll N^{s-k}B^{s/2}P^{-3/10} = E_2,$$

(6.4)

So, if we replace the integral $\int_{-\delta_q}^{\delta_q}$ in (6.3) by $\int_{1/2}^{1/2}$ we have the error $E_2$ given in (6.4).

Then by (6.1), (6.3), (4.1) we have

$$T_h(m) = J_h(m) \left\{ \sum_{q \leq P} \phi(q)^{-s} \sum_{\alpha} e \left( -\frac{ma}{q} \right) \prod_{j=1}^{h} W(ab_j, \chi_0) \prod_{j=h+1}^{s} W(ab_j, \chi_0) \right\}$$

$$+ E_2.$$ 

Similarly, by Lemma 5, if we replace the sum $\sum_{q \leq P, \alpha}$ in (6.5) by $\sum_{q \leq 1, \alpha}$ we have an error $\ll B^{1/2}P^{-3/10}$. So by (5.4), (6.5) we have

$$\bar{T}_h(m) = J_0(m) \left( \mathcal{S}_h(m) + E_3 \right) + E_2 \quad (h = 1, 2, \ldots, s),$$

where $E_3 = O(E_2 N^{k-s})$. By the same argument we have

$$T_0(m) = J_0(m) \left( \mathcal{S}_0(m) + E_3 \right) + E_2.$$ 

Note that each representative in either category (A) or (B) defined in §4 represents at most $O(1)$ terms. It follows from (4.2), (4.10), (6.7), (6.6) that

$$R_1(m) = J_0(m) (\mathcal{S}_0(m) + E_3) + O \left( \sum_{h=1}^{s} J_h(m) (\mathcal{S}_h(m) + E_3) \right)$$

$$+ O(E_2 + E_1).$$
By (4.10), (6.4), Lemmas 9(a) and 11(b) we see that for \( j = 1, 2 \)
\[(6.9) \quad E_{3} / s_{0}(n) \quad \text{and} \quad E_{j} / j_{0}(m) s_{0}(m) \ll B^{2s + 3/10}.\]
It follows from (6.8), (6.9), (2.1), Lemmas 11(a) and 9(b) that
\[(6.10) \quad R_{1}(m) > \frac{1}{2} j_{0}(m) s_{0}(m).\]

7. Minor arcs.

**Lemma 12.** If \( \alpha \in m \) then
\[\sum_{p \leq N} e(abp^{k}) \ll N |b| P^{-\omega(k)}\]
where \( \omega(k)^{-1} = 4^{(k+2)}(k + 1) \).

**Proof.** This is essentially Lemma 11 [9] (see also Lemma 5 [1]).

Let
\[R_{2}(m) = \int_{m}^{1} \prod_{j=1}^{s} S(b_{j}, \alpha) e(-m \alpha) d \alpha.\]
Then by Lemma 12 and the same argument as Lemma 12 in [9] we have
\[R_{2}(m) \ll N^{s-k} B^{-s} P^{-\omega(k)} (\log N)^{c_{4}}.\]

By (4.2), (6.10), Lemmas 9(a), 11(b) and (2.1)
\[(7.1) \quad \int_{Q^{-1}}^{1} \prod_{j=1}^{s} S(b_{j}, \alpha) e(-m \alpha) d \alpha = R_{1}(m) + R_{2}(m) \]
\[\gg N^{s-k} B^{-s} \{1 - c_{5} B^{s(s+1)} P^{-\omega(k)} (\log N)^{c_{4}} \} > 0.\]
Choose the least \( N \) satisfying (2.1) and (6.2). So (7.1) implies the existence of a solution of \( \sum_{j=1}^{s} b_{j} p_{j}^{k} = m \) in primes \( p_{j} \) and
\[\max_{1 \leq j \leq s} p_{j} \ll N \ll C_{1}(k) m^{1/k} + C_{2}(k) (\log B)^{s}.\]
This completes the proof of Theorem 1.

**Remark.** Combining the Circle Method with the Sieve Method, when \( k = 1 \) and \( s = 3 \), the author [10] is able to obtain a bound for solutions of (1.6) to be \( B^{4} \) where \( A > 0 \) is an absolute constant. However, for \( k \geq 2 \) it seems that these two methods do not combine well to replace the \( (\log B)^{2} \) in (1.7) by \( \log B \).

**References**


