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CAUCHY PROBLEM FOR NONLINEAR HYPERBOLIC SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We proved the sharp Sobolev estimate for Cauchy data for the general type of hyperbolic systems of nonlinear partial differential equations, which leads to a local existence and uniqueness theorem for solutions of the Cauchy problem in Sobolev spaces.

0. Introduction. This paper gives an analysis of the local solvability of the Cauchy problem for hyperbolic systems of nonlinear partial differential equations of the general type:

$$(*) \qquad \vec{e}(\vec{x}, D^m\vec{u}(\vec{x})) = 0, \quad D^j_{x_0}\vec{u}(x_0 = 0) = \vec{w}_j, \qquad j = 0, \dots, m-1,$$

where $x \in \mathbf{R} \times \mathbf{R}^N$, $D^m\vec{u} = \{D^\alpha\vec{u} \mid |\alpha| \le m\}$, $\vec{w}_j \in H^{M-j+1}_{\mathrm{loc}}(\mathbf{R}^N)$, $\vec{e} = (e_1, \dots, e_n)$, and $\vec{u} = (u_1, \dots, u_n)$. In the scalar case it has been proven by P. A. Dionne [**4**] that for any integer $M > N/2 + 1 + m$ there exists a unique solution of $(*)$ in $L^\infty([-T,T], H^{M+N/2+1}_{\mathrm{loc}}(\mathbf{R}^N))$ for T small enough. In Hughes, Kato, and Marsden [**6**] it is shown that local solutions for symmetric hyperbolic first order differential equations exist for $M > N/2 + 1$.

Bona and Scott have demonstrated in [**1**] that for $s \ge 2$ the initial value problem for the Korteweg-de Vries equation

$$u_t + uu_x + u_{xxx} = 0, \qquad t > 0, \qquad u(x, 0) = g(x) \in H^s(\mathbf{R})$$

has a unique solution $u \in C_0([0,\infty), H^s(\mathbf{R}))$.

All these results imply that for $M > N/2 + 1$ the general Cauchy problem $(*)$ may have a unique solution. Indeed, our major result is

THEOREM 4. *For $M > N/2+1$ the Cauchy problem $(*)$ for hyperbolic nonlinear systems has a unique local solution $\vec{u}$, such that*

$$\frac{\partial^j}{\partial t^j}\vec{u} \in C([-T,T], H^{M-j}_{\mathrm{loc}}(\mathbf{R}^N)) \cap L^\infty([-T,T], H^{M-j+1}_{\mathrm{loc}}(\mathbf{R}^N)), \qquad j = 0, \dots, m,$$

if T is small enough, provided

(a)
$$\vec{e}(\vec{x}, \vec{y}) \in C^\infty(\mathbf{R} \times \mathbf{R}^N, \underbrace{\mathbf{R}^r \times \cdots \times \mathbf{R}^r}_{n \text{ times}})$$

where

$$\vec{y} = D^m\vec{u}, \qquad r = \binom{m+N+1}{N+1}$$

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and

(b)

$$\text{the matrix } E = (\partial e_i/\partial y_{j\alpha}), \qquad 1 \le i, j \le n,$$

is invertible for $\alpha_0 = m$, *where* $y_{j\alpha} = D^\alpha u_j$.

This result is sharp since, as was proved by DiPerna [**5**] in a much more general context, a continuous (with respect to t) solution to the Cauchy problem $u_t + uu_x = 0$, $u(0) = g$ does not exist if $g \in H^\sigma(R)$, $\frac{1}{2} < \sigma \le \frac{3}{2}$.

Establishing Theorem 4 required the following steps. In §II we construct a quasilinear system equivalent to $(*)$, applying the method of [**4**], generalized for systems. Then in §III we further reduce it to a first order system which is solved in §IV, where we apply an iterative method and derive energy estimates in order to prove the existence and uniqueness of a solution.

The essential tools that made evaluation of M possible were the techniques of pseudodifferential operators with H^M coefficients which were suggested by M. E. Taylor in [**7**] and paradifferential operators introduced by J. M. Bony in [**2**].

We consider the Cauchy problem for an $n \times n$ system:

$$(1)\qquad \begin{aligned} &(*)\quad \vec{e}(x, D^m\vec{u}(x)) = 0, && x = (x_0, \dots, x_N) \in \mathbf{R}^{N+1},\\ & && \vec{e}, \vec{u} \text{ take values in } \mathbf{R}^n,\\ &D_0^i u_j(x_0 = 0) = w_{ji}, && i = 0, \dots, m-1, j = 1, \dots, n, \end{aligned}$$

with the following.

I. Hypotheses and notation. We denote

$$D_i = \frac{\partial}{\partial x_i}, \quad D^\alpha = D_0^{\alpha_0} D_1^{\alpha_1} \dots D_N^{\alpha_N},$$

$$D_i^{\alpha_i} = \left(\frac{\partial}{\partial x_i}\right)^{\alpha_i}, \qquad i = 0, 1, \dots, N,$$

$$D^m u_j = \{D^\alpha u_j |\, 0 \le |\alpha| = |\alpha_0 + \alpha_1 + \dots + \alpha_N| \le m\}, \qquad j = 1, \dots, n.$$

Note that for each j, $D^m u_j$ can be considered as a function taking values in $\mathbf{R}^r$, where

$$r = \binom{m + N + 1}{N + 1} = \frac{(m + N + 1)!}{(N + 1)! m!}.$$

Let $y_i = D^m u_i$ for $i = 1, \dots, n$. Then $y_i = (y_{i_1}, \dots, y_{i_r})$, $y_{i\alpha} = D^\alpha u_i$. Then we can rewrite the system $(*)$ as

$$(2)\qquad \vec{e}(x, y(x)) = 0, \quad \text{where } y = (y_1, \dots, y_n) \text{ takes values in } \underbrace{\mathbf{R}^r \times \dots \times \mathbf{R}^r}_{n \text{ times}}.$$

Let $e_{ky_{l\alpha}}$ be the partial derivative of e_k by $y_{l\alpha}$, $k = 1, \dots, n$; $l = 1, \dots, n$; $|\alpha| \le m$. Define $e_{kx_i}(x, y)$ by the equation

$$(3)\qquad \frac{\partial}{\partial x_i} e_k(x, y(x)) = e_{kx_i}(x, y(x)) + \sum_{l=1}^{n} \sum_{|\alpha|=0}^{m} e_{ky_{l\alpha}} \frac{\partial y_{l\alpha}}{\partial x_i}.$$

Since we are interested in local properties of the solution of (1), let

$$(x, y) \in \mathbf{R} \times \mathbf{T}^N \times \underbrace{\mathbf{R}^r \times \cdots \times \mathbf{R}^r}_{n \text{ times}}.$$

We suppose

(a) $e_k(x, y) \in C^\infty(X, Y), k = 1, \ldots, n,$ where $X = \mathbf{R} \times \mathbf{T}^n$ and $Y = \underbrace{\mathbf{R}^r \times \cdots \times \mathbf{R}^r}_{n \text{ times}}$.

(b) The matrix

$$E = \begin{pmatrix} e_{1y_{1\alpha}} & e_{1y_{2\alpha}} & \cdots & e_{1y_{n\alpha}} \\ e_{2y_{1\alpha}} & e_{2y_{2\alpha}} & \cdots & e_{2y_{n\alpha}} \\ & \vdots & & \\ e_{ny_{1\alpha}} & e_{ny_{2\alpha}} & \cdots & e_{ny_{n\alpha}} \end{pmatrix},$$

where $\alpha = (m, 0, \ldots, 0)$, i.e. for $\alpha_0 = m$, is invertible.

(c) The hyperbolicity condition is satisfied, i.e.

$$\det \left[\sum_{k=0}^{m} (i\tau)^{m-k} \tilde{A}_k \right]$$

has m purely imaginary, distinct roots $i\tau_j(\vec{x}, \vec{\xi})$, $j = 1, \ldots, m$, where

$$\tilde{A}_k = \sum_{\substack{|\alpha|=m \\ \alpha_0 = m-k}} \begin{pmatrix} e_{1y_{1\alpha}} & e_{1y_{2\alpha}} & \cdots & e_{1y_{n\alpha}} \\ & \vdots & & \\ e_{ny_{1\alpha}} & e_{ny_{2\alpha}} & \cdots & e_{ny_{n\alpha}} \end{pmatrix} \xi^{(\alpha_1, \ldots, \alpha_N)}.$$

(d) $w_{ij} \in H^{M-j+1}(\mathbf{T}^N)$, for $i = 1, \ldots, n$, $j = 0, \ldots, m-1$, with $M \in \mathbf{R}$ to be determined.

II. Construction of a quasilinear system. We want to construct a Cauchy problem for a quasilinear hyperbolic system equivalent to (1). We use the method of [**4**], generalized for systems. Let

(4)
$$A(x, y, D) = \sum_{|\alpha|=0}^{m} A_\alpha(x, y, D) = \sum_{|\alpha|=0}^{m} \begin{pmatrix} e_{1y_{1\alpha}} & e_{1y_{2\alpha}} & \cdots & e_{1y_{n\alpha}} \\ e_{2y_{1\alpha}} & e_{2y_{2\alpha}} & \cdots & e_{2y_{n\alpha}} \\ & \vdots & & \\ e_{ny_{1\alpha}} & e_{ny_{2\alpha}} & \cdots & e_{ny_{n\alpha}} \end{pmatrix} D^\alpha,$$

(5)
$$B(x, y) = (\vec{b}_{-1}(x, y), \ldots, \vec{b}_N(x, y)) = \begin{pmatrix} e_{10}, & -e_{1x_0}, & -e_{1x_1}, & \cdots, & -e_{1x_N} \\ e_{20}, & -e_{2x_0}, & -e_{2x_1}, & \cdots, & -e_{2x_N} \\ & \vdots & & & \\ e_{n0}, & -e_{nx_0}, & -e_{nx_1}, & \cdots, & -e_{nx_N} \end{pmatrix},$$

where

$$e_{k0}(x,y) = \sum_{|\alpha|=0}^{m} \sum_{i=1}^{n} e_{ky_{i\alpha}} y_{i\alpha} - e_k, \qquad k = 1, \dots, n. \tag{6}$$

Let $\alpha = \beta(\alpha) + \gamma(\alpha)$, where $\beta(\alpha) = (\beta_0, \beta_1, \dots, \beta_N)$ and $\gamma(\alpha) = (\gamma_0, \gamma_1, \dots, \gamma_N)$, such that

$$\begin{aligned} &|\beta(\alpha)| \le m - 1, \\ &|\gamma(\alpha)| = 0 \quad \text{if} \quad |\alpha| \le m - 1, \\ &|\gamma(\alpha)| = 1 \quad \text{if} \quad |\alpha| = m, \end{aligned}$$

and for each $\gamma(\alpha)$ define $k(\alpha)$ as follows:
if $|\alpha| = m$, then $\gamma(\alpha) = (0, \dots, 0, \gamma_{k(\alpha)}, 0, \dots, 0)$,
if $|\alpha| \le m-1$, then $k(\alpha) = -1$.
Let

$$U(x) = (U_{ik}(x)) = (\vec{U}_{-1}(x), \dots, \vec{U}_N(x)) = \begin{pmatrix} u_1 & u_{1x_0} & u_{1x_1} & \cdots & u_{1x_N} \\ u_2 & u_{2x_0} & u_{2x_1} & \cdots & u_{2x_N} \\ & & \vdots & & \\ u_n & u_{nx_0} & u_{nx_1} & & u_{nx_N} \end{pmatrix}, \tag{7}$$

$$i = 1, \dots, n; k = -1, 0, 1, \dots, N;$$

and

$$W_j(x_1, \dots, x_N) = (\vec{W}_{-1j}, \dots, \vec{W}_{Nj}) = \begin{pmatrix} w_{1j} & w_{1j+1} & D_{x_1} w_{1j} & \cdots & D_{x_N} w_{1j} \\ w_{2j} & w_{2j+1} & D_{x_1} w_{2j} & \cdots & D_{x_N} w_{2j} \\ & & \vdots & & \\ w_{nj} & w_{nj+1} & D_{x_1} w_{nj} & \cdots & D_{x_N} w_{nj} \end{pmatrix}, \tag{8}$$

$$j = 0, \dots, m-1,$$

be vector valued functions in $\mathbf{R} \times \mathbf{T}^N$ and $\mathbf{T}^N$ respectively. (Here functions $w_{1m} = D_0^m u_1(x_0 = 0), \dots, w_{nm} = D_0^m u_n(x_0 = 0)$, which appear in (8) when $j = m-1$, are determined by (1).)

Now consider the Cauchy problem for the following quasilinear system of order m:

$$\begin{cases} A(x, D^{\beta(\alpha)} \vec{U}_{k(\alpha)}, D) U = B(x, D^{\beta(\alpha)} \vec{U}_{k(\alpha)}), \\ D_0^j U(x_0 = 0) = W_j, \quad \text{where} \quad j = 0, \dots, m-1; W_j \in H^{M-j}(\mathbf{T}^N). \end{cases} \tag{9}$$

Here the components, $U_{ik(\alpha)}$, of $\vec{U}_{k(\alpha)}$ are entries of the column $k(\alpha)$ in the matrix (7), $k(\alpha) = -1, 0, 1, \dots, N$, and $D^{\beta(\alpha)} U_{ik(\alpha)}$ are substituted for $y_{i\alpha}$ in (4) and (5), $i = 1, \dots, n$.

LEMMA 1. *Cauchy problems* (1) *and* (9) *are equivalent.*

PROOF. The $n(N+2)$ equations in (9) can be separated into two different types.

First type:

$$\sum_{|\alpha|=0}^{m}\sum_{i=1}^{n} e_{ly_{i\alpha}}(x, D^{\beta(\alpha)}U_{k(\alpha)})D^{\alpha}u_i = e_{l0}(x, D^{\beta(\alpha)}U_{k(\alpha)}) \tag{10}$$

for $l = 1, \ldots, n$, which is

$$\sum_{|\alpha|=0}^{m-1}\sum_{i=1}^{n} e_{ly_{i\alpha}}(x, D^{\alpha}\vec{u})y_{i\alpha} + \sum_{|\alpha|=m}\sum_{i=1}^{n} e_{ly_{i\alpha}}(x, D^{\beta(\alpha)}U_{k(\alpha)})D^{\alpha}u_i$$

$$= \sum_{|\alpha|=0}^{m-1}\sum_{i=1}^{n} e_{ly_{i\alpha}}y_{i\alpha} + \sum_{|\alpha|=m}\sum_{i=1}^{n} e_{ly_{i\alpha}}(x, D^{\beta(\alpha)}U_{k(\alpha)})D^{\beta(\alpha)}U_{k(\alpha)} - e_l.$$

Here the first term on the left is equal to the first term on the right side of the equation. By definition of $\beta(\alpha)$ and $k(\alpha)$ the second term on the left and right sides are also equal.

Second type:

$$\sum_{|\alpha|=0}^{m}\sum_{i=1}^{n} e_{ly_{i\alpha}}(x, D^{\beta(\alpha)}U_{k(\alpha)})D^{\alpha}u_{ix_j} = -e_{lx_j}(x, D^{\alpha}\vec{u}) \tag{11}$$

where $l = 1, \ldots, n$; $j = 0, \ldots, N$. We rewrite the last equation as

$$\sum_{|\alpha|=0}^{m-1}\sum_{i=1}^{n} e_{ly_{i\alpha}}(x, D^{\alpha}\vec{u})D^{\alpha}u_{ix_j}$$

$$+ \sum_{|\alpha|=m}\sum_{i=1}^{n} e_{ly_{i\alpha}}(x, D^{\beta(\alpha)}\vec{U}_{k(\alpha)})D^{\alpha}u_{ix_j}$$

$$= -e_{lx_j}(x, D^{\alpha}\vec{u}).$$

Since $D^{\beta(\alpha)}\vec{U}_{k(\alpha)} = D^{\alpha}\vec{u}$ for $|\alpha| = m$, where $\vec{u}$ is a column $(u_1, \ldots, u_n)^t$, we use (3) to see that equation (11) means

$$\frac{\partial}{\partial x_i} e_l(x, D^{\alpha}\vec{u}) = 0.$$

We conclude that (10) and (11) are equivalent to (1). The second condition in (9) is equivalent to hypothesis I(d). Q.E.D.

III. Reduction to a first order system. In order to solve (9) we reduce it to a first order system as follows. Let us change notations: $(x_0, x_1, \ldots, x_N) = (t, x)$. Multiply both sides of (9) by E^{-1} on the left (see hypothesis I(b)) and rewrite it in the form

$$\left[\frac{\partial^m}{\partial t^m}I_n - \sum_{i=0}^{m-1} A_{m-i}(t, x, D^{\beta(\alpha)}U_{k(\alpha)}, D_x)\frac{\partial^i}{\partial t^i}\right] U = f(t, x, D^{\beta(\alpha)}U_{k(\alpha)}), \tag{12}$$

where $A_{m-i}(t, x, y, D_x)$ is a differential operator of order $m - i$ with top order symbol $\tilde{A}_{m-i}$ and

$$f(t, x, D^{\beta(\alpha)}U_{k(\alpha)}) = E^{-1}B(t, x, D^{\beta(\alpha)}U_{k(\alpha)}).$$

Since the next step in the transformation of equation (12) will involve a pseudodifferential operator of a certain class (see [**7**]), we will give the following

DEFINITION A. Let Ω be an open subset of $\mathbf{R}^N$, $m \in \mathbf{R}$. We define the symbol class $S^m_{1,0}(\Omega)$ to consist of the set of $p(x,\xi) \in C^\infty(\Omega \times \mathbf{R}^N)$ with the property that, for any compact $K \subset \Omega$ and any multi-indices α, β, there exists a constant $C_{K,\alpha,\beta}$ such that

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \leq C_{K,\alpha,\beta}(1+|\xi|)^{m-|\alpha|}$$

for all $x \in K$, $\xi \in \mathbf{R}^N$. In this case the operator $p(x,D)$ is said to belong to $OPS^m_{1,0}(\Omega)$.

If, moreover, there are smooth functions $p_{m-j}(x,\xi)$, homogeneous of degree $m-j$ in ξ for $|\xi| \geq 1$ such that

$$p(x,\xi) \sim \sum_{j \geq 0} p_{m-j}(x,\xi)$$

where the asymptotic condition means that

$$p(x,\xi) - \sum_{j=0}^{M} p_{m-j}(x,\xi) \in S^{m-M-1}_{1,0}(\Omega),$$

then we say that symbol $p(x,\xi) \in S^m(\Omega)$ and the operator $p(x,D) \in OPS^m(\Omega)$.

We can now define the concept of a pseudodifferential operator on a manifold M.

DEFINITION B. $p(x,D)\colon C_0^\infty(M) \to C^\infty(M)$ belongs to $OPS^m_{1,0}(M)$ $(OPS^m(M))$ if $p(x,\xi)$ is smooth on M and if for any coordinate neighborhood U in M with $\chi\colon U \to \mathcal{O}$ a diffeomorphism onto an open subset $\mathcal{O}$ of $\mathbf{R}^N$, the map of $C_0^\infty(\mathcal{O})$ into $C^\infty(\mathcal{O})$ given by $u \mapsto p(x,D)(u \circ \chi) \circ \chi^{-1}$ belongs to $OPS^m_{1,0}(\mathcal{O})$ $(OPS^m(\mathcal{O}))$.

Now let $U^j = (\partial/\partial t)^{j-1}\Lambda^{m-j}U$, $j = 1, \ldots, m$, and $\Lambda = (1-\Delta)^{1/2} \in OPS^1$. Then (12) is equivalent to

$$\frac{\partial}{\partial t}\begin{pmatrix} U^1 \\ \cdot \\ \cdot \\ \cdot \\ U^m \end{pmatrix} = \begin{pmatrix} O_n & \Lambda_n & O_n & \cdots & O_n \\ & O_n & \Lambda_n & & \vdots \\ & & & \ddots & \\ & & & & \Lambda_n \\ (b_1) & (b_2) & (b_3) & \cdots & (b_m) \end{pmatrix}\begin{pmatrix} U^1 \\ \cdot \\ \cdot \\ \cdot \\ U^m \end{pmatrix} + \begin{pmatrix} O_n \\ \cdot \\ \cdot \\ \cdot \\ O_n \\ f \end{pmatrix}, \tag{13}$$

where $b_j = A_{m-j+1}(t,x,P_{j1}(U^1,\ldots,U^m)^t,\ldots,P_{j\nu}(U^1,\ldots,U^m)^t,D_x)\Lambda^{j-m}$ with $P_{j\mu} \in OPS^s$, $s \leq 0$.

Each U^j in (13) is an n-vector, hence

$$\begin{pmatrix} U^1 \\ \cdot \\ \cdot \\ U^m \end{pmatrix}$$

is an mn-vector. Each entry in the matrix of equation (13) is an $n \times n$ matrix, for example, $\Lambda_n = \Lambda I_n$, $O_n = 0I_n$, so (13) is an $(m \cdot n) \times (m \cdot n)$ system. Changing notation, calling the column vector $(U^1,\dots,U^m)^t$ U and the matrix in (13) K, write (13) with the initial data as

$$\begin{aligned}(14)\qquad &\frac{\partial}{\partial t}U = K(t,x,P_1U,\dots,P_lU,D_x)U + f(t,x,P_1U,\dots,P_lU),\\ &U(0)=g,\end{aligned}$$

where

$$g=\begin{pmatrix} g_1\\ \vdots \\ g_m\end{pmatrix} = \begin{pmatrix}\Lambda^{m-1}W_0\\ \vdots\\ W_{m-1}\end{pmatrix}, \quad \text{i.e.}\quad g_j=\Lambda^{m-j}W_{j-1},$$

and the W_j's are given by (8), and where $P_j \in OPS^0$ and $K(t,x,U_1,\dots,U_l,\xi)\in S^1$.

IV. Solution of the first order hyperbolic system. By hypothesis I(c) system (14) is hyperbolic, and in order to solve it we will use an iterative method as follows. Given U on $\mathbf{R}\times\mathbf{T}^N$ with $U(0)=g$, we define $FU=V$ to be the solution to the system

$$\begin{aligned}(15)\qquad &\frac{\partial}{\partial t}V = K(t,x,P_1U,\dots,P_lU,D_x)V + f(t,x,P_1U,\dots,P_lU),\\ &V(0)=g\end{aligned}$$

and find a fixed point of F, i.e. a function U such that $FU=U$ on $(-T,T)\times\mathbf{T}^N$. In order to treat this problem, it will be necessary to introduce pseudodifferential operators with less than C^∞ symbols (see [**7**]) and paradifferential operators (see [**2**]).

DEFINITION 1. We say $p(x,\xi)\in H^MS^m_{1,0}(\mathbf{T}^N)$ and $p(x,D)\in OPH^MS^m_{1,0}(\mathbf{T}^N)$ provided

$$\|D^\alpha_\xi p(x,\xi)\|_{H^M(\mathbf{T}^N)} \le C(1+|\xi|)^{m-|\alpha|} \quad \text{for} |\alpha|\le M.$$

If, morover, there are functions $p_{m-j}(x,\xi)\in H^MS^{m-j}_{1,0}(\mathbf{T}^N)$, homogeneous of degree $m-j$ in ξ for $|\xi|\ge 1$ such that

$$p(x,\xi)\sim\sum_{j\ge 0}p_{m-j}(x,\xi),$$

then we say that $p(x,\xi)\in H^MS^m(\mathbf{T}^N)$ and $p(x,D)\in OPH^MS^m(\mathbf{T}^N)$.

DEFINITION 2. Let $\chi(\theta,\eta)$ be a C^∞ function defined on $\mathbf{R}^N\times\mathbf{R}^N\backslash O$, homogeneous of degree 0, such that for some small $\varepsilon_1,\varepsilon_2$, $0\le\varepsilon_1<\varepsilon_2$, $\chi(\theta,\eta)=1$ for $|\theta|\le\varepsilon_1|\eta|$, $\chi(\theta,\eta)=0$ for $|\theta|\ge\varepsilon_2|\eta|$, and let $s(\eta)$ be a C^∞ function from $\mathbf{R}^N$ to $\mathbf{R}$ which is 0 in some neighborhood of O and 1 outside of some compact subset of $\mathbf{R}^N$. Let $l(x,\xi)$ be a function homogeneous of degree m in ξ, C^∞ in ξ for $\xi\ne 0$, with compact support with respect to x, and C^ρ in x for ρ noninteger. Define the operator T_l on the space of distributions, $\mathcal{D}'$, as

$$(T_lu)^\frown(\xi)=\int\chi(\xi-\eta,\eta)\hat{l}(\xi-\eta,\eta)s(\eta)\hat{u}(\eta)\,d\eta,$$

where $\hat{l}(\theta,\xi)$ is the Fourier transform of $l(x,\xi)$ with respect to the first variable.

DEFINITION 3. (a) For any $\Omega \subset \mathbf{R}^N$, $m \in \mathbf{R}$, $\rho > 0$ and noninteger, we define $\Sigma_\rho^m(\Omega)$ as the set of all functions $l(x,\xi)$ in $\Omega \times (\mathbf{R}^N \backslash O)$ such that

$$l(x,\xi) = l_m(x,\xi) + l_{m-1}(x,\xi) + \cdots + l_{m-[\rho]}(x,\xi),$$

where $l_{m-k}(x,\xi)$ is homogeneous of degree $m-k$ in ξ, is C^∞ in ξ, and $C_{\mathrm{loc}}^{\rho-k}$ in x.

(b) If $l^{(i)} \in \Sigma_\rho^{m_i}$, $i = 1, 2$, we define $l^1 \# l^2 \in \Sigma_\rho^{m_1+m_2}$ as

$$l^1 \# l^2 = \sum_{|\alpha|+k_1+k_2 \leq [\rho]} \sum \sum \frac{1}{\alpha!} D_\xi^\alpha l^1_{m_1-k_1} D_x^\alpha l^2_{m_2-k_2}.$$

(c) If $l \in \Sigma_\rho^m$, we define $l^* \in \Sigma_\rho^m$ as

$$l^* = \sum_{|\alpha|+k \leq [\rho]} \sum \frac{1}{\alpha!} D_\xi^\alpha D_x^\alpha \bar{l}_{m-k}.$$

DEFINITION 4. Let Ω be an open subset of $\mathbf{R}^N$ and let L be a linear transformation in $\mathcal{D}'(\Omega)$, which is properly supported, i.e. for any compact K in Ω there is a compact $\hat{K}$ in Ω such that

$$\operatorname{supp} u \subset K \Rightarrow \operatorname{supp} Lu \subset \hat{K}$$

and

$$(\operatorname{supp} u) \cap \hat{K} = \varnothing \Rightarrow \operatorname{supp} Lu \cap K = \varnothing.$$

Then we call L a paradifferential operator of order m and of class C^ρ in Ω and write $L \in OP(\Sigma_\rho^m)(\Omega)$ if there exists $l \in \Sigma_\rho^m(\Omega)$ such that for any compact $K \subset \Omega$ and any $\chi \in C_0^\infty(\Omega)$ which is equal to 1 in a neighborhood of K, the operator $L - \chi T_{\chi \cdot l}$ is a continuous map of elements of H^s with support in K to $H^{s-m+\rho}$. In this case l is called a symbol of L.

The proofs of the following results are given in [**2**].

THEOREM 1. (a) *If* $L \in OP(\Sigma_\rho^m)(\Omega)$, *then* $L\colon H_{\mathrm{loc}}^s(\Omega) \to H_{\mathrm{loc}}^{s-m}(\Omega)$.

(b) *If* $L \in OP(\Sigma_\rho^m)(\Omega)$, *there exists a unique symbol* l *of* L, *i.e.* $\sigma(L) = l$.

(c) *If* $L^j \in OP(\Sigma_\rho^{m_j})(\Omega)$, $j = 1, 2$, *then* $L^1 L^2 \in OP(\Sigma_\rho^{m_1+m_2})(\Omega)$ *and* $\sigma(L^1 \cdot L^2) = \sigma(L^1) \# \sigma(L^2)$.

(d) *If* $L \in OP(\Sigma_\rho^m)(\Omega)$, *then* $\sigma(L^*) = (\sigma(L))^*$, *where* L^* *is the adjoint of* L.

(e) *If* L *is a classical pseudodifferential operator of order* m, *properly supported in* Ω, *with symbol*

$$l(x,\xi) \sim \sum_{j \geq 0} l_{m-j}(x,\xi),$$

then for any $\rho > 0$, $L \in OP(\Sigma_\rho^m)$ *with symbol* $\sigma(L) = \sum_{0 \leq j \leq [\rho]} l_{m-j}$.

NOTE. If $l(x,\xi)$ is such that $l_{m-j}(x,\xi)$ is in $C^{\gamma-j}$ as a function of x, then $L \in OP(\Sigma_\gamma^m)$ and $\sigma(L) = \sum_{0 \leq j \leq [\gamma]} l_{m-j}$ where γ is not an integer.

To return to our iterative method (15), we assume that $U \in C([-T,T], H^M(\mathbf{T}^N))$ and $\partial U/\partial t \in C([-T,T], H^M(\mathbf{T}^N))$ for M to be estimated. Then

$$K(t,x,P_1U,\ldots,P_lU,D_x) \in OPH^M S_{1,0}^1$$

and in order to solve the quasilinear hyperbolic system

$$\frac{\partial}{\partial t} V = KV + f, \quad V(0) = g \tag{16}$$

with $f \in C([-T,T], H^M(\mathbf{T}^N))$, $g \in H^M(\mathbf{T}^N)$, we need to construct what is called a symmetrizer for K.

DEFINITION 5. Let $K \in OPS^1_{1,0}$. Then a symmetrizer for $(\partial/\partial t) - K$ is a smooth one parameter family of operators $R = R(t) \in OPS^0$, such that

$$R_0(t,x,\xi) \text{ is a positive definite matrix for } |\xi| \geq 1, \tag{17}$$

$$RK + (RK)^* \in OPS^0_{1,0}. \tag{18}$$

If such a symmetrizer exists, one says $\partial/\partial t - K$ is symmetrizable.

The proof of the following proposition can be found in [**2**].

PROPOSITION 1. *Any strictly hyperbolic first order system $\partial/\partial t - K$, where $K \in OPS^1_{1,0}$, has a symmetrizer R and we have*

$$R(t,x,D_x) = \sum_{j=1}^{k} P_j(t,x,D)^* P_j(t,x,D), \tag{19}$$

$$\sigma_{RK}(t,x,\xi) = i \sum_{j=1}^{k} \lambda_j(t,x,\xi) P_j(t,x,\xi)^* P_j(t,x,\xi) \mod S^0, \tag{20}$$

where $i\lambda_\nu(t,x,\xi)$ are eigenvalues of $K_1(t,x,\xi)$, the principal symbol of K, $\lambda_1(t,x,\xi) < \lambda_2(t,x,\xi) < \cdots < \lambda_k(t,x,\xi)$, and $P_j(t,x,\xi) \in S^0$ are the projections onto the associated eigenspaces of $i\lambda_j(t,x,\xi)$:

$$P_j = \frac{1}{2\pi i} \int_{\gamma_j} (\varsigma - K_1(t,x,\xi))^{-1}\, d\varsigma$$

where γ_j is the circle around λ_j only.

Now we are ready to prove the following result.

PROPOSITION 2. *If $M > N/2 + 1$, then, given $K \in OPH^M S^1(\mathbf{T}^N)$, the hyperbolic system*

$$\partial V/\partial t = KV + f, \quad V(0) = g \tag{21}$$

has a unique solution $V \in C([-T,T], H^M(\mathbf{T}^N))$ for $g \in H^M(\mathbf{T}^N)$ and $f \in C([-T,T], H^M(\mathbf{T}^N))$. Such a solution satisfies the estimate

$$||V(t)||^2_{H^M} \leq (1 + C|t|) \left[||g||^2_{H^M} + \int_0^t ||f(\tau)||^2_{H^M}\, d\tau \right], \qquad |t| \leq T,$$

where C depends on finitely many seminorms of $K \in OPH^M S^1(\mathbf{T}^N)$, $R \in OPH^M S^0(\mathbf{T}^N)$, $\partial R/\partial t \in OPH^M S^0(\mathbf{T}^N)$, and $RK + (RK)^* \in OPH^L S^0(\mathbf{T}^N)$ for some $L < M$, and on N but not on the order of the system.

The following inequality from the theory of ordinary differential equations will be useful in the proof of Proposition 2.

LEMMA 2 (GRONWALL'S INEQUALITY). *If $y \in C^1$ and $y^1(t) + f(t)y \leq g(t)$, then*

$$y(t) \leq e^{-\int_0^t f(\tau)\,d\tau} \left[y_0 + \int_0^t g(\tau) e^{\int_0^\tau f(\sigma)\,d\sigma}\,d\tau \right].$$

PROOF OF PROPOSITION 2. Since $K \in OPH^M S^1(\mathbf{T}^N)$, it is clear from (19) that the symmetrizer $R \in OPH^M S^0(\mathbf{T}^N)$. Then we have the following asymptotic expansions of the symbols $K(t,x,\xi) \in H^M S^1$ and $R(t,x,\xi) \in H^M S^0$:

$$K(t,x,\xi) \sim \sum_{j\geq 0} K_{1-j}(t,x,\xi), \quad R(t,x,\xi) \sim \sum_{j \geq 0} R_{-j}(t,x,\xi)$$

where $K_{1-j}(t,x,\xi)$ and $R_{-j}(t,x,\xi)$ are homogeneous functions in ξ of degree $1-j$ and $-j$ respectively and are in $H^M(\mathbf{T}^N)$ as functions of x. Then, by the Sobolev imbedding theorem, K_{1-j} and R_{-j} belong to $C^{M-N/2}(\mathbf{T}^N)$ and we can apply Theorem 1 to conclude that

$$K(t,x,D_x) \in OP(\Sigma^1_{M-N/2})$$

with $\sigma(K) = \sum_{0\leq j \leq [M-N/2]} K_{1-j}(t,x,\xi)$ and

$$R(t,x,D_x) \in OP(\Sigma^0_{M-N/2})$$

with $\sigma(R) = \sum_{0\leq j\leq [M-N/2]} R_{-j}(t,x,\xi)$ for $M - N/2$ noninteger.

NOTE. In the case when $M - N/2$ is an integer we can find some small $\varepsilon > 0$, such that $M - N/2 - \varepsilon > 1$ (since by hypothesis $M - N/2 > 1$) and replace $M - N/2$ by noninteger $M - N/2 - \varepsilon$ for all symbols above.

It follows that $RK \in OP(\Sigma^1_{M-N/2})$ with

$$\sigma(RK) = R_0 K_1 + \sum\sum\sum_{1\leq|\alpha|+j_1+j_2\leq[M-N/2]} \frac{1}{\alpha!} D_\xi^\alpha R_{-j_1} D_x^\alpha K_{1-j_2}$$

where for each $|\alpha|$, j_1, and j_2 the corresponding term is homogeneous of degree $1 - (|\alpha| + j_1 + j_2)$ in ξ and belongs to $C^{[M-N/2]-(|\alpha|+j_1+j_2)}(\mathbf{T}^N)$ in x. Hence

$$\begin{aligned}\sigma((RK)^*) &= \sum\sum\sum\sum_{|\beta|+|\alpha|+j_1+j_2\leq[M-N/2]} \frac{1}{\alpha!\beta!} D_\xi^\beta D_x^\beta \overline{D_\xi^\alpha R_{-j_1} D_x^\alpha K_{1-j_2}} \\ &= R_0\overline{K}_1 + \sum\sum\sum\sum_{1\leq|\beta|+|\alpha|+j_1+j_2\leq[M-N/2]} \frac{1}{\alpha!\beta!} D_\xi^\beta D_x^\beta D_\xi^\alpha R_{-j_1} D_x^\alpha \overline{K}_{1-j_2}\end{aligned}$$

and since $K_1(t,x,\xi)$ has purely imaginary distinct eigenvalues for each (t,x,ξ), we have

(22)

$$\begin{aligned}\sigma(RK + (RK)^*) &= \sigma(RK) + \sigma((RK)^*) \\ &= \sum\sum\sum_{1\leq|\alpha|+j_1+j_2\leq[M-N/2]} \frac{1}{\alpha!} D_\xi^\alpha R_{-j_1} D_x^\alpha K_{1-j_2} \\ &\quad + \sum\sum\sum\sum_{1\leq|\beta|+|\alpha|+j_1+j_2\leq[M-N/2]} \frac{1}{\alpha!\beta!} D_\xi^\beta D_x^\beta D_\xi^\alpha R_{-j_1} D_x^\alpha \overline{K}_{1-j_2}.\end{aligned}$$

Thus $RK + (RK)^* \in OP(\Sigma^0_{M-N/2-1})$, which justifies the hypothesis that $M > N/2 + 1$. Applying Theorem 1 again, we conclude that

$$\begin{aligned}
&RK + (RK)^*\colon H^s(\mathbf{T}^N) \to H^s(\mathbf{T}^N) \text{ as well as} \\
&R\colon H^s(\mathbf{T}^N) \to H^s(\mathbf{T}^N), \\
&\frac{\partial}{\partial t}R\colon H^s(\mathbf{T}^N) \to H^s(\mathbf{T}^N), \text{ and} \\
&K\colon H^s(\mathbf{T}^N) \to H^{s-1}(\mathbf{T}^N) \quad \text{for any } s \in \mathbf{R}.
\end{aligned} \tag{23}$$

Now we can obtain the energy estimate for V solving (21). We write

$$\begin{aligned}
\frac{d}{dt}(RV, V) &= (RV_t, V) + (RV, V_t) + (R_tV, V) \\
&= (RKV + Rf, V) + (RV, KV + f) + (R_tV, V) \\
&= ((RK + K^*R)V, V) + (Rf, V) + (RV, f) + (R_tV, V) \\
&\le C_1\|V\|^2_{L^2} + C_2\|f\|^2_{L^2} \le C(RV, V) + C\|f\|^2_{L^2}.
\end{aligned} \tag{24}$$

Applying Gronwall's inequality (Lemma 2) to this yields

$$\|V(t)\|^2_{L^2} \le (1 + C|t|)\left[\|g\|^2_{L^2} + \int_0^t \|f(\tau)\|^2_{L^2}\, d\tau\right] \tag{25}$$

for $t \in [-T, T]$.

More generally, differentiating

$$\|V(t)\|^2_{H^s(\mathbf{T}^N)} = \|\Lambda^s V(t)\|^2_{L^2(\mathbf{T}^N)}$$

yields

$$\|V(t)\|^2_{H^s} \le (1 + C|t|)\left[\|g\|^2_{H^s} + \int_0^t \|f(\tau)\|^2_{H^s}\, d\tau\right] \tag{26}$$

for $t \in [-T, T]$, which is bounded in a bounded interval $-T \le t \le T$. C is a function of finitely many seminorms of $K \in OPH^MS^1$, $R \in OPH^MS^0$, $\partial R/\partial t \in OPH^MS^0$, and $RK + (RK)^* \in OPH^LS^0$ for some $L < M$ which we need not specify, and of N. Inequality (26) is valid whenever $u \in H^1([-T, T], H^{s+1}(\mathbf{T}^N))$, $f \in L^2([-T, T], H^s(\mathbf{T}^N))$, $g \in H^{s+1}(\mathbf{T}^N)$, and (21) is satisfied. We shall obtain the solution of the initial value problem (21) as a limit of solutions to the problem

$$\frac{\partial}{\partial t}V = KJ_\varepsilon V + f, \quad V(0) = g \tag{27}$$

where J_ε is a Friedrich's mollifier on $\mathbf{T}^N$ defined as the following:

DEFINITION 6. A Friedrich's mollifier on $\mathbf{T}^N$ is a family, J_ε, of scalar pseudodifferential operators, $0 \le \varepsilon \le 1$, such that

$$\begin{aligned}
&J_\varepsilon \in OPS^{-\infty}(\mathbf{T}^N) \text{ for each } \varepsilon \in (0, 1], \\
&\{J_\varepsilon : 0 < \varepsilon \le 1\} \text{ is a bounded subset of } OPS^0_{1,0}(\mathbf{T}^N), \text{ and} \\
&J_\varepsilon u \to u \text{ in } L^2(\mathbf{T}^N) \text{ as } \varepsilon \to 0 \text{ for each } u \in L^2(\mathbf{T}^N).
\end{aligned}$$

To construct a Friedrich's mollifier, we multiply $\varepsilon^{-|\varepsilon\xi|^2}$ by a partition of unity for some coordinate patches on $\mathbf{T}^N$.

Now for each $\varepsilon > 0$, $K_\varepsilon = KJ_\varepsilon$ is a continuous linear operator on H^s. Hence (27) can be considered as a Banach-space valued ordinary differential equation and to solve it we apply the Picard iteration method (see Dieudonné [3]). Thus, given $g \in H^{s+1}(\mathbf{T}^N)$ and $f \in C([-T,T], H^{s+1}(\mathbf{T}^N))$, we can solve (27), producing a solution $V_\varepsilon \in C^1([-T,T], H^{s+1}(\mathbf{T}^N))$. Now since $\{K_\varepsilon : 0 < \varepsilon \le 1\}$ is a bounded subset of $OPH^M S^1_{1,0}(\mathbf{T}^N)$ and $\{RK_\varepsilon + (RK_\varepsilon)^* : 0 < \varepsilon \le 1\}$ is a bounded subset of $OPH^L S^0_{1,0}(\mathbf{T}^N)$ for some $L < M$, we get the estimate (26) for V_ε:

$$||V_\varepsilon||^2_{H^s} \le (1 + C|t|)\left[||g||^2_{H^s} + \int_0^t ||f(\tau)||^2_{H^s}\, d\tau\right] \tag{28}$$

with C independent of ε, $0 < \varepsilon \le 1$. Now by this estimate, $\{V_\varepsilon : 0 < \varepsilon \le 1\}$ is a bounded subset of $C([-T,T], H^s(\mathbf{T}^N))$ given $g \in H^{s+1}(\mathbf{T}^N)$ and $f \in C([-T,T], H^{s+1}(\mathbf{T}^N))$. Since $V'_\varepsilon = K_\varepsilon V_\varepsilon + f$ and $\{K_\varepsilon : 0 < \varepsilon \le 1\}$ is a bounded subset of $OPH^M S^1_{1,0}(\mathbf{T}^N)$, it follows that $\{V'_\varepsilon : 0 < \varepsilon \le 1\}$ is a bounded subset of $C([-T,T], H^{s-1}(\mathbf{T}^N))$. Hence $\{V_\varepsilon : 0 < \varepsilon \le 1\}$ is a bounded subset of $C^1([-T,T], H^{s-1}(\mathbf{T}^N))$. Furthermore, for each $t_0 \in [-T,T]$, $\{V_\varepsilon(t_0) : 0 < \varepsilon \le 1\}$, being a bounded subset of $H^s(\mathbf{T}^N)$, is a relatively compact subset of $H^{s-1}(\mathbf{T}^N)$. Hence, by Ascoli's theorem [3], there is a sequence $\varepsilon_n \to 0$ such that V_{ε_n} converges in $C([-T,T], H^{s-1}(\mathbf{T}^N))$ to a limit we call V, which satisfies (21) in the sense of distributions.

Now let $g_j \in H^{s+4}(\mathbf{T}^N)$ with $g_j \to g$ in $H^s(\mathbf{T}^N)$ and let

$$f_j \in C([-T,T], H^{s+4}(\mathbf{T}^N)),$$

with $f_j \to f$ in $C([-T,T], H^s(\mathbf{T}^N))$. The argument above also produces solutions V_j to (21) with g, f replaced by g_j, f_j with $V_j \in C([-T,T], H^{s+2}(\mathbf{T}^N))$. Since $V'_j = KV_j + f_j$, it follows that $V_j \in C^1([-T,T], H^{s+1}(\mathbf{T}^N))$. Hence we can apply the energy inequality (26) and conclude that $\{V_j\}$ is a Cauchy sequence in $C([-T,T], H^s(\mathbf{T}^N))$. The limit V solves our system. As for uniqueness, since any $V \in C([-T,T], H^s(\mathbf{T}^N))$, solving (21) must belong to $C^1([-T,T], H^{s-1}(\mathbf{T}^N)) \subset H^1([-T,T], H^{s-1}(\mathbf{T}^N))$, the energy inequality (26) with s replaced by $s-2$ applies for a difference of two solutions and we see that the solution is unique. Q.E.D. (for Proposition 2).

To return to our iterative method (15), we suppose $g \in H^{M+1}(\mathbf{T}^N)$, $U \in C([-T,T], H^M(\mathbf{T}^N))$, and $\partial U/\partial t \in C([-T,T], H^{M-1}(\mathbf{T}^N))$, where $M > N/2 + 1$. Then

$$K(t, x, P_1 U, \ldots, P_l U, D_x) \in OPH^M S^1,$$

$f(t, x, P_1 U, \ldots, P_l U) \in C([-T,T], H^M(\mathbf{T}^N))$, and by Proposition 2 system (15) has a unique solution

$$V(t,x) \in C([-T,T], H^M(\mathbf{T}^N)),$$

such that $(\partial/\partial t)V(t,x) \in C([-T,T],\ H^{M-1}(\mathbf{T}^N))$. In order to prove convergence of the iterative method (15), we will construct equations for various derivatives of V. Set

$$V_{0\alpha} = D^\alpha_x V, \quad V_{1\alpha} = \frac{\partial}{\partial t} D^\alpha_x V, \quad U_{0\alpha} = D^\alpha_x U, \quad U_{1\alpha} = \frac{\partial}{\partial t} D^\alpha_x U.$$

Applying the chain rule to (15) yields

$$(29)\quad \frac{\partial}{\partial t}V_{0\alpha} = K(t,x,P_1U_{00},\dots,P_lU_{00},D_x)V_{0\alpha} + \sum_{\substack{\gamma+\sigma+\delta=\alpha,\sigma<\alpha\\ \delta=\delta_1^1+\cdots+\delta_{\mu_1}^1+\cdots+\delta_1^l+\cdots+\delta_{\mu_l}^l}} C_{\sigma\gamma\mu_1\dots\mu_l\delta_1^1\dots\delta_{\mu_1}^1\dots\delta_1^l\dots\delta_{\mu_l}^l} \cdot P_1U_{0\delta_1^1}\cdots P_1U_{0\delta_{\mu_1}^1}\cdots P_lU_{0\delta_1^l}\cdots P_lU_{0\delta_{\mu_l}^l}K_{\gamma\mu_1\dots\mu_l}V_{0\sigma} + \sum_{\substack{\gamma+\delta=\alpha\\ \delta=\delta_1^1+\cdots+\delta_{\mu_1}^1+\cdots+\delta_1^l+\cdots+\delta_{\mu_l}^l}} C'_{\sigma\gamma\mu_1\dots\mu_l\delta_1^1\dots\delta_{\mu_1}^1\dots\delta_1^l\dots\delta_{\mu_l}^l} \cdot P_1U_{0\delta_1^1}\cdots P_1U_{0\delta_{\mu_1}^1}\cdots P_lU_{0\delta_1^l}\cdots P_lU_{0\delta_{\mu_l}^l}f_{\gamma\mu_1\dots\mu_l}$$

and,

(30)

$$\begin{aligned}
\frac{\partial}{\partial t}V_{1\beta} &= K(t,x,P_1U_{00},\dots,P_lU_{00},D_x)V_{1\beta}\\
&+\sum_{i=1}^{l}P_iU_{10}K_iV_{0\beta}+K_tV_{0\beta}\\
&+\sum_{\gamma+\delta+\sigma=\beta} C_{\sigma\gamma\mu_1\dots\mu_l\delta_1^1\dots\delta_{\mu_1}^1\dots\delta_1^l\dots\delta_{\mu_l}^l}\\
&\cdot\Big[(P_1U_{1\delta_1^1}P_1U_{0\delta_2^1}\cdots P_1U_{0\delta_{\mu_1}^1}\cdots P_lU_{0\delta_1^l}\cdots P_lU_{0\delta_{\mu_l}^l}\\
&\quad+\cdots+P_1U_{0\delta_1^1}\cdots P_lU_{0\delta_{\mu_l-1}^l}P_lU_{1\delta_{\mu_l}^l})K_{\gamma\mu_1\cdots\mu_l}V_{0\sigma}\\
&\quad+P_1U_{0\delta_1^1}\cdots P_lU_{0\delta_{\mu_l}^l}\left(\sum_{i=1}^{l}P_iU_{10}K_{\gamma\mu_1\cdots\mu_i+1\cdots\mu_l}+K_{\gamma\mu_1\cdots\mu_l t}\right)V_{0\sigma}\\
&\qquad+P_1U_{0\delta_1^1}\cdots P_lU_{0\delta_{\mu_l}^l}K_{\gamma\mu_1\cdots\mu_l}V_{1\sigma}\Big]\\
&+\sum_{\gamma+\delta=\beta} C'_{\sigma\gamma\mu_1\cdots\mu_l\delta_1^1\cdots\delta_{\mu_1}^1\cdots\delta_1^l\cdots\delta_{\mu_l}^l}\\
&\cdot\Big[(P_lU_{1\delta_1^1}P_1U_{0\delta_2^1}\cdots P_1U_{0\delta_{\mu_1}^1}\cdots P_lU_{0\delta_1^l}\cdots P_lU_{0\delta_{\mu_l}^l}\\
&\quad+\cdots+P_1U_{0\delta_1^1}\cdots P_lU_{0\delta_{\mu_l-1}^l}P_lU_{1\delta_{\mu_L}^l})\\
&\cdot f_{\gamma\mu_1\cdots\mu_l}(t,x,P_1U,\dots,P_lU)\\
&\quad+P_1U_{0\delta_1^1}\cdots P_1U_{0\delta_{\mu_1}^1}\cdots P_lU_{0\delta_1^l}\cdots P_lU_{0\delta_{\mu_l}^l}\\
&\qquad\cdot\left(\sum_{i=1}^{l}P_iU_{10}f_{\gamma\mu_1\dots\mu_i+1\dots\mu_l}+f_{\gamma\mu_1\dots\mu_l t}\right)\Big].
\end{aligned}$$

Here

$$K_{\gamma\mu_1\dots\mu_l} = K_{\gamma\mu_1\dots\mu_l}(t,x,\phi_1,\dots,\phi_l) = D_x^\gamma D_{\phi_1}^{\mu_1}\dots D_{\phi_l}^{\mu_l} K(t,x,\phi_1,\dots,\phi_l),$$
$$K_i = K_i(t,x,\phi_1,\dots,\phi_l) = D_{\phi_i} K(t,x,\phi_1,\dots,\phi_l), \qquad i=1,\dots,l,$$

and

$$K_{\gamma\mu_1\dots\mu_l t} = D_t K_{\gamma\mu_1\dots\mu_l}.$$

$f_{\gamma\mu_1\dots\mu_l}$ and $f_{\gamma\mu_1\dots\mu_l t}$ are defined similarly.

Now we replace $U_{j\sigma}$ and $V_{j\sigma}$ by $P_{j\sigma}(\tilde U)$ and $P_{j\sigma}(\tilde V)$ respectively where

$$\tilde U = \{U_{0\alpha}, U_{1\beta}\colon 0\le |\alpha|\le M, 0\le|\beta|\le M-1\},$$
$$P_{j\sigma}(\tilde U) = \Lambda^{-(M-j-|\sigma|)} \sum_{|\beta|=M-j} C_{\alpha\beta}(x,D_x) U_{j\beta}. \tag{31}$$

$\tilde V$ and $P_{j\sigma}(\tilde V)$ are defined similarly. After these replacements system (29), (30) for $\tilde V$ becomes

$$\frac{\partial}{\partial t}\tilde V = K(t,x,P_1P_{00}\tilde U,\dots,P_lP_{00}\tilde U, D_x)\tilde V + \Phi(t,x,P_1P_{00}\tilde U,\dots,P_lP_{00}\tilde U,\tilde V). \tag{32}$$

Here

$$K(t,x,P_1P_{00}\tilde U,\dots,P_lP_{00}\tilde U,D_x)\in OPH^MS^1$$

and

$$\Phi(t,x,P_1P_{00}\tilde U,\dots,P_lP_{00}\tilde U,D_x)$$

contain terms of order ≤ 0.

Now since $U,V\in \mathbf{R}^{mn}$, it follows that $\tilde U,\tilde V\in\mathbf{R}^K$, where

$$K = mn\left\{\binom{M+N}{N} + \binom{M+N-1}{N}\right\}$$

and we have the following.

LEMMA 3. *Assuming $M > N/2+1$, we find that*

$$\Phi\colon \mathbf{R}\times\mathbf{T}^N\times[L^2(\mathbf{T}^N)]^{2mn\{\binom{M+N}{N}+\binom{M+N-1}{N}\}} \to [L^2(\mathbf{T}^N)]^{mn\{\binom{M+N}{N}+\binom{M+N-1}{N}\}} \tag{33}$$

is a Lipschitz continuous map.

PROOF. Consider the components of Φ of the form

$$\psi_1 = \prod_{i=1}^{l}\prod_{\mu=1}^{\mu_i} P_iP_{0\delta_\mu^i}\tilde U K_{\gamma\mu_1\dots\mu_l}(t,x,P_1P_{00}\tilde U,\dots,P_lP_{00}\tilde U,D_x)P_{0\sigma}\tilde V$$

and

$$\psi_2 = \prod_{i=1}^{l}\prod_{\mu=1}^{\mu_i} P_iP_{0\delta_\mu^i}\tilde U f_{\gamma\mu_1\dots\mu_l}(t,x,P_1P_{00}\tilde U,\dots,P_lP_{00}\tilde U),$$

where

$$\left|\gamma+\sigma+\sum_{i=1}^{l}\sum_{\mu=1}^{\mu_i}\delta_\mu^i\right|\le M, \qquad |\sigma|<M.$$

The map $\tilde{U} \to K_{\gamma\mu_1\dots\mu_l}(t,x,P_1P_{00}\tilde{U},\dots,P_lP_{00}\tilde{U},D_x)$ is a Lipschitz map of $L^2(\mathbf{T}^N)$ into OPH^MS^1 since $K_{\gamma\mu_1\dots\mu_l}(t,x,P_1P_{00}\tilde{U},\dots,P_lP_{00}\tilde{U},D_x)$ is a C^∞ function of its arguments. Consequently, according to (23),

$$K_{\gamma\mu_1\dots\mu_l}(t,x,P_1P_{00}\tilde{U},\dots,P_lP_{00}\tilde{U},D_x)\colon H^s(\mathbf{T}^N) \to H^{s-1}(\mathbf{T}^N)$$

for any $s \in \mathbf{R}$. Since $P_{0\sigma}\tilde{V} \in H^{M-|\sigma|}(\mathbf{T}^N)$ and $P_{0\delta^i_\mu}\tilde{U} \in M^{M-|\delta^i_\mu|}(\mathbf{T}^N)$, we conclude that $\psi_1 \in L^2(\mathbf{T}^N)$ and, being multilinear, is a Lipschitz function of its arguments. Similarly, the map $\tilde{U} \to f(t,x,P_1P_{00}\tilde{U},\dots,P_lP_{00}\tilde{U})$ is a Lipschitz map of $L^2(\mathbf{T}^N)$ into itself since $f(t,x,P_1P_{00}\tilde{U}(x),\dots,P_lP_{00}\tilde{U}(x)) \in H^M(\mathbf{T}^N) \subset L^2(\mathbf{T}^N)$. Hence, $\psi_2 \in L^2(\mathbf{T}^N)$ and is a Lipschitz function of its arguments. A similar argument controls all the other terms of Φ. Q.E.D.

Next, we construct a positive definite symmetrizer $R(t,x,w_1,\dots,w_l,D_x)$ for $K(t,x,w_1,\dots,w_l,D_x)$ and substitute $P_iP_{00}\tilde{U}$ for w_i, $i=1,\dots,l$, which gives us $R(t,x,P_1P_{00}\tilde{U},\dots,P_lP_{00}\tilde{U},D_x) \in OPH^MS^0$. Now we can write

$$\begin{aligned}(34)\qquad \frac{d}{dt}(R\tilde{V},\tilde{V}) &= (R(K\tilde{V}+\Phi),\tilde{V}) + (R\tilde{V},K\tilde{V}+\Phi) + \left(\left(\frac{\partial}{\partial t}R\right)\tilde{V},\tilde{V}\right)\\ &= ((RK+K^*R)\tilde{V},\tilde{V}) + (R\Phi,\tilde{V}) + (R\tilde{V},\Phi)\\ &\quad + \left(\left(R_t + \sum_{i=1}^{l} P_iP_{10}\tilde{U}R_{w_i}\right)\tilde{V},\tilde{V}\right).\end{aligned}$$

By the argument (23) the last three terms are bounded by

$$B(\|\tilde{U}\|_{L^2})\|\tilde{V}\|^2_{L^2} + B(\|\tilde{U}\|_{L^2})$$

for $t \in [-T,T]$, where B is some function of its arguments. By the same argument and since

$$\tilde{U} \to (RK+(RK)^*)(t,x,P_1P_{00}\tilde{U},\dots,P_lP_{00}\tilde{U},D_x)$$

is a Lipschitz map of $L^2(\mathbf{T}^N)$ into OPH^LS^0 for some $L<M$, we obtain the bound

$$((RK+K^*R)\tilde{V},\tilde{V}) \le B(\|\tilde{U}\|_{L^2})\|\tilde{V}\|^2_{L^2}.$$

Consequently, (34) becomes

$$\frac{d}{dt}(R\tilde{V},\tilde{V}) \le B'(\|\tilde{U}\|_{L^2})(R\tilde{V},\tilde{V}) + B'(\|\tilde{U}\|_{L^2})$$

and Gronwall's inequality yields

$$(35)\qquad \|\tilde{V}(t)\|^2_{L^2} \le e^{\int_0^t B'(\|\tilde{U}(\tau)\|_{L^2})\,d\tau}\left[\|\tilde{V}(0)\|^2_{L^2} + \int_0^t B'(\|\tilde{U}(\tau)\|_{L^2})\,d\tau\right]$$

where $\tilde{V}(0)$ is defined similar to (31) with

$$V_{0\alpha}(0) = D_x^\alpha g,\quad V_{1\beta}(0) = D_x^\beta(K(t,g)g) + D_x^\beta f,\qquad |\alpha| \le M,\ |\beta| \le M-1.$$

Note that $V_{0\alpha}(0) \in L^2(\mathbf{T}^N)$ since $g \in H^M(\mathbf{T}^N)$; and $V_{1\beta}(0) \in L^2(\mathbf{T}^N)$ since $f \in H^M(\mathbf{T}^N)$ and by the argument used in the proof of Lemma 3. So, we can rewrite (35) as

$$(36)\qquad \|\tilde{V}(\tau)\|_{L^2} \le e^{\int_0^t B'(\|\tilde{U}(\tau)\|_{L^2})\,d\tau}\left[C_0^2\|g\|^2_{H^M} + \int_0^t B'(\|\tilde{U}(t)\|_{L^2})\,d\tau\right].$$

Since by assumption $U \in C([-T,T],\ H^M(\mathbf{T}^N)) \cap C^1([-T,T], H^{M-1}(\mathbf{T}^N))$, then $\tilde{U} \in C([-T,T], L^2(\mathbf{T}^N))$. Suppose the norm of $\tilde{U}$ in this space is $\leq A_0$ where we pick $A_0 \geq 2C_0||g||_{H^M} + 1$. Pick the t interval fairly small, as follows. Suppose $T \leq T_0$ where T_0 is so small that, with $B_1 = \sup\{|B'(\lambda)|: |\lambda| \leq A_0\}$,

$$e^{T_0 B_1}[C_0^2||g||^2_{H^M} + T_0 B_1] \leq A_0^2. \tag{37}$$

It follows from (36) that, under this assumption, $||\tilde{V}||_{L^2} \leq A_0$, $|t| \leq T_0$. Consequently, for such a small t interval, the mapping $\tilde{V} = \tilde{F}\tilde{U}$ arising from (15) maps the set

$$\{\tilde{U} \in C([-T,T], L^2(\mathbf{T}^N)): ||\tilde{U}||_{L^2} \leq A_0\}$$

into itself.

To check the convergence of $V_K = F^K U$, we need to estimate the difference between $\tilde{V} = \tilde{F}(\tilde{U})$ and $\tilde{V}_1 = \tilde{F}(\tilde{U}_1)$. From (32) we get

$$\begin{aligned} \frac{\partial}{\partial t}(\tilde{V} - \tilde{V}_1) &= K(t,x,P_1P_{00}\tilde{U},\dots,P_lP_{00}\tilde{U},D_x)\tilde{V} \\ &\quad - K(t,x,P_1P_{00}\tilde{U}_1,\dots,P_lP_{00}\tilde{U}_1,D_x)\tilde{V}_1 \\ &\quad + \Phi(t,x,\tilde{U},\tilde{V}) - \Phi(t,x,\tilde{U}_1,\tilde{V}_1) \\ &= K(t,x,P_1P_{00}\tilde{U},\dots,P_lP_{00}\tilde{U},D_x)(\tilde{V} - \tilde{V}_1) \\ &\quad + [K(t,x,P_iP_{00}\tilde{U},D_x) - K(t,x,P_iP_{00}\tilde{U}_1,D_x)]\tilde{V}_1 \\ &\quad + \Phi(t,x,\tilde{U},\tilde{V}) - \Phi(t,x,\tilde{U}_1,\tilde{V}_1) \\ &= K(t,x,P_iP_{00}\tilde{U},D_x)(\tilde{V} - \tilde{V}_1) + \Delta \end{aligned} \tag{38}$$

where, with the aid of Lemma 3, we have

(39)
$$||\Delta||_{L^2(\mathbf{T}^N)} \leq C||\tilde{U}-\tilde{U}_1||_{L^2(\mathbf{T}^N)}[||\tilde{V}_1||_{H^1(\mathbf{T}^N)}+C_1]+C_2[||\tilde{V}||_{L^2(\mathbf{T}^N)}+||\tilde{V}_1||_{L^2(\mathbf{T}^N)}].$$

Now, if we set $\hat{U} = \{U_{0\alpha}, U_{1\beta}: |\alpha| \leq M+1,\ |\beta| \leq M\}$ and define $\hat{V}$ similarly, (15) gives rise to $\hat{V} = \hat{F}\hat{U}$ and the proof of (35) extends to

$$||\hat{V}(t)||_{L^2} \leq e^{\int_0^t B'(||\hat{U}(\tau)||_{L^2})\,d\tau}\left[C_1^2||g||^2_{H^{M+1}} + \int_0^t B'(||\hat{U}(\tau)||_{L^2})\,d\tau\right].$$

Pick $A_1 \geq 2C_1||g||_{H^{M+1}} + 1$, supposing $g \in H^{M+1}(\mathbf{T}^N)$. Let T_1 be such that if $||\hat{U}(t)||_{L^2} \leq A_1$ for $|t| \leq T_1$, then

$$||\hat{V}(t)||_{L^2} \leq A_1, \qquad |t| \leq T_1. \tag{40}$$

This implies

$$||\tilde{V}(t)||_{H^1} \leq CA_1, \qquad |t| \leq T_1, \tag{41}$$

and furthermore $\tilde{F}^\nu\tilde{U}$ satisfies the estimate (41) for $\nu = 1,2,3,\dots$. Now (38), (39), and (41) yield

$$\begin{aligned} ||\tilde{V}(t) - \tilde{V}_1(t)||_{L^2} &\leq A_2\int_0^t ||\tilde{U}(\tau) - \tilde{U}_1(\tau)||_{L^2}\,d\tau, \qquad |t| \leq T_1 \\ &\leq A_2T_1 \sup_{|\tau|\leq T_1} ||\tilde{U}(\tau) - \tilde{U}_1(\tau)||_{L^2}. \end{aligned} \tag{42}$$

It follows from equation (42) that $\tilde{F}^\nu \tilde{U}$ will converge to a limit as $\nu \to \infty$, in $C([-T_0, T_0], L^2(\mathbf{T}^N))$, provided $T_0 \leq \min(T, T_1)$ and $A_2 T_0 < 1$. The limit, $\tilde{w}$, must be of the form $\{w_{0\alpha}, w_{1\beta}: |\alpha| \leq M, |\beta| \leq M-1\}$ for some

$$w \in C([-T_0, T_0], H^M(\mathbf{T}^N)) \cap C^1([-T_0, T_0], H^{M-1}(\mathbf{T}^N))$$

and w must solve (14). Since the terms $\hat{F}^\nu \hat{U}$ are bounded in $L^\infty([-T_0, T_0], L^2(\mathbf{T}^N))$, we can see that $\hat{w} \in L^\infty([-T_0, T_0], L^2(\mathbf{T}^N))$, i.e.

$$w \in L^\infty([-T_0, T_0],\ H^{M+1}(\mathbf{T}))$$

and

$$\partial w/\partial t \in L^\infty([-T_0, T_0],\ H^M(\mathbf{T}^N)).$$

To prove uniqueness, let w_1 be another solution to (14) with similar regularity. Then (42) implies

$$||\tilde{w}(t) - \tilde{w}_1(t)||_{L^2} \leq A_2 \int_0^t ||\tilde{w}(\tau) - \tilde{w}_1(\tau)||_{L^2}\, d\tau$$

which immediately gives us $\tilde{w}(t) = \tilde{w}_1(t)$ or $w(t) = w_1(t)$.

We summarize the following.

THEOREM 2. *Given $M > N/2+1$, let $g \in H^{M+1}(\mathbf{T}^N)$. Then for T sufficiently small, the iterative method* (15) *converges to a unique solution U of* (14) *with*

$$U \in C([-T, T], H^M(\mathbf{T}^N)) \cap L^\infty([-T, T], H^{M+1}(\mathbf{T}^N)),$$
$$\partial U/\partial t \in C([-T, T], H^{M-1}(\mathbf{T}^N)) \cap L^\infty([-T, T], H^M(\mathbf{T}^N))$$

provided (14) *is either symmetric hyperbolic or strictly hyperbolic.*

V. Conclusion. Now we can solve the Cauchy problem (9).

THEOREM 3. *System* (9), *being symmetric hyperbolic or strictly hyperbolic, has a unique solution U on $[-T, T] \times \mathbf{T}^N$, provided $M > N/2+1$ and T small enough, and*

$$\frac{\partial^j}{\partial t^j} U \in C([-T, T], H^{M-j-1}(\mathbf{T}^N)) \cap L^\infty([-T, T], H^{M-j}(\mathbf{T}^N)), \qquad 0 \leq j \leq m-1.$$

Coming back to the original problem (1), we have the following result:

THEOREM 4. *Provided $M > N/2+1$, the Cauchy problem for a system* (1) *of nonlinear equations with hypotheses* (a), (b), (c), (d) *has a unique solution $\vec{u}$ on $[-T, T] \times \mathbf{T}^N$, such that*

$$\frac{\partial j}{\partial t^j} u_i \in C([-T, T], H^{M-j}(\mathbf{T}^N)) \cap L^\infty([-T, T], H^{M-j+1}(\mathbf{T}^N)),$$
$$i = 1, \ldots, n,\ j = 0, \ldots, m,$$

if T is small enough.

References

1. J. Bona and R. Scott, *Solutions of the Korteweg-de Vries equation in fractional order Sobolev spaces*, Duke Math. J. **43** (1976), 87–99.
2. J. Bony, *Calcul symbolique et propagation des singularités les équations aux derivées partielles non linéaires*, Ann. Sci. École Norm. Sup. **14** (1981), 209–246.
3. J. Dieudonné, *Foundations of modern analyis*, Academic Press, New York, 1964.
4. P. Dionne, *Sur les problèmes de Cauchy hyperboliques bien posés*, J. Anal. Math. **10** (1962–1963), 1–90.
5. R. DiPerna, *Singularities of solutions of nonlinear hyperbolic systems of conservation laws*, Arch. Rational Mech. Anal. **60** (1976), 75–100.
6. T. Hughes, T. Kato and J. Marsden, *Well posed quasilinear second order hyperbolic systems with applications to nonlinear elastodynamics and general relativity*, Arch. Rational Mech. Anal. **63** (1976), 273–294.
7. M. Taylor, *Pseudodifferential operators*, Princeton Univ. Press, Princeton, N.J., 1981.

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