HYPOELLIPTIC CONVOLUTION EQUATIONS
IN THE SPACE $K'_e$

DAE HYEON PAHK

Abstract. We consider convolution equations in the space $K'_e$ of distributions which “grow” no faster than $\exp(e^{k|x|})$ for some constant $k$. Our main results are to find conditions for convolution operators to be hypoelliptic in $K'_e$ in terms of their Fourier transforms.

1. Introduction. In [6] G. Sampson and Z. Zielézny studied hypoelliptic convolution equations in the space $K'_e$ of distributions which “grow” no faster than $\exp(k|x|^p)$ for some constant $k$. We extend these investigations to the space $K'_e$ of distributions which grow no faster than $\exp(e^{k|x|})$ for some constant $k$.

More precisely, we study convolution equations of the form

$$S \ast U = V$$

where $S$ is a distribution of $\mathcal{O}'(K'_e, K'_e)$ the space of convolution operators in $K'_e$ and $U, V \in K'_e$. The space $\mathcal{E}K'_e$ of $C^\infty$-functions in $K'_e$ is defined in a natural way and equation (1) (or $S$) is said to be hypoelliptic in $K'_e$ if all solutions $U \in K'_e$ are in $\mathcal{E}K'_e$ whenever $V \in \mathcal{E}K'_e$.

Our main results are the following theorems.

Theorem 1. The following conditions are necessary for a convolution operator $S \in \mathcal{O}'(K'_e, K'_e)$ to be hypoelliptic in $K'_e$:

(h1) There exist positive constants $B$ and $M$ such that
$$|\hat{S}(\xi)| \geq |\xi|^{-B} \quad \text{if } \xi \in \mathbb{R}^n \text{ and } |\xi| \geq M.$$ (h2) $\Omega(\eta)/\log|\varsigma| \to \infty$ as $|\varsigma| \to \infty$, $\varsigma \in \mathbb{C}^n$ and $\hat{S}(\varsigma) = 0$, where
$$\Omega(x) = (|x| + 1) \log(|x| + 1) - |x|.$$ (h3) For all positive constants $m, \varepsilon$, there exist positive constants $B, C$ such that $|\hat{S}(\varsigma)| \geq |\varsigma|^{-B} e^{-\Omega(\varepsilon \varsigma)}$ whenever $\varsigma = \xi + i\eta \in \mathbb{C}^n$, $\Omega(\eta) \leq m \log|\varsigma|$ and $|\varsigma| \geq C$.

Theorem 2. The following condition is sufficient for a distribution $S$ in $\mathcal{O}'(K'_e, K'_e)$ to be hypoelliptic in $K'_e$:

(h4) Given $\varepsilon > 0$ one can find a $B > 0$ such that for every $m$ there exists a constant $C_m > 0$ so that $|\hat{S}(\varsigma)| \geq |\varsigma|^{-B} \exp(-\Omega(\varepsilon \varsigma))$ whenever $\varsigma = \xi + i\eta \in \mathbb{C}^n$, $\Omega(\eta) \leq m \log|\varsigma|$ and $|\varsigma| \geq C_m$.
Before proving these results, we briefly recall all the spaces and facts involved in this paper. See [4] for details.

The spaces $\mathcal{K}_e$ and $\mathcal{K}_e'$. We denote $\mathcal{K}_e$ the space of all functions $\phi \in C^\infty(\mathbb{R}^n)$ such that

$$\nu_k(\phi) = \sup_{x \in \mathbb{R}^n, |\alpha| \leq k} \exp(e^{|x|})|D^\alpha \phi(x)| < \infty, \quad k = 1, 2, \ldots,$$

or equivalently,

$$\sup_{x \in \mathbb{R}^n, |\alpha| \leq k} \exp(M(kx))|D^\alpha \phi(x)| < \infty \quad \text{where} \quad M(x) = e^{|x|} - |x| - 1.$$

By $\mathcal{K}_e'$ we mean the space of continuous linear functionals on $\mathcal{K}_e$ which are represented by $D^m[\exp(e^{|x|})f(x)]$ for some positive integers $m, k$ and a bounded continuous function in $\mathbb{R}^n$, where $D = D_1 D_2 \cdots D_n$.

The spaces $O_c(\mathcal{K}_e', \mathcal{K}_e')$ and $\mathcal{E}_c(\mathcal{K}_e', \mathcal{K}_e')$. We denote by $O_c(\mathcal{K}_e', \mathcal{K}_e')$ the space of convolution operators $S$ in $\mathcal{K}_e'$ with the following structure: for every integer $k > 0$ there exists an integer $m \geq 0$ such that $S = \sum_{|\alpha| \leq m} D^\alpha f_\alpha$, where $f_\alpha$ are continuous functions in $\mathbb{R}^n$ whose product with $\exp(e^{|x|})$ is bounded. We also denote by $\mathcal{E}_c(\mathcal{K}_e', \mathcal{K}_e')$ the spaces of all $C^\infty$-functions $f$ in $\mathbb{R}^n$ such that $D^\alpha f(x) = O(\exp(e^{|x|})$ as $|x| \to \infty$, for some constants $a$ (depending on $f$) and all multi-indices $\alpha$.

Furthermore, we have Paley-Wiener type theorems for functions in $\mathcal{K}_e$ and distributions in $O_c(\mathcal{K}_e', \mathcal{K}_e')$. An entire function $F(\xi)$ is a Fourier transform of a function in $\mathcal{K}_e$ if and only if, for every integer $N \geq 0$ and every $\epsilon > 0$ there exists a constant $C$ such that

$$|F(\xi + i\eta)| \leq C(1 + |\xi|)^{-N} e^{\Omega(\epsilon\eta)}$$

where $\zeta = \xi + i\eta \in \mathbb{C}^n$, and an entire function $F(\xi)$ is a Fourier transform of a distribution $S$ in $O_c(\mathcal{K}_e', \mathcal{K}_e')$ if and only if for every $\epsilon > 0$ there exist constants $N$ and $C$ such that

$$|F(\xi + i\eta)| \leq C(1 + |\xi|)^N e^{\Omega(\epsilon\eta)}$$

where $\zeta = \xi + i\eta \in \mathbb{C}^n$.

We also use the following relations between dual functions $M(x)$ and $\omega(x)$ in the sense of Young, i.e. the generating functions $\mu(x) = e^{|x|} - 1$ and $\omega(x) = \log(|x| + 1)$ are mutually inverse;

$$\sup_{x \in \mathbb{R}^n} \exp(-M(kx) + |x| |\eta|) = \exp \left( \Omega \left( \frac{1}{k} \eta \right) \right).$$

2. Necessary conditions. Proofs of the necessary conditions are based on an idea similar to that used in [8]. We begin with a lemma.

**Lemma 1.** Let $T$ be a distribution whose Fourier transform is of the form

$$\hat{T} = \sum_{j=1}^{\infty} a_j \delta(\zeta_j)$$

where $\zeta_j = \xi_j + i\eta_j \in \mathbb{C}^n$ satisfy the conditions

(3) $\Omega(\eta_j) \leq m \log |\zeta_j|,$

(4) $|\zeta_j| > 2|\zeta_{j-1}| > 2^j, \quad j = 1, 2, \ldots,$
for a given positive integer \( m \) and
\[
a_j = O(|\xi_j|^\mu) \quad \text{as } j \to \infty
\]
for some positive integer \( \mu \). Then the series in (2) converges in \( K'_e \). We assert that
\[
T \in \mathcal{E}K'_e \text{ if and only if }
\]
(6) \( a_j = O(|\xi_j|^{-\nu}) \quad \text{as } j \to \infty \)
for every \( \nu \in \mathbb{N} \).

PROOF. By (2), (5), and the fact that a set \( B \) is bounded in \( K_e \) if and only if, for every \( N \) and \( \varepsilon > 0 \), there exists a constant \( C > 0 \) such that
\[
|\hat{\phi}(\xi)| \leq C(1 + |\xi|)^{-N}e^{\Omega(\varepsilon)}
\]
for all \( \xi \in \mathbb{C}^n \) and all \( \phi \in B \), the series \( T = \sum_{j=1}^{\infty} a_j \exp(2\pi i \langle x, \xi_j \rangle) \) converges in \( K'_e \). If the coefficients \( a_j \) satisfy condition (6),
\[
|D^\alpha T(x)| = \left| \sum_{j=1}^{\infty} a_j (2\pi i \xi_j)^\alpha \exp(2\pi i \langle x, \xi_j \rangle) \right|
\]
\[
\leq C_{\nu, x} \sum_{j=1}^{\infty} |\xi_j|^{|\alpha| - \nu} \exp(2\pi |x| |\eta_j|)
\]
\[
\leq C_{\nu, x} \sum_{j=1}^{\infty} |\xi_j|^{|\alpha| - \nu + m} \exp(2\pi |x| |\eta_j| - \Omega(\eta_j))
\]
\[
\leq C_{\nu, x} \exp(M(2\pi |x|)) \sum_{j=1}^{\infty} |\xi_j|^{|\alpha| - \nu + m}
\]
in view of (3). If we choose \( \nu \) greater than \(|\alpha| + m + 2\) and make use of (4), \( T \) is in \( \mathcal{E}K'_e \).

Conversely, assume that \( T \) is in \( \mathcal{E}K'_e \). Then, for every \( \nu \in \mathbb{N} \) and every \( \phi \in K_e \),
\[
\langle \exp(i \langle u, x \rangle) \Delta^\nu T(x), \phi(x) \rangle \to 0 \quad \text{as } |u| \to \infty, \quad u \in \mathbb{C}^n \text{ and } \Omega(\text{Im } u) \leq m \log |\xi|.
\]
In fact,
\[
|\langle \exp(i \langle u, x \rangle) \Delta^\nu T(x), \phi(x) \rangle| = \left| \frac{1}{(iu)^i} \int_{\mathbb{R}^n} \Delta^\nu T(x) \phi(x) D_x^i \exp(i \langle u, x \rangle) dx \right|
\]
\[
\leq \frac{1}{|u|^i} \int_{\mathbb{R}^n} |D_x(\Delta^\nu T(x) \phi(x))| \exp(|\text{Im } u| |x|) dx
\]
\[
\leq \frac{C}{|u|^i} \int_{\mathbb{R}^n} \exp(-M(2x) + |\text{Im } u| |x|) dx
\]
\[
\leq \frac{C}{|u|^i} \sup_{x \in \mathbb{R}^n} \exp(-M(x) + |\text{Im } u| |x|) \int_{\mathbb{R}^n} \exp(-M(x)) dx
\]
\[
\leq \frac{C}{|u|^i} \exp(\Omega(\text{Im } u)) \leq \frac{C|u|^m}{|u|^l} \to 0
\]
as \( |u| \to \infty, \ u \in \mathbb{C}^n \text{ and } \Omega(\text{Im } u) \leq m \log |u|, \) provided that \( l \) is greater than \( m \).

Passing to the Fourier transform, we get
\[
\langle \tau_u (\xi, \xi') \nu \hat{T}(\xi), \hat{\phi}(\xi) \rangle = \sum_{j=1}^{\infty} a_j \langle \xi_j, \xi_j \rangle^\nu \hat{\phi}(\xi_j - u) \to 0
\]
(7)
as $|u| \to \infty$, $u \in \mathbb{C}^n$ and $\Omega(\text{Im} u) \leq m \log |u|$. We fix a function $\phi$ in $\mathcal{K}_e$ such that $\phi(0) \geq 1$.

Suppose now that condition (6) is not satisfied. Then there are a $\rho > 0$ and a $\nu_0 \in \mathbb{N}$ such that

$$|\zeta_j|^{2\nu_0} |a_j| \geq \rho$$

for a subsequence of $\{a_j\}$, which we may take as the whole sequence without loss of generality. Using a Paley-Wiener type theorem for the $\phi$, we get

$$\hat{\phi}(\zeta) = O(|\zeta|^{-k}) \quad \text{for every } k \text{ when } \zeta \in \mathbb{C}^n \text{ and } \Omega(\text{Im } \zeta) \leq m \log |\zeta|.$$

Making use of (4), (5) and (9), we obtain the estimate

$$\sum_{j=1}^{\infty} a_j \langle \zeta_j, \zeta_j \rangle^{\nu_0} \hat{\phi}(\zeta_j - \zeta_k) = O(2^{-k}).$$

On the other hand, in view of (8), we have $|a_k| |\zeta_k|^{2\nu_0} \hat{\phi}(0) \geq \rho$. This contradicts the convergence of (7). Our assertion is thus established.

**Proof of Theorem 1.** It is sufficient to prove (h3), since (h3) implies (h1) and (h2). Assume (h3) is not satisfied. Then there exist constants $\varepsilon_0$ and $m_0$ such that for every $k = 1, 2, \ldots$, there is a $\zeta_k \in \mathbb{C}^n$ such that

$$|\zeta_k| \geq 2|\zeta_{k-1}| \geq 2^k, \quad \Omega(\eta_k) \leq m_0 \log |\zeta_k| \quad \text{and}$$

$$|\hat{S}(\zeta_k)| \leq |\zeta_k|^{-k} \exp(-\Omega(\varepsilon_0 \eta_k)), \quad k = 1, 2, \ldots.$$

Then the series $\sum_{j=1}^{\infty} \exp(2\pi i \langle x, \zeta_j \rangle)$ converges to $U$, say, in $\mathcal{K}'_e$ and it is not in $\mathcal{E}\mathcal{K}'_e$. The convolution $S * U$ is transformed according to the formula

$$S*U = \hat{S}\hat{U} = \sum_{j=1}^{\infty} \hat{S}(\zeta_j) \delta(\zeta_j).$$

By (10) and Lemma 1, $S * U$ is in $\mathcal{E}\mathcal{K}'_e$. This contradicts the hypoellipticity of $S$ in $\mathcal{K}'_e$.

**3. Sufficient condition.** We intend to prove that condition (h4) is sufficient for a distribution $S$ in $\mathcal{O}'(\mathcal{K}'_e, \mathcal{K}'_e)$ to be hypoelliptic in $\mathcal{K}'_e$. In order to prove our assertion we define suitable parametrices for a distribution $S$ in $\mathcal{O}'(\mathcal{K}'_e, \mathcal{K}'_e)$ and prove that these parametrices exist if $S$ fulfills the condition (h4).

In what follows $b$ and $k$ are positive integers.

**Definition.** A distribution $P$ in $\mathcal{K}'_e$ is said to be a $(b, k)$-parametrix for $S$ if it has the following properties:

(P1) There exists an integer $m > 0$ such that $P = \sum |\alpha| \leq mD^\alpha f_\alpha$ where $f_\alpha$, $|\alpha| \leq m$, are continuous functions in $\mathbb{R}^n$ such that $f_\alpha(x) = O(\exp(-M(bx)))$ as $|x| \to \infty$.

(P2) $S * P = \delta - W$ where $\delta$ is the Dirac measure and $W$ is a function in $C^k(\mathbb{R}^n)$ satisfying the growth condition $D^\alpha W(x) = O(\exp(-M(bx)))$ as $|x| \to \infty$ when $|\alpha| \leq k$.

We first show that this definition of a parametrix is suitable for our purpose.
Theorem 3. Let $S$ be a distribution in $\mathcal{O}'(\mathcal{K}'_e, \mathcal{K}'_e)$ such that for every pair $(b, k)$ of positive integers there exists a $(b, k)$-parametrix for $S$. Then $S$ is hypoelliptic in $\mathcal{K}'_e$.

Proof. Suppose that $U$ is a solution in $\mathcal{K}'_e$ of the equation $S \ast U = V$ where $V$ is in $\mathcal{E}'\mathcal{K}'_e$. By the structure theorem, we can write $U = D^\beta f$ for some $\beta$ where $f$ is a continuous function in $\mathbb{R}^n$ such that

$$f(x) = O(\exp(M(b_1 x)))$$

as $|x| \to \infty$, for some integer $b_1 > 0$. On the other hand, $V$ is a $C^\infty$-function in $\mathbb{R}^n$ such that for all multi-index $\alpha$

$$D^\alpha V(x) = O(\exp(M(b_2 x)))$$

as $|x| \to \infty$, for some integer $b_2 > 0$.

Suppose now that $l$ is any given positive integer. By assumption there exists a $(b, k)$-parametrix $P$ for $S$ with $b = 2b_1 + 2b_2 + 1$ and $k = l + |\beta|$; i.e.

$$S \ast P = \delta - W$$

where $P$ and $W$ satisfy the growth conditions in (P1) and (P2).

From (13) it follows that

$$U = U \ast \delta = U \ast (S \ast P) + U \ast W = V \ast P + U \ast W$$

where the convolutions are well defined and the associativity is legitimate because of the rate of decrease of $P$ and $W$.

But $V \ast P$ is in $\mathcal{E}'\mathcal{K}'_e$, since, by (P1), $D^\alpha ((V \ast P) = \sum_{|\beta| \leq m} (D^{\alpha + \beta} V) \ast f_\beta$ where $f_\beta(x) = O(\exp(-M(\beta x)))$ as $|x| \to \infty$, for $|\beta| \leq m$, so that $V \ast P$ is a $C^\infty$-function and, by (12), $D^\alpha (V \ast P)(x) = O(\exp(M(b_2 x)))$ as $|x| \to \infty$, for all $\alpha$.

Also $U \ast W = f \ast D^\beta W$, which shows, from (P2) and (11), that $U \ast W$ is a $C^l$-function and $D^\alpha (U \ast W)(x) = O(\exp(M(2b_1 x)))$ as $|x| \to \infty$, for all $|\alpha| \leq l$.

Consequently $U$ is a $C^l$-function and

$$D^\alpha U(x) = O(\exp(M(2b_2 x))) + O(\exp(M(2b_1 x))) = O(\exp(M(bx)))$$

as $|x| \to \infty$, for all $|\alpha| \leq l$. But $l$ was arbitrary and therefore $U$ must be in $\mathcal{E}'\mathcal{K}'_e$.

From this theorem all that remains is to show that condition (14) implies the existence of such $(b, k)$-parametrices. In order to simplify the notation we present the proof of existence of such parametrices for $n = 1$. The general case can be handled in similar way although there are notational difficulties (see [6]).

The proof of existence and parametrices. We apply condition (h4) with $\epsilon$ and $m$ to be fixed later. Suppose that (h4) holds for some given $\epsilon$, $m$, $B > 0$ and $C_m \geq 1$. Then the function

$$F(x, \zeta) = \{2\pi \hat{S}(\xi, \zeta)^m \}^{-1} \exp(i(x, \zeta))$$

is analytic in $\zeta$, when $\Omega(\eta) \leq m \log |\zeta|$ and $|\zeta| \geq C_m$, provided that $C_m$ is sufficiently large. If $\mu > B/2 + 1$, then $F(x, \zeta)$ is integrable over $R - I$ where $I = \{x \in R: |x| \leq C_m\}$. Moreover, if $\mu$ is even and

$$h(x) = \int_{R - I} F(x, \xi) d\xi,$$

We use $M(x) + M(y) \leq M(x + y)$ and $M(x + y) \leq M(2x) + M(2y)$ for all $x, y \in \mathbb{R}^n$. 

then the distribution $H = \Delta^\mu h$ satisfies the equation

$$S \ast H = \delta - \frac{1}{2\pi} \int_I \exp(i\alpha \xi) \, d\xi.$$  

We now shift the integral (14) over a suitable contour in the complex plane.

Let $\sigma(t)$ be a $C^\infty$-function defined for $t > 0$ in such a way that $\sigma(t) = C_m$ for $0 < t \leq C_m$, increases for $t \geq C_m$ and $\sigma(t) = \exp(aM_1(bt))$ for $t \geq 2C_m$ where $a$ is a sufficiently small positive constant which we will specify later and $M_1(t) = t(e^t - t - 1)$, and $\sigma(t)$ can be extended to the negative values of $t$ by setting $\sigma(t) = -\sigma(-t)$ for $t < 0$.

Furthermore, let $\tau(t)$ be an even $C^\infty$-function on $\mathbb{R}$ such that $\tau(t) = 0$ for $|t| \leq C_m$, increases for $|t| \geq C_m$, and $\tau(t) = b^2\mu(bt)$ for $|t| \geq 2C_m$, where $c$ is the same positive constant as in $\sigma(t)$.

We can choose a positive integer $m$ depending on $b$ and $a$ such that

$$\Omega(\tau(t)) \leq m \log |\sigma(t)|$$

for $|t| \geq C_m$ and $C_m$ sufficiently large.

Given any $x \in \mathbb{R}$ we denote by $\Gamma$ the contour in the complex plane defined by $\zeta(t) = \sigma(t) + i \text{sgn}(x)\tau(t)$ where $t$ runs from $-\infty$ to $-C_m$ and $C_m$ to $\infty$. By (16) the contour $\Gamma$ lies in the domain $\Omega(\eta) \leq m \log |\eta|$. If, in addition, $\mu > B + \varepsilon m + 1$, then we can write

$$h(x) = \int_{\Gamma} F(x, \zeta) \, d\zeta.$$ 

In fact, $F(x, \zeta)$ is an analytic function in the domain $\Omega(\eta) \leq m \log |\eta|$ and, by (16), we obtain

$$\frac{1}{2\pi} \int_0^{\tau(t)} \frac{\exp(i\alpha(\sigma(t) + i \text{sgn}(x)\eta))}{S(\sigma(t) + i \text{sgn}(x)\eta) |\sigma(t) + i \text{sgn}(x)\eta|^{2\mu}} \, d\eta$$

$$\leq \frac{1}{2\pi} \exp(\Omega(\varepsilon \tau(t))) \int_0^{\tau(t)} |\sigma(t) + i\eta|^{B-2\mu} \exp(-|x|\eta) \, d\eta$$

$$\leq \frac{1}{2\pi} \exp(\varepsilon \Omega(\tau(t))) \sigma(t)^{B-2\mu+2} \int_0^{\tau(t)} |\sigma(t) + i\eta|^{-2} \exp(-|x|\eta) \, d\eta$$

$$\leq C \exp(\varepsilon m + B - 2\mu + 2aM_1(bt)) \to 0$$

as $t \to \infty$, provided that $\mu > B + \varepsilon m + 1$. Thus our claim follows from the Cauchy integral formula.

We denote by $\Gamma_0$ the part of contour $\Gamma$ obtained by restricting the values of the parameter $t$ to the open interval $(-|x|, |x|)$ and by $\Gamma_1$ the remaining portion of $\Gamma$.

If $h_1(x) = \int_{\Gamma_1} F(x, \zeta) \, d\zeta$ and $P = \Delta^\mu h_1$, then, by (15) and (17), we have

$$S \ast P = \delta - W$$

where

$$W = S \ast \Delta^\mu h_2 + \frac{1}{2\pi} \int_I \exp(i\alpha \xi) \, d\xi$$

and

$$h_2(x) = \int_{\Gamma_0} F(x, \zeta) \, d\zeta.$$ 

The proof of the existence of parametrices follows immediately from the next two lemmas.
LEMMA 2. The function \( h_1 \) satisfies the growth condition

\[
(19) \quad h_1(x) = O(\exp(-M(bx)))
\]
as \( |x| \to \infty \).

PROOF. Consider the integral

\[
\int_{\Gamma_1} F(x, \zeta) d\zeta = \int_{|t| \geq |x|} F(x, \zeta(t)) \zeta'(t) dt.
\]

For sufficiently large \( |t| \), we have

\[
|\zeta(t)|^2 \geq \sigma(t)^2 + \tau(t)^2 \geq \sigma(t)^2 \geq \exp(2\mu M_1(bt))
\]
and

\[
|\zeta'(t)| = \sqrt{(b\mu(bt) \exp(\alpha M_1(bt)))^2 + (b\mu(bt))^2} \geq C \exp(2\alpha M_1(bt))
\]
for some constant \( C \) and for sufficiently large \( |t| \).

Also, from (h4) and (16), it follows that

\[
|\tilde{S}(\zeta)|^{-1} \leq |\zeta(t)| \leq \exp(\varepsilon \tau(t)) \leq |\zeta(t)|^2 \leq \exp(\varepsilon \alpha M_1(bt)) \leq \exp((2\sigma(t))^2 \exp(\varepsilon \alpha M_1(bt)) \leq C \exp\left((\varepsilon \alpha M_1(bt))\right),
\]
provided that \( |t| \) is sufficiently large.

Further, if \( |t| \geq |x| \), from Young's inequality we have

\[
|\exp(ix\zeta(t))| = \exp(-|x|\tau(t)) = \exp(-b^2|x|^2 \mu(bx)) \leq \exp(-bM(bx)).
\]

Consequently, for \( |t| \) sufficiently large and greater than \( |x| \),

\[
|h_1(x)| \leq \frac{1}{2\pi} \int_{|t| \geq |x|} \frac{|\exp(ix\zeta(t))||\zeta'(t)|}{|\tilde{S}(\zeta(t))||\zeta(t)|^{2\mu}} \ dt
\]

\[
\leq C \exp(-bM(bx)) \int_{|t| \geq |x|} \exp((\varepsilon m + B - 2\mu + 2)\alpha M_1(bt)) \ dt
\]

\[
\leq C \exp(-bM(bx))
\]

for some constant \( C \), provided that \( \mu > \varepsilon m + B + 1 \).

This is the desired estimate for \( h_1(x) \).

LEMMA 3. For any given pair \((b, k)\) we can choose the constants \( \varepsilon, a \) (sufficiently small) and \( m \) (sufficiently large) so that

\[
(20) \quad D^\alpha W(x) = O(\exp(-M(bx))) \quad \text{as} \quad |x| \to \infty
\]

for all \( |\alpha| \leq k \).

PROOF. Assume that \( |x| \to \infty \) through \( x \geq 0 \); otherwise we could modify our argument.
By definition

\[ D^\alpha W = S \ast D^\alpha \Delta^\mu h_2 + \frac{1}{2\pi} D^\alpha \int \exp(iz\xi) d\xi \]

where

\[ h_2(x) = \frac{1}{2\pi} \int_{|x|} \frac{\exp(ixz(t))\zeta'(t)}{S(|z(t)|)\zeta(t)|^2\mu} dt. \]

It is easy to verify that \( h_2 \) is a \( C^\infty \)-function such that \( h_2(x) = 0 \) for \( |x| \leq C_m \) and

\[ D^\alpha h_2(x) = O(\exp(a(|\alpha| + 1)))M_1(bx)) \quad \text{as} \quad |x| \to \infty \]

for all \( \alpha \).

On the other hand, by the structure theorem of distributions in \( \mathcal{K}'_\rho \), for every positive integer \( p \) there is an integer \( l \geq 0 \) such that \( S = \sum_{|\beta| \leq l} D^\beta f_\beta \) where \( f_\beta, |\beta| \leq l \), are continuous functions in \( \mathbb{R} \) satisfying the growth condition

\[ f_\beta(x) = O(\exp(-M(px))) \quad \text{as} \quad |x| \to \infty. \]

Therefore, if we choose \( \rho \geq 4b \) and \( a \) so small that \( (2\mu + k + l + 1)a < \rho/4b \), we can write

\[ S \ast D^\alpha \Delta^\mu h_2 = \sum_{|\beta| \leq l} (-1)^{|\alpha + \beta|} \int_{-\infty}^{\infty} f_\beta(y)D^\alpha Y^\alpha + \beta \Delta^\mu Y h_2(x - y) dy \]

where \( |\alpha| \leq k \).

To estimate (24) we decompose \( h_2(x-y) \) as follows; \( h_2(x-y) = g_1(x,y) + g_2(x,y) \) and

\[ g_1(x,y) = \int_{|t| \leq |x|} F(x-y,\zeta(t))\zeta'(t) dt \]

where \( \zeta(t) = \sigma(t) + isgn(x-y)\tau(t) \). Using the Cauchy integral theorem the contribution of \( g_1(x,y) \) toward the right-hand side of (24) is

\[ \frac{1}{2\pi} D^\alpha \int_{\Gamma_0} \exp(iz\zeta) d\zeta + \sum_{|\beta| \geq l} (-1)^{|\alpha + \beta|} \int_{x}^{\infty} f_\beta(y)D^\alpha Y^\alpha + \beta \Delta^\mu Y \]

\[ \times \int_{-\tau(|x|)}^{\tau(|x|)} \{F(x-y,\zeta_1(t))\zeta_1'(t) - F(x-y,\zeta_2(t))\zeta_2'(t)\} dt dy \]

where \( \zeta_1(t) = -\sigma(|x|) + it \) and \( \zeta_2(t) = \sigma(|x|) + it \).

For sufficiently large \( |x| \) each of the integrals in the second term of (25) can be estimated as follows. Given \( b > 0 \) we can choose \( \varepsilon \) and \( \rho \) so that \( \varepsilon b^2 < 1 \) and
\[ \rho > b^2 + 1. \] Then
\[
\left| \int_{-\infty}^{\infty} f_\beta(y) D_y^{\alpha+\beta} \Delta^m_y \int_{-\tau(|x|)}^{\tau(|x|)} F(x - y, \zeta(t)) \zeta_1(t) \, dt \right|
\leq C \int_{x}^{\infty} \exp(-M(\rho y)) \int_{-\tau(|x|)}^{\tau(|x|)} e^{(y-x)t}(\sigma(|x|)^2 + t^2)^{(k+l+B+2)/2} \times \exp(\Omega(\epsilon t)) \frac{1}{\sigma(|x|)^2 + t^2} \, dt \, dy
\leq C \exp(-|x|\tau(|x|)) \sigma(|x|)^{k+l+B+2} \exp(\Omega(\epsilon \tau(|x|))) \times \int_{x}^{\infty} \exp(-M(\rho y) + y\tau(|x|)) \, dy
\leq C \exp\{-b^2|x|\mu(b|x|) + a(k + l + B + 2)M_1(bx) + \Omega(\mu(b|x|))\}
\times \sup_y \exp(-M(b^2 y) + b^2|y|\mu(b|x|))
\leq C \exp\{(-b + a(k + l + B + 2))b|x|\mu(b|x|) + 2\Omega(\mu(b|x|))\}
\leq C \exp(-2M(bx)) \exp\{(-(b - 2) + a(k + l + B + 2))b|x|\mu(b|x|)\}
\leq C \exp(-2M(bx)) \quad \text{as } |x| \to \infty,
\]
provided that \( a(k + l + B + 2) < b - 2 \). Similarly we can get the same estimation for the remaining part.

For the first term in (24), we can write
\[
(25) \quad D^\alpha \int_{\Gamma_0} e^{iz\xi} \, d\zeta = D^\alpha \int_{\Gamma_2} e^{iz\xi} \, d\zeta - D^\alpha \int_{I} e^{iz\xi} \, d\zeta
\]
where the curve \( \Gamma_2 \) is defined by \( \zeta(t) \) for \( C_m < |t| < |x| \) and \( t \) for \( -C_m \leq |t| \leq C_m \). Applying the Cauchy integral theorem with the curve \( \Gamma_3 \) defined by \( t + i\tau(|x|) \), we have
\[
\left| D^\alpha \int_{\Gamma_2} e^{iz\xi} \, d\zeta \right| = \left| D^\alpha \int_{\Gamma_3} e^{iz\xi} \, d\zeta \right|
\leq \int_{\sigma(|x|)}^{\tau(|x|)} \exp(-|x|\tau(|x|))(t^2 + \tau(|x|)^2)^{k/2} \, dt
\leq C \exp(-b^2|x|\mu(b|x|) + a(k + 2)M_1(bx))
\leq C \exp\{-b + a(k + 2))b|x|\mu(b|x|)\}
\leq C \exp(-M(bx)) \quad \text{as } |x| \to \infty.
\]
Therefore, combining all of these estimations we conclude that the contribution of \( g_1(x, y) \) in the right-hand side of (24) is
\[
O(\exp(-M(bx))) - D^\alpha \int_{I} \exp(iz\xi) \, d\xi \quad \text{as } |x| \to \infty.
\]
The latter term will be canceled with the second term of \( D^\alpha W \) in (21).

The proof of the lemma will be complete if we can choose \( \epsilon, a \) sufficiently small and \( m \) sufficiently large so that
\[
\left| \int_{-\infty}^{\infty} f_\beta(y) D_y^{\alpha+\beta} g_2(x, y) \, dy \right| = O(\exp(-M(bx))) \quad \text{as } |x| \to \infty,
\]
for all $|\alpha| \leq k$ and $|\beta| \leq l$. From the definition of $g_2(x, y)$ we only need to estimate $g_2(x, y)$ for $|x - y|$ sufficiently large and $|x - y| \geq |x|$. The contribution of $g_2(x, y)$ toward the right-hand side of (24) is

$$
(26) \quad \int_{-\infty}^{\infty} f_\beta(y) D_y^{\alpha+\beta} \int_{|x-y| \geq |t| \geq |x|} F(x - y, \zeta(t)) \varepsilon'(t) dt \, dy
$$

$$
= \int_{-\infty}^{\infty} f_\beta(y) D_y^{\alpha+\beta} \int F(x - y, \zeta_1(t)) \varepsilon_1'(t) dt \, dy
$$

$$
+ \int_{-\infty}^{\infty} f_\beta(y) D_y^{\alpha+\beta} \int F(x - y, \zeta_2(t)) \varepsilon_2'(t) dt \, dy
$$

where $\zeta_1(t) = \sigma(t) - i\tau(t)$ and $\zeta_2(t) = \sigma(t) + i\tau(t)$. We now estimate the first term in the right side of (26) as before.

$$
\left| \int_{-\infty}^{\infty} f_\beta(y) D_y^{\alpha+\beta} \int F(x - y, \zeta_1(t)) \zeta_1'(t) dt \, dy \right|
$$

$$
\leq \int_{-\infty}^{\infty} |f_\beta(y)| \int_{|x-y| \geq |t| \geq |x|} \exp\{(x - y)\tau(t) + \Omega(\varepsilon\tau(t))\} \times |\zeta_1(t)|^{\alpha+\beta+B} |\zeta_1'(t)| dt \, dy
$$

$$
\leq C \exp\{-(b - 1) + a(b + l + B + 4)\} b|x| |\mu(b|x|)
$$

$$
\leq C \exp(-M(bx)) \quad \text{as } |x| \to \infty,
$$

provided that $a$ is so small that $a(k + l + B + 4) \leq b - 2$. Similarly we have the same estimation for the second term in (26), which proves the lemma.

**EXAMPLE 1.** Consider the entire function $\hat{S}(\xi) = \exp(i\xi)$ in the complex plane. We can easily show that $S$ is a hypoelliptic convolution operator in $K'_e$.

**REMARK 1.** When we switch the roles of $M(x)$ and $Q(x)$, we have the same inequality as (16) when $\sigma(t) = \exp(aM(bt))$. We have the same results in the space of distributions which "grow" no faster than $\exp(k|x| \log |k|x|)$ for some integer $k > 0$, i.e. we can get all dual results.

In the space $K'_l$, where "$^l$" means logarithm, obtained by changing the roles of $M(x)$ and $Q(x)$ in above argument (see Remark 1), we have two examples of convolution operators in $K'_l$, one of which is hypoelliptic and the other is not.

**EXAMPLE 2.** Let us consider the entire function $\hat{S}(\xi) = e^{-\xi^2}$. For given $\varepsilon > 0$, taking $C_\varepsilon = \sup_{\eta^2 \geq \Omega(\varepsilon\eta)} \{e^{\eta^2}\}$ when $\Omega(\eta) = e^{|\eta|} - |\eta| - 1$, we have

$$
|\hat{S}(\xi)| = e^{-\xi^2 + \eta^2} \geq e^{\eta^2} \leq C_\varepsilon \exp(\Omega(\varepsilon\eta))
$$

and so $S$ is in $O'_\varepsilon(K'_l, K'_l)$. But, from (h$_1$), it is not hypoelliptic.

On the other hand, the distribution $T$ whose Fourier transform $\hat{T}(\xi) = 1 + e^{-\xi^2}$ is in $O'_\varepsilon(K'_l, K'_l)$ as $S$ and it is hypoelliptic. Because, for given $\varepsilon > 0$ and $m$, taking $C_m$ so large that $\xi^2 - C - m \log |\xi| \geq 2$, where $C = \sup_{\eta^2 \geq \Omega(\eta)} \eta^2$, if $\Omega(\eta) \leq m \log |\xi|$ and $|\xi| \geq C_m$, we have

$$
|\hat{T}(\xi)| = (1 + 2e^{-\xi^2 + \eta^2} \cos(2\xi\eta) + e^{2(-\xi^2 + \eta^2)})^{1/2}
$$

$$
\geq 1 - e^{-\xi^2 + \eta^2} \geq 1 - e^{-\xi^2 + C + \Omega(\eta)}
$$

$$
\geq 1 - e^{-\xi^2 + C + m \log |\xi|} \geq 1 - e^{-2}
$$

$$
\geq |\xi|^{-1} \exp(-\Omega(\varepsilon\eta))
$$
if \( \Omega(\eta) \leq m \log |\zeta| \) and \( |\zeta| \geq C_m \). That is, it satisfies (h₄).

**REMARK 2.** In [6], they showed that the necessary conditions and sufficient condition are equivalent in the space of distributions which grow no faster than \( \exp(k|x|^p) \), \( p > 1 \), for some integer \( k > 0 \). To show this equivalence they proved the same kind of result as the lemma in [3] using the homogeneity of \( |x|^p \). In our spaces we cannot prove the same equivalence which we expect.

**REFERENCES**


**DEPARTMENT OF MATHEMATICS, YONSEI UNIVERSITY, SEOUL 120, KOREA**