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AROUND EFFROS' THEOREM

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ABSTRACT. Some stronger versions of the Effros theorem are discussed in the paper, not only for homeomorphisms but also for some other mappings, e.g. for open ones. Equivalent formulations of the theorem are presented as the ε-push property and the existence of a so-called Effros' metric.

1. Introduction. In 1965 when Effros' paper [**10**] on transformation groups and C^*-algebras appeared nobody had foreseen how important an influence a theorem would have (viz. Theorem 2.1 of [**10**, p. 39]) upon the development of the theory of homogeneous continua. It was not till nine years later that Ungar [**49**] (and also Hagopian [**12**, **13**]) employed the theorem in the study of homogeneous continua. Namely it has been shown (see [**49**, (1), p. 397 and **13**, Lemma 4, p. 37]) that these continua have the following property (where ρ denotes a metric on X):

(1.1) For each positive number ε and for each point x of the continuum X there is a positive number δ such that for every two points y, z of a δ-neighborhood of x there exists a homeomorphism h of X onto X satisfying $h(y) = z$ and $\rho(v, h(v)) < \varepsilon$ for all points v in X.

The property has been called the ε-push property [**14**, p. 651] or the Effros property [**46**, p. 198]. It is not too much to say that the formulation of the Effros property can be considered as a starting point of a new period of development of studies in homogeneous continua theory. The value and the important role of this property in further investigations of homogeneity were at once perceived by F. Burton Jones, who expressed his opinion in the title of [**20**]: *Use of a new technique in homogeneous continua.*

This increasing role of the result—saying that

(1.2) each homogeneous continuum X has the Effros property (1.1)

—is revealed in use of the result to prove many significant theorems which describe various properties of homogeneous continua, and which are well known to researchers in the field. Many applications of this result are listed in the references. In particular, the Effros theorem was applied to obtain some fixed point theorems [**12**,

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14] and to solve several questions associated with the notion of n-homogeneity [**49, 51**]. The Effros property was, and still is, extensively used to investigate the structure and various properties of homogeneous continua (see e.g. a survey article [**48**, and also **4, 5, 15–18, 24, 32–35, 37, 42–47**]).

Note that formulation of (1.2) and of (1.1) is topological (or metric), while the original proof of (1.2) uses the Effros theorem and therefore exploits some algebraic methods and properties. This is a reason why topologists tried to find a purely topological proof of (1.2) and of the Effros result. One such endeavor is contained in F. D. Ancel's paper [**1**]. Another one is enclosed here. It can be observed (simply by examining the proofs) that the completeness of the considered group of homeomorphisms is essentially used in Ancel's proof, while the authors use only the fact that the group is a Borel subset of the space of all self-mappings of a continuum. It is worth emphasizing however that all three proofs very strongly use the algebraic structure of a topological group.

Another reason motivated our work for such a topological proof. Namely we try to generalize some results which concern homogeneity of continua to ones which concern homogeneity with respect to wider classes of mappings than that of homeomorphisms. Recall that a topological space X is said to be homogeneous with respect to a subset M of the space X^X of all continuous mappings of X onto itself provided for each two points p and q of X there is a mapping f in M such that $f(p) = q$ [**8**, p. 267]. It seems to the authors that the following research program is worth further study

(1.3) What properties of homogeneous continua hold for continua that are homogeneous with respect to wider classes of continuous mappings (e.g. monotone, open, confluent, weakly confluent, etc.)?

Progress in this direction is stopped for lack of a pendant to the Effros property (1.1) for other (i.e. different than homeomorphisms) classes of mappings. It seems to the authors that even in the case when the Effros property is not true for a given class M of mappings (i.e. when the Effros number δ does not exist if a homeomorphism h in (1.1) is replaced by a mapping $h \in M$ and if homogeneity of X in (1.2) is replaced by its homogeneity with respect to M), in expected proofs of corresponding theorems, a weaker property can be used in place of the Effros property. This weaker property is formulated and proved (for homogeneous continua) below. The authors hope that the proof presented here will be useful enough to obtain some of the needed results for larger classes of mappings. One of them is the class of all open mappings. For this class a version of the Effros theorem has been established and has been applied to get some results related to the Kelley property and to local connectedness of a hyperspace. In the last part of the paper a concept of the Effros metric considered on a compact metric space is introduced, and some properties of this metric are discussed. The Effros metric on X is strongly related to homeomorphisms of X onto itself, in particular the equivalence of this new metric with the original one coincides with the homogeneity of the space. Introducing the Effros

metric the authors did not expect any essentially new results: they consider this concept only as a comfortable language to describe some properties connected with the homogeneity of spaces.

The paper consists of six sections. §2, which plays an auxiliary role, contains an evaluation of the Borel class of the collection of all open mappings from a compact metric space into a metric space. §3, a key to the whole paper, contains a proof of a version of the Effros theorem. The Effros property related not only to homeomorphisms but also to wider collections of mappings, is discussed in §4. §5 considers the class of open mappings, for which a version of the Effros property is proved. §6 is devoted to the Effros metric.

The authors do not collect definitions, notions and symbols used in the paper in a separate chapter as preliminaria. The needed concepts are recalled in their proper places, where they are used. However, we fix now that all considered spaces are assumed to be metric and all mappings are continuous. Furthermore, the following standard notation is used. (X, ρ) denotes a metric space X equipped with a metric ρ, and $\operatorname{diam}(X, \rho)$ denotes its diameter. The abbreviations cl and int mean the closure and the interior respectively of a subset of a space. $B(p, r)$ stands for an open ball of the radius r about a point p, and we let $B(S, r)$ denote an open ball around a subset S of a space, i.e., $B(S, r) = \bigcup\{B(p, r)\colon p \in S\}$. dist denotes the Hausdorff distance between two bounded nonempty closed subsets K and L of a metric space with a metric ρ, i.e.,

$$\begin{aligned}\operatorname{dist}(K, L) &= \inf\{r > 0\colon K \subset B(L, r) \text{ and } L \subset B(K, r)\}\\ &= \max\{\sup\{\rho(x, L)\colon x \in K\}, \sup\{\rho(y, K)\colon y \in L\}\}\end{aligned}$$

(see [**26**, §21, VII, p. 214]; cf. [**41**, (0.1), p. 1]). The composition of two mappings f and g is denoted by juxtaposition—fg. A collection of bounded mappings from a space X into a space Y with a metric ρ is metrized by the sup metric denoted by $\hat{\rho}$:

$$\hat{\rho}(f, g) = \sup\{\rho(f(x), g(x))\colon x \in X\}. \tag{1.4}$$

2. Borel sets of mappings. Let Y^X denote a collection of all continuous mappings from a compact metric space X onto a metric space Y, equipped with the compact-open topology (see [**27**, §44, I, p. 75] for the definition). The space Y^X is known to be separable and topologically complete [**27**, §44, V, Theorem 3, p. 90]. Recall that a subset of a complete space is topologically complete if and only if it is a G_δ-set [**26**, §35, III, p. 430].

Note that each of the following classes of mappings forms a G_δ-subset of the space Y^X,

(i) homeomorphisms [**27**, §44, VI, Theorem 1, p. 91];

(ii) monotone mappings [**28**, 4, p. 285]; cf. [**29**, 4, p. 800];

(iii) confluent mappings [**40**, Theorem (2.10), p. 569];

(iv) weakly confluent mappings [**40**, (2.5), p. 567]; cf. [**36**, 5, Theorem (5.54), p. 44];

(v) light mappings [**27**, §45, IV, Theorem 5, p. 109, and **51**, (4.4), p. 130].

Unfortunately, the authors do not know whether the class of open mappings forms a G_δ-subset of the space Y^X. An affirmative answer to this question is known under some additional conditions about X and Y, namely if X is locally connected and $Y = [0,1]$ (see [**39**, Theorem 3, p. 44], where other classes of mappings, as light, nonalternating and their compositions are also discussed). An evaluation of the Borel class of the set of all open mappings is given by the following

PROPOSITION 2.1. *The collection of all open mappings from a compact metric space X onto a metric space Y is an $F_{\sigma\delta}$-set in Y^X.*

In fact, given a point x of X and two positive real numbers r and s, put

$$G(x;\, r,s) = \{ f \in Y^X \colon \operatorname{cl} B(f(x), r) \subset f(\operatorname{cl} B(x,s)) \}.$$

Obviously $G(x;\, r,s)$ is a closed subset of the space Y^X. Now let P be a countable dense subset of X and let Q denote the set of all positive rationals. If G is the collection of all open mappings of X onto Y, we have

$$G = \bigcap_{x \in P} \bigcap_{s \in Q} \bigcup_{r \in Q} G(x;\, r,s),$$

which completes the proof.

Since the intersection of a G_δ-set and an $F_{\sigma\delta}$-set is an $F_{\sigma\delta}$-set, we have

COROLLARY 2.2. *The collections of all monotone open and of all light open mappings from a compact metric space X onto a metric space Y are $F_{\sigma\delta}$-sets in Y^X.*

3. Effros' theorem. Let a metric space X be compact. Assume $Y = X$ and let M be an arbitrary subset of X^X. For a mapping $g \in X^X$ we put $gM = \{ gf \colon f \in M \} \subset X^X$ and similarly $Mg = \{ fg \colon f \in M \} \subset X^X$. Fix a point $a \in X$ and denote by $M(a)$ the orbit of this point, i.e., $M(a) = \{ f(a) \colon f \in M \}$. More generally, for $A \subset X$ we let $M(A)$ denote the set of all points $f(a)$ for some $f \in M$ and for some $a \in A$.

If a group G is given, then for any two subsets E and F of G we put $EF = \{ xy \in G \colon x \in E \text{ and } y \in F \}$, and $E^{-1} = \{ x^{-1} \in G \colon x \in E \}$. Further, if a point a of the space X is fixed, we define $T_a \colon X^X \to X$ putting $T_a(f) = f(a)$. Note that T_a is continuous.

For a set $A \subset X$ we define the quasi-interior A^* of A as follows (see [**10**, p. 39]; cf. [**23**, $P(b)$, p. 211]).

$$A^* = \bigcup \{ V \colon V \text{ is open in } X \text{ and } V \setminus A \text{ is of the first category} \}. \tag{3.1}$$

Observe that

$$\text{the set } A^* \setminus A \text{ is of the first category.} \tag{3.2}$$

In fact, by separability of X, the union in (3.1) can be replaced by the union of countably many open sets V_i such that $A^* = \bigcup \{ V_i \colon i \in \{1,2,\dots\} \}$ and $V_i \setminus A$ are of the first category. Thus $A^* \setminus A = \bigcup \{ V_i \setminus A \colon i \in \{1,2,\dots\} \}$ and (3.2) follows from the Baire category theorem, the space X being complete.

Further,

$$\text{if } A \subset B, \text{ then } A^* \subset B^*. \tag{3.3}$$

Indeed, if $A \subset B$, then $A^* \setminus B \subset A^* \setminus A$ is of the first category by (3.2). Since A^* is open, the conclusion follows from the definition of B^*.

As an immediate consequence of (3.3) we get, for an arbitrary set I of indices, that

$$\text{if } \bigcup\{A_i\colon i \in I\} \subset A, \text{ then } \bigcup\{A_i^*\colon i \in I\} \subset A^*. \tag{3.4}$$

Recall that a set A contained in a metric space X is said to be analytic provided that it is a continuous image of a Borel set (see e.g. [**26**, §38, I, p. 453 and §39, I, p. 478]). If a set $A \subset X$ is analytic, then it has the Baire property in the restricted sense (see [**26**, §39, II, Corollary 1, p. 482]), which means that for each subset $S \subset X$ the set $A \cap S$ has the Baire property relative to S [**26**, §11, VI, Definition, p. 92]). In particular A has the Baire property, i.e., there exists an open set V such that $A \setminus V$ and $V \setminus A$ are first category sets [**26**, §11, I, p. 87]. Further we have

(3.5) if A has the Baire property, then $A \setminus A^*$ is of the first category.

In fact, let V be an open set such that the sets $A \setminus V$ and $V \setminus A$ are of the first category. Then $V \subset A^*$, whence $A \setminus A^* \subset A \setminus V$ and the conclusion follows.

Consider an arbitrary Borel subset M of X^X. Note that M is separable as a subspace of the metric separable space X^X. Let $\{H_i\colon i \in \{1,2,\dots\}\}$ be a countable base for M. Then

$$\begin{aligned}\bigcup\left\{T_a(H_i)\setminus(T_a(H_i))^*\colon i \in \{1,2,\dots\}\right\} \\ = \bigcup\left\{T_a(H)\setminus(T_a(H))^*\colon H \text{ is open in } M\right\}.\end{aligned} \tag{3.6}$$

In fact, the left member of the equality is obviously contained in the right. To prove the opposite inclusion let H be open in M. Then there is a sequence $\{i_j\colon j \in \{1,2,\dots\}\}$ of natural numbers such that $H = \bigcup\{H_{i_j}\colon j \in \{1,2,\dots\}\}$. So by (3.4) we have

$$\begin{aligned}&T_a(H)\setminus(T_a(H))^* \\ &\quad\subset \bigcup\left\{T_a(H_{i_j})\colon j \in \{1,2,\dots\}\right\}\setminus\bigcup\left\{\left(T_a(H_{i_j})\right)^*\colon j \in \{1,2,\dots\}\right\} \\ &\quad= \bigcup\left\{T_a(H_{i_j})\setminus\bigcup\left\{\left(T_a(H_{i_k})\right)^*\colon k \in \{1,2,\dots\}\right\}\colon j \in \{1,2,\dots\}\right\} \\ &\quad\subset \bigcup\left\{T_a(H_{i_j})\setminus\left(T_a(H_{i_j})\right)^*\colon j \in \{1,2,\dots\}\right\} \\ &\quad\subset \bigcup\left\{T_a(H_i)\setminus(T_a(H_i))^*\colon i \in \{1,2,\dots\}\right\},\end{aligned}$$

which gives (3.6).

Put

$$\tilde{M} = X\setminus\bigcup\left\{T_a(H)\setminus(T_a(H))^*\colon H \text{ is open in } M\right\}. \tag{3.7}$$

We claim that

$$\tilde{M} \text{ is dense in } X. \tag{3.8}$$

Indeed, since the sets $T_a(H_i)$ are analytic, they have the Baire property, whence $T_a(H_i)\setminus(T_a(H_i))^*$ are of the first category, and so the claim follows by completeness of X and by (3.6) and (3.7).

We now show two consequences of (3.8). We start with

PROPOSITION 3.1. *If a compact space X is homogeneous with respect to a Borel set $M \subset X^X$, then there is a mapping $h \in M$ such that $h(a) \in (T_a(H))^*$ for each H open in M with $h \in H$.*

PROOF. By homogeneity of X with respect to M and by (3.8) the set $T_a^{-1}(\tilde{M})$ is nonempty. So pick $h \in T_a^{-1}(\tilde{M})$ and note that if $h \in H$, where H is an open set in M, then $h(a) \in T_a(H) \cap \tilde{M}$, whence by (3.7) we have $h(a) \in (T_a(H))^*$.

Further we have another consequence of (3.8).

PROPOSITION 3.2. *If a compact space X is homogeneous with respect to a Borel subgroup G of the group of all homeomorphisms of X onto itself, then*

(i) *for each set H open in G we have $T_a(H) \subset (T_a(H))^*$, whence*

(ii) $\tilde{G} = X$.

PROOF. Put $M = G$ in (3.8) and suppose on the contrary that there is a set H open in G such that $T_a(H) \setminus (T_a(H))^* \neq \varnothing$. Let $b \in T_a(H) \setminus (T_a(H))^*$ and let x be an arbitrary point of $\tilde{G}$. Since X is homogeneous with respect to G, there is a mapping g in G such that $g(b) = x$. Thus

$$x \in g\big(T_a(H) \setminus (T_a(H))^*\big) = T_a(gH) \setminus (T_a(gH))^*,$$

and since gH is open in G, we see that x is not in $\tilde{G}$.

We are ready now to prove the main result of the present section. A proof of this theorem, presented below, is a slight modification of Effros' original proof of Theorem 2.1 of [**10**, p. 39].

THEOREM 3.3. *Let a space X be compact, a point $a \in X$ be fixed, and let G denote a Borel subgroup of the group of all homeomorphisms of X onto itself. If X is homogeneous with respect to G, then the partial mapping $T_a \mid G\colon G \to X$ is open.*

PROOF. By Proposition 3.2 we have

(3.9) if a nonempty set H is open in the group G, then $H(a) \subset (H(a))^*$, whence $H(a)$ is of the second category in X.

Moreover,

(3.10) if a nonempty set H is open in G, then $H^{-1}((H(a))^*) \subset (H^{-1}H)(a)$.

In fact, by homogeneity of X with respect to G we may assume each point of X has the form $g(a)$ for some homeomorphism $g \in G$. So let $g(a) \in H^{-1}((H(a))^*)$. Then there is an $h \in H$ and a point $p \in (H(a))^*$ such that $g(a) = h^{-1}(p)$, whence $hg(a) = p$, i.e., $p \in (Hg)(a)$. Thus the intersection $(Hg)(a) \cap (H(a))^*$ is nonempty. Further note that, since H is nonempty and open, so is Hg, whence $(Hg)(a)$ is of the second category by (3.9), and consequently the same holds for its intersection with the open set $(H(a))^*$. Therefore, $(Hg)(a) \cap H(a) \neq \varnothing$. Indeed, in the opposite case we would have $(Hg)(a) \cap (H(a))^* \subset (H(a))^* \setminus H(a)$, and since the latter set is of the first category by (3.9), we get a contradiction. Taking a point $q \in (Hg)(a) \cap H(a)$ we see there are homeomorphisms h and h' in H such that

$q = hg(a) = h'(a)$, whence $g(a) = h^{-1}h'(a)$, and so $g(a) \in (H^{-1}H)(a)$. Thus (3.10) is established.

Now let e denote the identity mapping of X onto itself and let N be an arbitrary neighborhood of e in the group G. We will show that

(3.11) for each neighborhood N of the identity e we have $\operatorname{int} N(a) \neq \varnothing$.

Indeed, since G is a topological group, for each element of G there is an open set H containing this element and such that $H^{-1}H \subset N$. Thus $(H^{-1}H)(a) \subset N(a)$. Further, $(H(a))^*$ is open by the definition, and is nonempty by (3.9); so the same is satisfied for $H^{-1}((H(a))^*)$, and (3.11) follows by (3.10).

As an immediate consequence of (3.11) we have

(3.12) if a nonempty set H is open in G, then $\operatorname{int} H(a) \neq \varnothing$.

Furthermore,

(3.13) if N is a neighborhood of the identity in G, then $a \in \operatorname{int} N(a)$.

In fact, find a nonempty open set H such that $H^{-1}H \subset N$. Since $\operatorname{int} H(a) \neq \varnothing$ by (3.12), let $g(a) \in \operatorname{int} H(a)$ for some $g \in H$. Then

$$\begin{aligned}\{a\} &= g^{-1}g(a) \subset g^{-1}(\operatorname{int} H(a)) = \operatorname{int} g^{-1}(H(a)) \\ &= \operatorname{int}(g^{-1}H)(a) \subset \operatorname{int}(H^{-1}H)(a) \subset \operatorname{int} N(a).\end{aligned}$$

Now let a nonempty set H be open in the group G. For each $h \in H$ we can find an open set N such that $e \in N$ and $hN \subset H$. Then by (3.13) we have $a \in \operatorname{int} N(a)$, whence

$$h(a) \in h(\operatorname{int} N(a)) = \operatorname{int} h(N(a)) = \operatorname{int}(hN)(a) \subset \operatorname{int} H(a).$$

Thus $h(a)$ is an interior point of $H(a)$, which implies that the set $H(a) = T_a(H)$ is open. The proof is complete.

It is worth emphasizing that in the above proof of Theorem 3.3 we have very strongly employed the algebraic structure of a topological group. Observe however that among topological properties of the group the above proof uses only the fact that the considered group is a Borel set, while the proof of a version of Effros' theorem presented in [**1**] essentially uses completeness of the group.

4. Effros' property. A mapping $f\colon S \to S'$ of a topological space S onto S' is said to be interior at a point $s \in S$ provided that for every open set V in S containing s, the point $f(s)$ is an interior point of $f(V)$ (see [**51**, p. 149]). Obviously f is open if and only if it is interior at each point of its domain.

As before let G denote a Borel subgroup of the group of all homeomorphisms of a compact space X onto itself. Note that interiority of the mapping $T_a \mid G\colon G \to X$ at the identity $e \in G$ is equivalent [**49**, p. 397] to the Effros property (i.e. ε-push property). Hence this property is a corollary to Theorem 3.3.

COROLLARY 4.1 (EFFROS' PROPERTY). *If a compact metric space X is homogeneous, then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\rho(x, y) < \delta$ for some $x, y \in X$, then there is an ε-homeomorphism h from X onto itself (i.e., $\rho(z, h(z)) < \varepsilon$ for all $z \in X$) such that $h(x) = y$.*

Let us mention that the inverse implication to that of Corollary 4.1, i.e., an implication from the Effros property to homogeneity of a compact metric space holds for connected spaces. In fact, let a continuum X have the Effros property and let ε be a positive number. Thus there is a $\delta > 0$ such that if two points of X are within a distance δ of each other, then there is a homeomorphism from X onto itself, mapping one of these points to the other and moving no point of X more than ε. Take an open cover of X consisting of all $\delta/2$-balls, choose a finite subcover of it, and consider the Lebesgue number λ of this finite subcover [**27**, §41, VI, Corollary 4c, p. 23]. Since X is connected, for each two points x and y of X there is a finite sequence of some $n+1$ points $x_0, x_1, \ldots, x_n$ of X such that $x_0 = x$, $x_n = y$, and $\rho(x_i, x_{i+1}) < \lambda$ for every $i \in \{0, 1, \ldots, n-1\}$. Thus each two consecutive members of this sequence are in the same element of the subcover, i.e., $\rho(x_i, x_{i+1}) < \delta$, whence there are homeomorphisms h_i of X onto itself with $h_i(x_i) = x_{i+1}$. Then the composition $h = h_n h_{n-1} \cdots h_0$ is a homeomorphism which maps x to y. So we have the following (well-known) proposition.

PROPOSITION 4.2. *If a metric continuum has the Effros property, then it is homogeneous.*

Note that connectedness of a compact space is necessary in the above proposition, as the disjoint union of the circle and the pseudo-arc shows.

Corollary 4.1 and Proposition 4.2 imply

COROLLARY 4.3. *For metric continua homogeneity and the Effros property are equivalent.*

In light of Proposition 4.2 one can ask the following

QUESTION 4.4. For what compact metric spaces does the Effros property imply their homogeneity?

It is easy to observe that the implication is true if the space is the disjoint union of finitely many homeomorphic homogeneous continua. But if the number of components of the space is infinite, then these conditions do not suffice to attain homogeneity of the whole space.

A number is related to the Effros property, namely the number δ mentioned in Corollary 4.1, and it is called the Effros number for X and ε (see [**34**, p. 373]). Up to now the Effros number was defined and considered for homeomorphisms only. However, if one considers homogeneity of a space with respect to some wider classes of mappings of the space onto itself, then it makes sense to introduce a concept of the Effros number for these classes in the same way as it was done for homeomorphisms.

We shall say that a compact metric space X with a metric ρ has the Effros property for a class M of mappings of X onto itself provided that for each positive ε there exists a positive δ (called the Effros number for M, X, and ε) such that if $\rho(x, y) < \delta$ for some $x, y \in X$, then there is an ε-mapping $f \in M$ (i.e., such that $\rho(z, f(z)) < \varepsilon$ for all $z \in X$) satisfying $f(x) = y$.

Note that—exactly as for homeomorphisms—the Effros property for a class $M \subset X^X$ is equivalent to interiority of the partical mapping $T_a \mid M$: $M \to X$ at the identity. However, in contrast to the class of homeomorphisms, homogeneity of a compact space X (even of a continuum) with respect to a class $M \subset X^X$ does not imply—in general—the Effros property for M. For example, the cone over a continuum is homogeneous with respect to the class $M = X^X$ of all continuous mappings [**25**, Proposition 3, p. 346], but X does not have the Effros property for M. Similarly, the Sierpiński plane universal curve is homogeneous with respect to the class of monotone mappings [**9**, Theorem 5, p. 131], while it can rather easily be observed that the considered monotone mappings must be far from the identity, and thus the Effros number does not exist if ε is sufficiently small. So we have the following

QUESTION 4.5. Let a compact space X be given which is homogeneous with respect to a class $M \subset X^X$. For what classes M does the space X have the Effros property for M?

It will be seen in the next section that the one-point union of two Menger universal curves is homogeneous with respect to light open mappings (see Example 5.5 below).

QUESTION 4.6. Has the one-point union of two Menger universal curves the Effros property for the class of mappings which are both light and open?

The inverse implication to that discussed in Question 4.5 is true for connected spaces, and the proof presented above for homeomorphisms is valid for other classes of mappings provided that the class is closed for compositions of mappings (i.e., if f, $g \in M$ implies $fg \in M$). So we have the following generalization of Proposition 4.2.

PROPOSITION 4.7. *If a continuum X has the Effros property for a class M of mappings that is closed for compositions, then X is homogeneous with respect to M.*

5. Consequences of openness of mappings. We shall formulate and prove the Effros theorem for the class of open mappings, and we shall show how it can be applied to study some properties of continua that are homogeneous with respect to this class of mappings. We start with propositions and examples showing the existence of nonhomogeneous curves that are homogeneous with respect to open mappings. As suggested to the authors by H. Kato and L. G. Oversteegen, the existence of continua having the above property is a consequence of Anderson-Wilson theorems (see [**52**, Theorems 1 and 2, p. 497]). H. Kato presents in [**21**] an example of a two-dimensional continuum having the considered property.

Recall that a mapping f between topological spaces X and Y is said to be monotone (light) if $f^{-1}(y)$ is connected (totally disconnected) for each point y of Y. For compact spaces the definition is equivalent to one saying that $f^{-1}(y)$ is a continuum (is zero-dimensional, respectively). Thus light mappings are sometimes called zero-dimensional ones (see [**51**, (4.4), p. 130]).

PROPOSITION 5.1. *Let a continuum X be locally connected. If there exists a monotone (a light) open mapping f from X onto the Menger universal curve U, then X is homogeneous with respect to the class of monotone (of light) open mappings.*

Indeed, take two arbitrary points p and q in X. By the Anderson-Wilson theorem [**52**, Theorem 1 (Theorem 2, respectively), p. 497] there exists a monotone (light) open surjection $g\colon U \to X$. Take a point $q' \in g^{-1}(q) \subset U$. Since U is homogeneous [**2**, Theorem III, p. 322; **3**, Theorem IV, p. 3; **38**], there is a homeomorphism $h\colon U \to U$ with $h(f(p)) = q'$. Then the composition $ghf\colon X \to X$ is a monotone (light) open mapping, and it maps p onto q.

PROPOSITION 5.2. *Let a subset F of the Menger universal curve U be finite. Then the quotient space U/F obtained by shrinking of F to a point is a locally connected curve which can be mapped onto U under a monotone open mapping and under a light open mapping. Furthermore, for each two finite sets F and F' of the same cardinality, the obtained quotient spaces U/F and U/F' are homeomorphic.*

PROOF. We show the uniqueness first. Fix in U two finite subsets F and F' of the same cardinality. It follows from the Anderson theorem [**2**, Theorem III, p. 322; **3**, Theorem IV, p. 3]) that there exists a homeomorphism $h\colon U \to U$ such that $h(F) = F'$. Thus if $\varphi\colon U \to U/F$ and $\varphi'\colon U \to U/F'$ are natural quotient mappings, then the composition $\varphi' h \varphi^{-1}$ is a homeomorphism from U/F onto U/F'.

To show the existence put $n = \operatorname{card} F$ and take a monotone (light) open mapping $g\colon U \to U$ with point-inverse sets homeomorphic to U (the Cantor set, respectively) (see [**52**, Theorems 1 and 2, p. 497]). Pick a point $y \in U$ and choose a set F' of n points in $g^{-1}(U)$. Shrinking of F' to a point we get the natural quotient mapping $\varphi'\colon U \to U/F'$. As a continuous image of U, the resulting space U/F' is a locally connected continuum. Its one-dimensionality is a consequence of that of U and of finiteness of F'. Finally, the mapping $g(\varphi')^{-1}\colon U/F' \to U$ is a monotone (light) open one.

COROLLARY 5.3. *For each finite nondegenerate subset F of the Menger universal curve U the quotient space U/F is a locally connected nonhomogeneous curve that is homogeneous with respect to the classes of monotone open and of light open mappings.*

In fact, U/F is nonhomogeneous because the point F is the only local separating point of U/F. The homogeneity of U/F with respect to the two discussed classes of mappings is an immediate consequence of Propositions 5.1 and 5.2.

To show some properties of an example below we need a lemma.

LEMMA 5.4. *The notion of a cut point is an invariant of monotone open mappings of continua.*

PROOF. Let X and Y be continua, and let a surjection $f\colon X \to Y$ be monotone open. Take a cut point p of X and note that since p is in the continuum $f^{-1}(f(p))$, the set $X \setminus f^{-1}(f(p))$ is the union of two disjoint nonempty open sets, say A and B. Then $f(A)$ and $f(B)$ are open subsets of Y. Further, they are disjoint, because if $y \in f(A) \cap f(B)$, then the continuum $f^{-1}(y)$ intersects both A and B, so it must contain the point p, a contradiction.

EXAMPLE 5.5 (L. G. OVERSTEEGEN). The one-point union of two Menger universal curves is not homogeneous with respect to monotone open mappings (thus it is not homogeneous) and it is homogeneous with respect to light open mappings.

PROOF. Let U_1 and U_2 be two copies of the Menger universal curve, the common part of which is a point c. Put $X = U_1 \cup U_2$. Since c is a cut point of X, and since it is the only one, the nonhomogeneity of X with respect to monotone open mappings follows from Lemma 5.4. To see homogeneity of X with respect to the class of light open mappings, consider a natural projection of X onto U_1, i.e. a mapping defined as follows: $f|U_1$ is the identity and $f|U_2$ is a homeomorphism with $(f|U_2)(c) = c$. Note that $f\colon X \to U_1$ defined in this way is both light and open. Thus the conclusion follows from Proposition 5.1.

The authors are looking for an example of a curve having symmetric properties to the continuum of Example 5.5. Namely we have

QUESTION 5.6. Does there exist a locally connected curve that is homogeneous with respect to monotone open mappings and is not homogeneous with respect to light open mappings?

To prove a further result we need a lemma.

LEMMA 5.7. *For each open mapping the inverse image of a first category set is of the first category.*

PROOF. Let a mapping $f\colon X \to Y$ be open. It is enough to show that the inverse image of a nowhere dense subset B of Y is nowhere dense in X. So take an open subset A of X. Then $f(A)$ is open in Y. Thus there is a nonempty open set $V \subset f(A)$ such that $B \cap V = \varnothing$. Therefore $A \cap f^{-1}(V)$ is an open subset of A which is disjoint from $f^{-1}(B)$.

The following proposition plays a key role in further considerations.

PROPOSITION 5.8. *Let a mapping $f\colon X \to Y$ between compact metric spaces X and Y be open, and let $A \subset X$ be such that $f(A)$ has the Baire property. Then*

$$f(A^*) \subset (f(A))^*. \tag{5.1}$$

PROOF. Since the set A^* is open by its definition (3.1), the set $f(A^*)$ is open. Thus to prove (5.1) it is enough to show—again by the definition of the quasi-interior—that the set $f(A^*)\setminus f(A)$ is of the first category. To this end note that since $f(A^*)$ is open and $f(A)$ has the Baire property, and since the family of all subsets of Y having the Baire property is a field [**26**, §11, III, Theorem 1, p. 88], the set $f(A^*)\setminus f(A) = f(A^*) \cap (Y\setminus f(A))$ has the Baire property. Therefore there is an open set $V \subset f(A^*)$ such that the sets

$$V\setminus\big(f(A^*)\setminus f(A)\big) \quad \text{and} \quad \big(f(A^*)\setminus f(A)\big)\setminus V \tag{5.2}$$

are of the first category. Note that the partial mapping $g = f|A^*\colon A^* \to f(A^*)$ is open. Thus, by Lemma 5.7, the inverse images under g of the sets (5.2) are of the first category in A^* and, therefore, since A^* is open in X, they are also of the first category in X. Obviously

$$V = \big(V\setminus\big(f(A^*)\setminus f(A)\big)\big) \cup \big(V \cap \big(f(A^*)\setminus f(A)\big)\big),$$

whence

$$g^{-1}(V) = g^{-1}\big(V\setminus\big(f(A^*)\setminus f(A)\big)\big) \cup g^{-1}\big(V\cap\big(f(A^*)\setminus f(A)\big)\big)$$
$$\subset g^{-1}\big(V\setminus\big(f(A^*)\setminus f(A)\big)\big) \cup \big(A^*\setminus A\big).$$

Therefore the open set $g^{-1}(V)$ is a subset of the union of two first category sets (note that $A^*\setminus A$ is of the first category by (3.2)), which is possible only in the case $V = \varnothing$. So the set $f(A^*)\setminus f(A)$ equals the latter of the two sets of (5.2) that are of the first category, and the proof is complete.

THEOREM 5.9. *If a compact metric space X is homogeneous with respect to the class $M \subset X^X$ of all open mappings, then for each pair of points a, b in X there exists an open mapping $f\colon X \to X$ such that $b = f(a) \in (T_a(H))^*$ for each H open in M with $f \in H$.*

PROOF. Observe that M is a Borel subset of X^X by Proposition 2.1. Thus there exists a mapping $h \in M$ satisfying the conclusion of Proposition 3.1. Further, by homogeneity of X with respect to M there is a mapping $g \in M$ with $g(h(a)) = b$. Put $f = gh$. Thus f is open, and $f(a) = b$. Let H be an open subset of M containing f. Then there exists an open set H' in M containing h and such that $gH' \subset H$. Therefore $T_a(gH') = g(T_a(H')) \subset T_a(H)$, which implies by (3.3) that

$$\big(g(T_a(H'))\big)^* \subset \big(T_a(H)\big)^*. \tag{5.3}$$

By Proposition 3.1 we have $h(a) \in (T_a(H'))^*$, whence

$$b = g(h(a)) \in g\big((T_a(H'))^*\big). \tag{5.4}$$

Note that the set $g(T_a(H'))$ is analytic, and therefore it has the Baire property. Thus Proposition 5.8 can be applied to the mapping g and the set $T_a(H')$, and we infer from (5.1) that $g((T_a(H'))^*) \subset (g(T_a(H')))^*$. Now the conclusion follows from (5.4) and (5.3).

Theorem 5.9 can be applied in a direct proof of the following result.

PROPOSITION 5.10. *If a continuum X is homogeneous with respect to the class $M \subset X^X$ of all open mappings, then*

(5.5) *for each subcontinuum K of X with nonempty interior, and for each open subset V of X containing K, there exists a subcontinuum L of X such that $K \subset \operatorname{int} L \subset V$ and $L \subset \operatorname{cl} V$.*

PROOF. Let a subcontinuum K have nonempty interior, and let an open set V in X contain K. Take a point $d \in \operatorname{int} K$ and put

$$\varepsilon = \tfrac{1}{2}\min\big\{\operatorname{dist}(K, X\setminus V), \operatorname{dist}(\{d\}, X\setminus \operatorname{int} K)\big\}.$$

Thus

$$B(K,\varepsilon) \subset V \tag{5.6}$$

and

$$B(d,\varepsilon) \subset \operatorname{int} K. \tag{5.7}$$

Denote by C the component of V which contains K. Thus C is closed relative to V, i.e.,

$$V \cap \operatorname{cl} C \subset C. \tag{5.8}$$

We claim that

$$K \subset \operatorname{int} C. \tag{5.9}$$

Suppose on the contrary that there is a point b in $K \setminus \operatorname{int} C$. By the definition of ε we have

$$\operatorname{cl} B(b, \varepsilon) \subset V. \tag{5.10}$$

Fix a point a in X. It follows from Theorem 5.9 that there is an open mapping f: $X \to X$ such that $b = f(a) \in (T_a(H))^*$ for each H open in the class M of open mappings with $f \in H$. Put $H = M \cap B(f, \varepsilon)$, where the ball is considered as a subset of the space X^X with the metric $\hat{\rho}$ defined by (1.4). Observe that

$$\text{if } g \in B(f, \varepsilon), \text{ then } \rho(f(x), g(x)) < \varepsilon \text{ for all } x \in X. \tag{5.11}$$

In particular, if $g \in B(f, \varepsilon)$, then $g(a) \in B(b, \varepsilon)$. Hence we conclude $T_a(H) \subset B(b, \varepsilon)$ by the definition of the mapping T_a: $X^X \to X$. Thus letting $A = T_a(H)$ we have $b \in A^*$ and $A \subset B(b, \varepsilon)$. This inclusion means $A \cap (X \setminus B(b, \varepsilon)) = \varnothing$, whence we see that

$$A^* \cap (X \setminus \operatorname{cl} B(b, \varepsilon)) = A^* \cap (X \setminus \operatorname{cl} B(b, \varepsilon)) \setminus A. \tag{5.12}$$

Since A^* is open by its definition (3.1), the left member of (5.12) is open as the intersection of two open sets. On the other hand, the right member of (5.12), as a subset of $A^* \setminus A$, is of the first category by (3.2). So we conclude that the intersection in the left member of (5.12) is empty, which means that $A^* \subset \operatorname{cl} B(b, \varepsilon)$, whence by (5.10) we have

$$A^* \subset V. \tag{5.13}$$

Therefore

$$A^* \cap (V \setminus \operatorname{cl} C) \neq \varnothing, \tag{5.14}$$

because otherwise $A^* \subset V \cap \operatorname{cl} C$ by (5.13), and since A^* is an open set containing the point b, we conclude that b is an interior point of C by (5.8), a contradiction to the choice of b. So (5.14) is established. Further,

$$A \cap (A^* \cap (V \setminus \operatorname{cl} C)) \neq \varnothing. \tag{5.15}$$

In fact, if not, then $A^* \cap (V \setminus \operatorname{cl} C) = (A^* \cap (V \setminus \operatorname{cl} C)) \setminus A$, and—using exactly the same arguments as before—we see that the left member of this equality is nonempty (see (5.14)) and open, while the right one is of the first category by (3.2), a contradiction.

It follows from (5.15) that there is a component C' of V distinct from C that intersects A. So by the definition of A there exists a mapping $g \in H \subset B(f, \varepsilon)$ such that $T_a(g) = g(a) \in C'$. Consider now the component Q of $f^{-1}(K)$ containing the point a, and note $g(a) \in g(Q) \cap C'$. Observe further that

$$g(Q) \subset V. \tag{5.16}$$

Indeed, let $y \in g(Q)$. Thus there is a point x in Q with $g(x) = y$. It follows from (5.11) that $y \in B(f(x), \varepsilon)$. Since $f(Q) = K$ by openness of f (see [**51**, Theorem 7.5, p. 148]), we have $f(x) \in K$, whence $B(f(x), \varepsilon) \subset B(K, \varepsilon)$ and $y \in V$ follows from (5.6). So (5.16) holds.

Using the equality $f(Q) = K$ once again we see there is a point $c \in Q$ with $f(c) = d$. Since $g \in B(f, \varepsilon)$, we conclude from (5.11) that $\rho(f(c), g(c)) < \varepsilon$, i.e., $g(c) \in B(d, \varepsilon)$, whence by (5.7) we have $g(c) \in \operatorname{int} K \subset K \subset C$. Thus $g(Q) \cap C \neq \varnothing \neq g(Q) \cap C'$, and therefore $g(Q)$ is a subcontinuum of V (see (5.16)) intersecting two distinct components of V. The contradiction completes the proof of claim (5.9).

Now it is enough to define $L = \operatorname{cl} C$. In fact, we have $K \subset \operatorname{int} C \subset \operatorname{int} \operatorname{cl} C = \operatorname{int} L$, and $C \subset V$ implies $L \subset \operatorname{cl} V$. The proof is finished.

Proposition 5.10 can also be shown in some other way. Namely the proposition is a corollary to a more general result which will be formulated and proved below using hyperspace techniques.

Given a continuum X, we denote by 2^X the hyperspace of all nonempty closed subsets of X, the topology on which is induced by the Hausdorff distance dist (see [**27**, §42, II, p. 47]). Recall that the hyperspace 2^X of a continuum X is an arcwise connected continuum (see e.g. [**41**, (1.13), p. 65]). A continuum X is said to be connected im kleinen at a point $p \in X$ provided that for each open neighborhood V of p in X there exists an open neighborhood W of p contained in V and such that if $x \in W$, then there is a connected subset Z of V containing both p and x (see [**41**, (1.88.2) and (1.89), pp. 132 and 133]). It is known [**19**, Theorem 1, p. 388]) that the hyperspace 2^X of a continuum X is connected im kleinen at a continuum K if and only if for each open subset V of X containing K there is a component of V that contains K in its interior. Since the above condition is obviously equivalent to that mentioned in (5.5), Proposition 5.10 can be reformulated as

PROPOSITION 5.11. *If a continuum X is homogeneous with respect to the class of all open mappings, then*

(5.17) *2^X is connected im kleinen at each continuum K of X with nonempty interior.*

Note that Propositions 5.10 and 5.11 are also equivalent to the following one.

PROPOSITION 5.12. *If a continuum X is homogeneous with respect to the class of all open mappings, and K is a subcontinuum of X with nonempty interior, then X/K is connected im kleinen at K.*

Recall that a continuum X with a metric ρ is said to have the property of Kelley provided that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for each two points a and b of X satisfying $\rho(a, b) < \delta$ and for each subcontinuum A of X containing the point a there exists a subcontinuum B of X containing the point b and satisfying $\operatorname{dist}(A, B) < \varepsilon$. The property, defined by J. L. Kelley in [**22**, (3.2), p. 26], is known as a tool of great importance in the theory of hyperspaces of continua. For a survey of results in this topic see e.g. Nadler's book [**41**, pp. 538ff].

If a continuum is homogeneous with respect to open mappings, then it has the property of Kelley [**7**, Statement, p. 380]. Therefore Proposition 5.11 is a corollary to the following result which is due to W. J. Charatonik.

PROPOSITION 5.13 (W. J. CHARATONIK). *If a continuum X has the property of Kelley, then condition* (5.17) *holds.*

PROOF. Let a subcontinuum K of X have nonempty interior. Since (5.17) is equivalent to (5.5), it is enough to show that, given any $\varepsilon > 0$, there exists a continuum L in X such that $K \subset \operatorname{int} L$ and $L \subset B(K, \varepsilon)$. Take a point $p \in \operatorname{int} K$ and let a number $\eta > 0$ be such that $B(p, \eta) \subset \operatorname{int} K$ and $\eta < \varepsilon/2$. Further, let a number $\delta > 0$ be properly chosen for this η as in the definition of the property of Kelley. Then for each point $x \in B(K, \delta)$ there exists a continuum K_x in X such that $x \in K_x$ and $\operatorname{dist}(K, K_x) < \eta$. Thus $K_x \cap B(p, \eta) \neq \varnothing$, whence $K \cap K_x \neq \varnothing$. Define L as the closure of the union $\bigcup\{K_x\colon\ x \in B(K, \delta)\}$. Therefore L is a continuum containing $B(K, \delta)$. So $K \subset \operatorname{int} L$, and $\operatorname{dist}(K, L) \leqslant \eta < \varepsilon$. The proof is complete.

We end the present section with some remarks which concern confluent mappings between compact spaces. Recall that a mapping $f\colon X \to Y$ from a compact space X onto a space Y is said to be—confluent, provided for each continuum K in Y and each component C of $f^{-1}(K)$ we have $f(C) = K$ [**6**, p. 213];—weakly confluent, provided for each continuum K in Y there exists a component C of $f^{-1}(K)$ such that $f(C) = K$ [**30**, p. 98];—locally confluent, provided for each point $y \in Y$ there exists a closed neighborhood V of y in Y such that the partial mapping $f \mid f^{-1}(V)$ is a confluent one of $f^{-1}(V)$ onto V [**31**, 4, p. 106];—an OM-mapping, provided f is representable as the composition $f = gh$ such that h is monotone and g is open [**31**, 3, p. 104].

It is known that

(5.18) All open mappings and all monotone mappings are confluent [**6**, V and VI, p. 214].

(5.19) The composition of confluent (weakly confluent) mappings is confluent (weakly confluent, respectively) [**6**, III, p. 214 and **31**, 4.4, p. 107].

(5.20) All confluent mappings are both weakly confluent and locally confluent [**31**, 4.3, p. 107].

(5.21) All locally confluent mappings onto locally connected spaces are OM-mappings [**31**, Corollary 5.2, p. 109].

Thus we have

(5.22) Every light locally confluent mapping onto a locally connected space is light open.

Indeed, let a compact space Y be locally connected and let a mapping $f\colon X \to Y$ of a compact space X onto Y be light and locally confluent. By (5.21) there are mappings g and h such that g is open, h is monotone, and $f = gh$. Since f is light

and h is monotone, the point-inverses under h must be singletons, and thus h is one-to-one; so f is open, and being light it is light open.

It can easily be observed that the argumentation presented in the proof of Proposition 5.1 for light open mappings is valid also in the case if "open" is replaced by "confluent" (but not if we substitute "locally confluent" for "open", since the composition of locally confluent mappings need not be locally confluent, see [**31**, Example 4.5, p. 107]. However, even if such a proposition for locally confluent mappings would be true, neither proposition (for confluent and for locally confluent mappings) obtained in this way is interesting enough, because of (5.22): no essentially new class of mappings would be considered in such a proposition.

The situation is not the same if we consider the class of weakly confluent mappings, for there are locally connected continua which are homogeneous with respect to light weakly confluent mappings without being homogeneous with respect to open ones. Namely we have

EXAMPLE 5.14. The one-point union of the Menger universal curve and of an arc is homogeneous with respect to the class of light weakly confluent mappings, but it is not homogeneous with respect to the class of open mappings.

In fact, let $X = U \cup A$ be the union of the Menger universal curve U and of an arc A with endpoints a and b such that $U \cap A = \{a\}$. Now an argument for "not" is well known: the point b is an endpoint of X (in the sense of Menger-Urysohn order of a point, see [**27**, §51, I, p. 274] which is an invariant under open mappings (see [**51**, Corollary 7.31, p. 147]), while no other point of X is its endpoint. For the rest of the argument note that there is a light weakly confluent mapping from X onto U, namely such a mapping f that $f|U$ is the identity and $f|A$ is an embedding of A into U. Then an argument for "yes" is now contained in the following

PROPOSITION 5.15. *If a locally connected continuum X can be mapped onto the Menger universal curve under a light weakly confluent mapping, then X is homogeneous with respect to the class of light weakly confluent mappings.*

A proof of this proposition is exactly the same as one of Proposition 5.1, using (5.19) for weakly confluent mappings.

Propositions 5.1 and 5.15 have a common generalization which runs as follows.

PROPOSITION 5.16. *Let a class M of mappings between locally connected continua be such that for each triad (f, h, g) of surjections—f from a domain space X onto the Menger universal curve U, h: $U \to U$, and g: $U \to X$—if f is in M, h is a homeomorphism and g is light (monotone) open, then the composition ghf: $X \to X$ is in M. If there is in M a light (a monotone) mapping from X onto U, then X is homogeneous with respect to the class of light (of monotone) mappings belonging to M.*

A proof runs like one of Proposition 5.1, and so it is left to the reader. Note that the classes of light, monotone, open, confluent or weakly confluent mappings may be substituted for M in Proposition 5.16.

In connection with the version of the Effros theorem for open mappings which has been presented as Theorem 5.9, one can ask if openness of mappings considered in that theorem is an essential assumption. In other words we have

QUESTION 5.17. Can the assumption of openness of mappings be relaxed in Theorem 5.9 and in Propositions 5.10 and 5.11?

Note that the class of confluent mappings (in place of open ones) seems to be the most natural class for such a discussion. However, it can easily be observed that neither proof of Proposition 5.10 (or equivalently of Proposition 5.11) is good enough to be applied (with some necessary changes) to the class of confluent mappings. In fact, the former is an application of Theorem 5.9, whose proof uses openness in many essential places and whose analog for confluent mappings (if true) needs, as it seems to the authors, a different argument. In the latter proof of Proposition 5.10 a statement is strongly exploited saying that if a continuum is homogeneous with respect to the class of open mappings, then it has the property of Kelley [**7**, p. 380], which is known to be false if one replaces openness by confluency [**21**].

6. Effros' metric. In this section we define, for an arbitrary compact metric space (X, ρ) a special metric σ on X, which will be called the Effros metric. Using the Effros property we shall show that in case X is a homogeneous space the Effros metric is equivalent to the original one. The inverse implication will be shown for connected spaces X and therefore a characterization will be obtained of homogeneous continua as such continua which have an equivalent Effros metric. Recall that two metrics ρ_1 and ρ_2 on X are said to be equivalent provided that for each positive number ε there exists a positive number δ such that for every two points x and y of X if $\rho_j(x, y) < \delta$ then $\rho_k(x, y) < \varepsilon$ for $j, k \in \{1, 2\}$.

Let a metric space X with a metric ρ be compact, and let G denote the group of all homeomorphisms of X onto itself with the identity element e. We consider G as a metric space equipped with the sup metric $\hat{\rho}$ (see (1.4)). It is known that the topology induced by $\hat{\rho}$ coincides with the previously considered compact-open topology on G (see [**27**, §44, V, Theorem 2, p. 89]).

Observe that for each $h \in G$ we have $\{\rho(h(x), x)\colon x \in X\} = \{\rho(h^{-1}(x), x)\colon x \in X\}$, whence we infer

$$\hat{\rho}(h, e) = \hat{\rho}(h^{-1}, e). \tag{6.1}$$

Similarly, since for each $g, h \in G$ we have $\{\rho(hg(x), x)\colon x \in X\} = \{\rho(h(x), g^{-1}(x))\colon x \in X\}$, we get

$$\hat{\rho}(hg, e) = \hat{\rho}(h, g^{-1}). \tag{6.2}$$

Obviously

$$\rho(h(x), x) \leqslant \hat{\rho}(h, e) \quad \text{for each } x \in X. \tag{6.3}$$

Given any two points $x, y \in X$ put $G(x, y) = \{h \in G\colon h(x) = y\}$. Define now a function σ from $X \times X$ into the reals, putting for each $x, y \in X$

$$\sigma(x, y) = \begin{cases} \inf\{\hat{\rho}(h, e)\colon h \in G(x, y)\}, & \text{if } G(x, y) \neq \varnothing, \\ \operatorname{diam}(X, \rho), & \text{if } G(x, y) = \varnothing. \end{cases} \tag{6.4}$$

The definitions of σ and of $\hat{\rho}$ imply

$$\sigma(x, y) \leqslant \operatorname{diam}(X, \rho) \quad \text{for all } x,\ y \in X. \tag{6.5}$$

According to (6.3) we have $\rho(x, y) \leqslant \hat{\rho}(h, e)$ for all $x,\ y \in X$ and all $h \in G(x, y)$, whence by (6.4) we conclude

$$\rho(x, y) \leqslant \sigma(x, y) \quad \text{for all } x,\ y \in X. \tag{6.6}$$

The definition of σ and (6.6) imply

$$\sigma(x, y) = 0 \quad \text{iff} \quad x = y. \tag{6.7}$$

It follows from (6.1) that

$$\sigma(x, y) = \sigma(y, x) \quad \text{for all } x,\ y \in X. \tag{6.8}$$

Further,

$$\sigma(x, z) \leqslant \sigma(x, y) + \sigma(y, z) \quad \text{for all } x,\ y,\ z \in X. \tag{6.9}$$

Indeed, consider first the case when there are homeomorphisms $g,\ h \in G$ with $g(x) = y$ and $h(y) = z$. Then their composition hg is in G and maps x onto z. Thus for an arbitrary positive number ε by the definition of σ we have $\hat{\rho}(g, e) \leqslant \sigma(x, y) + \varepsilon$ and $\hat{\rho}(h, e) \leqslant \sigma(y, z) + \varepsilon$. Thus by (6.2) and (6.1) we get

$$\begin{aligned}\sigma(x, z) &\leqslant \hat{\rho}(hg, e) = \hat{\rho}(h, g^{-1}) \leqslant \hat{\rho}(h, e) + \hat{\rho}(g^{-1}, e)\\ &= \hat{\rho}(h, e) + \hat{\rho}(g, e) \leqslant \sigma(x, y) + \sigma(y, z) + 2\varepsilon,\end{aligned}$$

whence (6.9) follows in this case. In the opposite case at least one summand of the right member of the inequality in (6.9) is equal to $\operatorname{diam}(X, \rho)$, and so the inequality holds by (6.5). Thus (6.9) is established.

Conditions (6.7)–(6.9) imply

$$\text{the function } \sigma \text{ is a metric on } X. \tag{6.10}$$

Furthermore, since for each $h \in G$ and for each $x \in X$ we have $\sigma(h(x), x) \leqslant \hat{\rho}(h, e)$ and $\rho(h(x), x) \leqslant \sigma(h(x), x)$, hence we get

$$\hat{\rho}(h, e) = \hat{\sigma}(h, e) \quad \text{for each homeomorphism } h \text{ in } G, \tag{6.11}$$

where $\hat{\sigma}$ is the sup metric on G determined by σ, i.e., $\hat{\sigma}(g, h) = \sup\{\sigma(g(x), h(x)) : x \in X\}$, for all $g,\ h \in G$. Therefore the following proposition has just been proved.

PROPOSITION 6.1. *Let a metric space (X, ρ) be compact, and let σ be the Effros metric on X. Then for each positive number $\varepsilon < \operatorname{diam}(X, \rho)$ and for each two points x and y of X the following conditions are equivalent*:

(6.12) $\sigma(x, y) < \varepsilon$,

(6.13) *there is a homeomorphism $h \in G(x, y)$ such that $\hat{\rho}(h, e) < \varepsilon$,*

(6.14) *there is a homeomorphism $h \in G(x, y)$ such that $\hat{\sigma}(h, e) < \varepsilon$.*

As an immediate consequence of Proposition 6.1 we get

COROLLARY 6.2. *Let the Effros metric σ be defined by* (6.4) *on a compact metric space (X, ρ). Then the space (X, σ) is discrete if and only if the identity mapping e is an isolated element of the group G of all homeomorphisms of (X, ρ) onto itself.*

To better understand how the Effros metric acts, consider now three examples. In the first one let $X = [0,1]$ and $\rho(x, y) = |x - y|$. Then (X, σ) is the disjoint union of three components: two of them correspond to the endpoints $\{0\}$ and $\{1\}$, and the other is homeomorphic to the open unit interval $(0,1)$. For the second example take X as the unit circle with its natural topology induced by the euclidean metric ρ in the plane, and note that (X, σ) is homeomorphic to (X, ρ). The third example is the well-known $\sin 1/x$ curve, i.e., the closure (in the plane) of the set $\{(x, \sin 2\pi/x)$: $0 < x \leqslant 1\}$ equipped with the euclidean metric ρ. Then (X, σ) is homeomorphic to the disjoint union of five components: of three singletons that correspond to the endpoints of (X, ρ), namely to $(0,1)$, $(0,-1)$ and $(1,0)$, and of two open intervals which correspond to the sets $\{(0, y)$: $-1 < y < 1\}$ and $\{(x, \sin 2\pi/x)$: $0 < x < 1\}$.

For an arbitrary compact metric space (X, ρ) let us denote by i the identity from (X, σ) onto (X, ρ) and observe by (6.6) that

$$i\colon (X, \sigma) \to (X, \rho) \text{ is continuous.} \tag{6.15}$$

Note that in general i is neither open nor closed, and that i^{-1} need not be continuous, as the first and the third examples above show.

Given two compact metric spaces (X_1, ρ_1) and (X_2, ρ_2), let σ_1 and σ_2 denote the Effros metrics on X_1 and X_2 respectively, and let i_1: $(X_1, \sigma_1) \to (X_1, \rho_1)$ and i_2: $(X_2, \sigma_2) \to (X_2, \rho_2)$ be the identities. For each surjection f: $(X_1, \rho_1) \to (X_2, \rho_2)$ we define f_*: $(X_1, \sigma_1) \to (X_2, \sigma_2)$ putting $f_* = i_2^{-1} f i_1$; and for each surjection g: $(X_1, \sigma_1) \to (X_2, \sigma_2)$ we define g^*: $(X_1, \rho_1) \to (X_2, \rho_2)$ putting $g^* = i_2 g i_1^{-1}$. Note that f_* and g^* are well-defined surjections and that $(f_*)^* = f$ and $(g^*)_* = g$. So by the definitions the following diagrams commute:

$$\begin{array}{ccc} (X_1, \rho_1) & \xrightarrow{f} & (X_2, \rho_2) \\ {\scriptstyle i_1}\uparrow & & \uparrow{\scriptstyle i_2} \\ (X_1, \sigma_1) & \xrightarrow[f_*]{} & (X_2, \sigma_2) \end{array} \tag{6.16}$$

$$\begin{array}{ccc} (X_1, \rho_1) & \xrightarrow{g^*} & (X_2, \rho_2) \\ {\scriptstyle i_1}\uparrow & & \uparrow{\scriptstyle i_2} \\ (X_1, \sigma_1) & \xrightarrow[g]{} & (X_2, \sigma_2) \end{array} \tag{6.17}$$

i.e., we have $f i_1 = i_2 f_*$ for (6.16) and $g^* i_1 = i_2 g$ for (6.17).

Furthermore, observe that continuity of f in (6.16) does not imply that of f_* in general. Indeed, let X_1 and X_2 be the unit circle and the unit closed interval, respectively, with their natural topologies (stemming from the euclidean topology of the plane), and let σ_1 and σ_2 be the corresponding Effros metrics. As we already know, (X_1, σ_1) is homeomorphic to (X_1, ρ_1) (so it is connected), while (X_2, σ_2) is

composed of three components. Therefore for no continuous surjection $f\colon (X_1, \rho_1) \to (X_2, \rho_2)$ can $f_*\colon (X_1, \sigma_1) \to (X_2, \sigma_2)$ be continuous. So it makes sense to ask the following

QUESTION 6.3. Under what conditions does continuity of f in (6.16) imply that of f_*?

Below (see Proposition 6.6) we give a partial answer to this question: the implication holds if f is a homeomorphism. Similarly, continuity of g in (6.17) does not imply that of g^* even if $X_1 = X_2$ and g is a homeomorphism. In fact, let $X_1 = X_2 = [0,1]$ and let $\rho_1 = \rho_2 = \rho$ be the natural metric on $[0,1]$. As we know $(X_1, \sigma_1) = (X_2, \sigma_2)$ is the disjoint union of three components: $\{0\}$, $\{1\}$, and $[0,1] \setminus \{0,1\}$, each of which is considered with the natural metric. Let g interchange the ends in $[0,1]$ with the Effros metric, i.e., let $g(x) = 1 - x$ for $x \in \{0,1\}$ and $g(x) = x$ for $x \in [0,1] \setminus \{0,1\}$. Thus g is a homeomorphism of $([0,1], \sigma)$ onto itself, while $g^*\colon ([0,1], \rho) \to ([0,1], \rho)$ which is described by the same formulas as g is, but which is defined on $([0,1], \rho)$, is obviously not continuous. So we again have a question.

QUESTION 6.4. Under what conditions does continuity of g in (6.17) imply that of g^*?

To prove Proposition 6.6 mentioned above we need a lemma. The lemma is not connected with the Effros metric and runs as follows.

LEMMA 6.5. *Let a homeomorphism $f\colon (X_1, \rho_1) \to (X_2, \rho_2)$ between two compact metric spaces be given. For $j \in \{1,2\}$ denote by G_j the group of all homeomorphism of (X_j, ρ_j) onto itself, having the identity element e_j. If a sequence $h_n \in G_1$ tends to e_1 (with respect to the* sup metric *$\hat{\rho}_1$ on G_1), then the sequence $fh_nf^{-1} \in G_2$ tends to e_2 (with respect to $\hat{\rho}_2$).*

Indeed, this is an immediate consequence of statements (1) and (2) of §3.4 of [**11**, p. 206].

PROPOSITION 6.6. *If, in diagram* (6.16), *the mapping f is a homeomorphism, then f_* is also a homeomorphism.*

PROOF. Note that if f in (6.16) is one-to-one, then so is f_* by its definition. We claim that

(6.18) if f in (6.16) is a homeomorphism, then f_* is continuous.

In fact, let a sequence of points x_n of (X_1, σ_1) tend to a point x. By the equivalence of (6.12) and (6.13) in Proposition 6.1 we conclude there is a sequence of homeomorphisms $h_n \in G_1(i_1(x), i_1(x_n))$ of (X_1, ρ_1) onto itself which tends to the identity $e_1 \in G_1$ (with respect to the metric $\hat{\rho}_1$ on G_1). Putting $g_n = fh_nf^{-1} \in G_2$ we see by Lemma 6.5 that g_n tends to e_2 (with respect to the metric $\hat{\rho}_2$ on G_2). Since $g_n(f(i_1(x))) = f(i_1(x_n))$ (i.e., $g_n \in G_2(f(i_1(x)), f(i_1(x_n))))$ simply by the definitions, we infer—again from the equivalence between (6.13) and (6.12) of Proposition 6.1—that the sequence of points $i_2^{-1}(f(i_1(x_n))) = f_*(x_n)$ of (X_2, σ_2) tends to the point $i_2^{-1}(f(i_1(x))) = f_*(x)$. Thus claim (6.18) is proved.

Now to show continuity of $(f_*)^{-1}$ it is enough to apply (6.18) to the mapping f^{-1} in place of f and to note that $(f_*)^{-1} = (f^{-1})_*$. The proof is complete.

As a consequence of Proposition 6.6 we conclude that the concept of the Effros metric σ is in a way independent of the original metric on the compact space X. More precisely, we have

COROLLARY 6.7. *If metrics ρ_1 and ρ_2 on an underlying compact space X are equivalent, then the corresponding Effros metrics σ_1 and σ_2 on X also are equivalent.*

Recall that a space X is said to be homogeneous between two points $p, q \in X$ if there exists a homeomorphism h of X onto itself such that $h(p) = q$ (i.e., if $h \in G(p, q)$).

PROPOSITION 6.8. *If a subset S of a compact metric space (X, ρ) is connected with respect to the Effros metric σ, then both spaces (X, ρ) and (X, σ) are homogeneous between each two points of S.*

PROOF. Since $(S, \sigma \mid S \times S)$ is a connected subspace of the space (X, σ), it is well chained, i.e., for each two points $x, y \in S$ and for each positive number ε there exists a finite sequence of some $n + 1$ points $x_0, x_1, \ldots, x_n$ of S such that $x_0 = x$, $x_n = y$ and $\sigma(x_j, x_{j+1}) < \varepsilon$ for every $j \in \{0, 1, \ldots, n - 1\}$ (see [**51**, p. 13 and (8.3), p. 14]). Let $\varepsilon < \operatorname{diam}(X, \rho)$. Thus, by Proposition 6.1, for each $j \in \{0, 1, \ldots, n - 1\}$ there is a homeomorphism h_j of (X, ρ) onto itself such that $h_j \in G(x_j, x_{j+1})$. Then the composition $h = h_n h_{n-1} \cdots h_1 h_0$ is the needed homeomorphism of (X, ρ) onto itself which is in $G(x, y)$. So (X, ρ) is homogeneous between x and y.

Now consider S as a subset of (X, σ) and take two arbitrary points x' and y' of S. Putting $x = i(x')$ and $y = i(y')$, and using the result we have just proved, we find a homeomorphism h: $(X, \rho) \to (X, \rho)$ with $h(x) = y$. Applying Proposition 6.6 we see that h_* is a homeomorphism of (X, σ) onto itself. Finally we have $h_*(x') = i^{-1}hi(x') = i^{-1}h(x) = i^{-1}(y) = y'$. So (X, σ) is homogeneous between x' and y', which completes the proof.

COROLLARY 6.9. *Given a compact metric space (X, ρ), let σ be the Effros metric on X. If C is a component of (X, σ), then $(C, \rho \mid C \times C)$ and $(C, \sigma \mid C \times C)$ are both homogeneous subspaces of (X, ρ) and of (X, σ) respectively.*

In fact, apply Proposition 6.8 and note that, since C is a component of (X, σ), we have $h_*(C) \subset C$, where h_* is the homeomorphism considered in the final part of the proof of Proposition 6.8. Since the inclusion holds true for each component of (X, σ) and since h_* is a surjection, we have $h_*(C) = C$, and therefore $(C, \sigma \mid C \times C)$ is homogeneous. Now a similar equality for h holds (if C is considered to be a subset of (X, ρ)), i.e., $h(C) = C$, simply by the definition of h_*, whence we conclude that $(C, \rho \mid C \times C)$ also is homogeneous.

Assume now the space (X, ρ) under consideration is homogeneous. Then we conclude that σ and ρ are equivalent: one of the two implications considered in the definition of equivalent metrics follows from (6.6), and the other one is a consequence of the Effros property (Corollary 4.1). So we have

COROLLARY 6.10. *If a compact metric space X is homogeneous, then the Effros metric σ on X is equivalent to the original metric ρ on X.*

Note that, by Proposition 6.1, we can consider Corollary 6.10 as a reformulation of the Effros property for the space X.

As a consequence of Corollaries 6.9 and 6.10 we have

COROLLARY 6.11. *Given a compact metric space* (X, ρ), *let* σ *be the Effros metric on* X. *If the space* (X, σ) *is connected, then* (X, ρ) *and* (X, σ) *are both homogeneous continua which are homeomorphic.*

In fact, if (X, σ) is connected, then—applying notations of Corollary 6.9—we have $C = X$, whence (X, ρ) is a continuum; further, the continuum (X, ρ) is homogeneous again by Corollary 6.9, and thus the conclusion follows from Corollary 6.10.

As we have seen from Proposition 4.2, the implication from the Effros property to homogeneity of a compact metric space (and conversely, see Corollary 4.3) holds if the space is connected. The same holds for a converse to Corollary 6.10. Namely we have

PROPOSITION 6.12. *If for a given continuum the Effros metric is equivalent to the original one, then the continuum is homogeneous.*

In fact, assume that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for every two points x and y of a continuum X if $\rho(x, y) < \delta$, then $\sigma(x, y) < \varepsilon$. The last inequality is equivalent by Proposition 6.1 to (6.13), which obviously means the Effros property for X. Now the conclusion holds by Proposition 4.2.

Observe that a previous example of the disjoint union of the circle and the pseudo-arc shows that connectedness of the space is an essential assumption in Proposition 6.12.

Corollaries 4.3, 6.9, 6.11, and Proposition 6.12 can be summarized as

THEOREM 6.13. *The following conditions are equivalent for a continuum* (X, ρ):

(i) *X is homogeneous,*

(ii) *X has the Effros property,*

(iii) *X has an equivalent Effros metric σ defined by* (6.4),

(iv) (X, σ) *is connected.*

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