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DOUBLY SLICED KNOTS WHICH ARE NOT THE DOUBLE OF A DISK

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ABSTRACT. In this paper we show that double disk knots can be distinguished from general doubly sliced knots in dimensions $4n+1$.

A double disk knot is formed by unioning two identical disk knots along their boundary. J. P. Levine has demonstrated that these knots are all doubly sliced [**9**], i.e., they can be realized as a slice of the trivial knot. Spun knots, high dimensional ribbon knots [**2**], and Sumners' knots constructed in [**13**] are all examples of double disk knots.

In [**9**] Levine also gives an example of a classical knot and a 2-knot that are doubly sliced but not the double of a disk. In this paper we show that doubly sliced knots are distinct from double disk knots in dimensions $4n+1$. Our method of distinguishing double disk knots produces obstructions from the Casson-Gordon invariants. This paper is the main result of the author's Ph.D. thesis and he wishes to thank his advisor, J. P. Levine, for his help and encouragement.

An n dimensional knot is a codimension two spherical knot or a smooth oriented pair (S^{n+2}, K) where K is a submanifold which is homeomorphic to S^n. An n dimensional disk knot is a smooth oriented pair (B^{n+2}, D) where D is homeomorphic to the n-disk and $\partial B^{n+2} \cap D = \partial D$.

We apply disk knots to the study of knots in two distinct ways. First, the boundary of an n-disk knot (B^{n+2}, D) is the $(n-1)$-knot $(\partial B^{n+2}, \partial D)$. Second, we may join two n-disk knots that have diffeomorphic boundaries along their boundaries. Here we obtain an n-knot. If two disk knots (B_1^{n+2}, D_1) and (B_2^{n+2}, D_2) have diffeomorphic boundaries by some orientation preserving diffeomorphism, $f\colon \partial(B_1, D_1) \to \partial(B_2, D_2)$, then we may form the n-knot

$$-(B_1, D_1) \cup_f (B_2, D_2) = (-B_1 \cup_f B_2, -D_1 \cup_f D_2).$$

Given an n-disk knot (B^{n+2}, D), one can construct an $(n+1)$-disk knot $\Sigma(B, D)$ called the suspension of D. In the P.L. category the suspension of D may be realized as $(B^{n+2} \times I, D \times I)$ [**9**]. We use the smooth version, obtained by rounding the corners. The boundary of ΣD is $(-B^{n+2}, -D) \cup_I (B^{n+2}, D)$, the n-knot formed by doubling the disk (B^{n+2}, D). Knots formed by doubling a disk are called *double disk knots.*

A knot which is the boundary of a disk knot is called null cobordant. Some knots bound particularly nice disk knots and are slices of the trivial knot. The disk knot $(B^{n+2}, D) = D^{n+2,n}$ is *invertible* if there exists a disk knot $(B^{n+2}, \Delta) = \Delta^{n+2,n}$ and a diffeomorphism, $f\colon \partial D^{n+2,n} \to \partial \Delta^{n+2,n}$ such that $-D^{n+2,n} \cup_f \Delta^{n+2,n}$ is the

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trivial knot. Knots which bound invertible disks are called doubly-null-cobordant or doubly sliced [**13**]. "Doubly" refers only to a higher order of slicing and is not an operational word as in double disk knot.

Our observations rely upon the Casson-Gordon invariants [**4**] as generalized by Gilmer [**5**]. We now define these invariants.

Let M be a $(2k-1)$-manifold and d an integer. Isomorphism classes of Z_d-covers of M with a specified generator for the covering translations correspond to elements of $[M, BZ_d] = H^1(M, Z_d) = \operatorname{Hom}(H_1(M), Z_d)$. If $\psi \in \operatorname{Hom}(H_1(M), Z_d)$ we get $\langle \tilde{M}, \tau\rangle$, such a covering with a generator. There exists an n and a $2k$-manifold with boundary W such that $\langle \tilde{W}, T\rangle$ is a Z_d cover of W and $\partial\langle \tilde{W}, T\rangle = n \cdot \langle \tilde{M}, \tau\rangle$,

$$\partial\begin{pmatrix} \tilde{W} \\ \downarrow \\ W \end{pmatrix} = n \begin{pmatrix} \tilde{M} \\ \downarrow \\ M \end{pmatrix}.$$

This fact is essentially that $\Omega_{2k-1}(BZ_d)$ is torsion. Let $\overline{H}_k(W, \psi) = e^{2\pi i/d}$ eigenspace of T_* in $H_k(\tilde{W}) \otimes \mathbf{C}$. If k is even then the intersection form, $\langle x, y\rangle = x \cdot y$, is Hermitian and if k is odd the form $\langle x, y\rangle = ix \cdot y$ is Hermitian. Let $\overline{\sigma}(W, \psi)$ denote the signature of $\langle\ ,\ \rangle|_{\overline{H}_k(W,\psi)}$. We define the Casson-Gordon invariant as

$$\sigma(M, \psi) = \frac{1}{n}(\overline{\sigma}(W, \psi) - \sigma(W)).$$

Gilmer [**5**] shows this invariant to be well defined.

An m component link of dimension n or an m-link is an ordered collection of m disjoint smooth oriented submanifolds of S^{n+2}, each of which is homeomorphic to S^n. We assume $n > 2$ and our links will be ordered. A link is denoted by L or $(S^{n+2}; L_1, \ldots, L_m)$. Every m-link is bounded by a Seifert surface, W. A link is a boundary link if it has an m-component Seifert surface $W = W_1 \cup \cdots \cup W_m$ with $\partial W_i = L_i$.

The following construction and invariant appears in [**11**] and is further developed in [**12**]. We state the needed results. These results generalize [**15**].

Let $V_1, \ldots, V_m \subset D^{n+3}$ be disjoint codimension two submanifolds with trivial normal bundles such that $V_i \cap \partial D^{n+3} = \partial V_i = L_i$. Denote such an ordered set by $(D^{n+3}; V_1, \ldots, V_m)$ and call this collection a special m-tuple. For every link there are special m-tuples. By Alexander duality $H_1(D^{n+3} - (V_1 \cup \cdots \cup V_m)) = Z^m$, so let μ_i be the meridian to V_i. Let $G = Z_{a_1} \oplus \cdots \oplus Z_{a_m}$. We really wish to consider m-tuples of cyclic groups $Z_{a_1}, \ldots, Z_{a_m}$). The order is important and we indicate it by ordering the summands of G. We also choose preferred generators for G; let g_i be the generator of Z_{a_i}.

THEOREM 1. *Let $(D^{n+3}; V_1, \ldots, V_m)$ be a special m-tuple and let $G = Z_{a_1} \oplus \cdots \oplus Z_{a_m}$, one summand corresponding to one submanifold of D^{n+3}. Then there is a canonical G-manifold M_V that has the following properties*:

(1) $M \xrightarrow{\pi} M/G = D^{n+3}$.

(2) $M - \pi^{-1}(V_1 \cup \cdots \cup V_m) \to D^{n+3} - (V_1 \cup \cdots \cup V_m)$ *is the regular G-covering space arising from the map*

$$\varphi\colon H_1(D^{n+1} - (V_1 \cup \cdots \cup V_m)) \to G, \qquad \varphi(\mu_i) = g_i.$$

(3) $\pi^{-1}(V_i) \xrightarrow{\pi} V_i$ *is a regular covering space with group G/Z_{a_i}.*

Suppose $n = 2q-1$. Let $\langle\ ,\ \rangle$ again denote the Hermitian pairing on $H_{q+1}(M_V)\otimes \mathbf{C}$. If ω_i is an a_ith root of unity we let $E = \bigcap_{i=1}^{m}$ (ω_i eigenspace of g_{i*}). Define the link invariant $\operatorname{sig}_L(\omega_1,\dots,\omega_m)$ as signature $(\langle\ ,\ \rangle|_E)$. If χ is the irreducible characer $\chi(g_i) = \omega_i$, the inner product on characters is $[\ ,\ ]$ and $\operatorname{Sign}(G, M_V)$ is the character from the G-signature representation [**3**], then

$$\operatorname{sig}_L(\omega_1,\dots,\omega_m) = [\operatorname{Sign}(G, M_V), \chi].$$

This invariant is independent of the chosen V_i's and is a generalization of the Levine-Tristram signature function, $\sigma_L(\ )$ [**8, 14**]. In particular, $\sigma_L(\omega) = \operatorname{sig}_L(\omega,\dots,\omega)$.

We suppose L is a $(2q-1)$-dimensional boundary link and $\hat{V} = \hat{V}_1 \cup \cdots \cup \hat{V}_m$ is a collection of disjoint Seifert surfaces for the L_i. Let $S = (A_{ij})$ be a Seifert matrix for $\hat{V}$. The matrix S is composed of blocks of matrices. A diagonal block A_{ii} is an $l_i \times l_i$ Seifert matrix for $\hat{V}_i$. The off-diagonal blocks record the linking information between the various $\hat{V}_i$. More details appear in [**7**].

THEOREM 2. *If $\omega_1,\dots,\omega_m$ is a collection of roots of unity and L is the link described above, then*

$$\operatorname{sig}_L(\omega_1,\dots,\omega_m) = \begin{cases} \operatorname{sign}(i(I-W)(-SW^{-1} - S^T)), & q \text{ even},\\ \operatorname{sign}((I-W)(-SW^{-1} + S^T)), & q \text{ odd},\end{cases}$$

where

$$W = \begin{pmatrix} [\omega_1] & & & \\ & [\omega_2] & & 0 \\ & 0 & \ddots & \\ & & & [\omega_m] \end{pmatrix}, \qquad [\omega_i] = \omega_i I_{l_i\times l_i}.$$

We also remark that if L is a boundary link then sig_L is an invariant of its boundary link cobordism class and if L is null boundary link cobordant then $\operatorname{sig}_L \equiv 0$.

Detecting double disk knots. If K is a knot and D a disk knot we write N_K and N_D to denote the cyclic cover of the sphere or disk branched along the knot or disk knot which corresponds to the map $H_1(\text{exterior}) \to Z_d$ by $\mu \to [1]$.

THEOREM 3. *Let (S^{2q+1}, K) be the double of the disk knot (B^{2q+1}, Δ). Also let d and a be positive integers. If N_K is the d-fold branched cyclic cover of (S^{2q+1}, K) then there is a direct sum decomposition $H_1(N_K) = A \oplus B$ satisfying:*
(1) *There is an epimorphism $A \twoheadrightarrow B$.*
(2) *If $\phi: H_1(N_K) \to Z_a$ is a map such that $\phi|_B = 0$ then $\sigma(N_K, \phi) = 0$.*

REMARK. The obstructions in Theorem 3 differ from Ruberman's obstructions [**10**] on three points. First, the existence of an epimorphism allows us to distinguish summands of $H_1(N_K)$. Second, for the Casson-Gordon invariant bound, Ruberman shows $|\sigma(N_K, \phi)| \le \dim(H_{q+1}(N_K; Z_a))$ while ours is zero. Third, Ruberman must assume $a = p^r$ for some prime p while our a is arbitrary.

PROOF. According to [**9**] every suspension in invertible. Let (B^{2q+2}, D) be an inverse to $\Sigma\Delta$ so that $D \cup_f \Sigma\Delta$ is unknotted. The usual Mayer-Vietoris sequence argument [**6**] yields

$$0 = H_2(S^{2q+2}) \to H_1(N_K) \to H_1(N_{\Sigma\Delta}) \oplus H_1(N_D) \to H_1(S^{2q+2}) = 0$$

since $N_{\text{unknot}} = S^{2q+2}$. Let $B = \operatorname{Ker}\{H_1(N_K) \xrightarrow{i_*} H_1(N_{\Sigma\Delta})\} \approx H_1(N_D)$ and $A = \operatorname{Ker}\{H_1(N_K) \to H_1(N_D)\} \approx H_1(N_{\Sigma\Delta})$. So $H_1(N_K) = A \oplus B$.

Property 1. Consider the sequence of the pair $N_{\partial\Delta} \subset N_\Delta$,

(i) $H_1(N_{\partial\Delta}) \to H_1(N_\Delta) \xrightarrow{p_*} H_1(N_\Delta, N_{\partial\Delta}) \to 0$

where $N_{\partial\Delta}$ is the d-fold branched cover of $(\partial B^{2q+1}, \partial\Delta)$. First note that $N_{\Sigma\Delta}$ is homeomorphic to $I \times N_\Delta$ and the composite map $N_\Delta \to N_K \to N_{\Sigma\Delta}$ is, up to a homeomorphism of $N_{\Sigma\Delta}$, $N_\Delta \to 1 \times N_\Delta \subset I \times N_\Delta$.

Consider the diagram,

$$\begin{tikzcd}
 & & 0 & & \\
 & & H_1(N_K, N_\Delta) \arrow[u] & & \\
0 \arrow[r] & B \arrow[r] \arrow[ur, "f"] & H_1(N_K) \arrow[r] \arrow[u] & H_1(N_{\Sigma\Delta}) \arrow[r] \arrow[dl, "\approx"] & 0 \\
 & & H_1(N_\Delta) \arrow[u] & & \\
 & & 0 \arrow[u] & &
\end{tikzcd}$$

A simple diagram chase yields that $f: B \to H_1(N_K, N_\Delta)$ is an isomorphism. By excision, $H_1(N_K, N_\Delta) \approx H_1(N_\Delta, N_{\partial\Delta})$ and so by p_* in (i) we have

$$A \approx H_1(N) \overset{p_*}{\twoheadrightarrow} H_1(N_\Delta, N_{\partial\Delta}) \approx B$$

is an epimorphism.

Property 2. The sequence $0 \to B \to H_1(N_K) \xrightarrow{i_*} H_1(N_{\Sigma\Delta}) \to 0$ is split exact. So if $\phi: H_1(N_K) \to Z_a$ is zero on B then ϕ extends to $\psi: H_1(N_{\Sigma\Delta}) \to Z_a$. The maps ϕ and ψ represent classes in $H^1(N_K; Z_a)$ and $H^1(N_{\Sigma\Delta}; Z_a)$ and are related by $i^*\psi = \phi$. Since $H^1(X; Z_a) = [X; BZ_a]$, we get covering spaces $\tilde{N}_K$ and $\tilde{N}_{\Sigma\Delta}$ so that

$$\begin{matrix} \tilde{N}_K \\ \downarrow \\ N_K \end{matrix} = \partial \begin{pmatrix} \tilde{N}_{\Sigma\Delta} \\ \downarrow \\ N_{\Sigma\Delta} \end{pmatrix}.$$

$N_{\Sigma\Delta}$ is homeomorphic to $N_\Delta \times I$ so the covering

$$\begin{matrix} \tilde{N}_{\Sigma\Delta} \\ \downarrow \\ N_{\Sigma\Delta} \end{matrix} \quad \text{is homeomorphic to} \quad \begin{matrix} \tilde{N}_\Delta \times I \\ \downarrow \\ N_\Delta \times I \end{matrix}$$

for some covering $\tilde{N}_\Delta \to N_\Delta$. Now, $\sigma(N_K, \phi) = \overline{\sigma}(N_{\Sigma\Delta}, \phi) - \sigma(N_{\Sigma\Delta})$ which is zero, since both $\tilde{N}_{\Sigma\Delta}$ and $N_{\Sigma\Delta}$ are homeomorphic to products and all intersections are zero. □

Doubly sliced knots. We now give a construction which produces examples of doubly sliced knots (compare with [**10**]). Let $(S^{n+2}; L_1, L_2) = L$ be a link with $\pi_1(\text{exterior}) = Z * Z$. Imbed $L \times [-1, 1]$ in S^{n+2} using the unique (up to homotopy) untwisted normal vector field to L. Let r and s be integers and $p_1, p_2 \in L_1$ and $q_1, q_2 \in L_2$. We form a class of knots $K(L; r, s)$. Connect $p_1 \times 1$ to $p_1 \times (-1)$

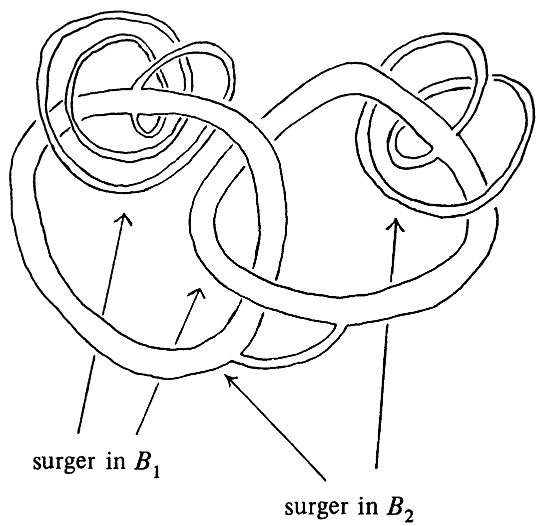


FIGURE 1

by a one handle which follows the meridian to L_1 for $r + \frac{1}{2}$ times around $L_1 \times 0$ and connect $q_1 \times 1$ to $q_1 \times (-1)$ by a 1-handle which wraps around $L_2 \times 0$ $s + \frac{1}{2}$ times following the meridian. The 1-handles are imbedded without twisting, i.e., the induced framing on $p_1 \times I$ or $q_1 \times I$ union the core of the handle is trivial and extends over a disk. Denote these 1-handles h_1^1 and h_2^1. Add a third 1-handle which connects $p_2 \times 1$ to $q_2 \times 1$. This construction depends upon the path from $p_2 \times 1$ to $q_2 \times 1$ and for each path we get a knot

$$K = \partial(L_1 \times I \cup L_2 \times I \cup \text{ the three 1-handles}).$$

The bounded manifold (i.e., the handlebody) is a Seifert surface for K and is diffeomorphic to $(S^n \times S^1 - D^{n+1})\#_\partial(S^n \times S^1 - D^{n+1})$.

PROPOSITION 4. (1) *K is a slice knot.*

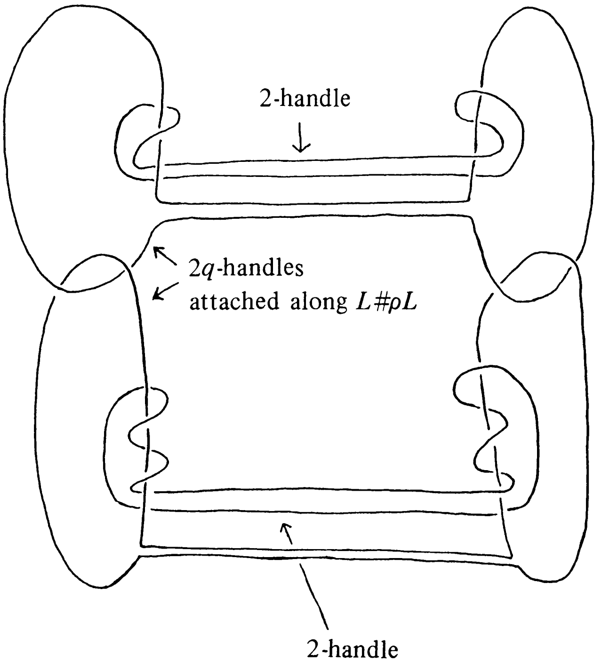
(2) *If the knots (S^{n+2}, L_1) and (S^{n+2}, L_2) are slice knots then K is a doubly sliced knot.*

PROOF. The two 1-handles, h_1^1 and h_2^1, may be surgered in B^{n+3}. The surgery changes each $S^n \times S^1 - D^n$ into D^{n+2} and we get $D^{n+2} \#_\partial D^{n+2}$ as a spanning disk for K.

(2) If L_1 and L_2 are slice knots then we can surger L_1 in B_1^{n+3} and L_2 in B_2^{n+3}. We can also surger h_2^1 in B_1^{n+3} and h_1^1 in B_2^{n+3} (see Figure 1). The result is a knot in $S^{n+3} = B_1 \cup B_2$ of which $(\partial B_1, K)$ is a slice. The trace of the above surgeries is an $(n+3)$-disk and so the knot in S^{n+3} is trivial. Therefore K is doubly sliced. □

We wish to apply Theorem 3 to the knots $K(L; r, s)$ so we now confront the problem of computing the associated Gasson-Gordon invariants.

Construction of the 2-fold cover. We denote by N the 2-fold branched cyclic cover of K. Let $\hat{V}$ denote the Seifert surface V with its interior pushed into $\overset{\circ}{B}{}^{2q+2}$. If W is the 2-fold branched cyclic cover of $(B^{2q+2}, \hat{V})$ then $\partial W = N$.



Surgery on a Link Description

FIGURE 2

The following description of W is from [**1**] where a detailed analysis is given. W is constructed from two copies of the $(2q+2)$-ball, B and B', both containing copies of the Seifert surface, V and V', in their boundaries. Identify the Seifert surfaces to obtain W. These identifications are accomplished by identifying one handle at a time. As a handlebody, V is constructed with two 0-handles, p and q; three 1-handles, an S^1 at p, an S^1 at q and an arc from p to q; and two $(2q-1)$-handles, L_1 and L_2. Identify p and p' with a 1-handle and q with q'. Next identify the 1-handle between p and q with the corresponding 1-handle in $\partial B'$. This partially constructed W is a $2q+2$ ball. To obtain W we add 2-handles and $2q$-handles. The $2q$-handles are attached along $L \# \rho L$ where $\rho(S^{2q+1}, L) = (S^{2q+1}, -L)$. One 1-handle attaches along an S^1 which links $2r+1$ times with $L_1 \# \rho L_1$ and does not link with $L_2 \# \rho L_2$. The other 1-handle links $2s+1$ times with $L_2 \# \rho L_2$ and does not link with $L_1 \# \rho L_1$. If $2r+1 = a$ and $2s+1 = b$ then $H_1(N) = Z_a \oplus Z_b$. We have a"surgery on a link" description of N (see Figure 2).

Computation of Casson-Gordon invariants. We use the G-signature theorem to compute Casson-Gordon invariants. The use of the G-signature theorem for this type of invariant is documented in [**3**, §7] and specifically for the Casson-Gordon invaraint in [**4** and **5**].

If C is a closed $2q+2$ dimensional Z_d-manifold so that C/Z_d is a manifold then the G-signature theorem dictates a formula $F(C, Z_d)$ such that

$$F(C, Z_d) + \sigma(C/Z_d) = \operatorname{sign}\langle\ ,\ \rangle|_{E_1(C)}$$

where $E_r(C) = e^{2\pi ri/d}$ eigenspace of the generator in $H_{q+1}(C)\otimes \mathbf{C}$ and $F(C, Z_d)$ is a formula involving only data about the fixed points of elements other than the identity and their normal bundles. This formula $F(C, Z_d)$ may be defined abstractly on any Z_d-manifold whether or not it is closed.

If $\phi\colon H_1(N^{2q+1}) \to Z_d$ is realized by the covering space $\tilde{N} \to N$ then $\sigma(N,\phi)$ may be computed using a bounding Z_d-manifold $\hat{M}$ where $\hat{M} \to \hat{M}/Z_d = M$ and $\partial\hat{M} \to \partial M$ is the covering $\hat{N} \to N$. An argument similar to that in [**4**] yields,

$$\sigma(N,\phi) = \operatorname{sign}\langle\ ,\ \rangle|_{E_1(\hat{M})} - \sigma(M) - F(\hat{M}, Z_d).$$

We compute $F(\hat{M}, Z_d)$ for certain types of branched covers but not the standard branched cyclic covers. Let $G = Z_{d_1}\oplus\cdots\oplus Z_{d_m}$ with $(d_i, d_j) = 1$ if $i \neq j$. Let $|G| = d$ so that $G = Z_d$. Further, write g_i for the generator of Z_{d_i} and τ for $\prod_{i=1}^m g_i$. We wish our branched G-covers, $p\colon \tilde{M} \to M$, to satisfy the following properties:

(1) There exist $V_i \subset M$, $i \leq m$, disjoint closed codimension two submanifolds.

(2) The fixed points of g_i are equivariantly $p^{-1}(V_i) = V_i \times G/Z_{d_i}$.

(3) $\nu(p^{-1}(V_i))$ is a plane bundle and the action of g_i is multiplication by a primitive d_ith root of unity.

For integers s and r we let $s \equiv s_i \bmod d_i$, $s_i < d_i$; $r \equiv r_i \bmod d_i$, $r_i < d_i$ and $\overline{s_ir_i} \equiv s_ir_i \bmod d_i$, $\overline{s_ir_i} < d_i$. The primitive dth root of unity may be written uniquely as $\omega = e^{2\pi i/d} = \omega_1\omega_2\cdots\omega_m$ where $\omega_i = e^{(2\pi i/d_i)p_i}$ is a d_ith root of unity.

LEMMA 5. *Let $\tilde{M}^{2q+2} \to M^{2q+2}$ be a closed branched Z_d-cover as described above, so τ generates the covering translations. Then,*

$$\operatorname{sign}\langle\ ,\ \rangle|_{E_r(\tilde{M})} = \sigma(M) - \sum_i 2^{2q}\frac{d_i - 2\overline{p_ir_i}}{d_i}\sigma(V_i).$$

PROOF. Let χ be the character so that $\chi(\tau) = \omega^r$. Then,

$$\begin{aligned}\operatorname{sign}\langle\ ,\ \rangle|_{E_r(\tilde{M})} &= [\operatorname{Sign}(G,\tilde{M}),\chi] = \frac{1}{d}\sum_{s=0}^{d-1}\operatorname{Sign}(\tau^s,\tilde{M})\overline{\chi(\tau^s)}\\ &= \frac{1}{d}\sum_{s=1}^{d-1}\operatorname{Sign}(\tau^s,\tilde{M})\omega^{-rs} + \sigma(M)\end{aligned}$$

since $\operatorname{Sign}(I,\tilde{M}) = \sigma(\tilde{M})$ and a transfer argument yields $\sigma(\tilde{M}) = d\sigma(M)$. The calculation of the last sum is similar to the calculation in Lemma 2.1 of [**4**] or 3.4 in [**10**]. We refer the reader to these references for details. □

PROPOSITION 6. *Let $L \subset S^{2q+1}$ be a boundary m-link and let $W = B^{2q+2} \cup \{h_i^2\}\cup\{h_j^{2q}\}$ where the h_i^2 are 2-handles and the h_j^{2q} are $2q$-handles attached along L_j. Let $M = \partial W$ and $\mu_i \in H_1(M)$ represent the ith meridian. If $\phi\colon H_1(M) \to G$ by $\phi(\mu_i) = g_i$ is a well defined map then for $0 < r < d$ we have*

$$\sigma(M, r\phi) = \begin{cases} \operatorname{sig}_L(\omega_1^{r_1},\ldots,\omega_m^{r_m}) & \text{if } q \text{ is odd,}\\ \operatorname{sig}_L(\omega_1^{r_1},\ldots,\omega_m^{r_m}) + \sum_{i=1}^m 2^{2q}\dfrac{d_i - 2\overline{p_ir_i}}{d_i}\sigma(A_{ii} + A_{ii}^T) & \text{if } q \text{ is even,}\end{cases}$$

where A_{ii} is a Seifert matrix for L_i.

The proof of this theorem proceeds in a similar fashion to Theorem 3.5 in [**10**] but uses our formula $F(\ ,\ )$ and the $Z_{d_1} \oplus \cdots \oplus Z_{d_m}$ manifold associated to L (see Theorem 1) instead of the branched Z_d cover.

PROPOSITION 7. *If $L \subset S^{2q+1}$ is boundary sliced, then $\sigma(M, r\phi) = 0$.*

REMARK. More general results are possible than those of Propositions 6 and 7 but the proofs are more complicated and the results are not required in this paper. We still wish to note the following:

(1) If $L \subset S^{2q+1}$ is any m-link and A_{ii} is the matrix of a Seifert surface for the knot (S^{2q+1}, L_i) then the formulas of Proposition 6 are valid.

(2) If L is composed of slice knots then $\sigma(M, r\phi) = \mathrm{sig}_L(\omega_1^{r_1} \cdots \omega_m^{r_m})$.

We write Ω for an m-tuple of roots of unity, $(\omega_1, \ldots, \omega_m)$.

PROOF OF PROPOSITION 7. If L is boundary sliced then $\mathrm{sig}_L(\Omega) = 0$ for all m-tuples Ω (Theorem 2). Since each knot (S^{2q+1}, L_i) is sliced all of its Seifert surfaces have signature zero. □

An example.

PROPOSITION 8. *Suppose $2r + 1 = a$ and $2s + 1 = b$ are distinct primes, $L \subset S^{2q+1}$ is a 2-component boundary link with Seifert surface $V_1 \cup V_2$ and $\sigma(V_1) = \sigma(V_2) = 0$. If $K \in K(L; r, s)$ is a double disk knot then $\mathrm{sig}_{L \,\#\, \rho L}(\omega_1, \omega_2) = 0$ for all ω_1 an ath root of unity and ω_2 a bth root of unity.*

PROOF. Let M be the 2-fold branched cyclic cover of (S^{2q+1}, K) as it was previously constructed. By Theorem 3, $H_1(M) = A \oplus B$ and there is an epimorphism $A \twoheadrightarrow B$. But $H_1(M) = Z_a \oplus Z_b$ and $(a, b) = 1$ so that the only possible decomposition is $A = Z_a \oplus Z_b$ and $B = 0$. Therefore by (2) of Theorem 3, all the Casson-Gordon invariants of M must vanish. Let $\phi\colon H_1(M) \to Z_a \oplus Z_b$ be $\phi(\mu_1) = (1, 0)$ and $\phi(\mu_2) = (0, 1)$. If $\omega_1\omega_2 = e^{2\pi r i/ab}$ then by Proposition 6 and the condition $\sigma(V_i) = \sigma(A_{ii} + A_{ii}^T) = 0$, $\sigma(M, r) = \mathrm{sig}_{L \,\#\, \rho L}(\omega_1, \omega_2)$ so the theorem follows. □

LEMMA 9. *If the boundary m-link $J \subset S^{2q+1}$ has Seifert matrix (A_{ij}) then ρJ has Seifert matrix (B_{ij}) where $B_{ij} = (-1)^{q+1} A_{ji}^T$.*

PROOF. J and ρJ have the same Seifert surface except for orientation. The $+$ direction for J is the $-$ direction for ρJ. If x and y are qth dimensional homology classes of the Seifert surface and i_+ denotes the push off map in the $+$ direction for J then

$$\lambda(x, i_+y) = \lambda(i_-x, y) = (-1)^{q+1}\lambda(y, i_-x)$$

so $A_{ij} = (-1)^{q+1} B_{ji}^T$. □

LEMMA 10. *If L is a boundary link in S^{2q+1} then $\mathrm{sig}_{\rho L}(\Omega) = (-1)^{q+1}\,\mathrm{sig}_L(\Omega)$.*

PROOF. Let W be a root of unity matrix as in Theorem 2. We first show that $\mathrm{sig}_{\rho L}(\Omega) = \mathrm{sig}_L(\overline{\Omega})$. By Theorem 2 we can compute $\mathrm{sig}_L(\Omega)$ as $\sigma(iA)$ or $\sigma(A)$ where

$$A = (I - W)(-SW^{-1} + (-1)^{q+1} S^T)$$

and so by Theorem 2 $\operatorname{sig}_{\rho L}(\Omega) = \sigma(B)$ or $\sigma(iB)$ where

$$B = (I - W)(-(-1)^{q+1} S^T W^{-1} + S).$$

Now,

$$\begin{aligned} W^{-1}BW &= W^{-1}(I - W)(-(-1)^{q+1} S^T W^{-1} + S)W \\ &= (W^{-1} - I)(-(-1)^{q+1} S^T + SW) = (I - W^{-1})(-SW + (-1)^{q+1} S^T) \end{aligned}$$

so $\operatorname{sig}_L(\overline{\Omega}) = \operatorname{sig}_{\rho L}(\Omega)$. Now, if X is Hermitian then $\sigma(X) = \sigma(\overline{X})$ since if $OXO^* = R$ is a real matrix then $\overline{O}\,\overline{X}\,\overline{O}^* = \overline{R} = R$. If q is odd

$$\operatorname{sig}_L(\Omega) = \sigma(A) = \sigma(\overline{A}) = \operatorname{sig}_L(\overline{\Omega}) = \operatorname{sig}_{\rho L}(\Omega).$$

If q is even

$$\operatorname{sig}_L(\Omega) = \sigma(iA) = \sigma(\overline{iA}) = -\sigma(i\overline{A}) = -\operatorname{sig}_L(\overline{\Omega}) = -\operatorname{sig}_{\rho L}(\Omega). \quad \square$$

Let L be the simple boundary link given by the following matrix (see [**7**] to realize this matrix as a link):

$$\begin{bmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{bmatrix}.$$

Let K be a knot of the form $K(L; 2, 1)$. Each component of L is unknotted so by Proposition 4, K is doubly sliced. If K is also a double disk knot then by Proposition 8, $\operatorname{sig}_{L \# \rho L}(\omega, \eta) = 0$ for ω a 5th root of unity and η a 3rd root of unity. Now, $\operatorname{sig}_L(\omega, \eta)$ is equal to

$$\operatorname{sign}\left\{\begin{bmatrix} 0 & 1-\omega^{-1} & (1-\omega)(1-\eta^{-1}) & (1-\omega)(1-\eta^{-1}) \\ 1-\omega & 0 & (1-\omega)(1-\eta^{-1}) & (1-\omega)(1-\eta^{-1}) \\ (1-\omega^{-1})(1-\eta) & (1-\omega^{-1})(1-\eta) & 0 & 1-\eta^{-1} \\ (1-\omega^{-1})(1-\eta) & (1-\omega^{-1})(1-\eta) & 1-\eta & 0 \end{bmatrix}\right\}$$

which is

$$\operatorname{sign}\left\{\begin{bmatrix} -(1-\omega)(1-\omega^{-1}) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1-(1-\omega)(1-\omega^{-1})(1-\eta)(1-\eta^{-1}) & 0 \\ 0 & 0 & 0 & -(1-\eta)(1-\eta^{-1}) \end{bmatrix}\right\}$$

$\operatorname{sig}_L(\omega, \eta)$ is nonzero for $\eta = e^{2\pi i/3}$ and $\omega = e^{2\pi \cdot 3i/5}$ since $\|1-\omega\|$ and $\|1-\eta\|$ are greater than 1. By Lemma 24, $\operatorname{sig}_{L \# \rho L}(\omega, \eta) = 2 \operatorname{sig}_L(\omega, \eta)$ and so K is not a double disk knot.

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