

MINIMAL SUBMANIFOLDS OF A SPHERE WITH BOUNDED SECOND FUNDAMENTAL FORM

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ABSTRACT. Let h be the second fundamental form of an n -dimensional minimal submanifold M of a unit sphere S^{n+p} ($p \geq 2$), S be the square of the length of h , and $\sigma(u) = \|h(u, u)\|^2$ for any unit vector $u \in TM$. Simons proved that if $S \leq n/(2 - 1/p)$ on M , then either $S \equiv 0$, or $S \equiv n/(2 - 1/p)$. Chern, do Carmo, and Kobayashi determined all minimal submanifolds satisfying $S \equiv n/(2 - 1/p)$. In this paper the analogous results for $\sigma(u)$ are obtained. It is proved that if $\sigma(u) \leq \frac{1}{3}$, then either $\sigma(u) \equiv 0$, or $\sigma(u) \equiv \frac{1}{3}$. All minimal submanifolds satisfying $\sigma(u)$ are determined. A stronger result is obtained if M is odd-dimensional.

1. Introduction. Let M be a smooth (i.e. C^∞) compact n -dimensional Riemannian manifold minimally immersed in a unit sphere S^{n+p} of dimension $n + p$. Let h be the second fundamental form of the immersion. h is a symmetric bilinear mapping $T_x \times T_x \rightarrow T_x^\perp$ for $x \in M$, where T_x is the tangent space of M at x and T_x^\perp is the normal space to M at x . We denote by $S(x)$ the square of the length of h at x . By the equation of Gauss, $S(x) = n(n - 1) - \rho(x)$, where $\rho(x)$ is the scalar curvature of M at x . Therefore, $S(x)$ is an intrinsic invariant of M . Let $\Pi: UM \rightarrow M$ and UM_x be the unit tangent bundle of M and its fiber over $x \in M$, respectively. We set $\sigma(u) = \|h(u, u)\|^2$ for any u in UM . $\sigma(u)$ is not an intrinsic invariant of M . However, like $S(x)$, $\sigma(u)$ is a measure of an immersion from being totally geodesic.

J. Simons in [6] proved that if $S(x) \leq n/(2 - 1/p)$ everywhere on M , then either $S(x) \equiv 0$ (i.e. M is totally geodesic), or $S(x) \equiv n/(2 - 1/p)$. In [1], S.-S. Chern, M. do Carmo, and S. Kobayashi determined all minimal submanifolds M of S^{n+p} satisfying $S(x) \equiv n/(2 - 1/p)$ (for $p = 1$ it was also obtained by B. Lawson [2]). The purpose of the present paper is to obtain the analogous results for $\sigma(u)$.

To present our results we first describe the following examples of minimal immersions [1, 5].

A. Let $S^m(r)$ be an m -dimensional sphere in \mathbf{R}^{m+1} of radius r . We imbed $S^m(\sqrt{\frac{1}{2}}) \times S^m(\sqrt{\frac{1}{2}})$ into $S^{2m+1} = S^{2m+1}(1)$ as follows. Let $\xi, \eta \in S^m(\sqrt{\frac{1}{2}})$. Then ξ and η are vectors in \mathbf{R}^{m+1} of length $\sqrt{\frac{1}{2}}$. We can consider (ξ, η) as a unit vector in $\mathbf{R}^{2m+2} = \mathbf{R}^{m+1} \times \mathbf{R}^{m+1}$. It is easy to see that $S^m(\sqrt{\frac{1}{2}}) \times S^m(\sqrt{\frac{1}{2}})$ is a minimal submanifold of S^{2m+1} .

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B. Let F be the field \mathbf{R} of real numbers, the field \mathbf{C} of complex numbers, or the field Q of quaternions. Define d by

$$d = \begin{cases} 1, & \text{if } F = \mathbf{R}, \\ 2, & \text{if } F = \mathbf{C}, \\ 4, & \text{if } F = Q. \end{cases}$$

Let FP^2 denote the projective plane over F . FP^2 is considered as the quotient space of the unit $(3d - 1)$ -dimensional sphere $S^{3d-1}(1) = \{x \in F^3: \bar{x} \cdot x = 1\}$ obtained by identifying x with λx where $\lambda \in F$ such that $|\lambda| = 1$. The canonical metric g_0 in FP^2 is the invariant metric such that the fibering $\pi: S^{3d-1}(1) \rightarrow FP^2$ is a Riemannian submersion. The sectional curvature of $\mathbf{R}P^2$ is 1, the holomorphic sectional curvature of $\mathbf{C}P^2$ is 4, and the Q -sectional curvature of QP^2 is 4, with respect to the metric g_0 . Let $\mathcal{M}(3, F)$ be the vector space of all 3×3 matrices over F and let

$$\mathcal{X}(3, F) = \{A \in \mathcal{M}(3, F): A^* = A, \text{ trace } A = 0\}$$

where $A^* = {}^t\bar{A}$. $\mathcal{X}(3, F)$ is a subspace of $\mathcal{M}(3, F)$ of real dimension $3d+2$. We define the inner product in $\mathcal{X}(3, F) = \mathbf{R}^{3d+2}$ by $\langle A, B \rangle = \frac{1}{2} \text{trace}(AB)$ for $A, B \in \mathcal{X}(3, F)$. Define a map $\bar{\psi}: S^{3d-1} \rightarrow \mathbf{R}^{3d+2} = \mathcal{X}(3, F)$ as follows.

$$\bar{\psi}(x) = \begin{bmatrix} |x_1|^2 - \frac{1}{3} & x_1\bar{x}_2 & x_1\bar{x}_3 \\ x_2\bar{x}_1 & |x_2|^2 - \frac{1}{3} & x_2\bar{x}_3 \\ x_3\bar{x}_1 & x_3\bar{x}_2 & |x_3|^2 - \frac{1}{3} \end{bmatrix}$$

for $x = (x_1, x_2, x_3) \in S^{3d-1}(1) \subset F^3$. Then, it is easily verified that $\bar{\psi}$ induces a map $\psi: FP^2 \rightarrow \mathbf{R}^{3d+2} = \mathcal{X}(3, F)$ such that $\bar{\psi} = \psi \circ \pi$. Direct computation shows that $\psi(FP^2) \subset S^{3d+1}(1/3)$. We blow up the metric g_0 by putting $g = 3g_0$ in FP^2 , so that the sectional curvature of $\mathbf{R}P^2$ is $\frac{1}{3}$ and the holomorphic sectional curvature (resp. Q -sectional curvature) of $\mathbf{C}P^2$ (resp. QP^2) is $\frac{4}{3}$, with respect to the metric g . Then ψ gives a map $\psi: FP^2 \rightarrow S^{3d+1}(1)$. It is proved in [5] that ψ is an isometric minimal imbedding. Thus, we have the following isometric minimal imbeddings:

$$\begin{aligned} \psi_1: \mathbf{R}P^2 &\rightarrow S^4(1) \quad (\text{the Veronese surface}), \\ \psi_2: \mathbf{C}P^2 &\rightarrow S^7(1), \\ \psi_3: QP^2 &\rightarrow S^{13}(1). \end{aligned}$$

In a similar manner one may obtain (see [5] for details) an isometric imbedding of the Cayley projective plane $\text{Cay } P^2$ furnished with the canonical metric (normalized such that the C -sectional curvature equals $\frac{4}{3}$) into $S^{25}(1)$:

$$\psi_4: \text{Cay } P^2 \rightarrow S^{25}(1).$$

In addition there is an immersion

$$\psi'_1: S^2(\sqrt{3}) \rightarrow S^4(1)$$

defined by $\psi'_1 = \psi_1 \circ \pi$.

For $n, m \geq 0$, let $S^n(1)$ be the great sphere in $S^{n+m}(1)$ given by

$$S^n(1) = \{(x_1, \dots, x_{n+m+1}) \in S^{n+m}(1): x_{n+2} = \dots = x_{n+m+1} = 0\},$$

and $\tau_{n,m}: S^n(1) \rightarrow S^{n+m}(1)$ be the inclusion. For $p = 0, 1, \dots$, we set

$$\begin{aligned} \phi_{1,p} &= \tau_{4,p} \circ \psi_1: \mathbf{R}P^2 \rightarrow S^{4+p}, \\ \phi_{2,p} &= \tau_{7,p} \circ \psi_2: \mathbf{C}P^2 \rightarrow S^{7+p}, \\ \phi_{3,p} &= \tau_{13,p} \circ \psi_3: \mathbf{Q}P^2 \rightarrow S^{13+p}, \\ \phi_{4,p} &= \tau_{25,p} \circ \psi_4: \text{Cay } P^2 \rightarrow S^{25+p}, \\ \phi'_{1,p} &= \tau_{4,p} \circ \psi'_1: S^2(\sqrt{3}) \rightarrow S^{4+p}. \end{aligned}$$

$\phi_{i,p}$ ($i = 1, \dots, 4; p = 0, 1, \dots$), is an isometric minimal imbedding and $\phi'_{1,p}$ ($p = 0, 1, \dots$), is an isometric minimal immersion.

We now state the results of the present paper.

THEOREM 1. *Let M be a compact n -dimensional manifold minimally immersed in a unit sphere S^{n+1} . Assume that n ($= 2m$) is even.*

(i) *If $\sigma(u) < 1$ for any $u \in UM$, then M is totally geodesic in S^{n+1} .*

(ii) *If $\max_{u \in UM} \sigma(u) = 1$, then M is $S^m(\frac{1}{2}) \times S^m(\frac{1}{2})$ minimally imbedded in S^{2m+1} as described above.*

THEOREM 2. *Let M be a compact n -dimensional manifold minimally immersed in a unit sphere S^{n+1} . Assume that n ($= 2m + 1$) is odd. If $\sigma(u) \leq 1/(1 - 1/n)$ for any $u \in UM$, then M is totally geodesic in S^{n+1} .*

REMARK. Theorems 1(i) and 2 are easy consequences of J. Simons' results [6]. The only nontrivial part of Theorem 1(ii) is that $\max_{u \in UM} \sigma(u) = 1$ implies $S(x) \equiv n$ on UM . The remaining part of Theorem 1(ii) readily follows from results of S.-S. Chern, M. do Carmo, S. Kobayashi [1], and B. Lawson [2]. We present Theorems 1 and 2 mainly for completeness. Our main results are Theorems 3 and 4.

THEOREM 3. *Let M be a compact n -dimensional manifold minimally immersed in a unit sphere S^{n+p} . Assume that $p \geq 2$ and n ($= 2m$) is even.*

(i) *If $\sigma(u) < \frac{1}{3}$ for any $u \in UM$, then M is totally geodesic in S^{n+p} .*

(ii) *If $\max_{u \in UM} \sigma(u) = \frac{1}{3}$, then $\sigma(u) \equiv \frac{1}{3}$ on UM , and the immersion of M into S^{n+p} is one of the imbeddings $\phi_{i,p}$ ($i = 1, \dots, 4; p = 0, 1, \dots$), or the immersions $\phi'_{1,p}$ ($p = 0, 1, \dots$), described above.*

THEOREM 4. *Let M be a compact n -dimensional manifold minimally immersed in a unit sphere S^{n+p} . Assume that $p \geq 2$ and n ($= 2m + 1$) is odd. If $\sigma(u) \leq 1/(3 - 2/n)$ for any $u \in UM$, then M is totally geodesic in S^{n+p} .*

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2. Maximal directions. Let M be a compact n -dimensional manifold minimally immersed in S^{n+p} . We choose a local field of adapted orthonormal frames in S^{n+p} , that is frames $\{e_1, \dots, e_{n+p}\}$ such that the vectors e_1, \dots, e_n are tangent to M . The vectors e_{n+1}, \dots, e_{n+p} are therefore normal to M . From now on let the indices a, b, c, \dots , run from $1, \dots, n$, and the indices $\alpha, \beta, \gamma, \dots$, run from $n + 1, \dots, n + p$. Let $h = (h_{ab}^\alpha)$ be the second fundamental form of the immersed

manifold M , and $\sigma(u) = \|h(u, u)\|^2$ for $u \in UM$. Since the immersion of M into S^{n+p} is minimal, $\sum_{\alpha} h_{\alpha\alpha}^{\alpha} = 0$ for all α .

Let $x \in M$. Suppose that $u \in UM_x$ satisfies $\sigma(u) = \max_{v \in UM_x} \sigma(v)$. We shall call u a *maximal direction* at x . Let $\{e_1, \dots, e_{n+p}\}$ be an adapted frame at x . Assume that e_1 is a maximal direction at x , $\sigma(e_1) \neq 0$, and $e_{n+1} = h(e_1, e_1)/\|h(e_1, e_1)\|$. Because of our choice of e_{n+1} ,

$$(2.1) \quad h_{11}^{\alpha} = 0, \quad \alpha \neq n + 1.$$

Since e_1 is a maximal direction, we have at the point x for any $t, x^2, \dots, x^n \in \mathbf{R}$

$$(2.2) \quad \left\| h \left(e_1 + t \sum_{a=2}^n x^a e_a, e_1 + t \sum_{a=2}^n x^a e_a \right) \right\|^2 \leq \left[1 + t^2 \sum_{a=2}^n (x^a)^2 \right]^2 (h_{11}^{n+1})^2.$$

Expanding in terms of t , we obtain

$$4th_{11}^{n+1} \sum_{a \neq 1} x^a h_{1a}^{n+1} + O(t^2) \leq 0.$$

It follows that

$$(2.3) \quad h_{1a}^{n+1} = 0, \quad a = 2, \dots, n.$$

We now choose an adapted frame at $x \in M$ such that in addition to (2.1) and (2.3),

$$(2.4) \quad h_{ab}^{n+1} = 0, \quad a \neq b.$$

Once more expanding (2.2) in terms of t , we obtain

$$(2.5) \quad -2t^2 \left\{ \sum_{a \neq 1} \left[h_{11}^{n+1} (h_{11}^{n+1} - h_{aa}^{n+1}) - 2 \sum_{\alpha \neq n+1} (h_{1a}^{\alpha})^2 \right] (x^a)^2 - 4 \sum_{\alpha \neq n+1} \sum_{\substack{a, b \neq 1 \\ a \neq b}} h_{1a} h_{1b} x^a x^b \right\} + O(t^3) \leq 0.$$

It follows that

$$(2.6) \quad 2 \sum_{\alpha \neq n+1} (h_{1a}^{\alpha})^2 \leq h_{11}^{n+1} (h_{11}^{n+1} - h_{aa}^{n+1}), \quad a = 2, \dots, n.$$

Let us define a tensor field $H = (H_{abcd})$ on M by the formula

$$(2.7) \quad H_{abcd} = \sum_{\alpha} h_{ab}^{\alpha} h_{cd}^{\alpha}.$$

It is clear that $\sigma(u) = H(u, u, u, u)$.

LEMMA 1. *Let u be a maximal direction at $x \in M$. Assume that $\sigma(u) \neq 0$. Let e_1, \dots, e_{n+p} be an adapted frame at x such that $e_1 = u$, $e_{n+1} = h(e_1, e_1)/\|h(e_1, e_1)\|$, and $h_{ab}^{n+1} = 0$ for $a \neq b$. At the point x*

(i) *if $p = 1$, then*

$$(2.8) \quad \frac{1}{2}(\Delta H)_{1111} \geq (h_{11}^{n+1})^2 \left[n - \sum_a (h_{aa}^{n+1})^2 \right].$$

(ii) if $p \geq 2$, then

$$(2.9) \quad \frac{1}{2}(\Delta H)_{1111} \geq (h_{11}^{n+1})^2 \left[n - n(h_{11}^{n+1})^2 - 2 \sum_a (h_{aa}^{n+1})^2 \right]$$

with equality attained if and only if

$$(2.10) \quad (h_{11}^{n+1} - h_{aa}^{n+1}) \left[h_{11}^{n+1}(h_{11}^{n+1} - h_{aa}^{n+1}) - 2 \sum_{\alpha \neq n+1} (h_{1a}^\alpha)^2 \right] = 0$$

and

$$(2.11) \quad \nabla_a h_{11}^\alpha = 0$$

for all a and all α , where Δ and ∇_a denote the Laplacian and the covariant derivative, respectively.

PROOF.

$$\frac{1}{2}(\Delta H)_{1111} = h_{11}^{n+1}(\Delta h)_{11}^{n+1} + \sum_{a,\alpha} (\nabla_a h_{11}^\alpha)^2.$$

Using Simons' formula [6] for the Laplacian of the second fundamental form (see also [1]), we obtain

$$(2.12) \quad \frac{1}{2}(\Delta H)_{1111} = (h_{11}^{n+1})^2 \left[n - \sum_a (h_{aa}^{n+1})^2 \right] + \sum_{a,\alpha} (\nabla_a h_{11}^\alpha)^2, \quad \text{if } p = 1,$$

and

$$(2.13) \quad \begin{aligned} \frac{1}{2}(\Delta H)_{1111} &= (h_{11}^{n+1})^2 \left[n - n(h_{11}^{n+1})^2 - 2 \sum_a (h_{aa}^{n+1})^2 \right] \\ &+ \sum_a h_{11}^{n+1}(h_{11}^{n+1} - h_{aa}^{n+1}) \left[h_{11}^{n+1}(h_{11}^{n+1} - h_{aa}^{n+1}) - 2 \sum_{\alpha \neq n+1} (h_{1a}^\alpha)^2 \right] \\ &+ \sum_{a,\alpha} (\nabla_a h_{11}^\alpha)^2, \quad \text{if } p \geq 2, \end{aligned}$$

from which the lemma follows readily by inequality (2.6). \square

LEMMA 2. Let an adapted frame $\{e_1, \dots, e_{n+p}\}$ at $x \in M$ be as in Lemma 1.

(i) Assume that $n (= 2m)$ is even. If

$$\sigma(u) \leq \begin{cases} 1, & \text{if } p = 1, \\ \frac{1}{3}, & \text{if } p \geq 2, \end{cases} \quad \text{for all } u \in UM_x,$$

then $(\Delta H)_{1111} \geq 0$. If equality $(\Delta H)_{1111} = 0$ is attained, then it is possible to renumber e_1, \dots, e_{2m} such that the following equalities hold

$$(2.14) \quad h_{11}^{n+1} = \dots = h_{mm}^{n+1} = -h_{m+1}^{n+1} = \dots = -h_{2m}^{n+1} = \begin{cases} 1, & \text{if } p = 1, \\ 1/\sqrt{3}, & \text{if } p \geq 2. \end{cases}$$

(ii) Assume that $n (= 2m + 1)$ is odd. If

$$\sigma(u) \leq \begin{cases} 1 - \frac{1}{n}, & \text{if } p = 1, \\ \frac{1}{3 - 2/n}, & \text{if } p \geq 2, \end{cases} \quad \text{for all } u \in UM_x,$$

then $(\Delta H)_{1111} \geq 0$. If equality $(\Delta H)_{1111} = 0$ is attained, then it is possible to renumber e_1, \dots, e_{2m+1} such that the following equalities hold.

$$(2.15) \quad \begin{aligned} h_{11}^{n+1} &= \dots = h_{mm}^{n+1} = -h_{m+1 \ m+1}^{n+1} = \dots = -h_{2m \ 2m}^{n+1} \\ &= \begin{cases} \left(1 - \frac{1}{n}\right)^{-1/2}, & \text{if } p = 1, \\ \left(3 - \frac{2}{n}\right)^{-1/2}, & \text{if } p \geq 2, \end{cases} \\ h_{2m+1 \ 2m+1}^{n+1} &= 0. \end{aligned}$$

PROOF. Since e_1 is a maximal direction

$$(2.16) \quad -h_{11}^{n+1} \leq h_{aa}^{n+1} \leq h_{11}^{n+1}, \quad a = 2, \dots, n.$$

Because of minimality of the immersion of M into S^{n+p} ,

$$(2.17) \quad \sum_{a=2}^n h_{aa}^{n+1} = -h_{11}^{n+1}.$$

It is easily seen that the convex function $f(h_{22}^{n+1}, \dots, h_{nn}^{n+1}) = \sum_{a=2}^n (h_{aa}^{n+1})^2$ of $(n - 1)$ variables $h_{22}^{n+1}, \dots, h_{nn}^{n+1}$ subject to the linear constraints (2.16), (2.17) attains its maximal value when (after suitable renumbering of e_1, \dots, e_n)

$$h_{11}^{n+1} = \dots = h_{mm}^{n+1} = -h_{m+1 \ m+1}^{n+1} = \dots = -h_{2m \ 2m}^{n+1}, \quad \text{if } n = 2m,$$

and

$$\begin{aligned} h_{11}^{n+1} = \dots = h_{mm}^{n+1} = -h_{m+1 \ m+1}^{n+1} = \dots = -h_{2m \ 2m}^{n+1}, \\ h_{2m+1 \ 2m+1}^{n+1} = 0, \quad \text{if } n = 2m + 1. \end{aligned}$$

Therefore, by inequalities (2.8), (2.9),

$$\frac{1}{2}(\Delta H)_{1111} \geq \begin{cases} n(h_{11}^{n+1})^2[1 - \sigma(e_1)], & \text{if } p = 1, \ n = 2m, \\ n(h_{11}^{n+1})^2[1 - 3\sigma(e_1)], & \text{if } p \geq 2, \ n = 2m, \\ (h_{11}^{n+1})^2[n - (n - 1)\sigma(e_1)], & \text{if } p = 1, \ n = 2m + 1, \\ (h_{11}^{n+1})^2[n - (3n - 2)\sigma(e_1)], & \text{if } p \geq 2, \ n = 2m + 1. \end{cases}$$

This proves the lemma. \square

Let $L(x)$ be a function on M defined by $L(x) = \max_{u \in UM_x} \sigma(u)$.

LEMMA 3. Assume that one of A_1, A_2, A_3, A_4 is satisfied.

- (A₁) $p = 1$, n is even, $\sigma(u) \leq 1$ for all $u \in UM$,
- (A₂) $p = 1$, n is odd, $\sigma(u) \leq 1/(1 - 1/n)$ for all $u \in UM$,
- (A₃) $p \geq 2$, n is even, $\sigma(u) \leq \frac{1}{3}$ for all $u \in UM$,
- (A₄) $p \geq 2$, n is odd, $\sigma(u) \leq 1/(3 - 2/n)$ for all $u \in UM$.

Then $L(x)$ is a constant function on M .

PROOF. Following an idea in [3] we prove the lemma using the maximum principle. Clearly $L(x)$ is a continuous function. It suffices to show that $L(x)$ is subharmonic in the generalized sense. Fix $x \in M$ and let e_1 be a maximal direction at x . In an open neighborhood U_x of x within the cut-locus of x we shall denote by $u(y)$ the tangent vector to M obtained by parallel transport of $e_1 = u(x)$ along the unique geodesic joining x to y within the cut-locus of x . Define $g_x(y) = \sigma(u(y))$. Then

$$\begin{aligned} \Delta g_x(x) &= \Delta[H(u(y), u(y), u(y), u(y))]_{y=x} \\ &= \sum_a (\nabla_a^2 H)(e_1, e_1, e_1, e_1) = (\Delta H)_{1111}(x). \end{aligned}$$

If $\|h(e_1, e_1)\| \neq 0$, then by Lemma 2, $(\Delta H)_{1111}(x) \geq 0$. If $\|h(e_1, e_1)\| = 0$, then $h \equiv 0$ at x . In this case the formula of Simons [6] for Δh shows that $\Delta h = 0$ at x , and therefore

$$(\Delta H)_{1111}(x) = \sum_{a,\alpha} (\nabla_a h_{11}^\alpha)^2 \geq 0.$$

Thus, we obtain that in any case $\Delta g_x(x) = (\Delta H)_{1111}(x) \geq 0$.

For the Laplacian of continuous functions, we have the generalized definition

$$\Delta L = C \lim_{r \rightarrow 0} \frac{1}{r^2} \left(\int_{B(x,r)} L / \int_{B(x,r)} 1 - L(x) \right),$$

where C is a positive constant and $B(x, r)$ denotes the geodesic ball of radius r with the center at x . With this definition L is subharmonic on M if and only if $\Delta L(x) \geq 0$ at each point $x \in M$. Since $g_x(x) = L(x)$ and $g_x \leq L$ on U_x , $\Delta L(x) \geq \Delta g_x(x) \geq 0$. Thus, $L(x)$ is subharmonic and hence constant on M . \square

3. Proofs of Theorems 1-4.

LEMMA 4. Assume that one of B_1, B_2, B_3, B_4 is satisfied.

- (B₁) $p = 1, n$ is even, $\sigma(u) < 1$ for all $u \in UM$,
- (B₂) $p = 1, n$ is odd, $\sigma(u) < 1/(1 - 1/n)$ for all $u \in UM$,
- (B₃) $p \geq 2, n$ is even, $\sigma(u) < \frac{1}{3}$ for all $u \in UM$,
- (B₄) $p \geq 2, n$ is odd, $\sigma(u) < 1/(3 - 2/n)$ for all $u \in UM$.

Then M is totally geodesic in S^{n+p} .

PROOF. Let $x \in M$ and e_1 be a maximal direction at x . Assume that $\sigma(e_1) \neq 0$. Let $g_x(y) = \sigma(u(y))$ be the function defined in the proof of Lemma 3. By Lemma 3, $g_x(x)$ is a maximum of g_x . Therefore, $(\Delta H)_{1111}(x) = \Delta g_x(x) \leq 0$. On the other hand, by Lemma 2, $(\Delta H)_{1111}(x) \geq 0$. Therefore, $(\Delta H)_{1111} = 0$ on M . Hence, by (2.14) and (2.15),

$$\sigma(e_1) = \begin{cases} 1, & \text{if } p = 1, n \text{ is even,} \\ \frac{1}{1 - 1/n}, & \text{if } p = 1, n \text{ is odd,} \\ \frac{1}{3}, & \text{if } p \geq 2, n \text{ is even,} \\ \frac{1}{3 - 2/n}, & \text{if } p \geq 2, n \text{ is odd,} \end{cases}$$

contradicting the assumptions B_1, B_2, B_3, B_4 . Hence, $h(u, u) = 0$ for all $u \in UM$, that is M is totally geodesic in S^{n+p} . \square

PROOF OF THEOREM 1. (i) follows from Lemma 4. We prove (ii). As in the poof of Lemma 4, we obtain $(\Delta H)_{1111} = 0$. Hence, by (2.4) and (2.14),

$$S(x) = \sum_{\alpha, a, b} (h_{ab}^\alpha)^2 = \sum_a (h_{aa}^{n+1})^2 = n.$$

All minimal immersions into S^{n+1} satisfying $S(x) \equiv n$ were found by S.-S. Chern, M. do Carmo, and S. Kobayashi in [1] and B. Lawson in [2]. It is easy to see that among their immersions only $S^m(\sqrt{\frac{1}{2}}) \times S^m(\sqrt{\frac{1}{2}})$ imbedded in S^{2m+1} satisfies the condition $\max_{u \in UM} \sigma(u) = 1$. This completes the proof of Theorem 1. \square

PROOF OF THEOREM 2. By Lemmas 3 and 4, we have to consider only the case $L(x) = \max_{u \in UM_x} \sigma(u) \equiv 1/(1 - 1/n)$ on M . As in the proof of Lemma 4, $(\Delta H)_{1111} = 0$. Hence, by (2.15),

$$S(x) \equiv \sum_{\alpha, a, b} (h_{ab}^\alpha)^2 \equiv \sum_{a=1}^{n+1} \frac{1}{(1 - 1/n)} \equiv n.$$

It is shown in [1] that if M is minimally immersed in S^{n+1} and $S(x) \equiv n$, then h_{aa}^{n+1} may attain at most two different values for $a = 1, \dots, n$. However, since by (2.15),

$$h_{11}^{n+1} = \left(\frac{n}{n-1}\right)^{1/2}, \quad h_{m+1, m+1}^{n+1} = -\left(\frac{n}{n-1}\right)^{1/2}, \quad h_{2m+1, 2m+1}^{n+1} = 0,$$

we obtain a contradiction, so the equality $\max_{u \in UM} \sigma(u) \equiv 1/(1 - 1/n)$ on UM is impossible. This completes the proof of Theorem 2. \square

PROOF OF THEOREM 3. (i) follows from Lemma 4. We prove (ii). As in the proof of Lemma 4, we obtain $(\Delta H)_{1111} = 0$. Let the indices i, j, k, \dots , run from $1, \dots, m$, and let $\bar{i}, \bar{j}, \bar{k}, \dots$, denote $i + m, j + m, k + m, \dots$, respectively. By (2.14) we have

$$(3.1) \quad h_{ii}^{n+1} = -h_{\bar{i}\bar{i}}^{n+1} = -1/\sqrt{3}, \quad i = 1, \dots, m.$$

Since $\|h(e_i, e_i)\|^2 \leq \frac{1}{3}$ and $\|h(e_{\bar{i}}, e_{\bar{i}})\|^2 \leq \frac{1}{3}$, we obtain

$$(3.2) \quad h_{ii}^\alpha = h_{\bar{i}\bar{i}}^\alpha = 0, \quad \alpha \neq n + 1; \quad i = 1, \dots, m.$$

By (2.10), $\sum_{\alpha \neq n+1} (h_{ij}^\alpha)^2 = \frac{1}{3}$. Since each vector e_a , ($a = 1, \dots, n$), is a maximal direction,

$$(3.3) \quad \sum_{\alpha \neq n+1} (h_{ij}^\alpha)^2 = \frac{1}{3}, \quad i, j = 1, \dots, m.$$

Let $u = (e_i + e_j)/\sqrt{2}$. Then

$$\begin{aligned} \sigma(u) &= \frac{1}{4} \|h(e_i + e_j, e_i + e_j)\|^2 \\ &= \frac{1}{4} \|(h_{ii}^{n+1} + h_{jj}^{n+1})e_{n+1} + 2 \sum_{\alpha \neq n+1} h_{ij}^\alpha e_\alpha\|^2 \\ &= \frac{1}{3} + \sum_{\alpha \neq n+1} (h_{ij}^\alpha)^2 \leq \frac{1}{3}. \end{aligned}$$

Therefore,

$$(3.4) \quad h_{ij}^\alpha = 0, \quad \alpha \neq n + 1; \quad i, j = 1, \dots, m.$$

Similarly,

$$(3.5) \quad h_{\bar{i}\bar{j}}^\alpha = 0, \quad \alpha \neq n + 1; \quad i, j = 1, \dots, m.$$

Expansion (2.5) now takes the form

$$t^2 \left(-4 \sum_{\alpha} \sum_{j \neq k} h_{ij}^\alpha h_{i\bar{k}}^\alpha x^{\bar{j}} x^{\bar{k}} \right) + O(t^3) \leq 0.$$

It follows that $\sum_{\alpha} h_{ij}^\alpha h_{i\bar{k}}^\alpha = 0$ for $j \neq k$. Since each vector e_a is a maximal direction,

$$(3.6) \quad \sum_{\alpha} h_{ij}^\alpha h_{i\bar{k}}^\alpha = 0, \quad j \neq k,$$

$$(3.7) \quad \sum_{\alpha} h_{i\bar{k}}^\alpha h_{j\bar{k}}^\alpha = 0, \quad i \neq j.$$

Once more expanding (2.2) in terms of t ,

$$2t^3 \sum_{\alpha, j, k, l} (h_{i\bar{k}}^\alpha h_{j\bar{l}}^\alpha + h_{i\bar{l}}^\alpha h_{j\bar{k}}^\alpha) x^j x^{\bar{k}} x^{\bar{l}} + O(t^4) \leq 0,$$

from which

$$(3.8) \quad \sum_{\alpha} (h_{i\bar{k}}^\alpha h_{j\bar{l}}^\alpha + h_{i\bar{l}}^\alpha h_{j\bar{k}}^\alpha) = 0, \quad i \neq j \text{ or } k \neq l.$$

Using (2.4) and (3.1)–(3.8), we obtain by direct computation that $\sigma(u) = \frac{1}{3}$ for any $u \in UM$. B. O’Neill [4] calls an immersion λ -isotropic if $\|h(u, u)\| = \lambda$ for any $u \in UM$. Therefore, *the immersion under consideration is $1/\sqrt{3}$ -isotropic.*

By Lemma 1, $\nabla_a h_{11}^\alpha = 0$. It follows that $\nabla_a h_{bb}^\alpha = 0$. By polarization, $\nabla_a h_{bc}^\alpha = 0$ for all α, a, b, c . Therefore, *the second fundamental form of the immersion is parallel.* All λ -isotropic minimal immersions into a unit sphere with parallel second fundamental form were completely classified by K. Sakamoto in [5]. Among his immersions only $\phi_{1,p}, \phi_{2,p}, \phi_{3,p}, \phi_{4,p}$ and $\phi'_{1,p}$ described in §1, are $1/\sqrt{3}$ -isotropic. This completes the proof of the theorem. \square

PROOF OF THEOREM 4. By Lemmas 3 and 4, we need only consider the case $L(x) \equiv 1/(3 - 2/n)$ on M . We show that this case cannot occur. Thus, assume that $L(x) \equiv 1/(3 - 2/n)$ on M . As in the proof of Lemma 4, $(\Delta H)_{1111} = 0$. Let the indices i, j, k, \dots , run from $1, \dots, m$, and let $\bar{i}, \bar{j}, \bar{k}, \dots$, denote $i + m, j + m, k + m, \dots$, respectively. By (2.15),

$$(3.9) \quad \begin{aligned} h_{ii}^{n+1} &= -h_{\bar{i}\bar{i}}^{n+1} = (3 - 2/n)^{1/2}, \quad i = 1, \dots, m, \\ h_{nn}^{n+1} &= 0. \end{aligned}$$

As in the proof of Theorem 3,

$$(3.10) \quad h_{ij}^\alpha = h_{i\bar{j}}^\alpha = 0, \quad \alpha \neq n + 1; \quad i, j = 1, \dots, m.$$

Since $h_{nn}^\alpha = -\sum_i h_{ii}^\alpha - \sum_i h_{i\bar{i}}^\alpha$,

$$(3.11) \quad h_{nn}^\alpha = 0.$$

By (2.10),

$$(3.12) \quad \sum_\alpha (h_{ij}^\alpha)^2 = \frac{1}{3 - 2/n}, \quad i, j = 1, \dots, m$$

$$(3.13) \quad \sum_\alpha (h_{in}^\alpha)^2 = \frac{1}{2(3 - 2/n)}, \quad i = 1, \dots, m,$$

$$(3.14) \quad \sum_\alpha (h_{i\bar{n}}^\alpha)^2 = \frac{1}{2(3 - 2/n)}, \quad i = 1, \dots, m.$$

As in the proof of Theorem 3, we obtain with the help of expansion (2.2) the following equalities:

$$(3.15) \quad \sum_\alpha h_{ij}^\alpha h_{i\bar{k}}^\alpha = 0,$$

$$(3.16) \quad \sum_\alpha h_{i\bar{k}}^\alpha h_{j\bar{k}}^\alpha = 0,$$

$$(3.17) \quad \sum_\alpha h_{i\bar{j}}^\alpha h_{in}^\alpha = 0,$$

$$(3.18) \quad \sum_\alpha h_{i\bar{j}}^\alpha h_{n\bar{j}}^\alpha = 0,$$

$$(3.19) \quad \sum_\alpha (h_{i\bar{k}}^\alpha h_{j\bar{i}}^\alpha + h_{i\bar{i}}^\alpha h_{j\bar{k}}^\alpha) = 0, \quad i \neq j \text{ or } k \neq 1,$$

$$(3.20) \quad \sum_\alpha (h_{i\bar{k}}^\alpha h_{jn}^\alpha + h_{j\bar{k}}^\alpha h_{in}^\alpha) = 0, \quad i \neq j,$$

$$(3.21) \quad \sum_\alpha (h_{i\bar{j}}^\alpha h_{n\bar{k}}^\alpha + h_{i\bar{k}}^\alpha h_{n\bar{j}}^\alpha) = 0, \quad j \neq k,$$

$$(3.22) \quad \sum_\alpha h_{in}^\alpha h_{jn}^\alpha = 0, \quad i \neq j,$$

$$(3.23) \quad \sum_\alpha h_{i\bar{n}}^\alpha h_{j\bar{n}}^\alpha = 0, \quad i \neq j,$$

$$(3.24) \quad \sum_\alpha h_{i\bar{n}}^\alpha h_{n\bar{j}}^\alpha = 0.$$

Let $u = \sum_\alpha u^\alpha e_\alpha \in UM$. Direct computation with the help of (2.4) and (3.9)–(3.24) shows that

$$(3.25) \quad \sigma(u) = [1 - (u^n)^4](3 - 2/n)^{-1}.$$

It follows from (3.25) that for any $x \in M$, the tangent space T_x of M at x is a direct sum of two mutually orthogonal subspaces $T_x = P_x + Q_x$, where P_x is $2m$ -dimensional and is defined by

$$(3.26) \quad P_x = \{X \in T_x : \|h(X, X)\| = (3 - 2/n)^{-1/2} \|X\|^2\},$$

and Q_x is 1-dimensional and is defined by

$$(3.27) \quad Q_x = \{X \in T_x : h(X, X) = 0\}.$$

LEMMA 5. *The distributions $P : x \rightarrow P_x$ and $Q : x \rightarrow Q_x$ are smooth distributions on M .*

PROOF. It is sufficient to prove that Q is smooth. Let $x_0 \in M$ and $\{e_1, \dots, e_{n+p}\}$ be a smooth local field of orthonormal adapted frames in a neighborhood U of x_0 such that $e_n(x_0) \in Q_{x_0}$. If U is sufficiently small, there is a unique vector X of the form $X = \sum_{a=1}^{2m} X^a e_a + e_n$ which belongs to Q_x at each point $x \in U$. We prove that $X^a, a = 1, \dots, 2m$, are smooth functions of x .

By (3.27), $X^a(x), a = 1, \dots, 2m$, are a unique solution of the system of equations

$$(3.28) \quad h^\alpha(X, X) = \sum_{a,b=1}^{2m} h_{ab}^\alpha(x) X^a X^b + 2 \sum_{a=1}^{2m} h_{an}^\alpha(x) X^a = 0, \\ \alpha = n + 1, \dots, n + p.$$

At the point x_0 the Jacobian of system (3.28) is

$$(\partial h^\alpha / \partial X^a) = 2(h_{an}^\alpha), \quad \alpha = n + 1, \dots, n + p; a = 1, \dots, 2m.$$

By (3.13), (3.14) and (3.22)–(3.24), the rows of the matrix (h_{an}^α) are mutually orthogonal nonzero vectors. Hence, $\text{rank}(\partial h^\alpha / \partial X^a) = 2m$ at x_0 . Therefore, $X^a, a = 1, \dots, 2m$, are smooth functions of x in a sufficiently small neighborhood of x_0 . \square

We now return to the proof of Theorem 4. Let $x \in M$. By Lemma 5, we may choose a smooth family of orthonormal adapted frames $\{e_1, \dots, e_{n+p}\}$ in some neighborhood U of x such that equations (2.4), (3.9)–(3.24) are satisfied on U . Set

$$N_a = \left[2 \left(3 - \frac{2}{n} \right) \right]^{1/2} \sum_{\alpha} h_{an}^\alpha e_\alpha, \quad a = 1, \dots, 2m.$$

By (2.4), (3.13), (3.14), and (3.22)–(3.24), the vectors $e_{n+1}, N_1, \dots, N_{2m}$ are orthonormal. Therefore, with no loss of generality, we may assume that $e_{n+1+a} = N_a, a = 1, \dots, 2m$. Then,

$$(3.29) \quad h_{in}^{n+1+i} = h_{in}^{n+1+\bar{i}} = \left[2 \left(3 - \frac{2}{n} \right) \right]^{1/2}, \quad i = 1, \dots, m,$$

$$(3.30) \quad h_{in}^\alpha = 0, \quad \alpha \neq n + 1 + i, \quad i = 1, \dots, m,$$

$$(3.31) \quad h_{in}^\alpha = 0, \quad \alpha \neq n + 1 + \bar{i}, \quad i = 1, \dots, m.$$

Let the indices A, B, C run from $1, \dots, n + p$, and let $\{\omega^A\}$ and $\{\omega_B^A\}$ be the coframe dual to the frame $\{e_A\}$ and the connection forms of the Riemannian connection on S^{n+p} , respectively. Then,

$$(3.32) \quad d\omega^A = \sum_B \omega^B \wedge \omega_B^A,$$

$$(3.33) \quad d\omega_B^A = \sum_C \omega^C \wedge \omega_C^A + \omega^A \wedge \omega^B,$$

$$(3.34) \quad \omega^\alpha = 0,$$

$$(3.35) \quad \omega_a^\alpha = \sum_b h_{ab}^\alpha \omega^b,$$

$$(3.36) \quad dh_{ab}^\alpha - \sum_c h_{cb}^\alpha \omega_a^c - \sum_c h_{ac}^\alpha \omega_b^c + \sum_\beta h_{ab}^\beta \omega_\beta^\alpha = \sum_c (\nabla_c h_{ab}^\alpha) \omega^c.$$

As in the proof of Theorem 3, we obtain

$$(3.37) \quad \nabla_c h_{ab}^\alpha = 0, \quad a, b = 1, \dots, 2m; \quad c = 1, \dots, n.$$

Let us take $\alpha = h + 1 + i, a = b = i$ in (3.36). By (2.4), (3.9)–(3.11), (3.29)–(3.31), and (3.37),

$$(3.38) \quad -2 \sum_k h_{ki}^{n+1+i} \omega_i^{\bar{k}} - \left[2 \left(3 - \frac{2}{n} \right) \right]^{-1/2} \omega_i^n + \left(3 - \frac{2}{n} \right)^{-1/2} \omega_{n+1}^{n+1+i} = 0.$$

Analogously, taking $\alpha = n + 1 + i, a = i, b = j \neq i$ in (3.36),

$$(3.39) \quad -2 \sum_k h_{kj}^{n+1+i} \omega_j^{\bar{k}} + \left(3 - \frac{2}{n} \right)^{-1/2} \omega_{n+1}^{n+1+i} = 0, \quad i \neq j.$$

Summing (3.39) with respect to j ($j \neq i$) and adding (3.38), we have

$$(3.40) \quad -2 \sum_{j,k} h_{kj}^{n+1+i} \omega_j^{\bar{k}} + m \left(3 - \frac{2}{n} \right)^{-1/2} \omega_{n+1}^{n+1+i} - \left[2 \left(3 - \frac{2}{n} \right) \right]^{-1/2} \omega_i^n = 0.$$

Let us now take $\alpha = n + 1 + i, a = b = \bar{k}$ in (3.36). Then,

$$(3.41) \quad -2 \sum_j h_{j\bar{k}}^{n+1+i} \omega_{\bar{k}}^j - \left(3 - \frac{2}{n} \right)^{-1/2} \omega_{n+1}^{n+1+i} = 0.$$

Summing (3.41) with respect to \bar{k} ,

$$(3.42) \quad -2 \sum_{j,k} h_{j\bar{k}}^{n+1+i} \omega_{\bar{k}}^j - m \left(3 - \frac{2}{n} \right)^{-1/2} \omega_{n+1}^{n+1+i} = 0.$$

Finally, adding (3.40) to (3.42), we get

$$(3.43) \quad \omega_i^n = 0.$$

Analogously, we obtain

$$(3.44) \quad \omega_{\bar{i}}^n = 0.$$

Differentiating (3.43) and using (2.4), (3.9)–(3.11), (3.29)–(3.31), and (3.4), we obtain

$$(3.45) \quad - \sum_{\alpha, a, b} h_{ia}^\alpha h_{bn}^\alpha \omega^a \wedge \omega^b + \omega^n \wedge \omega^i = 0$$

Taking the coefficient of $\omega^n \wedge \omega^i$ in (3.45) we have $-\sum_{\alpha} (h_{in}^\alpha)^2 + 1 = 0$. By (3.13), it gives $2(3 - 2/n) = 1$ and therefore $n = 5/4$, yielding a contradiction. Therefore, the equality $\max_{u \in UM_x} \sigma(u) \equiv 1/(3 - 2/n)$ on M is impossible. This completes the proof of Theorem 4.

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