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MINIMAL SUBMANIFOLDS OF A SPHERE WITH BOUNDED SECOND FUNDAMENTAL FORM

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ABSTRACT. Let h be the second fundamental form of an n-dimensional minimal submanifold M of a unit sphere S^{n+p} ($p \geq 2$), S be the square of the length of h, and $\sigma(u) = \|h(u,u)\|^2$ for any unit vector $u \in TM$. Simons proved that if $S \leq n/(2-1/p)$ on M, then either $S \equiv 0$, or $S \equiv n/(2-1/p)$. Chern, do Carmo, and Kobayashi determined all minimal submanifolds satisfying $S \equiv n/(2-1/p)$. In this paper the analogous results for $\sigma(u)$ are obtained. It is proved that if $\sigma(u) \leq \frac{1}{3}$, then either $\sigma(u) \equiv 0$, or $\sigma(u) \equiv \frac{1}{3}$. All minimal submanifolds satisfying $\sigma(u)$ are determined. A stronger result is obtained if M is odd-dimensional.

1. Introduction. Let M be a smooth (i.e. C^∞) compact n-dimensional Riemannian manifold minimally immersed in a unit sphere S^{n+p} of dimension $n+p$. Let h be the second fundamental form of the immersion. h is a symmetric bilinear mapping $T_x \times T_x \to T_x^\perp$ for $x \in M$, where T_x is the tangent space of M at x and $T_x^\perp$ is the normal space to M at x. We denote by $S(x)$ the square of the length of h at x. By the equation of Gauss, $S(x) = n(n-1) - \rho(x)$, where $\rho(x)$ is the scalar curvature of M at x. Therefore, $S(x)$ is an intrinsic invariant of M. Let $\Pi: UM \to M$ and UM_x be the unit tangent bundle of M and its fiber over $x \in M$, respectively. We set $\sigma(u) = \|h(u,u)\|^2$ for any u in UM. $\sigma(u)$ is not an intrinsic invariant of M. However, like $S(x)$, $\sigma(u)$ is a measure of an immersion from being totally geodesic.

J. Simons in [**6**] proved that if $S(x) \leq n/(2-1/p)$ everywhere on M, then either $S(x) \equiv 0$ (i.e. M is totally geodesic), or $S(x) \equiv n/(2-1/p)$. In [**1**], S.-S. Chern, M. do Carmo, and S. Kobayashi determined all minimal submanifolds M of S^{n+p} satisfying $S(x) \equiv n/(2-1/p)$ (for $p=1$ it was also obtained by B. Lawson [**2**]). The purpose of the present paper is to obtain the analogous results for $\sigma(u)$.

To present our results we first describe the following examples of minimal immersions [**1, 5**].

A. Let $S^m(r)$ be an m-dimensional sphere in $\mathbf{R}^{m+1}$ of radius r. We imbed $S^m(\sqrt{\frac{1}{2}}) \times S^m(\sqrt{\frac{1}{2}})$ into $S^{2m+1} = S^{2m+1}(1)$ as follows. Let $\xi, \eta \in S^m(\sqrt{\frac{1}{2}})$. Then ξ and η are vectors in $\mathbf{R}^{m+1}$ of length $\sqrt{\frac{1}{2}}$. We can consider (ξ,η) as a unit vector in $\mathbf{R}^{2m+2} = \mathbf{R}^{m+1} \times \mathbf{R}^{m+1}$. It is easy to see that $S^m(\sqrt{\frac{1}{2}}) \times S^m(\sqrt{\frac{1}{2}})$ is a minimal submanifold of S^{2m+1}.

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B. Let F be the field $\mathbf{R}$ of real numbers, the field $\mathbf{C}$ of complex numbers, or the field Q of quaternions. Define d by

$$d = \begin{cases} 1, & \text{if } F = \mathbf{R}, \\ 2, & \text{if } F = \mathbf{C}, \\ 4, & \text{if } F = Q. \end{cases}$$

Let FP^2 denote the projective plane over F. FP^2 is considered as the quotient space of the unit $(3d-1)$-dimensional sphere $S^{3d-1}(1) = \{x \in F^3 : {}^t\overline{x} \cdot x = 1\}$ obtained by identifying x with λx where $\lambda \in F$ such that $|\lambda| = 1$. The canonical metric g_0 in FP^2 is the invariant metric such that the fibering $\pi : S^{3d-1}(1) \to FP^2$ is a Riemannian submersion. The sectional curvature of $\mathbf{R}P^2$ is 1, the holomorphic sectional curvature of $\mathbf{C}P^2$ is 4, and the Q-sectional curvature of QP^2 is 4, with respect to the metric g_0. Let $\mathcal{M}(3, F)$ be the vector space of all 3×3 matrices over F and let

$$\mathcal{H}(3, F) = \{A \in \mathcal{M}(3, F) : A^* = A, \text{ trace } A = 0\}$$

where $A^* = {}^t\overline{A}$. $\mathcal{H}(3, F)$ is a subspace of $\mathcal{M}(3, F)$ of real dimension $3d+2$. We define the inner product in $\mathcal{H}(3, F) = \mathbf{R}^{3d+2}$ by $\langle A, B \rangle = \frac{1}{2}\operatorname{trace}(AB)$ for $A, B \in \mathcal{H}(3, F)$. Define a map $\overline{\psi} : S^{3d-1} \to \mathbf{R}^{3d+2} = \mathcal{H}(3, F)$ as folows.

$$\overline{\psi}(x) = \begin{bmatrix} |x_1|^2 - \frac{1}{3} & x_1\overline{x}_2 & x_1\overline{x}_3 \\ x_2\overline{x}_1 & |x_2|^2 - \frac{1}{3} & x_2\overline{x}_3 \\ x_3\overline{x}_1 & x_3\overline{x}_2 & |x_3|^2 - \frac{1}{3} \end{bmatrix}$$

for $x = (x_1, x_2, x_3) \in S^{3d-1}(1) \subset F^3$. Then, it is easily verified that $\overline{\psi}$ induces a map $\psi : FP^2 \to \mathbf{R}^{3d+2} = \mathcal{H}(3, F)$ such that $\overline{\psi} = \psi \circ \pi$. Direct computation shows that $\psi(FP^2) \subset S^{3d+1}(1/3)$. We blow up the metric g_0 by putting $g = 3g_0$ in FP^2, so that the sectional curvature of $\mathbf{R}P^2$ is $\frac{1}{3}$ and the holomorphic sectional curvature (resp. Q-sectional curvature) of $\mathbf{C}P^2$ (resp. QP^2) is $\frac{4}{3}$, with respect to the metric g. Then ψ gives a map $\psi : FP^2 \to S^{3d+1}(1)$. It is proved in [**5**] that ψ is an isometric minimal imbedding. Thus, we have the following isometric minimal imbeddings:

$$\begin{aligned} &\psi_1 : \mathbf{R}P^2 \to S^4(1) \quad \text{(the Veronese surface)}, \\ &\psi_2 : \mathbf{C}P^2 \to S^7(1), \\ &\psi_3 : QP^2 \to S^{13}(1). \end{aligned}$$

In a similar manner one may obtain (see [**5**] for details) an isometric imbedding of the Cayley projective plane Cay P^2 furnished with the canonical metric (normalized such that the C-sectional curvature equals $\frac{4}{3}$) into $S^{25}(1)$:

$$\psi_4 : \text{Cay } P^2 \to S^{25}(1).$$

In addition there is an immersion

$$\psi_1' : S^2\left(\sqrt{3}\right) \to S^4(1)$$

defined by $\psi_1' = \psi_1 \circ \pi$.

For $n, m \geq 0$, let $S^n(1)$ be the great sphere in $S^{n+m}(1)$ given by

$$S^n(1) = \{(x_1, \ldots, x_{n+m+1}) \in S^{n+m}(1) : x_{n+2} = \cdots = x_{n+m+1} = 0\},$$

and $\tau_{n,m}: S^n(1) \to S^{n+m}(1)$ be the inclusion. For $p = 0, 1, \ldots$, we set

$$\begin{aligned}
\phi_{1,p} &= \tau_{4,p} \circ \psi_1 : \mathbf{R}P^2 \to S^{4+p}, \\
\phi_{2,p} &= \tau_{7,p} \circ \psi_2 : \mathbf{C}P^2 \to S^{7+p}, \\
\phi_{3,p} &= \tau_{13,p} \circ \psi_3 : QP^2 \to S^{13+p}, \\
\phi_{4,p} &= \tau_{25,p} \circ \psi_4 : \mathrm{Cay}\ P^2 \to S^{25+p}, \\
\phi'_{1,p} &= \tau_{4,p} \circ \psi'_1 : S^2\left(\sqrt{3}\right) \to S^{4+p}.
\end{aligned}$$

$\phi_{i,p}$ ($i = 1, \ldots, 4$; $p = 0, 1, \ldots$), is an isometric minimal imbedding and $\phi'_{1,p}$ ($p = 0, 1, \ldots$), is an isometric minimal immersion.

We now state the results of the present paper.

THEOREM 1. *Let M be a compact n-dimensional manifold minimally immersed in a unit sphere S^{n+1}. Assume that n $(= 2m)$ is even.*

(i) *If $\sigma(u) < 1$ for any $u \in UM$, then M is totally geodesic in S^{n+1}.*

(ii) *If $\max_{u \in UM} \sigma(u) = 1$, then M is $S^m(\frac{1}{2}) \times S^m(\frac{1}{2})$ minimally imbedded in S^{2m+1} as described above.*

THEOREM 2. *Let M be a compact n-dimensional manifold minimally immersed in a unit sphere S^{n+1}. Assume that n $(= 2m+1)$ is odd. If $\sigma(u) \le 1/(1 - 1/n)$ for any $u \in UM$, then M is totally geodesic in S^{n+1}.*

REMARK. Theorems 1(i) and 2 are easy consequences of J. Simons' results [**6**]. The only nontrivial part of Theorem 1(ii) is that $\max_{u \in UM} \sigma(u) = 1$ implies $S(x) \equiv n$ on UM. The remaining part of Theorem 1(ii) readily follows from results of S.-S. Chern, M. do Carmo, S. Kobayashi [**1**], and B. Lawson [**2**]. We present Theorems 1 and 2 mainly for completeness. Our main results are Theorems 3 and 4.

THEOREM 3. *Let M be a compact n-dimensional manifold minimally immersed in a unit sphere S^{n+p}. Assume that $p \ge 2$ and n $(= 2m)$ is even.*

(i) *If $\sigma(u) < \frac{1}{3}$ for any $u \in UM$, then M is totally geodesic in S^{n+p}.*

(ii) *If $\max_{u \in UM} \sigma(u) = \frac{1}{3}$, then $\sigma(u) \equiv \frac{1}{3}$ on UM, and the immersion of M into S^{n+p} is one of the imbeddings $\phi_{i,p}$ ($i = 1, \ldots, 4$; $p = 0, 1, \ldots$), or the immersions $\phi'_{1,p}$ ($p = 0, 1, \ldots$), described above.*

THEOREM 4. *Let M be a compact n-dimensional manifold minimally immersed in a unit sphere S^{n+p}. Asume that $p \ge 2$ and n $(= 2m+1)$ is odd. If $\sigma(u) \le 1/(3 - 2/n)$ for any $u \in UM$, then M is totally geodesic in S^{n+p}.*

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2. Maximal directions. Let M be a compact n-dimensional manifold minimally immersed in S^{n+p}. We choose a local field of adapted orthonormal frames in S^{n+p}, that is frames $\{e_1, \ldots, e_{n+p}\}$ such that the vectors $e_1, \ldots, e_n$ are tangent to M. The vectors $e_{n+1}, \ldots, e_{n+p}$ are therefore normal to M. From now on let the indices $a, b, c, \ldots$, run from $1, \ldots, n$, and the indices $\alpha, \beta, \gamma, \ldots$, run from $n+1, \ldots, n+p$. Let $h = (h^\alpha_{ab})$ be the second fundamental form of the immersed manifold M, and $\sigma(u) = \|h(u,u)\|^2$ for $u \in UM$. Since the immersion of M into S^{n+p} is minimal, $\sum_a h^\alpha_{aa} = 0$ for all α.

Let $x \in M$. Suppose that $u \in UM_x$ satisfies $\sigma(u) = \max_{v \in UM_x} \sigma(v)$. We shall call u a *maximal direction* at x. Let $\{e_1, \dots, e_{n+p}\}$ be an adapted frame at x. Assume that e_1 is a maximal direction at x, $\sigma(e_1) \neq 0$, and $e_{n+1} = h(e_1,e_1)/\|h(e_1,e_1)\|$. Because of our choice of e_{n+1},

$$h^\alpha_{11} = 0, \qquad \alpha \neq n+1. \tag{2.1}$$

Since e_1 is a maximal direction, we have at the point x for any $t, x^2, \dots, x^n \in \mathbf{R}$

$$\left\| h\left(e_1 + t\sum_{a=2}^n x^a e_a, e_1 + t\sum_{a=2}^n x^a e_a\right)\right\|^2 \leq \left[1 + t^2 \sum_{a=2}^n (x^a)^2\right]^2 (h^{n+1}_{11})^2. \tag{2.2}$$

Expanding in terms of t, we obtain

$$4t h^{n+1}_{11} \sum_{a\neq 1} x^a h^{n+1}_{1a} + O(t^2) \leq 0.$$

It follows that

$$h^{n+1}_{1a} = 0, \qquad a = 2, \dots, n. \tag{2.3}$$

We now choose an adapted frame at $x \in M$ such that in addition to (2.1) and (2.3),

$$h^{n+1}_{ab} = 0, \qquad a \neq b. \tag{2.4}$$

Once more expanding (2.2) in terms of t, we obtain

$$\begin{aligned} -2t^2 \Bigg\{ &\sum_{a\neq 1}\left[h^{n+1}_{11}(h^{n+1}_{11} - h^{n+1}_{aa}) - 2\sum_{\alpha\neq n+1}(h^\alpha_{1a})^2\right](x^a)^2 \\ &- 4\sum_{\alpha \neq n+1}\sum_{\substack{a,b\neq 1\\ a\neq b}} h_{1a}h_{1b}x^a x^b\Bigg\} + O(t^3) \leq 0. \end{aligned} \tag{2.5}$$

It follows that

$$2\sum_{\alpha\neq n+1}(h^\alpha_{1a})^2 \leq h^{n+1}_{11}(h^{n+1}_{11} - h^{n+1}_{aa}), \qquad a = 2, \dots, n. \tag{2.6}$$

Let us define a tensor field $H = (H_{abcd})$ on M by the formula

$$H_{abcd} = \sum_\alpha h^\alpha_{ab} h^\alpha_{cd}. \tag{2.7}$$

It is clear that $\sigma(u) = H(u,u,u,u)$.

LEMMA 1. *Let u be a maximal direction at $x \in M$. Assume that $\sigma(u) \neq 0$. Let $e_1, \dots, e_{n+p}$ be an adapted frame at x such that $e_1 = u$, $e_{n+1} = h(e_1,e_1)/\|h(e_1,e_1)\|$, and $h^{n+1}_{ab} = 0$ for $a \neq b$. At the point x*

(i) *if $p = 1$, then*

$$\frac{1}{2}(\Delta H)_{1111} \geq (h^{n+1}_{11})^2\left[n - \sum_a (h^{n+1}_{aa})^2\right]. \tag{2.8}$$

(ii) *if* $p \geq 2$, *then*

$$\frac{1}{2}(\Delta H)_{1111} \geq (h_{11}^{n+1})^2 \left[n - n(h_{11}^{n+1})^2 - 2\sum_a (h_{aa}^{n+1})^2 \right] \tag{2.9}$$

with equality attained if and only if

$$(h_{11}^{n+1} - h_{aa}^{n+1}) \left[h_{11}^{n+1}(h_{11}^{n+1} - h_{aa}^{n+1}) - 2 \sum_{\alpha \neq n+1} (h_{1a}^{\alpha})^2 \right] = 0 \tag{2.10}$$

and

$$\nabla_a h_{11}^{\alpha} = 0 \tag{2.11}$$

for all a *and all* α, *where* Δ *and* ∇_a *denote the Laplacian and the covariant derivative, respectively.*

PROOF.

$$\frac{1}{2}(\Delta H)_{1111} = h_{11}^{n+1}(\Delta h)_{11}^{n+1} + \sum_{a,\alpha}(\nabla_a h_{11}^{\alpha})^2.$$

Using Simons' formula [**6**] for the Laplacian of the second fundamental form (see also [**1**]), we obtain

$$\frac{1}{2}(\Delta H)_{1111} = (h_{11}^{n+1})^2 \left[n - \sum_a (h_{aa}^{n+1})^2 \right] + \sum_{a,\alpha}(\nabla_a h_{11}^{\alpha})^2, \quad \text{if } p = 1, \tag{2.12}$$

and

$$\begin{aligned} \frac{1}{2}(\Delta H)_{1111} &= (h_{11}^{n+1})^2 \left[n - n(h_{11}^{n+1})^2 - 2\sum_a (h_{aa}^{n+1})^2 \right] \\ &+ \sum_a h_{11}^{n+1}(h_{11}^{n+1} - h_{aa}^{n+1}) \left[h_{11}^{n+1}(h_{11}^{n+1} - h_{aa}^{n+1}) - 2 \sum_{\alpha \neq n+1} (h_{1a}^{\alpha})^2 \right] \\ &+ \sum_{a,\alpha}(\nabla_a h_{11}^{\alpha})^2, \quad \text{if } p \geq 2, \end{aligned} \tag{2.13}$$

from which the lemma follows readily by inequality (2.6). □

LEMMA 2. *Let an adapted frame* $\{e_1, \dots, e_{n+p}\}$ *at* $x \in M$ *be as in Lemma* 1.
(i) *Assume that* n $(= 2m)$ *is even. If*

$$\sigma(u) \leq \begin{cases} 1, & \text{if } p = 1, \\ \dfrac{1}{3}, & \text{if } p \geq 2, \end{cases} \quad \text{for all } u \in UM_x,$$

then $(\Delta H)_{1111} \geq 0$. *If equality* $(\Delta H)_{1111} = 0$ *is attained, then it is possible to renumber* $e_1, \dots, e_{2m}$ *such that the following equalities hold*

$$h_{11}^{n+1} = \cdots = h_{mm}^{n+1} = -h_{m+1\ m+1}^{n+1} = \cdots = -h_{2m\ 2m}^{n+1} = \begin{cases} 1, & \text{if } p = 1, \\ 1/\sqrt{3}, & \text{if } p \geq 2. \end{cases} \tag{2.14}$$

(ii) *Assume that* n $(=2m+1)$ *is odd. If*

$$\sigma(u) \le \begin{cases} 1 - \dfrac{1}{n}, & \text{if } p = 1, \\ \dfrac{1}{3 - 2/n}, & \text{if } p \ge 2, \end{cases} \qquad \text{for all } u \in UM_x,$$

then $(\Delta H)_{1111} \ge 0$. *If equality* $(\Delta H)_{1111} = 0$ *is attained, then it is possible to renumber* $e_1, \dots, e_{2m+1}$ *such that the following equalities hold.*

$$h_{11}^{n+1} = \cdots = h_{mm}^{n+1} = -h_{m+1\ m+1}^{n+1} = \cdots = -h_{2m\ 2m}^{n+1}$$

$$= \begin{cases} \left(1 - \dfrac{1}{n}\right)^{-1/2}, & \text{if } p = 1, \\ \left(3 - \dfrac{2}{n}\right)^{-1/2}, & \text{if } p \ge 2, \end{cases} \tag{2.15}$$

$$h_{2m+1\ 2m+1}^{n+1} = 0.$$

PROOF. Since e_1 is a maximal direction

$$-h_{11}^{n+1} \le h_{aa}^{n+1} \le h_{11}^{n+1}, \qquad a = 2, \dots, n. \tag{2.16}$$

Because of minimality of the immersion of M into S^{n+p},

$$\sum_{a=2}^{n} h_{aa}^{n+1} = -h_{11}^{n+1}. \tag{2.17}$$

It is easily seen that the convex function $f(h_{22}^{n+1}, \dots, h_{nn}^{n+1}) = \sum_{a=2}^{n} (h_{aa}^{n+1})^2$ of $(n-1)$ variables $h_{22}^{n+1}, \dots, h_{nn}^{n+1}$ subject to the linear constraints (2.16), (2.17) attains its maximal value when (after suitable renumbering of $e_1, \dots, e_n$)

$$h_{11}^{n+1} = \cdots = h_{mm}^{n+1} = -h_{m+1\ m+1}^{n+1} = \cdots = -h_{2m\ 2m}^{n+1}, \quad \text{if } n = 2m,$$

and

$$h_{11}^{n+1} = \cdots = h_{mm}^{n+1} = -h_{m+1\ m+1}^{n+1} = \cdots = -h_{2m\ 2m}^{n+1},$$
$$h_{2m+1\ 2m+1}^{n+1} = 0, \quad \text{if } n = 2m+1.$$

Therefore, by inequalities (2.8), (2.9),

$$\frac{1}{2}(\Delta H)_{1111} \ge \begin{cases} n(h_{11}^{n+1})^2[1 - \sigma(e_1)], & \text{if } p = 1,\ n = 2m, \\ n(h_{11}^{n+1})^2[1 - 3\sigma(e_1)], & \text{if } p \ge 2,\ n = 2m, \\ (h_{11}^{n+1})^2[n - (n-1)\sigma(e_1)], & \text{if } p = 1,\ n = 2m+1, \\ (h_{11}^{n+1})^2[n - (3n-2)\sigma(e_1)], & \text{if } p \ge 2,\ n = 2m+1. \end{cases}$$

This proves the lemma. □

Let $L(x)$ be a function on M defined by $L(x) = \max_{u \in UM_x} \sigma(u)$.

LEMMA 3. *Assume that one of* A_1, A_2, A_3, A_4 *is satisfied.*
(A_1) $p = 1$, *n is even,* $\sigma(u) \le 1$ *for all* $u \in UM$,
(A_2) $p = 1$, *n is odd,* $\sigma(u) \le 1/(1 - 1/n)$ *for all* $u \in UM$,
(A_3) $p \ge 2$, *n is even,* $\sigma(u) \le \frac{1}{3}$ *for all* $u \in UM$,
(A_4) $p \ge 2$, *n is odd,* $\sigma(u) \le 1/(3 - 2/n)$ *for all* $u \in UM$.
Then $L(x)$ is a constant function on M.

PROOF. Following an idea in [3] we prove the lemma using the maximum principle. Clearly $L(x)$ is a continuous function. It suffices to show that $L(x)$ is subharmonic in the generalized sense. Fix $x \in M$ and let e_1 be a maximal direction at x. In an open neighborhood U_x of x within the cut-locus of x we shall denote by $u(y)$ the tangent vector to M obtained by parallel transport of $e_1 = u(x)$ along the unique geodesic joining x to y within the cut-locus of x. Define $g_x(y) = \sigma(u(y))$. Then

$$\begin{aligned}\Delta g_x(x) &= \Delta[H(u(y), u(y), u(y), u(y))]_{y=x} \\ &= \sum_a (\nabla_a^2 H)(e_1, e_1, e_1, e_1) = (\Delta H)_{1111}(x).\end{aligned}$$

If $\|h(e_1, e_1)\| \neq 0$, then by Lemma 2, $(\Delta H)_{1111}(x) \geq 0$. If $\|h(e_1, e_1)\| = 0$, then $h \equiv 0$ at x. In this case the formula of Simons [6] for Δh shows that $\Delta h = 0$ at x, and therefore

$$(\Delta H)_{1111}(x) = \sum_{a,\alpha} (\nabla_a h_{11}^\alpha)^2 \geq 0.$$

Thus, we obtain that in any case $\Delta g_x(x) = (\Delta H)_{1111}(x) \geq 0$.

For the Laplacian of continuous functions, we have the generalized definition

$$\Delta L = C \lim_{r \to 0} \frac{1}{r^2} \left(\int_{B(x,r)} L \Big/ \int_{B(x,r)} 1 - L(x) \right),$$

where C is a positive constant and $B(x, r)$ denotes the geodesic ball of radius r with the center at x. With this definition L is subharmonic on M if and only if $\Delta L(x) \geq 0$ at each point $x \in M$. Since $g_x(x) = L(x)$ and $g_x \leq L$ on U_x, $\Delta L(x) \geq \Delta g_x(x) \geq 0$. Thus, $L(x)$ is subharmonic and hence constant on M. □

3. Proofs of Theorems 1–4.

LEMMA 4. *Assume that one of B_1, B_2, B_3, B_4 is satisfied.*
(B_1) *$p = 1$, n is even, $\sigma(u) < 1$ for all $u \in UM$,*
(B_2) *$p = 1$, n is odd, $\sigma(u) < 1/(1 - 1/n)$ for all $u \in UM$,*
(B_3) *$p \geq 2$, n is even, $\sigma(u) < \frac{1}{3}$ for all $u \in UM$,*
(B_4) *$p \geq 2$, n is odd, $\sigma(u) < 1/(3 - 2/n)$ for all $u \in UM$.*
Then M is totally geodesic in S^{n+p}.

PROOF. Let $x \in M$ and e_1 be a maximal direction at x. Assume that $\sigma(e_1) \neq 0$. Let $g_x(y) = \sigma(u(y))$ be the function defined in the proof of Lemma 3. By Lemma 3, $g_x(x)$ is a maximum of g_x. Therefore, $(\Delta H)_{1111}(x) = \Delta g_x(x) \leq 0$. On the other hand, by Lemma 2, $(\Delta H)_{1111}(x) \geq 0$. Therefore, $(\Delta H)_{1111} = 0$ on M. Hence, by (2.14) and (2.15),

$$\sigma(e_1) = \begin{cases} 1, & \text{if } p = 1,\ n \text{ is even}, \\ \dfrac{1}{1 - 1/n}, & \text{if } p = 1,\ n \text{ is odd}, \\ \dfrac{1}{3}, & \text{if } p \geq 2,\ n \text{ is even}, \\ \dfrac{1}{3 - 2/n}, & \text{if } p \geq 2,\ n \text{ is odd}, \end{cases}$$

contradicting the assumptions B_1, B_2, B_3, B_4. Hence, $h(u,u) = 0$ for all $u \in UM$, that is M is totally geodesic in S^{n+p}. $\square$

PROOF OF THEOREM 1. (i) follows from Lemma 4. We prove (ii). As in the poof of Lemma 4, we obtain $(\Delta H)_{1111} = 0$. Hence, by (2.4) and (2.14),

$$S(x) = \sum_{\alpha,a,b} (h^\alpha_{ab})^2 = \sum_a (h^{n+1}_{aa})^2 = n.$$

All minimal immersions into S^{n+1} satisfying $S(x) \equiv n$ were found by S.-S. Chern, M. do Carmo, and S. Kobayashi in [**1**] and B. Lawson in [**2**]. It is easy to see that among their immersions only $S^m(\sqrt{\frac{1}{2}}) \times S^m(\sqrt{\frac{1}{2}})$ imbedded in S^{2m+1} satisfies the condition $\max_{u\in UM} \sigma(u) = 1$. This completes the proof of Theorem 1. $\square$

PROOF OF THEOREM 2. By Lemmas 3 and 4, we have to consider only the case $L(x) = \max_{u \in UM_x} \sigma(u) \equiv 1/(1-1/n)$ on M. As in the proof of Lemma 4, $(\Delta H)_{1111} = 0$. Hence, by (2.15),

$$S(x) \equiv \sum_{\alpha,a,b} (h^\alpha_{ab})^2 \equiv \sum_{a=1}^{n+1} \frac{1}{(1-1/n)} \equiv n.$$

It is shown in [**1**] that if M is minimally immersed in S^{n+1} and $S(x) \equiv n$, then h^{n+1}_{aa} may attain at most two different values for $a = 1, \ldots, n$. However, since by (2.15),

$$h^{n+1}_{11} = \left(\frac{n}{n-1}\right)^{1/2}, \quad h^{n+1}_{m+1\ m+1} = -\left(\frac{n}{n-1}\right)^{1/2}, \quad h^{n+1}_{2m+1\ 2m+1} = 0,$$

we obtain a contradiction, so the equality $\max_{u\in UM} \sigma(u) \equiv 1/(1-1/n)$ on UM is impossible. This completes the proof of Theorem 2. $\square$

PROOF OF THEOREM 3. (i) follows from Lemma 4. We prove (ii). As in the proof of Lemma 4, we obtain $(\Delta H)_{1111} = 0$. Let the indices $i, j, k, \ldots,$ run from $1, \ldots, m$, and let $\bar{i}, \bar{j}, \bar{k}, \ldots,$ denote $i+m$, $j+m$, $k+m, \ldots,$ respectively. By (2.14) we have

$$h^{n+1}_{ii} = -h^{n+1}_{\bar{i}\bar{i}} = -1/\sqrt{3}, \qquad i = 1, \ldots, m. \tag{3.1}$$

Since $\|h(e_i, e_i)\|^2 \le \frac{1}{3}$ and $\|h(e_{\bar{i}}, e_{\bar{i}})\|^2 \le \frac{1}{3}$, we obtain

$$h^\alpha_{ii} = h^\alpha_{\bar{i}\bar{i}} = 0, \qquad \alpha \ne n+1;\ i = 1, \ldots, m. \tag{3.2}$$

By (2.10), $\sum_{\alpha \ne n+1} (h^\alpha_{1\bar{j}})^2 = \frac{1}{3}$. Since each vector e_a, $(a = 1, \ldots, n)$, is a maximal direction,

$$\sum_{\alpha \ne n+1} (h^\alpha_{i\bar{j}})^2 = \frac{1}{3}, \qquad i, j = 1, \ldots, m. \tag{3.3}$$

Let $u = (e_i + e_j)/\sqrt{2}$. Then

$$\begin{aligned}\sigma(u) &= \frac{1}{4}\|h(e_i + e_j, e_i + e_j)\|^2 \\ &= \frac{1}{4}\|(h^{n+1}_{ii} + h^{n+1}_{jj})e_{n+1} + 2\sum_{\alpha \ne n+1} h^\alpha_{ij} e_\alpha\|^2 \\ &= \frac{1}{3} + \sum_{\alpha \ne n+1} (h^\alpha_{ij})^2 \le \frac{1}{3}.\end{aligned}$$

Therefore,

$$h^{\alpha}_{i\bar{j}} = 0, \qquad \alpha \neq n+1;\ i,j = 1,\dots,m. \tag{3.4}$$

Similarly,

$$h^{\alpha}_{\bar{i}\bar{j}} = 0, \qquad \alpha \neq n+1;\ i,j = 1,\dots,m. \tag{3.5}$$

Expansion (2.5) now takes the form

$$t^2\left(-4\sum_{\alpha}\sum_{j\neq k} h^{\alpha}_{i\bar{j}} h^{\alpha}_{i\bar{k}} x^{\bar{j}} x^{\bar{k}}\right) + O(t^3) \leq 0.$$

It follows that $\sum_\alpha h^{\alpha}_{1\bar{j}} h^{\alpha}_{1\bar{k}} = 0$ for $j \neq k$. Since each vector e_a is a maximal direction,

$$\sum_{\alpha} h^{\alpha}_{i\bar{j}} h^{\alpha}_{i\bar{k}} = 0, \qquad j \neq k, \tag{3.6}$$

$$\sum_{\alpha} h^{\alpha}_{i\bar{k}} h^{\alpha}_{j\bar{k}} = 0, \qquad i \neq j. \tag{3.7}$$

Once more expanding (2.2) in terms of t,

$$2t^3 \sum_{\alpha,j,k,l} (h^{\alpha}_{1\bar{k}} h^{\alpha}_{j\bar{l}} + h^{\alpha}_{1\bar{l}} h^{\alpha}_{j\bar{k}}) x^j x^{\bar{k}} x^{\bar{l}} + O(t^4) \leq 0,$$

from which

$$\sum_{\alpha} (h^{\alpha}_{i\bar{k}} h^{\alpha}_{j\bar{l}} + h^{\alpha}_{i\bar{l}} h^{\alpha}_{j\bar{k}}) = 0, \qquad i \neq j \text{ or } k \neq l. \tag{3.8}$$

Using (2.4) and (3.1)–(3.8), we obtain by direct computation that $\sigma(u) = \frac{1}{3}$ for any $u \in UM$. B. O'Neill [**4**] calls an immersion λ-isotropic if $\|h(u,u)\| = \lambda$ for any $u \in UM$. Therefore, *the immersion under consideration is $1/\sqrt{3}$-isotropic.*

By Lemma 1, $\nabla_a h^{\alpha}_{11} = 0$. It follows that $\nabla_a h^{\alpha}_{bb} = 0$. By polarization, $\nabla_a h^{\alpha}_{bc} = 0$ for all α, a, b, c. Therefore, *the second fundamental form of the immersion is parallel.* All λ-isotropic minimal immersions into a unit sphere with parallel second fundamental form were completely classified by K. Sakamoto in [**5**]. Among his immersions only $\phi_{1,p}, \phi_{2,p}, \phi_{3,p}, \phi_{4,p}$ and $\phi'_{1,p}$ described in §1, are $1/\sqrt{3}$-isotropic. This completes the proof of the theorem. □

PROOF OF THEOREM 4. By Lemmas 3 and 4, we need only consider the case $L(x) \equiv 1/(3-2/n)$ on M. We show that this case cannot occur. Thus, assume that $L(x) \equiv 1/(3-2/n)$ on M. As in the proof of Lemma 4, $(\Delta H)_{1111} = 0$. Let the indices $i, j, k, \dots,$ run from $1, \dots, m$, and let $\bar{i}, \bar{j}, \bar{k}, \dots,$ denote $i+m$, $j+m$, $k+m, \dots,$ respectively. By (2.15),

$$\begin{aligned} h^{n+1}_{ii} &= -h^{n+1}_{\bar{i}\bar{i}} = (3-2/n)^{1/2}, \qquad i = 1,\dots,m, \\ h^{n+1}_{nn} &= 0. \end{aligned} \tag{3.9}$$

As in the proof of Theorem 3,

$$h^{\alpha}_{ij} = h^{\alpha}_{\bar{i}\bar{j}} = 0, \qquad \alpha \neq n+1;\ i,j = 1,\dots,m. \tag{3.10}$$

Since $h^\alpha_{nn} = -\sum_i h^\alpha_{ii} - \sum_i h^\alpha_{\bar{i}\bar{i}}$,

$$h^\alpha_{nn} = 0. \tag{3.11}$$

By (2.10),

$$\sum_\alpha (h^\alpha_{i\bar{j}})^2 = \frac{1}{3-2/n}, \qquad i,j = 1,\dots,m \tag{3.12}$$

$$\sum_\alpha (h^\alpha_{in})^2 = \frac{1}{2(3-2/n)}, \qquad i = 1,\dots,m, \tag{3.13}$$

$$\sum_\alpha (h^\alpha_{\bar{i}n})^2 = \frac{1}{2(3-2/n)}, \qquad i = 1,\dots,m. \tag{3.14}$$

As in the proof of Theorem 3, we obtain with the help of expansion (2.2) the following equalities:

$$\sum_\alpha h^\alpha_{i\bar{j}} h^\alpha_{i\bar{k}} = 0, \tag{3.15}$$

$$\sum_\alpha h^\alpha_{i\bar{k}} h^\alpha_{j\bar{k}} = 0, \tag{3.16}$$

$$\sum_\alpha h^\alpha_{i\bar{j}} h^\alpha_{in} = 0, \tag{3.17}$$

$$\sum_\alpha h^\alpha_{i\bar{j}} h^\alpha_{n\bar{j}} = 0, \tag{3.18}$$

$$\sum_\alpha (h^\alpha_{i\bar{k}} h^\alpha_{j\bar{l}} + h^\alpha_{i\bar{l}} h^\alpha_{j\bar{k}}) = 0, \qquad i \neq j \text{ or } k \neq 1, \tag{3.19}$$

$$\sum_\alpha (h^\alpha_{i\bar{k}} h^\alpha_{jn} + h^\alpha_{j\bar{k}} h^\alpha_{in}) = 0, \qquad i \neq j, \tag{3.20}$$

$$\sum_\alpha (h^\alpha_{i\bar{j}} h^\alpha_{n\bar{k}} + h^\alpha_{i\bar{k}} h^\alpha_{n\bar{j}}) = 0, \qquad j \neq k, \tag{3.21}$$

$$\sum_\alpha h^\alpha_{in} h^\alpha_{jn} = 0, \qquad i \neq j, \tag{3.22}$$

$$\sum_\alpha h^\alpha_{\bar{i}n} h^\alpha_{\bar{j}n} = 0, \qquad i \neq j, \tag{3.23}$$

$$\sum_\alpha h^\alpha_{in} h^\alpha_{\bar{j}n} = 0. \tag{3.24}$$

Let $u = \sum_a u^a e_a \in UM$. Direct computation with the help of (2.4) and (3.9)–(3.24) shows that

$$\sigma(u) = [1 - (u^n)^4](3 - 2/n)^{-1}. \tag{3.25}$$

It follows from (3.25) that for any $x \in M$, the tangent space T_x of M at x is a direct sum of two mutually orthogonal subspaces $T_x = P_x + Q_x$, where P_x is $2m$-dimensional and is defined by

$$P_x = \{X \in T_x \colon \|h(X,X)\| = (3-2/n)^{-1/2}\|X\|^2\}, \tag{3.26}$$

and Q_x is 1-dimensional and is defined by

$$Q_x = \{X \in T_x : h(X,X) = 0\}. \tag{3.27}$$

LEMMA 5. *The distributions $P: x \to P_x$ and $Q: x \to Q_x$ are smooth distributions on M.*

PROOF. It is sufficient to prove that Q is smooth. Let $x_0 \in M$ and $\{e_1, \ldots, e_{n+p}\}$ be a smooth local field of orthonormal adapted frames in a neighborhood U of x_0 such that $e_n(x_0) \in Q_{x_0}$. If U is sufficiently small, there is a unique vector X of the form $X = \sum_{a=1}^{2m} X^a e_a + e_n$ which belongs to Q_x at each point $x \in U$. We prove that X^a, $a = 1, \ldots, 2m$, are smooth functions of x.

By (3.27), $X^a(x)$, $a = 1, \ldots, 2m$, are a unique solution of the system of equations

$$h^\alpha(X,X) = \sum_{a,b=1}^{2m} h^\alpha_{ab}(x) X^a X^b + 2\sum_{a=1}^{2m} h^\alpha_{an}(x) X^a = 0, \qquad \alpha = n+1, \ldots, n+p. \tag{3.28}$$

At the point x_0 the Jacobian of system (3.28) is

$$(\partial h^\alpha / \partial X^a) = 2(h^\alpha_{an}), \qquad \alpha = n+1, \ldots, n+p;\ a = 1, \ldots, 2m.$$

By (3.13), (3.14) and (3.22)–(3.24), the rows of the matrix (h^α_{an}) are mutually orthogonal nonzero vectors. Hence, $\operatorname{rank}(\partial h^\alpha/\partial X^a) = 2m$ at x_0. Therefore, X^a, $a = 1, \ldots, 2m$, are smooth functions of x in a sufficiently small neighborhood of x_0. □

We now return to the proof of Theorem 4. Let $x \in M$. By Lemma 5, we may choose a smooth family of orthonormal adapted frames $\{e_1, \ldots, e_{n+p}\}$ in some neighborhood U of x such that equations (2.4), (3.9)–(3.24) are satisfied on U. Set

$$N_a = \left[2\left(3 - \frac{2}{n}\right)\right]^{1/2} \sum_\alpha h^\alpha_{an} e_\alpha, \qquad a = 1, \ldots, 2m.$$

By (2.4), (3.13), (3.14), and (3.22)–(3.24), the vectors $e_{n+1}, N_1, \ldots, N_{2m}$ are orthonormal. Therefore, with no loss of generality, we may assume that $e_{n+1+a} = N_a$, $a = 1, \ldots, 2m$. Then,

$$h^{n+1+i}_{in} = h^{n+1+\bar{i}}_{\bar{i}n} = \left[2\left(3 - \frac{2}{n}\right)\right]^{1/2}, \qquad i = 1, \ldots, m, \tag{3.29}$$

$$h^\alpha_{in} = 0, \qquad \alpha \neq n+1+i,\ i = 1, \ldots, m, \tag{3.30}$$

$$h^\alpha_{\bar{i}n} = 0, \qquad \alpha \neq n+1+\bar{i},\ i = 1, \ldots, m. \tag{3.31}$$

Let the indices A, B, C run from $1, \ldots, n+p$, and let $\{\omega^A\}$ and $\{\omega_B^A\}$ be the coframe dual to the frame $\{e_A\}$ and the connection forms of the Riemannian connection on S^{n+p}, respectively. Then,

$$d\omega^A = \sum_B \omega^B \wedge \omega_B^A, \tag{3.32}$$

$$d\omega_B^A = \sum_C \omega^C \wedge \omega_C^A + \omega^A \wedge \omega^B, \tag{3.33}$$

$$\omega^\alpha = 0, \tag{3.34}$$

$$\omega_a^\alpha = \sum_b h_{ab}^\alpha \omega^b, \tag{3.35}$$

$$dh_{ab}^\alpha - \sum_c h_{cb}^\alpha \omega_a^c - \sum_c h_{ac}^\alpha \omega_b^c + \sum_\beta h_{ab}^\beta \omega_\beta^\alpha = \sum_c (\nabla_c h_{ab}^\alpha)\omega^c. \tag{3.36}$$

As in the proof of Theorem 3, we obtain

$$\nabla_c h_{ab}^\alpha = 0, \qquad a, b = 1, \ldots, 2m;\ c = 1, \ldots, n. \tag{3.37}$$

Let us take $\alpha = h+1+i$, $a = b = i$ in (3.36). By (2.4), (3.9)–(3.11), (3.29)–(3.31), and (3.37),

$$-2\sum_k h_{\bar{k}i}^{n+1+i}\omega_i^{\bar{k}} - \left[2\left(3-\frac{2}{n}\right)\right]^{-1/2}\omega_i^n + \left(3-\frac{2}{n}\right)^{-1/2}\omega_{n+1}^{n+1+i} = 0. \tag{3.38}$$

Analogously, taking $\alpha = n+1+i$, $a = i$, $b = j \neq i$ in (3.36),

$$-2\sum_k h_{\bar{k}j}^{n+1+i}\omega_j^{\bar{k}} + \left(3-\frac{2}{n}\right)^{-1/2}\omega_{n+1}^{n+1+i} = 0, \qquad i \neq j. \tag{3.39}$$

Summing (3.39) with respect to j ($j \neq i$) and adding (3.38), we have

$$-2\sum_{j,k} h_{\bar{k}j}^{n+1+i}\omega_j^{\bar{k}} + m\left(3-\frac{2}{n}\right)^{-1/2}\omega_{n+1}^{n+1+i} - \left[2\left(3-\frac{2}{n}\right)\right]^{-1/2}\omega_i^n = 0. \tag{3.40}$$

Let us now take $\alpha = n+1+i$, $a = b = \bar{k}$ in (3.36). Then,

$$-2\sum_j h_{j\bar{k}}^{n+1+i}\omega_{\bar{k}}^j - \left(3-\frac{2}{n}\right)^{-1/2}\omega_{n+1}^{n+1+i} = 0. \tag{3.41}$$

Summing (3.41) with respect to $\bar{k}$,

$$-2\sum_{j,k} h_{j\bar{k}}^{n+1+i}\omega_{\bar{k}}^j - m\left(3-\frac{2}{n}\right)^{-1/2}\omega_{n+1}^{n+1+i} = 0. \tag{3.42}$$

Finally, adding (3.40) to (3.42), we get

$$\omega_i^n = 0. \tag{3.43}$$

Analogously, we obtain

$$\omega_{\bar{i}}^n = 0. \tag{3.44}$$

Differentiating (3.43) and using (2.4), (3.9)–(3.11), (3.29)–(3.31), and (3.4), we obtain

$$-\sum_{\alpha,a,b} h_{ia}^{\alpha} h_{bn}^{\alpha} \omega^a \wedge \omega^b + \omega^n \wedge \omega^i = 0 \tag{3.45}$$

Taking the coefficient of $\omega^n \wedge \omega^i$ in (3.45) we have $-\sum_\alpha (h_{in}^\alpha)^2 + 1 = 0$. By (3.13), it gives $2(3-2/n) = 1$ and therefore $n = 5/4$, yielding a contradiction. Therefore, the equality $\max_{u \in UM_x} \sigma(u) \equiv 1/(3-2/n)$ on M is impossible. This completes the proof of Theorem 4.

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