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CONCAVITY OF SOLUTIONS OF THE POROUS MEDIUM EQUATION

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ABSTRACT. We consider the problem

$$\text{(P)} \qquad \begin{cases} u_t = (u^m)_{xx} & \text{with } x \in \mathbf{R}, t > 0, \\ u(x,0) = u_0(x) & \text{for } x \in \mathbf{R}, \end{cases}$$

where $m > 1$ and u_0 is a continuous, nonnegative function that vanishes outside an interval (a,b) and such that $(u_0^{m-1})'' \leq -C \leq 0$ in (a,b). Using a Trotter-Kato formula we show that the solution conserves the concavity in time: for every $t > 0$, $u(x,t)$ vanishes outside an interval $\Omega(t) = (\varsigma_1(t), \varsigma_2(t))$ and

$$(u^{m-1})_{xx} \leq -\frac{C}{1 + C(m(m+1)/(m-1))t}$$

in $\Omega(t)$. Consequently the interfaces $x = \varsigma_i(t)$, $i = 1,2$, are concave curves. These results also give precise information about the large time behavior of solutions and interfaces.

Introduction. We consider the Cauchy problem

$$\text{(P)} \qquad \begin{cases} u_t = (u^m)_{xx} & \text{in } Q = \mathbf{R} \times (0,\infty), \\ u(x,0) = u_0(x) & \text{for } x \in \mathbf{R}, \end{cases}$$

where $m > 1$ and u_0 is a nonnegative continuous function vanishing outside a bounded interval (a,b) with $u_0(x) > 0$ for $x \in (a,b)$.

It is well known (see for instance [**V2**] for general references on this problem) that there exists a unique generalized solution $u(x,t)$ of (P) that is nonnegative bounded and continuous on $\overline{Q}$. Moreover there exist two continuous monotone curves $x = \varsigma_1(t)$, $x = \varsigma_2(t)$ called the interfaces such that $\varsigma_1(0) = a$, $\varsigma_2(0) = b$, and

$$\Omega \stackrel{\text{def}}{=} \{(x,t) \in Q; u(x,t) > 0\} = \{(x,t) \in Q; \varsigma_1(t) < x < \varsigma_2(t)\}.$$

Also $u \in C^\infty(\Omega)$.

In the description of the flow of a gas through a porous medium, u represents the density of the gas and $v = mu^{m-1}/(m-1)$ represents the pressure. We shall retain this denomination in the sequel. Of course $v \in C^\infty(\Omega)$ and it satisfies the equation

$$\text{(0.1)} \qquad v_t = (m-1)vv_{xx} + v_x^2$$

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on Ω. We recall also the basic "convexity inequalities"

$$v_{xx} \geq -\frac{1}{(m+1)t} \quad \text{in } D'(Q), \tag{0.2}$$

$$(-1)^i \left(\varsigma_i'' + \frac{m}{(m+1)t} \varsigma_i' \right) > 0 \quad \text{in } D'(\mathbf{R}^+). \tag{0.3}$$

Now we assume that the initial pressure $v_0 = m u_0^{m-1}/(m-1)$ is concave in (a, b); more precisely we assume

$$v_{0,xx} \leq -C \quad \text{in } D'(a, b) \tag{0.4}$$

with some constant $C \geq 0$. The purpose of this paper is to study the concavity of the pressure v on Ω and of the interfaces ς_i on $\mathbf{R}^+$, and their consequences. The main result is

THEOREM 1. *Under assumption* (0.4), *the following "concavity inequalities" hold*:

$$v_{xx} \leq -\frac{C}{1+(m+1)Ct} \quad \text{on } \Omega, \tag{0.5}$$

$$(-1)^i \left(\varsigma_i'' + \frac{mC}{1+(m+1)Ct} \varsigma_i' \right) \leq 0 \quad \text{in } D'(\mathbf{R}^+) \text{ for } i = 1, 2. \tag{0.6}$$

In the case $C = 0$, the concavity (0.5) has already been proved by Graveleau and Jamet [**GJ**]. For proving (0.5) in the general case $C \geq 0$, we will follow their approach based on splitting (0.1) into the two equations

$$v_t = (m-1) v v_{xx}, \tag{0.7}$$

$$v_t = v_x^2. \tag{0.8}$$

We then recover (0.1) from (0.7), (0.8) via a Trotter-Kato formula.

In the case $C > 0$, the combination of (0.5) and (0.6) with (0.2) and (0.3) gives strong information about the asymptotic behavior of the pressure and the interfaces. It was proved in [**V1**] that this asymptotic behavior is determined in first approximation by the two invariants of the motion, namely the total mass

$$M = \int u_0(x)\, dx = \int u(x,t)\, dx \tag{0.9}$$

and the center of mass

$$x_0 = M^{-1} \int x u_0(x)\, dx = M^{-1} \int x u(x,t)\, dx. \tag{0.10}$$

In fact there is a self-similar (Barenblatt) solution given in terms of the pressure by

$$\overline{v}(x,t) = \frac{[r(t)^2 - (x - x_0)^2]_+}{2(m+1)t} \tag{0.11}$$

with

$$r(t) = c(M^{m-1} t), \qquad c = c(m) > 0, \tag{0.12}$$

such that as $t \to \infty$, $v - \overline{v} = o(t^{-m/(m+1)})$ uniformly in $x \in \mathbf{R}$, $v_x + x - x_0/(m+1)t = o(t^{-1})$ uniformly in Ω, $\varsigma_i(t) = x_0 + (-1)^i r(t) + o(1)$, and $\varsigma_i'(t) = (-1)^i r'(t) + o(t^{-1})$.

As a consequence of Theorem 1 these results can be improved as follows:

THEOREM 2. *Under the assumption* (0.4) *with* $C > 0$ *we have as* $t \to \infty$

$$v_{xx}(x,t) = -\frac{1}{(m+1)t} + O\left(\frac{1}{t^2}\right) \tag{0.13}$$

uniformly in x *for* $(x,t) \in \Omega$ *and*

$$\varsigma_i''(t) = (-1)^i r''(t)(1 + O(1/t)). \tag{0.14}$$

These rates of convergence are optimal as it can easily be checked on the time-delayed Barenblatt solutions $v(x,t;\tau) = \overline{v}(x,t+\tau)$, where $\overline{v}$ is given by (0.11) and $\tau > 0$. From (0.13), (0.14) sharp estimates follow for v, v_x, ς, and ς'.

The main problem in proving concavity results lies in the fact that v_x has a jump discontinuity across a moving interface of a solution to problem (P). Therefore v_{xx} is very singular and positive on the interfaces. Using equations (0.7), (0.8) allows us to overcome this difficulty because the interfaces in (0.7) do not move while in (0.8) we are reduced to studying the characteristic lines of a first-order equation.

On the contrary, in proving lower bounds for v_{xx} no such difficulty arises since we can always approximate our solution by smooth positive solutions and apply the maximum principle to the equation satisfied by v_{xx} as in the proof of (0.2) (cf. [**AB**]). Thus if v is a solution of (0.1) whose initial datum v_0 satisfies

$$v_{0,xx} \geq -C \quad \text{in } D'(\mathbf{R}), \tag{0.15}$$

then

$$v_{xx} \geq -\frac{C}{1+(m+1)Ct} \quad \text{in } D'(Q). \tag{0.16}$$

It is worth noting that we can always use the maximum principle method of [**AB**] when dealing with problem (P) for $0 < m < 1$ (the so-called fast-diffusion case) or $m = 1$ (the heat equation) and the above results (0.5), (0.16) are true if we define $v = -mu^{-(1-m)}/(1-m)$ if $m < 1$, $v = \log(u)$ if $m = 1$. We shall leave the verification of these facts to the reader, but let us recall that $v \in C^\infty(Q)$ in both cases and $v < 0$ if $m < 1$.

The plan of the paper is as follows: In §1 we study the Cauchy problems associated to (0.7) and (0.8) for v_0 satisfying (0.4). We prove the statement about v_{xx} in Theorem 1 via the Trotter-Kato formula in §2. Finally, §3 studies the interfaces and the asymptotic behavior.

1. Preliminaries. In this section we study the Cauchy problems for equations $v_t = \lambda v v_{xx}$ and $v_t = v_x^2$. To make their application in §2 easier it is convenient to introduce the following notations. Let $\mathcal{N}$ be the set of continuous nonnegative functions $f\colon \mathbf{R} \to \mathbf{R}$ satisfying for some bounded interval $I(f) = (a,b)$

$$f(x) = 0 \quad \text{if } x \notin (a,b), \qquad f'' \leq 0 \quad \text{in } D'(a,b).$$

It is clear that if $f \in \mathcal{N}$, then either $f \equiv 0$ or $I(f) = \{x \in \mathbf{R}; f(x) > 0\}$. $\mathcal{N}$ is a closed set in the space $C_c(\mathbf{R})$ of continuous real functions on $\mathbf{R}$ with compact support with the usual topology. To measure the concavity of a function $f \in \mathcal{N}$ we define

$$C(f) = \sup\{C \geq 0; f_{xx} \leq -C \text{ in } D'(I(f))\},$$
$$c(f) = \sup\{c \geq 0; cf_{xx} \geq -1 \text{ in } D'(\mathbf{R})\}$$

with the convention $C(f) = 0$ if $f \equiv 0$. Clearly $0 \le C(f) \le c(f)^{-1} \le \infty$ and the maps $f \mapsto C(f)$, $f \mapsto c(f)$ are upper semicontinuous functionals on $\mathcal{N}$ (endowed with the topology of $C_c(\mathbf{R})$).

Let $\mathcal{N}_1 = \{f \in \mathcal{N}; c(f) > 0\}$. The following results can be proved by elementary calculus.

LEMMA 1.1. *Let $f \in \mathcal{N}_1$ with $f \not\equiv 0$ and $I(f) = (a,b)$.*

(i) *$f \in W^{2,\infty}(a,b)$ and*

$$C(f) = \inf_{(a,b)} \operatorname{ess}(-f'') \le \sup_{(a,b)} \operatorname{ess}(-f'') = c(f)^{-1}. \tag{1.1}$$

(ii) *$f \in W^{1,\infty}(\mathbf{R})$ and*

$$f \le \frac{(b-a)^2}{c(f)}, \qquad (f')^2 \le \frac{2\|f\|_\infty}{c(f)}. \tag{1.2}$$

(iii) *$f' \in BV(\mathbf{R})$, $f'(a+) > 0 > f'(b-)$, and*

$$f'' = f''\chi_{(a,b)} + f'(a+)\delta_a - f'(b-)\delta_b \quad \text{in } D'(\mathbf{R}). \tag{1.3}$$

We solve first the Cauchy problem

$$v_t = \lambda v v_{xx} \quad \text{in } Q = \mathbf{R} \times (0,\infty), \tag{1.4a}$$

$$v(x,0) = v_0(x) \quad \text{for } x \in \mathbf{R}, \tag{1.4b}$$

where λ is a positive constant and $v_0 \in \mathcal{N}$. We obtain

PROPOSITION 1.2. *Let $v_0 \in \mathcal{N}$ and $I(v_0) = (a,b)$. There exists a unique function $v \in C(\overline{Q})$ that solves problem (1.4) in the sense that*

(a) *$v \ge 0$ in Q, v is positive precisely on the set $\Omega = (a,b) \times (0,\infty)$, and $v \in C^\infty(\Omega)$.*

(b) *$v_t = \lambda v v_{xx}$ in Ω.*

(c) *$v(x,0) = v_0(x)$ for every $x \in \mathbf{R}$.*

Moreover, if $v(t) = v(\cdot,t)$ then for every $t > 0$, $v(t) \in \mathcal{N}_1$ and

$$0 \le v(t) \le v_0 \quad \text{in } \mathbf{R}, \tag{1.5}$$

$$-\frac{1}{c(v_0) + \lambda t} \le v(t)_{xx} \le -\frac{C(v_0)}{1 + C(v_0)\lambda t} \quad \text{in } D'(a,b). \tag{1.6}$$

If $c(v_0) > 0$ then for every $(x,t) \in Q$

$$v_0(x) - v(x,t) \le \frac{\lambda t v_0(x)}{c(v_0)}. \tag{1.7}$$

PROOF. Let us assume first that $c(v_0) > 0$. To avoid the degeneracy of the equation (1.4a) at the level $v = 0$ we begin by considering the problems

$$(\mathrm{P}_\varepsilon) \qquad \begin{cases} v_t = \lambda v v_{xx} & \text{in } \Omega = \{a < x < b, t > 0\}, \\ v(x,0) = v_{0\varepsilon}(x) & \text{for } a \le x \le b, \\ v(a,t) = v(b,t) = \varepsilon(\lambda t C(v_0) + 1)^{-1} & \text{for } t > 0, \end{cases}$$

where ε is positive and $\{v_{0\varepsilon}\}_\varepsilon$ is a sequence of C^∞, positive functions in $[a,b]$ such that

(i) $(v_{0\varepsilon})_{xx} \le C(v_0)$ and $(v_{0\varepsilon})_{xx} \ge 1/c(v_0)$,

(ii) $v_{0\varepsilon} \to v_0$ uniformly in $[a,b]$ as $\varepsilon \to 0$,

(iii) $v_{0\varepsilon}(x) = \varepsilon$ and $v_{0\varepsilon,xx} = -C(v_0)$ at $x = a, b$.

Since (P_ε) is a quasilinear parabolic problem and we have the a priori estimate $\varepsilon < v(x,t) < \sup_x\{v_{0\varepsilon}(x)\}$ for the solutions, by the standard theory (cf. [**LSU**]) there exists a solution $v_\varepsilon \in C^\infty(\overline{\Omega})$ of problem (P_ε).

Consider now the function $p(x,t) = v_{\varepsilon xx}(x,t)$. It satisfies in Ω the equation

$$p_t = \lambda v_\varepsilon p_{xx} + 2\lambda v_{\varepsilon x} p_x + \lambda p^2,$$

with initial condition $p(x,0) = v_{0\varepsilon,xx}(x)$ and boundary conditions

$$p(a,t) = p(b,t) = -\frac{C(v_0)}{\lambda t C(v_0) + 1}.$$

(Note that $p = v_{\varepsilon t}/(\lambda v)$.) Since for every $k \ge 0$ the function

$$P(x,t;k) = -(\lambda t + k)^{-1}$$

is an explicit solution of the equation satisfied by p, it follows from the maximum principle that

$$-\frac{1}{\lambda t + c(v_0)} \le p(x,t) \le -\frac{C(v_0)}{\lambda t C(v_0) + 1} \tag{1.8}$$

in Ω. Since $p = v_{\varepsilon,xx}$ and $v_\varepsilon(a,t) = v_\varepsilon(b,t) \le \varepsilon$ it follows from Lemma 1.1 and (1.8) that the estimates

$$0 \ge v_{\varepsilon t} \ge -\frac{\lambda v_\varepsilon}{\lambda t + c(v_0)},$$

$$|v_\varepsilon| \le \varepsilon + \frac{(b-a)^2}{2(\lambda t + c(v_0))},$$

$$|v_{\varepsilon x}|^2 \le \frac{2\|v_0\|_\infty}{c(v_0) + \lambda t} \quad \text{and} \quad |v_{\varepsilon x}| \le \frac{(b-a)}{2(c(v_0) + \lambda t)}$$

hold in Ω. Therefore if we pass to the limit $\varepsilon \to 0$ there is a subsequence $\{v_{\varepsilon_k}\}$ of $\{v_\varepsilon\}$ that converges uniformly in $[a,b] \times [0,T]$ for every $T \ge 0$ to a function $\overline{v} \in C(\overline{\Omega})$ and the above estimates hold for $\overline{v}$ so that $\overline{v} \in W^{1,\infty}((a,b) \times (\tau,\infty))$ for every $\tau > 0$. Because of (1.8) the sequence $v_\varepsilon(x,t)$ is uniformly bounded away from 0 in a neighborhood of every point $(x,t) \in \Omega$, hence by standard bootstrap arguments (cf. [**LSU**] or [**A**]) we conclude that $\overline{v} \in C^\infty(\Omega)$ and $\overline{v}_t = \lambda \overline{v}\,\overline{v}_{xx}$ is satisfied in the classical sense. It is also clear that $\overline{v} \in C(\overline{\Omega})$ and both the initial and boundary conditions are satisfied.

To obtain a solution v in Q we extend $\overline{v}$ by $v(x,t) = 0$ if $x \notin (a,b)$ and $t \ge 0$.

The uniqueness of the solution is a consequence of the maximum principle (it follows in particular that all the sequence $\{v_\varepsilon\}$ converges to $\overline{v}$). In fact if we have two solutions v, w of (1.4) corresponding to initial data v_0, $w_0 \in \mathcal{N}_1$ and $v_0 \ge w_0$ then $v \ge w$ in Q.

In case $c(v_0) = 0$ we can still use the above procedure to obtain a solution and only the convergence at $t \to 0$ poses a problem that can be solved by approximating the initial data v_0 above and below with data in $\mathcal{N}_1$ and using the above comparison results. □

REMARK. Independently of the authors, M. Ughi [**U**] has studied the equation $u_t = uu_{xx} - u(1-u)$ in $\mathbf{R} \times (0,T)$ for initial data not necessarily concave. She obtains results about existence, uniqueness, and nonuniqueness for suitable classes of weak solutions. The results apply to equation (1.4a) with obvious changes.

Consider now the initial-value problem

$$v_t = |v_x|^2 \quad \text{in } Q, \tag{1.9a}$$

$$v(x,0) = v_0(x) \quad \text{for } x \in \mathbf{R}. \tag{1.9b}$$

Equation (1.9a) is one of the simplest and best-known examples of the Hamilton-Jacobi equation. There is now a general theory of viscosity solutions for equations of the type $u_t + H(u_x) = 0$ where H is a continuous function. In particular we have the following result proved by M. G. Crandall and P. L. Lions.

PROPOSITION 1.3 (CF. [**CL**, THEOREM VT.2]). *Let $v_0 \in BUC(\mathbf{R})$, the set of bounded uniformly continuous functions in $\mathbf{R}$. There exists a unique $v \in C(\overline{Q})$, which is a viscosity solution of $u_t + H(u_x) = 0$ in Q with initial data $v(x,0) = v_0(x)$.*

Moreover the maps $T(t)\colon BUC(\mathbf{R}) \to BUC(\mathbf{R})$ defined for $t \geq 0$ by $T(t)v_0 = v(\cdot,t)$ form a strongly continuous, order-preserving contraction semigroup in $BUC(\mathbf{R})$.

The fact that in problem (1.9) the nonlinearity $H(p) = p^2$ is convex considerably simplifies the construction of the solutions of (1.9), especially if the initial data $v_0 \in \mathcal{N}$. In this case we can use the classical method of characteristics (cf. [**Bt, L**]) as follows. Assume for simplicity that $v_0 \in C^1([a,b])$, where (a,b) is the support of v_0. Through every point $(\xi, 0)$ with $a < \xi < b$ we construct a characteristic line

$$x(\xi,t) = \xi - 2v_0'(\xi)t. \tag{1.10}$$

Along this line the derivative $w = v_x$ of every classical solution of $v_t = |v_x|^2$ must be constant. Therefore we get

$$v(x,t) = v(\xi,0) + \int_0^t \frac{d}{d\tau} v(x(\tau),\tau)\,dt = v_0(\xi) - |v_0'(\xi)|^2 t. \tag{1.11}$$

The concavity of v_0 in (a,b) implies that these characteristics do not cross each other. In that way a C^1 solution of (1.9a) can be constructed in the region where $v > 0$. It is easy to see that this region has the form

$$\Omega = \{(x,t) \in Q\colon s_1(t) < x < s_2(t)\} \tag{1.12}$$

where $x = s_1(t)$ and $x = s_2(t)$ are Lipschitz-continuous curves (the interfaces), $s_1(0) = a$, $s_2(0) = b$, and $(-1)^i s_i(t)$ is nondecreasing in time. Along the interfaces the derivative v_x is discontinuous, i.e. a shock propagates according to the well-known Rankine-Hugoniot conditions.

The following results about problem (1.9) will be needed in the proof of Theorem 1.

PROPOSITION 1.4. *Let v be the viscosity solution of (1.9) with $v_0 \in \mathcal{N}$. Then $v \in W^{1,\infty}(Q_\tau)$ for every $\tau > 0$, $v(t) = T(t)v_0 \in \mathcal{N}_1$ for every $t > 0$, and*

$$c(v(t)) \geq c(v_0) + 2t, \tag{1.13a}$$

$$C(v(t)) \geq \frac{C(v_0)}{1 + 2tC(v_0)}. \tag{1.13b}$$

Moreover if $c(v_0) > 0$, *then*

$$v(t) - v_0 \leq \frac{2t||v_0||_\infty}{c(v_0)}. \tag{1.14}$$

PROOF. By the contracting property of $T(t)$ we may assume that $C(v_0), c(v_0) > 0$ and $v_0 \in C^1([a,b])$. To prove (1.13) let $t > 0$ and let x_1 and x_2 be two points $s_1(t) \leq x_1 < x_2 \leq s_2(t)$. There are well-defined starting points ξ_1 and ξ_2 in (a,b) for the characteristic line (1.10) through (x_1,t) and (x_2,t) respectively. Setting $w_1 = v_x(x_1,t)$, $w_2 = v_x(x_2,t)$ we have

$$x_1 - x_2 = \xi_1 - \xi_2 + 2t(w_2 - w_1).$$

But v_x is constant along the characteristics, hence $w_2 - w_1 = v_0'(x_2) - v_0'(x_1) > 0$ since v_0 is strictly concave. Therefore if we write

$$\frac{x_1 - x_2}{w_2 - w_1} = \frac{\xi_1 - \xi_2}{w_2 - w_1} + 2t$$

and let $x_1 - x_2 \to 0$ we obtain the relationship

$$\frac{1}{v_{xx}(x,t)} = \frac{1}{v_0''(\xi)} + 2t \tag{1.15}$$

which is valid in Ω. (1.13) follows immediately. From this we get

$$v_t = v_x^2 \leq \frac{2||v_0||_\infty}{2t + c(v_0)}. \tag{1.16}$$

(1.14) is now immediate. □

2. The Trotter-Kato formula. Let us now consider the problem

$$v_t = (m-1)vv_{xx} + v_x^2 \quad \text{in } Q, \tag{2.1a}$$

$$v(\cdot,0) = v_0 \in \mathcal{N}. \tag{2.1b}$$

To solve (3.1) we shall use a Trotter-Kato formula based on the results of the previous section (with $\lambda = m-1$). It is interesting in that respect to think of the solution $v(x,t)$ of problem (1.4) as a semigroup $S(t)$: $\mathcal{N} \to \mathcal{N}$, $t > 0$, defined by $S(t)v_0 = v(t)$ if $v(t) = v(\cdot,t)$, v being the solution of (1.4) with initial data v_0. In the same way the solution to problem (1.9) defines another semigroup $T(t)$: $\mathcal{N} \to \mathcal{N}$, $t > 0$. Both are order-preserving continuous semigroups in $\mathcal{N}$. The concavity estimates (1.6), (1.13) can be reformulated as

$$c(S(t)v_0) \geq c(v_0) + (m-1)t, \tag{2.2a}$$

$$c(T(t)v_0) \geq c(v_0) + 2t, \tag{2.2b}$$

$$C(S(t)v_0) \geq \frac{C(v_0)}{1 + (m-1)C(v_0)t}, \tag{2.2c}$$

and

$$C(T(t)v_0) \geq \frac{C(v_0)}{1 + 2C(v_0)t}. \tag{2.2d}$$

Results of the type of Chernoff and Trotter-Kato formulas for pairs of continuous semigroups are known when both semigroups are contractive in some Banach space

X (cf. [**BCP**]). Unfortunately in our case, while the semigroup T is contractive in $L^\infty(\mathbf{R})$ or $BUC(\mathbf{R})$, the semigroup S is not. In fact the only contraction properties known for the solutions of (1.4) apply to the function $u = \log(v)$, which satisfies the equation $u_t = \lambda(\exp(u))_{xx}$ (cf. [**BC**]).

The convergence of our Trotter-Kato formula relies on the compactness due to (2.2). We proceed as follows. For every $\varepsilon > 0$ we define the approximate solution $v_\varepsilon(t)$ to problem (2.1) by

$$v_\varepsilon(t) = \frac{t-(n-1)\varepsilon}{\varepsilon} v_\varepsilon^n + \frac{n\varepsilon - t}{\varepsilon} v_\varepsilon^{n-1} \tag{2.3}$$

if $(n-1)\varepsilon \le t < n\varepsilon$, where for $n = 1, 2, \ldots$

$$v_\varepsilon^n = (S(\varepsilon)T(\varepsilon))^n(v_0). \tag{2.4}$$

We obtain the following convergence result.

PROPOSITION 2.1. *As $\varepsilon \to 0$, v_ε converges to a function $v \in (\overline{Q}) \cap W^{1,\infty}(Q_\tau)$ for every $\tau > 0$ uniformly on compact subsets of $\overline{Q} = R \times [0, \infty)$, where v satisfies*

$$v_t = \frac{m-1}{2}(v^2)_{xx} - (m-2)v_x^2 \tag{2.5}$$

in $D'(Q)$. Moreover if $v(t)\colon x \to v(x,t)$, then for every $t > 0$, $v(t) \in \mathcal{N}_1$ and

$$C(v(t)) \ge \frac{C(v_0)}{1+(m+1)t}, \tag{2.6a}$$

$$c(v(t)) \ge c(v_0) + (m+1)t. \tag{2.6b}$$

Also

$$v_x^2 \le \frac{2||v_0||_\infty}{c(v_0)+(m+1)t}, \tag{2.7}$$

$$\frac{(m-1)||v_0||_\infty}{c(v_0)+(m+1)t} \le v_t \le \frac{2||v_0||_\infty}{c(v_0)+(m+1)t}. \tag{2.8}$$

PROOF. By induction we have $v_\varepsilon^n \in \mathcal{N}_1$,

$$0 \le v_\varepsilon^n \le T(n\varepsilon)v_0 \le ||v_0||_\infty, \tag{2.9a}$$

$$C(v_\varepsilon^n) \ge \frac{C(v_0)}{1+(m+1)n\varepsilon C(v_0)}, \tag{2.9b}$$

$$c(v_\varepsilon^n) \ge c(v_0) + (m+1)n\varepsilon, \tag{2.9c}$$

$$\operatorname{supp}(v_\varepsilon^n) \subset \operatorname{supp}(T(n\varepsilon)v_0). \tag{2.9d}$$

Observe that (2.9a) follows from the fact that $S(\varepsilon)v \le v$ and $T(\varepsilon)$ and $S(\varepsilon)$ are order-preserving maps. (2.9b) and (2.9c) follow from (2.2) and finally (2.9d) is a consequence of (2.9a).

Assume now that $c(v_0) > 0$ and fix $t_0 > 0$. We have

$$v_\varepsilon \in W^{1,\infty}(\mathbf{R} \times (0, t_0)), \qquad 0 \le v_\varepsilon \le ||v_0||_\infty,$$

$$\operatorname{supp}(v_\varepsilon(t)) \subset \operatorname{supp}(T(t_0+\varepsilon)v_0) \quad \text{if } 0 < t < t_0,$$

$$v_{\varepsilon,x}^2 \le 2||v_0||_\infty / c(v_0).$$

Moreover

$$\frac{d^+ v_\varepsilon}{dt} = \frac{S(\varepsilon)w_\varepsilon - w_\varepsilon}{\varepsilon} + \frac{T(\varepsilon)\tilde{v}_\varepsilon - \tilde{v}_\varepsilon}{\varepsilon},$$

where $\tilde{v}_\varepsilon(t) = v_\varepsilon^{n-1}$ if $(n-1)\varepsilon \le t < n\varepsilon$ and $w_\varepsilon = T(\varepsilon)\tilde{v}_\varepsilon$. Since $c(w_\varepsilon(t)) \ge c(\tilde{v}_\varepsilon(t)) + 2\varepsilon \ge c(v_0)$ we have by (1.7) and (1.14)

$$-\frac{(m-1)||v||_\infty}{c(v_0)} \le \frac{d^+ v_\varepsilon}{dt} \le \frac{2||v||_\infty}{c(v_0)}.$$

It follows in particular that v_ε is relatively compact in $C(\mathbf{R} \times [0, t_0])$. If v_ε converges uniformly to a function $v \in C(\mathbf{R} \times [0, t_0])$ along a sequence $\varepsilon = \varepsilon_k \to 0$, we have $v \in W^{1,\infty}(\mathbf{R} \times [0, t_0])$,

$$0 \le v \le T(t)v_0, \qquad |v_x^2| \le \frac{2||v_0||_\infty}{c(v_0)},$$

and

$$-\frac{(m-1)||v_0||_\infty}{c(v_0)} \le v_t \le \frac{2||v_0||_\infty}{c(v_0)}.$$

Furthermore the estimates (2.6a) and (2.6b) follow from (2.9b) and (2.9c) and the upper-semicontinuity of the functionals c and C.

To end the proof of the proposition we have yet to check that equation (2.5) holds. For that we write

$$\begin{aligned} \frac{d^+ v_\varepsilon}{dt} &= \frac{1}{\varepsilon}\int_0^\varepsilon (S(\tau)w_\varepsilon(t))_\tau \, d\tau + \frac{1}{\varepsilon}\int_0^\varepsilon (T(\tau)\tilde{v}_\varepsilon(t))_\tau \, d\tau \\ &= \frac{m-1}{2\varepsilon}\int_0^\varepsilon \{(S(\tau)w_\varepsilon(t))^2\}_{xx} \, d\tau + \frac{1}{\varepsilon}\int_0^\varepsilon \{(T(\tau)\tilde{v}_\varepsilon(t))_x\}^2 \, d\tau \\ &\quad - \frac{m-1}{\varepsilon}\int_0^\varepsilon \{(S(\tau)w_\varepsilon(t))_x\}^2 \, d\tau \\ &\equiv I_1 + I_2 + I_3. \end{aligned} \tag{2.10}$$

We want to pass to the limit $\varepsilon \to 0$. Let us begin for instance with I_2. Since $v_{\varepsilon_k} \to v$ uniformly in $\mathbf{R} \times [0, t_0]$, using also (1.14) it follows that

$$\sup_{\substack{0 \le \tau \le \varepsilon \\ 0 \le t \le t_0}} ||T(\tau)\tilde{v}_\varepsilon(t) - v(t)||_\infty \not\equiv 0 \quad \text{as} \quad \varepsilon_k \to 0.$$

If we now observe that $c(T(\tau)\tilde{v}_\varepsilon(t)) \ge c(v_0)$ and

$$\operatorname{supp}(T(\tau)\tilde{v}_\varepsilon(t)) \subset \operatorname{supp}(T(t_0 + \varepsilon)v_0) \subset [-R, R]$$

for some $R > 0$ the convergence

$$|(T(\tau)\tilde{v}_\varepsilon(t))_x|^2 \to |v(t)_x|^2 \quad \text{in} \quad C([0, t_0]; L^1(\mathbf{R})) \tag{2.11}$$

will hold uniformly in $\tau \in (0, \varepsilon)$ as a consequence of the following compactness result applied to the family $\{f_{t,\tau,\varepsilon}(x) = T(\tau)\tilde{v}_\varepsilon(t)\}$, $0 \le t \le t_0$, $\varepsilon > 0$, $0 \le \tau \le \varepsilon$.

LEMMA 2.2. *For every constant c_0 and $R > 0$, the set*

$$\mathcal{N}_{c_0,R} = \{f \in \mathcal{N}: c(f) \geq c_0, \operatorname{supp}(f) \subset [-R, R]\}$$

is a compact subset of $W^{1,p}(\mathbf{R})$ for any $p \in [1, \infty)$. Moreover there exist constants $C = C(c_0, R) > 0$ and $\theta \in (0,1)$ such that

$$||f_1 - f_2||_{W^{1,p}(\mathbf{R})} \leq C||f_1 - f_2||^{\theta}_{L^p(\mathbf{R})} \tag{2.12}$$

for any two functions $f_1 f_2 \in \mathcal{N}_{c_0,R}$.

For a proof of this result see [**L**, Lemma 10.1]. It is based on proving that $\mathcal{N}_{c_0,R} \subset W^{s,p}(\mathbf{R})$ for some $s > 1$ and $p > 1$ and then using interpolation.

Continuing with the proof of Proposition 2.1 it follows from (2.11) that

$$I_2 = \frac{1}{\varepsilon}\int_0^{\varepsilon} \{(T(\tau)\tilde{v}_\varepsilon(t))_x\}^2 \to |v(t)_x|^2 \quad \text{in } C([0,t_0]; L^1(\mathbf{R})).$$

In the same way we prove that

$$\frac{1}{\varepsilon}\int_0^{\varepsilon} \{(S(\tau)w_\varepsilon(t))_x\}^2\, d\tau \to (v(t)_x)^2 \quad \text{in } C([0,t_0]; L^1(\mathbf{R})).$$

Finally since

$$F_\varepsilon(t) = \frac{1}{\varepsilon}\int_0^{\varepsilon} (S(\tau)w_\varepsilon(t))^2\, d\tau \to v(t)^2 \quad \text{in } C(\mathbf{R} \times [0,t_0])$$

it follows that $2I_1/(m-1) = F_\varepsilon(t)_{xx} \to (v(t)^2)_{xx}$ in $D'(\mathbf{R} \times (0,t_0))$. Therefore we may let $\varepsilon_k \to 0$ in (2.10) to obtain (2.5).

The case $c(v_0) = 0$ is easily dealt with by comparison since for every $\delta > 0$

(i) $S(\delta)v_0 \leq v_0 \leq T(\delta)v_0$ and $||T(\delta)v_0 - S(\delta)v_0||_\infty \to 0$ as $\delta \to 0$,

(ii) $c(S(\delta)v_0) > 0$, $c(T(\delta)v_0) > 0$. □

To conclude the proof of the first part of Theorem 1 we have to identify the function v as the pressure of the solution to problem (0.1).

PROPOSITION 2.3. *Let $v \in C(\overline{Q}) \cap W^{1,\infty}(Q_\tau)$ for every $\tau > 0$ satisfy:*

(i) *for every $t > 0$, $v(\cdot, t) \in \mathcal{N}$,*

(ii) *$v_t = (m-1)(v^2)_{xx}/2 + (2-m)|v_x|^2$ in $D'(Q)$,*

(iii) *$v(\cdot, 0) = mu_0^{m-1}/(m-1)$,*

where u_0 is nonnegative, continuous and bounded.

Then v is the pressure associated to the solution of (0.1).

PROOF. Let $u = ((m-1)v/m)^{1/(m-1)}$ and let $\overline{u}$ be the usual solution of (0.1). By the preceding results if Ω is the subregion of Q where v is positive, then Ω has the form $\{(x,t) \in Q: s_1(t) < x < s_2(t)\}$ and we have $u \in C(\overline{\Omega})$. Also (by standard regularity theory) $u \in C^\infty(\Omega)$ and $u_t = (u^m)_{xx}$ is satisfied in Ω in the classical sense. Since $u \leq \overline{u}$ on the parabolic boundary of Ω by the maximum principle we get $u \leq \overline{u}$ in Ω.

Moreover since $(u^m)_x = uv_x$ and $v(\cdot, t) \in \mathcal{N}$ it is easy to see that $(u^m)_x \to 0$ as (x,t) tends to the lateral boundary of Ω, hence for every $t > 0$

$$\frac{d}{dt}\int u(x,t)\, dx = [(u^m)_x]_{x=s_1(t)}^{x=s_2(t)} = 0$$

and we have

$$\int u(x,t)\, dx = \int u_0(x)\, dx = \int \overline{u}(x,t)\, dx.$$

Therefore $u = \overline{u}$ in Q. □

3. Interfaces and asymptotic behavior. In this section we complete the proof of Theorem 1 by studying the concavity of the interfaces of the solutions to problem (P) and derive its consequences for the asymptotic behavior of solutions and interfaces, Theorem 2.

Proof of the concavity statement (0.6) *of Theorem* 1. We may assume that v is a solution of (0.4) whose initial datum v_0 satisfies (0.4) with $C > 0$. Also we may consider only the right-hand interface and drop the subindex, $\varsigma(t) = \varsigma_2(t)$. Following the idea of the proof of the convexity inequality (0.3) [**CF**, **V1**] we compare our solution v with the Barenblatt solution w that has best contact with it, this time from above, at a given point of the interface, the main difference lying in the fact that we have to use a time-delayed Barenblatt. The outline of the proof is as follows: Let $t_1 > 0$. Since $v(\cdot, t_1)$ is strictly concave in its support (cf. (0.5)), $v_x(\varsigma(t_1) - 0, t_1)$ is necessarily negative, therefore by the interface equation

$$\varsigma'(t) = -v_x(\varsigma(t) - 0, t) \tag{3.1}$$

[**A**, **K**] we have $\varsigma'(t) > 0$. We consider the Barenblatt solution

$$w(x,t) = \frac{(r(t+\tau)^2 - (x - x_1)^2)_+}{2(m+1)(t+\tau)} \tag{3.2}$$

where $r(t)$ is as in (0.12). The parameters $M, \tau > 0$ and $x_1 \in \mathbf{R}$ shall be adjusted so as to have

(i) $v(\varsigma(t_1), t_1) = w(\varsigma(t_1), t_1)$,
(ii) $v_x(\varsigma(t_1) - 0, t_1) = w_x(\varsigma(t_1) - 0, t_1)$,
(iii) $v_{xx}(\xi, t_1) \le w_{xx}(\xi, t_1)$ whenever $v(\xi, t_1) > 0$.

By (0.5) and (3.2) the last inequality is implied by

$$-\frac{C}{1 + (m+1)Ct_1} \le -\frac{1}{(m+1)(t_1+\tau)},$$

namely $\tau \ge 1/((m+1)C)$. The best choice is $\tau = 1/((m+1)C)$. We can now choose M and x_1 so that (i) and (ii) are satisfied. It follows from (i)–(iii) that $v(\cdot, t_1) \le w(\cdot, t_1)$. Since both v and w are solutions of (0.7) in $S = \mathbf{R} \times (t_1, \infty)$ it follows that $v(x,t) \le w(x,t)$ in S. Therefore for every $t > t_1$ we have

$$\varsigma(t) \le \varsigma_w(t) \equiv x_1 + r(t+\tau).$$

Since at $t = t_1$ we have $\varsigma(t_1) = \varsigma_w(t_1)$, $\varsigma'(t_1) = \varsigma_w'(t_1)$ (by (i), (ii)), and

$$\varsigma_w''(t) = -\frac{m}{(m+1)(t+\tau)}\varsigma_w'(t), \tag{3.3}$$

we conclude as in [**CF**, **V1**] that

$$\varsigma''(t_1) \le \frac{m}{(m+1)(t_1+\tau)}\varsigma'(t_1) \tag{3.4}$$

in the sense of measures of $(0, \infty)$, i.e. (0.6). □

(0.3) and (0.6) together imply that ς'' is a locally bounded function. (0.6) can be reformulated as saying that $\varsigma'(t)/r'(t+\tau)$ is nonincreasing as a function of t.

We now study the asymptotic behavior. We state in detail the results for the interface.

PROPOSITION 3.1. *Let v be a solution of* (0.4) *such that v_0 satisfies* (0.4) *with $C > 0$ and let $\tau = 1/((m+1)C)$. Then as $t \to \infty$ we have*

$$r(t) + x_0 \le \varsigma(t) \le r(t+\tau) + x_0, \tag{3.5}$$

$$r'(t) \ge \varsigma'(t) \ge r'(t+\tau). \tag{3.6}$$

Moreover $\varsigma(t) - r(t+\tau) \uparrow x_0$ and $\varsigma'(t)/r'(t+\tau) \downarrow 1$.

PROOF. Since $\varsigma'(t)/r'(t+\tau)$ is nonincreasing and $\varsigma(t)/r(t) \to 1$ (cf. [**V1**]), it follows that $\varsigma'(t)/r'(t+\tau) \downarrow 1$. Therefore $\varsigma(t) - r(t+\tau)$ is nondecreasing. Since $r(t+\tau) - r(t) \to 0$ and $\varsigma(t) = r(t) + x_0$ as $t \to \infty$ [**V1**] we have $\varsigma(t) - r(t+\tau) \uparrow x_0$. This proves the right-hand inequalities. The left-hand inequalities were proved in [**V1**].

COROLLARY 3.2. *As $t \to \infty$*

$$|\varsigma(t) - (r(t) + x_0)| \le C_1 \tau t^{-m/(m+1)}, \tag{3.7}$$

$$|\varsigma'(t) - r'(t)| \le C_2 \tau t^{-(m/(m+1)+1)}, \tag{3.8}$$

$$|\varsigma''(t) - r''(t)| \le C_3 \tau t^{-(m/(m+1)+2)}, \tag{3.9}$$

where C_1, C_2, C_3 depend only on m.

PROOF. (3.7) and (3.8) follow respectively from (3.5) and (3.6). Then the estimate for ς'' follows from this, (0.3), and (0.6).

To end the proof of Theorem 2 we remark that (0.13) comes from (0.2) and (0.5). By integration we obtain estimates for v and v_x. It is to be remarked that the estimates for v, v_x, ς, and ς' are valid also in the case of symmetric solutions without the assumption of concavity (cf. [**V1**, Theorem B]). In that case the result extends to several space dimensions. Let us finally state another consequence of the above results.

COROLLARY 3.3. *The interface of a concave solution to problem* (P) *has zero waiting time. More precisely with the above notations we have*

$$\varsigma'(0) \ge r'_M(\tau) = (c/(m+1))(M^{m-1}\tau^{-m})^{1/(m+1)}, \tag{3.10}$$

$$\varsigma'(0) \ge C(b-a)/2. \tag{3.11}$$

REFERENCES

[**A**] D. C. Aronson, *Regularity properties of flows through porous media: the interface*, Arch. Rational Mech. Anal. **37** (1970), 1–10.

[**AB**] D. C. Aronson and Ph. Benilan, *Régularité des solutions de l'équation des milieux poreux dans* $\mathbf{R}^n$, C. R. Acad. Sci. Paris **288** (1979), 103–105.

[**BC**] Ph. Bénilan and M. G. Crandall, *The continuous dependence on ϕ of the solutions of $u_t - \Delta\phi(u) = 0$*, Indiana Univ. Math. J. **30** (1981), 161–177.

[**BCP**] Ph. Bénilan, M. G. Crandall and A. Pazy, *Evolution equations governed by accretive operators* (to appear).

[**BT**] S. H. Benton, *The Hamilton-Jacobi equation: a global approach*, Academic Press, New York, 1977.

[**CF**] L. A. Caffarelli and A. Friedman, *Regularity of the free-boundary for the one-dimensional flow of gas in a porous medium*, Amer. J. Math. **101** (1979), 1193–1218.

[**CL**] M. G. Crandall and P. L. Lions, *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. **277** (1983), 1–42.

[**GJ**] J. L. Graveleau and P. Jamet, *A finite-difference approach to some degenerate nonlinear parabolic equations*, SIAM J. Appl. Math. **29** (1971), 199–223.

[**K**] B. F. Knerr, *The porous medium equation in one dimension*, Trans. Amer. Math. Soc. **234** (1977), 381–415.

[**LSU**] O. A. Ladyzhenskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and quasilinear equation of parabolic type*, Transl. Math. Monos., vol. 23, Amer. Math. Soc., Providence, R.I., 1969.

[**L**] P. L. Lions, *Generalized solutions of Hamilton-Jacobi equations*, Research Notes in Math., no. 69, Pitman, Boston, Mass., 1982.

[**U**] M. Ughi, *A degenerate parabolic equation modelling the spread of an epidemic*, Ann. Mat. Pura Appl. (to appear).

[**V1**] J. L. Vazquez, *Asymptotic behaviour and propagation properties of the one-dimensional flow of gas in a porous medium*, Trans. Amer. Math. Soc. **277** (1983), 507–527.

[**V2**] ——, *The interfaces of one-dimensional flows in porous media*, Trans. Amer. Math. Soc. **285** (1984), 717–737.

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