THE SZEGÖ PROJECTION:
SOBOLEV ESTIMATES IN REGULAR DOMAINS

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ABSTRACT. The Szegö projection preserves global smoothness in weakly pseudoconvex domains that are regular in the sense of Diederich, Fornæss, and Catlin. It preserves local smoothness near boundary points of finite type.

The Bergman projection, being linked to the $\bar{\partial}$ problem, plays an important role in several complex variables, as demonstrated by recent applications to the boundary behavior of holomorphic maps [7, 16]. The Szegö projection is interesting both by analogy and by its connection with the $\bar{\partial}_b$ problem and hence with unsolvable equations of Lewy type. While in strictly pseudoconvex domains the Bergman and Szegö projections are both well understood, in general much more is known about the Bergman projection. This is no surprise, for the theory of the $\bar{\partial}$ problem is further advanced than the theory of the $\bar{\partial}_b$ problem. Nonetheless I believe that the Szegö projection ought not be any more mysterious than the Bergman projection. This paper is intended to support this contention.

The most important domains for which the Bergman projection is known to be regular are domains with a good $\bar{\partial}$-Neumann operator. Here it is shown that in such domains the Szegö projection also is regular. More precisely, in smooth bounded pseudoconvex domains satisfying Catlin's property (P) (the most general condition presently known that guarantees global regularity of the Neumann operator) the Szegö projection is shown to be globally regular (Theorem 4.1). Moreover at boundary points of finite type the Szegö projection is locally regular (Theorem 5.1). In both cases the Szegö projection exactly preserves the degree of differentiability, measured in Sobolev norms.

The method of proof is not the obvious one of directly generalizing from the strictly peudoconvex case, for there are serious obstacles to such a generalization. For instance the approach of analysis on the Heisenberg group, used by Phong and Stein [29] to derive estimates for both the Bergman and Szegö projections, cannot yet be extended because the theory of singular integrals on weakly pseudoconvex surfaces is still in its early stages [28, 15]. Similarly one does not know how to carry over the Fourier integral operator approach of Boutet de Monvel and Sjöstrand [10] to the weakly pseudoconvex case. Explicit construction of Cauchy-Fantappiè forms,

à la Kerzman and Stein [21], seems intractable in view of the complicated boundary geometry of weakly pseudoconvex domains.

In principle estimates for the Szegö projection can be derived from estimates for the $\bar{\partial}_b$ operator. Since the first version of this paper was written some details have become available [36, 37] of Kohn's microlocal techniques for proving subelliptic estimates for $\bar{\partial}_b$. These estimates provide an alternate approach to the local estimates at points of finite type given in §5. (However the exact relation between Kohn's definition of finite type and D'Angelo's definition—the one I use—is presently unknown.) As far as I am aware, the global estimates for the Szegö projection in §4 cannot be obtained directly from known estimates for $\bar{\partial}_b$.

This paper completely avoids the theory of the $\bar{\partial}_b$ operator by basing the proofs directly on the $\bar{\partial}$-Neumann problem. The idea is to transfer the Szegö projection to an interior operator (say by Green's theorem). This involves taking a derivative, so what has to be shown is that integrating against the Szegö kernel over the interior is smoothing by one derivative. This turns out to be equivalent to showing that integrating against the Bergman kernel over the boundary loses exactly one derivative. The latter property is demonstrated by relating the Bergman kernel to the Neumann operator in the usual way.

This method of relating a kernel function to the Bergman kernel is one I have used previously [9] to study Sobolev space analogues of the Bergman projection. This article is the promised companion paper, advertised in [9] under the title "Regularity of the Szegö projection in domains of finite type." In particular Appendix B contains proofs of some lemmas that were stated without proof in [9].

The main results of this paper are in §§4 and 5. §2 is devoted to a general lemma on stability of the Szegö projection. §1 sets notation, and §3 recapitulates the $\bar{\partial}$-Neumann theory. The appendices are technical: Appendix A states a useful trick for localizing estimates, and Appendix B discusses Sobolev norms of harmonic functions.

1. Preliminaries. Throughout the paper Ω denotes a smooth bounded domain in $\mathbf{R}^N$, so

$$\Omega = \left\{ x \in \mathbf{R}^N : \rho(x) < 0 \right\}$$

where $\rho: \mathbf{R}^N \to \mathbf{R}$ is a bounded C^∞ function with the gradient $\nabla\rho$ nonvanishing on the boundary $b\Omega$. Generally N equals $2n$ and $\mathbf{R}^N$ is identified with $\mathbf{C}^n$. It is implicitly assumed that n is bigger than one, since in $\mathbf{C}$ the results follow from classical potential theory. The usual Hilbert spaces of square-integrable functions $L^2(\Omega)$ and $L^2(b\Omega)$ carry the standard inner product and norm:

$$\langle f, g \rangle = \int_\Omega f\bar{g} \quad \text{and} \quad \|f\|_0^2 = \langle f, f \rangle \quad \text{for } f \text{ and } g \text{ in } L^2(\Omega);$$

$$\langle f, g \rangle_b = \int_{b\Omega} f\bar{g} \quad \text{and} \quad \|\|f\|\|_0^2 = \langle f, f \rangle_b \quad \text{for } f \text{ and } g \text{ in } L^2(b\Omega).$$

The same inner product notation is used to denote the action of a distribution f on a smooth function $\bar{g}$.

More generally the two-bar norm $\| \ \|_s$ denotes the norm in the interior Sobolev space $W^s(\Omega)$ and the three-bar norm denotes the norm in the boundary Sobolev space $W^s(b\Omega)$. (Here W^s is the standard Sobolev space [1, 26] which, when s is a positive integer, consists of functions with s square-integrable derivatives.) The union over all real s of $W^s(\Omega)$, with the usual inductive limit topology, is the space $W^{-\infty}(\Omega)$; the intersection, with the projective limit topology, is the space $C^\infty(\bar\Omega)$ of functions smooth up to the boundary.

When $s \geqslant 0$,

$$\|f\|_{-s} = \sup\{|\langle f, g\rangle| : g \in C_0^\infty(\Omega), \|g\|_s = 1\}.$$

Often one wants to pair not with compactly supported functions but with functions in $C^\infty(\bar\Omega)$. I use the nonstandard notation

$$(1.1) \qquad \|f\|_{-s}^* = \sup\{|\langle f, g\rangle| : g \in C^\infty(\bar\Omega), \|g\|_s = 1\}, \qquad \|f\|_s^* = \|f\|_s$$

when $s \geqslant 0$. The asterisk is a signal that close attention is required. It is an observation of Bell (for holomorphic [2] and pluriharmonic [5] functions) and of Straube (for the general case [33, 34]) that $\| \ \|_{-s}^*$ is equivalent to $\| \ \|_{-s}$ on harmonic functions. In Appendix B a local equivalence is demonstrated.

I use the prefixes Har and Hol to indicate the subspaces of a function space consisting of harmonic and holomorphic functions respectively. Thus the Bergman projection **B** is the orthogonal projection from $L^2(\Omega)$ onto the closed subspace Hol $L^2(\Omega)$; it is represented by integration against the Bergman kernel function $B(w, z)$:

$$\mathbf{B}f(w) = \langle f(\cdot), B(\cdot, w)\rangle.$$

The Szegö projection **S** is the orthogonal projection from $L^2(b\Omega)$ onto the closed subspace Hol $L^2(b\Omega)$ consisting of functions in $L^2(b\Omega)$ whose Poisson integrals are holomorphic in Ω. It is represented by integration against the Szegö kernel function $S(w, z)$:

$$\mathbf{S}f(w) = \langle f(\cdot), S(\cdot, w)\rangle_b.$$

Since for every real s the spaces Har $W^s(\Omega)$ and $W^{s-1/2}(b\Omega)$ are identified via the Poisson integral ([26]; also see Appendix B of this paper), the Szegö projection may be viewed as a map Har $W^{1/2}(\Omega) \to$ Hol $W^{1/2}(\Omega)$ of interior spaces.

Indeed the approach in this paper is to study the Szegö projection by working on the interior of Ω. It is useful to keep in mind the following version of Green's theorem. Fix a defining function ρ and a smooth function ψ equal to $|\nabla\rho|^{-1}$ in a neighborhood of $b\Omega$. For small nonnegative ε write

$$\Omega_\varepsilon = \{z \in \Omega : \rho(z) < -\varepsilon\}.$$

If u and f are functions smooth up to the boundary of Ω_ε and u is harmonic, then

$$(1.2) \qquad \int_{b\Omega_\varepsilon} \bar u f = \int_{\Omega_\varepsilon} \bar u \mathbf{L}_\varepsilon f$$

where

$$(1.3) \qquad \mathbf{L}_\varepsilon f = \Delta\big[f\psi(\rho + \varepsilon)\big].$$

If u is replaced by a holomorphic function h, then a further integration by parts gives

$$(1.4) \qquad \int_{b\Omega_\varepsilon} \bar{h}f = \int_{\Omega_\varepsilon} \bar{h}\mathbf{L}'_\varepsilon f$$

where

$$(1.5) \qquad \mathbf{L}'_\varepsilon f = f\Delta\big[\psi(\rho + \varepsilon)\big] + 4\sum \frac{\partial f}{\partial z_j}\frac{\partial}{\partial \bar{z}_j}\big[\psi(\rho + \varepsilon)\big].$$

The following definition is formulated for use in the discussion of local estimates.

DEFINITION 1.1. A pair of *standard cut-off functions* means a pair of functions ζ_1 and ζ_2 in $C_0^\infty(\mathbf{R}^N)$ such that ζ_2 is identically equal to 1 in a neighborhood of the support of ζ_1. The *distance d* between ζ_1 and ζ_2 is the lesser of 1 and the distance between the support of ζ_1 and the support of $(1 - \zeta_2)$.

The Titchmarsh convention on constants [35] is in effect: the letter C denotes a constant that may change its identity at each new occurrence.

2. Stability of the Szegö projection. For the proof of Theorem 4.1 it is necessary to know that the Szegö projections for subdomains converge appropriately to the Szegö projection for Ω. The analogous statement for the Bergman projection is known [30, 6, 19]. The proof for the Szegö kernel is slightly more involved, and for completeness I include it here.

THEOREM 2.1. *Let* Ω *be a smooth bounded domain in* $\mathbf{C}^n$ (*not necessarily pseudoconvex*) *with defining function* ρ. *Let* $\mathbf{S}_\varepsilon$ *be the Szegö projection for the subdomain* Ω_ε, *where*

$$\Omega_\varepsilon = \{z \in \Omega: \rho(z) < -\varepsilon\}.$$

Suppose that for the positive real number r the estimate

$$\|\mathbf{S}_\varepsilon u\|_{W^r(b\Omega_\varepsilon)} \leqslant C\|u\|_{W^r(b\Omega_\varepsilon)}$$

holds for all sufficiently small positive ε *and all* u *in* $W^r(b\Omega_\varepsilon)$, *with C independent of* ε *and u. Then*

$$\|\mathbf{S}u\|_{W^r(b\Omega)} \leqslant C\|u\|_{W^r(b\Omega)}$$

for all u in $W^r(b\Omega)$.

PROOF. The heart of the proof is the verification that the Szegö kernel function $S_\varepsilon(w, z)$ for the subdomain Ω_ε approximates $S(w, z)$ in the sense that

$$(2.1) \qquad \|S(w, \cdot) - S_\varepsilon(w, \cdot)\|_{L^2(b\Omega_\varepsilon)} \to 0$$

as $\varepsilon \to 0$, the convergence being uniform for w in any compact subset of Ω. Note that since the Szegö kernel function is holomorphic in the first variable, it will follow automatically from (2.1) that $\|D_w^\alpha S(w, \cdot) - D_w^\alpha S_\varepsilon(w, \cdot)\|_{L^2(b\Omega_\varepsilon)} \to 0$ as $\varepsilon \to 0$ for every multi-index α.

Suppose for the moment that (2.1) is known. Fix a positive number δ, and let u be in $\operatorname{Har} W^r(b\Omega)$. Then $\mathbf{S}u(w) - \mathbf{S}_\varepsilon u(w)$ converges to zero uniformly for w in Ω_δ, and so do all its derivatives. Therefore

$$\|\mathbf{S}u\|_{W^{r+1/2}(\Omega_\delta)} = \lim_{\varepsilon \to 0} \|\mathbf{S}_\varepsilon u\|_{W^{r+1/2}(\Omega_\delta)} \leqslant \lim\sup \|\mathbf{S}_\varepsilon u\|_{W^{r+1/2}(\Omega_\varepsilon)}$$

$$\leqslant C \lim\sup \|u\|_{W^{r+1/2}(\Omega_\varepsilon)} \leqslant C\|u\|_{W^{r+1/2}(\Omega)},$$

with C independent of δ and u. Now let δ tend to zero and use the identification of $W^r(b\Omega)$ and $\operatorname{Har} W^{r+1/2}(\Omega)$ to obtain the result. To prove the theorem it remains to check (2.1).

VERIFICATION OF (2.1). Fix w in Ω and let f_ε denote the unique function in $\operatorname{Hol} L^2(b\Omega_\varepsilon)$ minimizing the norm subject to the constraint that $f_\varepsilon(w) = 1$. Then

$$(2.2) \qquad S_\varepsilon(w, z) = \overline{f_\varepsilon(z)} / \|f_\varepsilon\|^2_{L^2(b\Omega_\varepsilon)}.$$

Part of what needs to be shown is that the f_ε converge to the solution f_0 of the corresponding extremal problem for Ω.

At any rate

$$(2.3) \qquad \int_{b\Omega_\varepsilon} |f_\varepsilon|^2 \leqslant \int_{b\Omega_\varepsilon} |f_0|^2$$

since f_0 is a candidate for the solution to the extremal problem for Ω_ε. The right-hand side of (2.3) tends to the square of the $L^2(b\Omega)$ norm of f_0, so

$$(2.4) \qquad \int_{b\Omega_\varepsilon} |f_\varepsilon|^2 \leqslant (1 + \alpha(\varepsilon)) \int_{b\Omega} |f_0|^2$$

where $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$. In particular (2.4) implies that the f_ε form a normal family in Ω. Let $f_{\varepsilon(j)}$ be a subsequence converging uniformly on compact sets. The limit function F is holomorphic and $F(w) = 1$.

Let f_ε^* denote the function in $\operatorname{Har} L^2(b\Omega)$ whose value on $b\Omega$ is obtained by pushing f_ε out from $b\Omega_\varepsilon$ normally. Then

$$(2.5) \qquad \int_{b\Omega} |f_\varepsilon^*|^2 \leqslant (1 + \beta(\varepsilon)) \int_{b\Omega_\varepsilon} |f_\varepsilon|^2$$

where $\beta(\varepsilon) \to 0$ as $\varepsilon \to 0$, and by (2.4) the f_ε^* are uniformly bounded in $L^2(b\Omega)$. Let $j(k)$ be a subsequence such that the $f_{\varepsilon(j(k))}^*$ converge weakly in $L^2(b\Omega)$. Stability of the Poisson kernel and the uniform estimate (2.4) imply that $f_\varepsilon(z) - f_\varepsilon^*(z)$ tends to zero for each fixed z in Ω. Therefore the $f_{\varepsilon(j(k))}^*$ converge to F in Ω. Accordingly $F \in \operatorname{Hol} L^2(b\Omega)$ and, in view of (2.4) and (2.5)

$$(2.6) \qquad \int_{b\Omega} |F|^2 \leqslant \lim\inf \int_{b\Omega_{\varepsilon(j(k))}} |f_{\varepsilon(j(k))}|^2$$

$$\leqslant \lim\sup \int_{b\Omega_\varepsilon} |f_\varepsilon|^2 \leqslant \int_{b\Omega} |f_0|^2.$$

The uniqueness of the solution to the extremal problem together with (2.6) implies that $F = f_0$. Since f_0 is the unique limit point of the normal family f_ε, the f_ε converge to f_0 uniformly on compact subsets of Ω. Moreover by (2.6) $\|f_\varepsilon\|_{L^2(b\Omega_\varepsilon)} \to \|f_0\|_{L^2(b\Omega)}$. It follows from the relation (2.2) between S_ε and f_ε that $S_\varepsilon(w, z) \to S(w, z)$ uniformly in z in compact sets for fixed w.

The reproducing property of S_ε implies, for each fixed w in Ω,

$$(2.7) \quad \|S(w, \cdot) - S_\varepsilon(w, \cdot)\|^2_{L^2(b\Omega_\varepsilon)} = S_\varepsilon(w, w) - 2S(w, w) + \int_{b\Omega_\varepsilon} |S(w, \cdot)|^2.$$

Now $S_\varepsilon(w, w) \to S(w, w)$ by the pointwise convergence just demonstrated, and since $S(w, \cdot) \in \mathrm{Har}\, L^2(b\Omega)$ the third term on the right-hand side of (2.7) tends to $\int_{b\Omega} |S(w, \cdot)|^2$, which also equals $S(w, w)$. Hence (2.7) implies (2.1) for fixed w in Ω.

The uniformity in w follows easily by reductio ad absurdum. Suppose there were a sequence of points $w_1, w_2, \ldots$ in Ω converging to a point w_∞ in Ω and a sequence $\varepsilon(j)$ converging to zero such that

$$(2.8) \quad \|S(w_j, \cdot) - S_{\varepsilon(j)}(w_j, \cdot)\|_{L^2(b\Omega_{\varepsilon(j)})} > c > 0$$

for some c independent of j. Since $\|\partial S(w, \cdot)/\partial w_k\|_{L^2(b\Omega)}$ is the norm of the functional $h \to \partial h(w)/\partial w_k$ on $\mathrm{Hol}\, L^2(b\Omega)$, it is bounded by a constant times an inverse power of the distance from w to the boundary $b\Omega$. Hence by the mean-value theorem the norm of $S(w_j, \cdot) - S(w_\infty, \cdot)$ in $L^2(b\Omega)$ tends to zero as $j \to \infty$, and a fortiori

$$(2.9) \quad \|S(w_j, \cdot) - S(w_\infty, \cdot)\|_{L^2(b\Omega_{\varepsilon(j)})} \to 0$$

as $j \to \infty$. For the same reason

$$(2.10) \quad \|S_{\varepsilon(j)}(w_j, \cdot) - S_{\varepsilon(j)}(w_\infty, \cdot)\|_{L^2(b\Omega_{\varepsilon(j)})} \to 0$$

as $j \to \infty$. The pointwise convergence already proved implies

$$(2.11) \quad \|S(w_\infty, \cdot) - S_{\varepsilon(j)}(w_\infty, \cdot)\|_{L^2(b\Omega_{\varepsilon(j)})} \to 0$$

as $j \to \infty$. Combining the last three statements (2.9), (2.10), and (2.11) gives a contradiction to the ansatz (2.8). This completes the proof of (2.1) and of the theorem.

3. Reminders on the $\bar{\partial}$-Neumann problem. In this section I summarize some known estimates for the $\bar{\partial}$-Neumann problem that are required to run the machine in the next two sections. The references are the seminal paper of Kohn and Nirenberg [24] and the important recent contributions of Catlin [11, 12, 39]. The reader is advised to skip to §§4 and 5 and refer to these results as needed.

Consider $\bar{\partial}$ as a closed densely defined operator acting on forms with square-integrable coefficients. The Hilbert space adjoint is denoted $\bar{\partial}^*$. When Ω is a smooth bounded pseudoconvex domain in $\mathbf{C}^n$ the complex Laplacian $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ acting on the space of $(0, 1)$-forms is surjective; its inverse is the Neumann operator N. The quadratic form Q is defined on $(0, 1)$-forms in the domains of $\bar{\partial}$ and $\bar{\partial}^*$ by

$$Q(f, f) = \|\bar{\partial}f\|^2_0 + \|\bar{\partial}^*f\|^2_0.$$

One says that a compactness estimate holds for the $\bar{\partial}$-Neumann problem if for every positive ε there exists $C(\varepsilon)$ such that

(3.1) $$\|f\|_0^2 \leqslant \varepsilon Q(f, f) + C(\varepsilon)\|f\|_{-1}^2.$$

THEOREM 3.1 (KOHN AND NIRENBERG). *Suppose that Ω is a smooth bounded pseudoconvex domain in $\mathbf{C}^n$ such that a compactness estimate holds for the $\bar{\partial}$-Neumann problem. Fix a positive real number r. Then for every positive ε there is $C(\varepsilon)$ such that*

$$\|\bar{\partial}^* N f\|_r \leqslant \varepsilon \|f\|_r + C(\varepsilon)\|f\|_0$$

for all $(0, 1)$-forms f such that $\bar{\partial} f = 0$.

In [12] Catlin found an important sufficient criterion for the existence of a compactness estimate. A domain Ω in $\mathbf{C}^n$ satisfies Catlin's property (P) if for every positive M there exists a plurisubharmonic function λ in $C^\infty(\overline{\Omega})$ such that $0 \leqslant \lambda \leqslant 1$ and there exists a positive δ so that

(3.2) $$\sum_{j, k=1}^n \frac{\partial^2 \lambda(z)}{\partial z_j \partial \bar{z}_k} t_j \bar{t}_k \geqslant M|t|^2$$

for all t in $\mathbf{C}^n$ and all points z in Ω at distance less than δ from the boundary.

THEOREM 3.2 (CATLIN). *If Ω is a smooth bounded pseudoconvex domain in $\mathbf{C}^n$ satisfying property (P) then a compactness estimate holds for the $\bar{\partial}$-Neumann problem. Moreover if (3.2) holds for a certain M then (3.1) holds for ε equal to $40/M$.*

Inspection of the proof shows that $C(\varepsilon)$ depends only on ε, the diameter of Ω, and the first derivatives of a cut-off function in $C_0^\infty(\Omega)$ that is identically equal to one at points at distance δ or greater from the boundary of Ω.

In §4 it will be necessary to know that Theorem 3.2 holds uniformly on approximating subdomains of Ω. This fact is implicit in Catlin's work, but for the reader's convenience I include an explicit statement and proof. First recall that by a construction of Diederich and Fornæss [17] there is a smooth defining function ρ for the pseudoconvex domain Ω such that $-(-\rho)^\eta$ is strictly plurisubharmonic inside Ω for all sufficiently small positive η. (Range later showed [31] that if $\theta(z)$ is any smooth defining function for Ω then $\theta(z)\exp(-K|z|^2)$ is a Diederich-Fornæss defining function for sufficiently large K.)

PROPOSITION 3.3. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbf{C}^n$ with Diederich-Fornæss defining function ρ. Suppose Ω satisfies Catlin's property (P). Then for all sufficiently small positive β the interior approximating domain Ω_β (consisting of all points z in Ω for which $\rho(z) < -\beta$) admits a Kohn-Nirenberg compactness estimate*

$$\|f\|_{L^2(\Omega_\beta)}^2 \leqslant \varepsilon Q_\beta(f, f) + C(\varepsilon)\|f\|_{W^{-1}(\Omega_\beta)}^2$$

with $C(\varepsilon)$ independent of β.

PROOF. What has to be shown, in view of Catlin's theorem, is that for every M and every sufficiently small positive β there exists a plurisubharmonic function λ_β in $C^\infty(\overline{\Omega}_\beta)$ such that $0 \leqslant \lambda_\beta \leqslant 1$ and

(3.3) $$\sum_{j, k=1}^n \frac{\partial^2 \lambda_\beta(z)}{\partial z_j \partial \bar{z}_k} t_j \bar{t}_k \geqslant M|t|^2$$

for all t in $\mathbf{C}^n$ and all z for which $-\beta - \delta < \rho(z) < -\beta$, where δ is a positive number independent of β.

By hypothesis there is a positive δ_0 and a plurisubharmonic function λ_0 in $C^\infty(\overline{\Omega})$ such that $0 \leqslant \lambda_0 \leqslant 1$ and (3.2) holds when $-\delta_0 < \rho(z) < 0$. If $\beta \leqslant \delta_0/2$ and λ_β is taken to be λ_0 then the desired inequality (3.3) holds when $-\beta - \delta < \rho(z) < -\beta$ as long as δ is no greater than $\delta_0/2$.

When $\beta > \delta_0/2$ the required λ_β can be constructed from the Diederich-Fornæss defining function ρ. Denote $-(-\rho)^\eta$ by μ, where η is a fixed number such that $0 < \eta < 1$ and such that μ is strictly plurisubharmonic in Ω. Let m denote the minimum eigenvalue of the complex Hessian of μ on the set of z in Ω such that $\rho(z) \leqslant -\delta_0/2$. Set λ_β equal to $\exp[2Mm^{-1}(\mu + \beta^\eta)]$. Then $0 \leqslant \lambda_\beta(z) \leqslant 1$ when $z \in \Omega_\beta$, the function $\lambda_\beta \in C^\infty(\overline{\Omega}_\beta)$, and (3.3) holds when

$$-\left(M^{-1}m \log \sqrt{2}\,\right)^{1/\eta} - \beta < \rho(z) < -\beta.$$

Combining the two cases shows the required λ_β exists, with δ independent of β, if δ is the minimum of $\delta_0/2$ and $(M^{-1}m \log \sqrt{2}\,)^{1/\eta}$. This proves the proposition.

It is a well-known consequence of the Kohn-Nirenberg theory that, since the Bergman projection $\mathbf{B} = \mathrm{Id} - \overline{\partial}^* N \overline{\partial}$, a compactness estimate for the $\overline{\partial}$-Neumann problem implies that the Bergman projection maps $W^r(\Omega)$ continuously into itself for each positive r. The proof of this fact was for some years a closely held secret among the cognoscenti, but a sketch has now appeared in print [23]. The constants involved in the estimates depend on the domain only through its diameter and a finite number of derivatives of the boundary defining function. Therefore the previous proposition implies the following estimate for $\mathbf{B}$, uniform over subdomains.

PROPOSITION 3.4. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbf{C}^n$ with Diederich-Fornæss defining function ρ, and suppose Ω satisfies Catlin's property* (P). *Then for every nonnegative real number r there exists a constant C such that for all sufficiently small positive ε the Bergman projection $\mathbf{B}_\varepsilon$ for the subdomain Ω_ε (the set of z for which $\rho(z) < -\varepsilon$) admits the estimate*

$$\|\mathbf{B}_\varepsilon f\|_{W^r(\Omega_\varepsilon)} \leqslant C\|f\|_{W^r(\Omega_\varepsilon)}.$$

A point q in the boundary of Ω is said to have finite type in the sense of D'Angelo [14] if the supremum of the order of contact of complex varieties with the boundary of Ω at q is finite. Catlin has shown [13, 39] that a point q in the boundary of a smooth bounded pseudoconvex domain is of finite type if and only if the $\overline{\partial}$-Neumann problem is subelliptic near q. This means that there is a neighborhood U of q and a positive ε such that

$$\|\phi\|_\varepsilon^2 \leqslant C\big(Q(\phi, \phi) + \|\phi\|_0^2\big)$$

for all smooth $(0, 1)$-forms ϕ in the domains of $\overline{\partial}$ and $\overline{\partial}^*$ and supported in U. According to the Kohn-Nirenberg theory such a subelliptic estimate implies a pseudo-local estimate for $\overline{\partial}^* N$. The result that will be used in §5 is the following.

PROPOSITION 3.5. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbf{C}^n$ for which the Neumann operator preserves the space of $(0, 1)$-forms with coefficients in $C^\infty(\overline{\Omega})$. Suppose q is a point of finite type in the boundary of Ω. Then $\overline{\partial}^*N$ gains a fractional derivative near q in the following sense: There is a neighborhood U of q and a positive number ε such that for all nonnegative real numbers r and m there exists a constant C such that for all standard cut-off functions η_1 and η_2 supported in U*

$$(3.4) \qquad \left\| \eta_1 \overline{\partial}^* Nf \right\|_{r+\varepsilon} \leqslant Cd^{-c} \|\eta_1\|_c \left(\|\eta_2 f\|_r + \|f\|^*_{-m} \right)$$

for all $\overline{\partial}$-closed $(0, 1)$-forms f with coefficients in $C^\infty(\overline{\Omega})$, where d is the distance between η_1 and η_2 (Definition 1.1).

The global regularity hypothesis is included to ensure that the global term in (3.4) can be taken with arbitrary negative index, rather than with L^2 norm. This hypothesis is of course satisfied if Catlin's property (P) holds. The dependence on η_1 and η_2 has the indicated form since in the proof derivatives of η_1 appear together with derivatives of a sequence of cut-off functions with supports intermediate between those of η_1 and η_2. The form of this dependence will be used in §5 in connection with the iteration trick of Appendix A.

As before, estimates for the Neumann operator lead to estimates for the Bergman projection. The following statement will suffice for the application in §5.

PROPOSITION 3.6. *With the same hypotheses as in Proposition 3.5 the Bergman projection admits the estimate*

$$\|\eta_1 \mathbf{B}f\|_r \leqslant C \left(\|\eta_2 f\|_r + \|f\|^*_{-m} \right)$$

for all f in $C^\infty(\overline{\Omega})$. The constant C depends on the nonnegative real numbers r and m and on the standard cut-off functions η_1 and η_2.

4. Global estimates for the Szegö projection. This section is devoted to the proof of the main result on global regularity of the Szegö projection.

THEOREM 4.1. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbf{C}^n$. Suppose Ω satisfies Catlin's property (P), discussed in §3. Then the Szegö projection for Ω maps $C^\infty(b\Omega)$ continuously into itself, and more generally maps $W^r(b\Omega)$ continuously into itself for every positive real number r.*

The theorem applies to the following types of domains:

(a) STRICTLY PSEUDOCONVEX DOMAINS. For such domains regularity of the Szegö projection is well known [29, 21, 10], and this known case will be used in the proof of the more general Theorem 4.1.

(b) WEAKLY PSEUDOCONVEX DOMAINS WITH REAL ANALYTIC BOUNDARY. An example is the set of points (z, w) in $\mathbf{C}^2$ such that $|z|^2 + (\operatorname{Re} z)^2 + |w|^4 < 1$. The theorem of course applies to the simpler example $|z|^2 + |w|^4 < 1$, but since this domain has bicircular symmetry an easier proof of the regularity theorem is known for it [8, 32].

(c) PSEUDOCONVEX DOMAINS OF FINITE TYPE. These domains include those of categories (a) and (b). For domains of finite type a stronger local regularity theorem holds. This is the main result of the next section.

(d) CATLIN'S "WEAKLY REGULAR" DOMAINS. These include the domains of category (c). An example that is not of finite type is the set of points (z, w) in $\mathbf{C}^2$ defined by the inequality

$$\left(|z|^2 + |w - 1|^2\right) \exp\left[-\left(|z|^2 + |w - 1|^2\right)^{-1}\right] < 1 - |w|^2.$$

This smooth bounded pseudoconvex domain is strictly pseudoconvex at every boundary point except the point $(0, 1)$, where the Levi form degenerates to infinite order.

PROOF OF THEOREM 4.1. Let ρ be a Diederich-Fornæss defining function for Ω, so that for all sufficiently small positive ε the interior domain Ω_ε such that $\Omega_\varepsilon = \{ z \in \Omega: \rho(z) < -\varepsilon \}$ has smooth strictly pseudoconvex boundary. In view of Theorem 2.1 what needs to be shown is that when $r > 0$

(4.1) $\|\mathbf{S}_\varepsilon u\|_{W^r(b\Omega_\varepsilon)} \leqslant C \|u\|_{W^r(b\Omega_\varepsilon)}$

for all u in $W^r(b\Omega_\varepsilon)$. Of course since Ω_ε is strictly pseudoconvex it is known that an estimate of the form (4.1) holds for each ε: the whole point here is that C may be taken independent of ε.

Since the Bergman kernel function $B_\varepsilon(w, z)$ for Ω_ε is smooth up to the boundary $b\Omega_\varepsilon$ off the boundary diagonal [20], the equation

$$T_\varepsilon f(w) = \int_{b\Omega_\varepsilon} B_\varepsilon(w, z) f(z)$$

defines a map T_ε from smooth functions on $b\Omega_\varepsilon$ to holomorphic functions on Ω_ε. The following a priori estimates, uniform in ε, are central to the proof of the theorem.

Key global estimates. For every positive real number r there is a constant C such that for all sufficiently small positive ε

(4.2) $\|T_\varepsilon f\|_{W^r(\Omega_\varepsilon)} \leqslant C \|f\|_{W^{r+1}(\Omega_\varepsilon)}$ for f in $C^\infty(\overline{\Omega}_\varepsilon)$,

(4.3) $\|h\|_{W^{r+1}(\Omega_\varepsilon)} \leqslant C\left(\|T_\varepsilon h\|_{W^r(\Omega_\varepsilon)} + \|h\|_{L^2(\Omega_\varepsilon)}\right)$ for h in $\mathrm{Hol}\, C^\infty(\overline{\Omega}_\varepsilon)$.

It is straightforward, granted these estimates, to prove (4.1). Since when $r = 0$ the inequality holds by definition with C equal to 1, it suffices by interpolation to prove (4.1) when r is large, say when $r > 1$. Suppose then $u \in \mathrm{Har}\, W^r(b\Omega_\varepsilon)$ and $r > 1$. Let $u_1, u_2, \ldots$ be a sequence of harmonic functions in $C^\infty(\overline{\Omega}_\varepsilon)$ such that $u_j \to u$ in $W^r(b\Omega_\varepsilon)$ as $j \to \infty$. Since Ω_ε is strictly pseudoconvex, it is known a priori that $\mathbf{S}_\varepsilon u_j \in C^\infty(\overline{\Omega}_\varepsilon)$ and that $\mathbf{S}_\varepsilon u_j \to \mathbf{S}_\varepsilon u$ in $W^r(b\Omega_\varepsilon)$ as $j \to \infty$. Rewriting key estimate (4.3) in terms of boundary norms (cf. (B.2) of Appendix B) gives

$$\|\mathbf{S}_\varepsilon u_j\|_{W^r(b\Omega_\varepsilon)} \leqslant C\left(\|T_\varepsilon \mathbf{S}_\varepsilon u_j\|_{W^{r-1}(b\Omega_\varepsilon)} + \|\mathbf{S}_\varepsilon u_j\|_{L^2(b\Omega_\varepsilon)}\right).$$

Since $T_\varepsilon \mathbf{S}_\varepsilon u_j = T_\varepsilon u_j$ by the definition of the Szegö projection, it follows from key estimate (4.2) that

$$\|\mathbf{S}_\varepsilon u_j\|_{W^r(b\Omega_\varepsilon)} \leqslant C \|u_j\|_{W^r(b\Omega_\varepsilon)}.$$

Now let $j \to \infty$ to obtain (4.1).

PROOF OF THE KEY GLOBAL ESTIMATES. To obtain (4.2) write $T_\varepsilon f = \mathbf{B}_\varepsilon \mathbf{L}_\varepsilon' f$, where $\mathbf{L}_\varepsilon'$ is the special first order differential operator defined by (1.5). Then (4.2) is immediate from the estimate for the Bergman projection stated in Proposition 3.4.

To obtain (4.3) write $T_\varepsilon h = \mathbf{B}\mathbf{L}_\varepsilon h = \mathbf{L}_\varepsilon h - \bar{\partial}^* N_\varepsilon \bar{\partial} \mathbf{L}_\varepsilon h$, where $\mathbf{L}_\varepsilon$ is the differential operator defined by (1.3). In view of Proposition 3.3 and Theorem 3.1 there is a compactness estimate for $\bar{\partial}^* N_\varepsilon$ that is independent of ε, so

$$\left\| \bar{\partial}^* N_\varepsilon \bar{\partial} \mathbf{L}_\varepsilon h \right\|_{W^r(\Omega_\varepsilon)} \leqslant \delta \left\| \bar{\partial} \mathbf{L}_\varepsilon h \right\|_{W^r(\Omega_\varepsilon)} + C(\delta) \left\| \bar{\partial} \mathbf{L}_\varepsilon h \right\|_{L^2(\Omega_\varepsilon)}.$$

Since $\Delta h = 0$, the operator $\mathbf{L}_\varepsilon$ looks first-order to h, and since $\bar{\partial} h = 0$, the operator $\bar{\partial} \mathbf{L}_\varepsilon$ also looks first-order to h. Therefore

$$\left\| \bar{\partial}^* N_\varepsilon \bar{\partial} \mathbf{L}_\varepsilon h \right\|_{W^r(\Omega_\varepsilon)} \leqslant K \left(\delta \| h \|_{W^{r+1}(\Omega_\varepsilon)} + C(\delta) \| h \|_{W^1(\Omega_\varepsilon)} \right).$$

Accordingly

$$\left\| \mathbf{L}_\varepsilon h \right\|_{W^r(\Omega_\varepsilon)} \leqslant \| T_\varepsilon h \|_{W^r(\Omega_\varepsilon)} + K \left(\delta \| h \|_{W^{r+1}(\Omega_\varepsilon)} + C(\delta) \| h \|_{W^1(\Omega_\varepsilon)} \right)$$

with K and $C(\delta)$ independent of ε. By part (a) of Lemma B.8 of Appendix B,

$$\| h \|_{W^{r+1}(\Omega_\varepsilon)} \leqslant C \left(\| \mathbf{L}_\varepsilon h \|_{W^r(\Omega_\varepsilon)} + \| h \|_{L^2(\Omega_\varepsilon)} \right).$$

Combining this estimate with the previous one and taking δ sufficiently small gives

$$\| h \|_{W^{r+1}(\Omega_\varepsilon)} \leqslant C \left(\| T_\varepsilon h \|_{W^r(\Omega_\varepsilon)} + \| h \|_{W^1(\Omega_\varepsilon)} \right)$$

with a new constant C independent of ε. Since $r > 0$,

$$C \| h \|_{W^1(\Omega_\varepsilon)} \leqslant \tfrac{1}{2} \| h \|_{W^{r+1}(\Omega_\varepsilon)} + C' \| h \|_{L^2(\Omega_\varepsilon)},$$

so (4.3) follows. This completes the proof of Theorem 4.1.

REMARK 4.2. The theorem has the following consequences:

(1) The Szegö kernel function for Ω has finite order growth at the boundary [8], and in particular the Szegö kernel function $S(w, z) \in C^\infty(\omega \times \bar{\Omega})$ for every relatively compact subset ω of Ω. In domains of finite type a stronger property holds (Corollary 5.2).

(2) The Szegö projection extends (by duality) to a continuous map of the space of distributions on $b\Omega$ into itself, and the extension satisfies the estimate $\||\mathbf{S}u\||_{-r} \leqslant C\||u\||_{-r}$. Moreover if $u \in \mathrm{Har}\, W^{-\infty}(b\Omega)$ then the distribution $\mathbf{S}u$ is the boundary value of the holomorphic function defined by

$$\mathbf{S}u(w) = \lim_{\varepsilon \to 0} \int_{b\Omega_\varepsilon} S(w, z) u(z).$$

The last statement follows from work of Straube [33] on distribution boundary values of harmonic functions.

(3) The operators T and U defined by

$$Th(w) = \int_{b\Omega} B(w, z) h(z), \quad Uh(w) = \int_\Omega S(w, z) h(z)$$

satisfy the identities

(4.4) $$\langle Th, g \rangle = \langle h, g \rangle_b, \qquad \langle h, g \rangle = \langle Uh, g \rangle_b$$

for h and g in $\operatorname{Hol} C^\infty(\overline{\Omega})$, as follows by approximating in the first case g and in the second case h by compactly supported functions and applying Fubini's theorem. Moreover these operators have the following properties.

COROLLARY 4.3. *Under the hypotheses of Theorem* 4.1
  (i) *T and U are inverse isomorphisms of $\operatorname{Hol} C^\infty(\overline{\Omega})$ onto itself*;
  (ii) *T and U extend (uniquely) to inverse isomorphisms of $\operatorname{Hol} W^{-\infty}(\Omega)$ onto itself*;
  (iii) *the extensions define mutually inverse isomorphisms*

$$T: \operatorname{Hol} W^{r+1}(\Omega) \to \operatorname{Hol} W^r(\Omega), \quad U: \operatorname{Hol} W^r(\Omega) \to \operatorname{Hol} W^{r+1}(\Omega)$$

for every real number r.

PROOF. The hypothesis implies that for every real number r there is a constant C such that for all h in $\operatorname{Hol} W^{-\infty}(\Omega)$

$$(4.5) \quad C^{-1}\|h\|_r \le \sup\left\{|\langle h, g\rangle| : g \in \operatorname{Hol} C^\infty(\overline{\Omega}), \|g\|_{-r} = 1\right\} \le C\|h\|_r,$$

$$(4.6) \quad C^{-1}\|\!|h|\!\|_r \le \sup\left\{|\langle h, g\rangle_b| : g \in \operatorname{Hol} C^\infty(\overline{\Omega}), \|\!|g|\!\|_{-r} = 1\right\} \le C\|\!|h|\!\|_r.$$

The first statement follows from the estimates for the Bergman projection and the density of $\operatorname{Hol} C^\infty(\overline{\Omega})$ in $\operatorname{Hol} W^{-\infty}(\Omega)$. It was first proved by Bell [4] in the strictly pseudoconvex case and generalized in [6, 25]. The second statement follows easily from the estimates of Theorem 4.1 for the Szegö projection and the same density.

When $h \in \operatorname{Hol} C^\infty(\overline{\Omega})$ identities (4.4) and estimates (4.5) and (4.6) together with the correspondence (B.2) between boundary and interior norms give

$$C^{-1}\|h\|_{r+1} \le \|Th\|_r \le C\|h\|_{r+1}, \qquad C^{-1}\|h\|_r \le \|Uh\|_{r+1} \le C\|h\|_r.$$

In particular T and U map $\operatorname{Hol} C^\infty(\overline{\Omega})$ continuously into itself and, since $\operatorname{Hol} C^\infty(\overline{\Omega})$ is dense in $\operatorname{Hol} W^{-\infty}(\Omega)$, they extend uniquely to continuous maps of $\operatorname{Hol} W^{-\infty}(\Omega)$ into itself. It is clear from (4.4) that $TU = UT = \operatorname{Id}$ on $\operatorname{Hol} C^\infty(\overline{\Omega})$, and this relation persists for the extension to $\operatorname{Hol} W^{-\infty}(\Omega)$.

5. Local estimates at points of finite type. In the previous section it was shown that the Szegö projection of a function that is smooth on the whole boundary is again smooth on the whole boundary. In this section it is shown that the Szegö projection of a function that is smooth near a point of finite type is again smooth near that point. The proof of this local result follows the same outline as the proof of the global result. The stronger property of subellipticity replaces compactness, which makes it possible to localize the argument. One simplification occurs: since the global regularity is known a priori, it is not necessary to work on approximating interior domains.

THEOREM 5.1. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbf{C}^n$ satisfying Catlin's property* (P). *Let q be a point of finite type in the boundary of Ω. Then the Szegö projection of every distribution that is smooth near q is again smooth near q.*

More precisely, there exists a neighborhood U of q such that for all cut-off functions ζ_1 and ζ_2 in $C_0^\infty(U)$ with ζ_2 identically equal to 1 in a neighborhood of the support of ζ_1, and for all real numbers r and m, there exists a constant C such that

$$(5.1) \qquad\qquad \|\!|\zeta_2 \mathbf{S} u|\!\|_r \le C\big(\|\!|\zeta_2 u|\!\|_r + \|\!|u|\!\|_{-m}\big)$$

for all distributions u in $W^{-\infty}(b\Omega)$.

COROLLARY 5.2. *In the above situation the Szegö kernel function*

$$S(w, z) \in C^\infty \left(U \cap \overline{\Omega} \times \overline{\Omega} \setminus \{ (w, z) \in b\Omega \times b\Omega : w = z \} \right).$$

That is, the Szegö kernel function is smooth in both variables jointly when one variable approaches a boundary point of finite type and the other variable approaches any other boundary point.

PROOF OF THEOREM 5.1. Choose the neighborhood U by Catlin's theorem so that a subelliptic estimate of order ε holds, as described in Proposition 3.5. As in §4 consider the operator T acting on functions in $C^\infty(b\Omega)$ via

$$Tu(w) = \int_{b\Omega} B(w, z) u(z).$$

The heart of the proof lies in the following a priori estimates.

KEY LOCAL ESTIMATES. For all standard cut-off functions ζ_1 and ζ_2 supported in U and for all nonnegative real numbers r and m there is a constant C such that

(5.2) $\qquad \||\zeta_1 Tu|\|_r \leqslant C \left(\||\zeta_2 u|\|_{r+1} + \||u|\|_{-m} \right)$ for all u in $C^\infty(b\Omega)$,

(5.3) $\qquad \||\zeta_1 h|\|_{r+1} \leqslant C \left(\||\zeta_2 Th|\|_r + \||h|\|_{-m} \right)$ for all h in $\operatorname{Hol} C^\infty(\overline{\Omega})$.

Assume for the moment the validity of these estimates. Let η_1 and η_2 be another pair of standard cut-off functions, with η_1 equal to 1 in a neighborhood of the support of ζ_1 and ζ_2 equal to 1 in a neighborhood of the support of η_2. If $u \in C^\infty(b\Omega)$ and $r \geqslant 1$ then (5.3) implies

$$\||\zeta_1 \mathbf{S} u|\|_r \leqslant C \left(\||\eta_1 T \mathbf{S} u|\|_{r-1} + \||\mathbf{S} u|\|_{-m} \right).$$

By the global regularity proved in §4,

$$\||\mathbf{S} u|\|_{-m} \leqslant C \||u|\|_{-m}.$$

Moreover $T\mathbf{S} = T$ on smooth functions, so by (5.2)

$$\||\eta_1 T \mathbf{S} u|\|_{r-1} \leqslant C \left(\||\eta_2 u|\|_r + \||u|\|_{-m} \right).$$

Thus (5.1) holds for smooth functions u when $r \geqslant 1$. To prove (5.1) for general u it suffices to take a sequence $u_1, u_2, \ldots$ of smooth functions such that $u_j \to u$ in $W^{-m}(b\Omega)$ and $\eta_2 u_j \to \eta_2 u$ in $W^r(b\Omega)$ and pass to the limit.

To extend the result to arbitrary r observe that, by the estimate just proved,

$$\||\zeta_1 \mathbf{S}[(1 - \zeta_2)u]|\|_1 \leqslant C \||u|\|_{-m}.$$

Combined with global regularity this inequality yields, when $r < 1$,

$$\||\zeta_1 \mathbf{S} u|\|_r \leqslant C \left(\||\mathbf{S}(\zeta_2 u)|\|_r + \||\zeta_1 \mathbf{S}[(1 - \zeta_2)u]|\|_1 \right)$$

$$\leqslant C \left(\||\zeta_2 u|\|_r + \||u|\|_{-m} \right).$$

PROOF OF THE KEY LOCAL ESTIMATES. It is convenient to shift from boundary to interior norms using Lemma B.5 of Appendix B. Thus

(5.4) $\qquad \||\zeta_1 Tu|\|_r \leqslant C \left(\|\eta_1 Tu\|_{r+1/2} + \|Tu\|_{-m-1/2} \right).$

If $u \in \operatorname{Har} C^\infty(\overline{\Omega})$ then $Tu = T\mathbf{S} u$ and the global estimates for T and $\mathbf{S}$ yield

$$\|Tu\|_{-m-1/2} \leqslant C \|\mathbf{S} u\|_{-m+1/2} \leqslant C \||u|\|_{-m}.$$

Therefore to prove (5.2) it is enough to estimate the first term on the right-hand side of (5.4). Write $Tu = \mathbf{B}\mathbf{L}'u$ where $\mathbf{B}$ is the Bergman projection and $\mathbf{L}'$ is the special first-order differential operator of (1.5) with ε set equal to zero. The estimate of Proposition 3.6 for $\mathbf{B}$ gives

$$\|\eta_1 Tu\|_{r+1/2} \leqslant C\Big(\|\eta_2 \mathbf{L}'u\|_{r+1/2} + \|\mathbf{L}'u\|^*_{-m-1/2}\Big).$$

By Lemmas B.6 and B.5

$$\|\eta_2 \mathbf{L}'u\|_{r+1/2} \leqslant C\big(\|\|\zeta_2 u\|\|_{r+1} + \|\|u\|\|_{-m}\big)$$

and

$$\|\mathbf{L}'u\|^*_{-m-1/2} \leqslant C\|\|u\|\|_{-m},$$

so (5.2) follows.

In proving (5.3) I assume that the intermediate cut-off functions η_1 and η_2 have been constructed as in Appendix A, so that their derivatives can be estimated by inverse powers of the distance d between ζ_1 and ζ_2. Write $Th = \mathbf{L}h - \bar{\partial}^* N \bar{\partial} \mathbf{L}h$, where $\mathbf{L}$ is the operator of (1.3) with ε set equal to zero. By Lemma B.8,

$$\|\zeta_1 h\|_{r+1} \leqslant Cd^{-c}\|\zeta_1\|_c\big(\|\eta_1 \mathbf{L}h\|_r + \|h\|_{-m}\big)$$
$$\leqslant Cd^{-c}\|\zeta_1\|_c\big(\|\eta_1 Th\|_r + \|\eta_1 \bar{\partial}^* N \bar{\partial} \mathbf{L}h\|_r + \|h\|_{-m}\big).$$

The subelliptic estimate (Proposition 3.5) now comes into play:

$$\|\eta_1 \bar{\partial}^* N \bar{\partial} \mathbf{L}h\|_r \leqslant Cd^{-c}\big(\|\eta_2 \bar{\partial} \mathbf{L}h\|_{r-\varepsilon} + \|\bar{\partial} \mathbf{L}h\|^*_{-m-1}\big)$$

when $r \geqslant \varepsilon$. Combining the last two estimates and invoking Lemma B.6 gives

$$\|\zeta_1 h\|_{r+1} \leqslant Cd^{-c}\|\zeta_1\|_c\big(\|\zeta_2 Th\|_r + \|\zeta_2 h\|_{r+1-\varepsilon} + \|h\|_{-m}\big).$$

By the iteration trick of Appendix A it is possible to absorb the middle term on the right-hand side. Passing from interior norms to boundary norms gives (5.3). This completes the proof of Theorem 5.1.

PROOF OF COROLLARY 5.2. Let ζ_1 and ζ_2 be cut-off functions as in Theorem 5.1, and let Z be a closed ball disjoint from the support of ζ_2. By Sobolev's lemma it is enough to show that for each multi-index α and each positive r

$$\sup\big\{\|\zeta_1 D_z^\alpha S(\cdot, z)\|_r : z \in Z \cap \Omega\big\} < \infty.$$

As observed in [21], if P denotes the Poisson kernel then $S(w, z) = \mathbf{S}[P(\cdot, z)](w)$. Therefore by Theorem 5.1 and the local correspondence of Lemma B.5 between boundary and interior norms

$$\|\zeta_1 D_z^\alpha S(\cdot, z)\|_r \leqslant C\big(\|\|\zeta_2 D_z^\alpha P(\cdot, z)\|\|_{r-1/2} + \|\|D_z^\alpha P(\cdot, z)\|\|_{-m}\big).$$

The first term is finite when $z \in Z \cap \Omega$ since the Poisson kernel is smooth on $b\Omega \times \bar{\Omega}$ off the diagonal. The second term is bounded by

$$\sup\big\{|D^\alpha u(z)| : z \in Z \cap \Omega, \ u \text{ harmonic}, \ \|\|u\|\|_m = 1\big\}.$$

Since the boundary norm of a harmonic function dominates the interior norm, this is bounded independently of z when $m > n + |\alpha|$. This completes the proof.

REMARK. The seminal paper on kernel smoothness is [20]. The analogue of Corollary 5.2 for the Bergman kernel function is implicit there. Essentially Kerzman's argument shows that a kernel smoothness result holds as soon as a local estimate of the form (5.1) holds.

Appendix A: An iteration trick. A common situation that arises in the study of pseudo-local estimates is that one can prove for a linear operator L an estimate of the form

$$(A.1) \qquad \|\zeta_1 L f\|_r \leqslant C \big(\|\zeta_2 f\|_r + \|\zeta_2 L f\|_{r-\varepsilon} \big),$$

where ζ_2 is identically equal to 1 on the support of ζ_1, and one would like to absorb the lower order error term to obtain

$$(A.2) \qquad \|\zeta_1 L f\|_r \leqslant C \big(\|\zeta_2 f\|_r + \|f\|_{-m} \big).$$

If (A.1) held for all r then (A.2) would follow by an obvious iteration, lowering the index of the error term by ε at each step. Typically, however, (A.1) is known only for positive r, so that the best obvious conclusion is (A.2) with $m = 0$.

Nonetheless it is possible to derive (A.2) from (A.1) by a more sophisticated iteration scheme. One has to determine the dependence of the constant on the cut-off functions and study what happens as their supports squeeze together. This ingenious idea essentially goes back to Hopf [27] and Schauder [40]. I learned the trick from the paper [18] of Greene and Krantz, and the formulation I give here is patterned on theirs.

Let Ω be a smooth bounded domain in $\mathbf{R}^N$ and suppose L is a continuous linear operator from $W^{-\infty}(\Omega)$ into itself. Fix real numbers r and m, a norm G (thought of as a global term) dominating $\| \ \|_{-m}$, a norm M (thought of as the main term) such that $M(gf) \leqslant \|g\|_c M(f)$, an open set U, a positive number ε, and an operator T.

HYPOTHESIS. There is a constant C such that for all standard cut-off functions ζ_1 and ζ_2 supported in U

$$(A.3) \qquad \|\zeta_1 L f\|_r \leqslant C d^{-c} \|\zeta_1\|_c \big(M(\zeta_2 T f) + G(f) + \|\zeta_2 L f\|_{r-\varepsilon} \big)$$

for all f in a certain subspace of $W^{-\infty}(\Omega)$. Here d is the distance between ζ_1 and ζ_2 (Definition 1.1). Note that C is independent of the cut-off functions.

CONCLUSION. There is a (new) constant C such that for all standard cut-off functions ζ_1 and ζ_2 supported in U

$$(A.4) \qquad \|\zeta_1 L f\|_r \leqslant C d^{-c} \|\zeta_1\|_c \big(M(\zeta_2 T f) + G(f) \big)$$

for all functions f in the given class for which $\|\zeta_2 L f\|_r < \infty$. The constant C is independent of the cut-off functions and of f.

PROOF. Fix cut-off functions ζ_1 and ζ_2 for which it is desired to prove (A.4). Also fix a nonnegative smooth function ϕ supported in the unit ball and with integral equal to one. For each positive integer j set

$$\phi_j(x) = \big(2^{j+2} d^{-1} \big)^N \phi \big(2^{j+2} d^{-1} x \big).$$

Let η_j denote the convolution of ϕ_j with the indicator function of the set of points in U at distance at least $2^{-j}d$ from the support of $(1 - \zeta_2)$. Then each $\eta_j \in C_0^\infty(U)$, the function η_1 is identically equal to one in a neighborhood of the support of ζ_1, each η_{j+1} is identically equal to one in a neighborhood of the support of η_j, and ζ_2 is identically equal to one in a neighborhood of the support of each η_j.

The hypothesis (A.3) implies that

$$
(A.5) \qquad \|\eta_j Lf\|_r \leqslant Cd^{-c}2^{jc}\Big(M(\zeta_2 Tf) + G(f) + \|\eta_{j+1}Lf\|_{r-\varepsilon}\Big)
$$

where C now depends on derivatives of the mollifier ϕ, but not on j nor on f. Since L is continuous on $W^{-\infty}(\Omega)$ there is m' such that

$$
(A.6) \qquad \|Lf\|_{-m'} \leqslant C\|f\|_{-m}.
$$

By a standard fact about Sobolev norms [1] there is C such that, when $0 < \lambda < 1$,

$$
(A.7) \qquad \|g\|_{r-\varepsilon} \leqslant C\big(\lambda\|g\|_r + \lambda^{-c}\|g\|_{-m'}\big).
$$

Combining the last three inequalities gives

$$
(A.8) \qquad \|\eta_j Lf\|_r \leqslant Cd^{-c}2^{jc}\Big(M(\zeta_2 Tf) + \lambda^{-c}G(f) + \lambda\|\eta_{j+1}Lf\|_r\Big)
$$

where C depends on L and m through (A.6) and on r and ε through (A.7) but is independent of λ, j, and f.

Assuming without loss of generality that the C in (A.8) is larger than $|r|$ plus the dimension N, define λ in terms of this C by

$$
\lambda = (2C)^{-1}d^c 2^{-jc - c(c+1)}
$$

and write

$$
N(j) = 2^{-jc(c+1)}\|\eta_j Lf\|_r
$$

to obtain

$$
(A.9) \qquad N(j) \leqslant \tfrac{1}{2}N(j+1) + \big[Cd^{-c}2^{c(c+1)}\big]^{c+1}\big[M(\zeta_2 Tf) + G(f)\big]
$$

with the same C as in (A.8). Since $\zeta_2 Lf \in W^r(\Omega)$ by hypothesis, it follows that $N(j) \to 0$ when $j \to \infty$. Therefore (A.9) implies

$$
N(j) \leqslant 2\big[Cd^{-c}2^{c(c+1)}\big]^{c+1}\big[M(\zeta_2 Tf) + G(f)\big]
$$

for all j. In particular setting j equal to 1 gives

$$
\|\eta_1 Lf\|_r \leqslant Cd^{-c}\big(M(\zeta_2 Tf) + G(f)\big)
$$

with a new C. Since η_1 is identically equal to 1 on the support of ζ_1, the conclusion (A.4) follows.

Appendix B: harmonic functions in $\mathbf{R}^N$. This appendix contains explicit statements and proofs of some facts about Sobolev norms of harmonic functions for which there does not seem to be a suitable reference, although some of the results are implicit in [26]. The applications in the body of this paper are only to holomorphic functions, but nonetheless it seems more natural to state the estimates for harmonic functions. The astute reader will realize that similar results hold for elliptic operators

other than the Laplacian, but since this generalization leads far afield from the main goal I do not consider it here. In a previous paper [9] I stated and used without proof a number of the results of this appendix.

Three basic principles are central to the discussion. The first is that in questions of duality it suffices to pair harmonic functions with functions of compact support. This idea comes from work of Bell and of Straube, and I include some remarks on Bell's Φ^s and Straube's T^k operators. The second principle is the well known fact that boundary and interior norms of harmonic functions are equivalent, with a shift of one-half degree of differentiability. The third principle is that to estimate the Sobolev norm of a harmonic function it suffices to consider derivatives in the direction normal to the boundary.

Throughout this appendix Ω denotes a smooth bounded domain in $\mathbf{R}^N$. The Sobolev spaces are defined as in §1. The basic estimate of elliptic theory says that if $r \geqslant 0$ then

$$(\mathrm{B.1}) \qquad \|f\|_{r+2} \leqslant C\big(\|\Delta f\|_r + \|\|f\|\|_{r+3/2}\big)$$

for all f in $C^\infty(\overline{\Omega})$. It will be useful to have a version of this estimate for negative r. Lions and Magenes introduced a new norm in [26] to study elliptic theory in W^{-s} spaces. A consequence of their theory is the key inequality

$$(\mathrm{B.2}) \qquad C^{-1}\|u\|_r \leqslant \|\|u\|\|_{r-1/2} \leqslant C\|u\|_r, \qquad u \text{ harmonic,}$$

for *every* real number r. However I prefer to extend (B.1) by duality, using the norm $\|\ \|^*$ defined in (1.1).

In the following lemma the first statement generalizes the elliptic estimate (B.1), the second statement is a converse, and the third compares the norm (1.1) to the usual one.

LEMMA B.1. *For every real number r there is a constant C such that*

$$(\mathrm{B.3}) \qquad \|f\|_r^* \leqslant C\big(\|\|f\|\|_{r-1/2} + \|\Delta f\|_{r-2}^*\big);$$

$$(\mathrm{B.4}) \qquad \|\|f\|\|_{r-1/2} \leqslant C\big(\|f\|_r + \|\Delta f\|_{r-2}^*\big);$$

$$(\mathrm{B.5}) \qquad \|f\|_r^* \leqslant C\big(\|f\|_r + \|\Delta f\|_{r-2}^*\big)$$

for all f in $C^\infty(\overline{\Omega})$.

For harmonic functions estimate (B.5) reduces to the statement

$$(\mathrm{B.6}) \qquad \|u\|_{-r}^* \leqslant C\|u\|_{-r} \quad \text{when } r \geqslant 0 \text{ and } u \text{ is harmonic.}$$

The existence of such an inequality was first discovered by Bell [2] for holomorphic functions. Its proof for harmonic functions results from the following ideas of Bell and Straube.

In [3] Bell observed that the harmonic Bergman projection $\mathbf{B}_{\mathrm{har}}$ from $L^2(\Omega)$ onto the subspace of harmonic functions is bounded from $W^r(\Omega)$ into itself when $r \geqslant 0$. Moreover he constructed a linear differential operator Φ^s (called L^s in [3]) of order

$s(s + 3)/2$ with $C^\infty(\overline{\Omega})$ coefficients such that for all f in $C^\infty(\overline{\Omega})$

(B.7) $\mathbf{B}_{\mathrm{har}}\Phi^s f = \mathbf{B}_{\mathrm{har}} f$;

(B.8) $\Phi^s f$ vanishes to order s at $b\Omega$, that is, $\Phi^s f \in W_0^s(\Omega)$;

(B.9) $\Phi^s f$ is a sum of terms of the form $b\rho^k D^\alpha f$ where ρ is the defining function for the domain, $b \in C^\infty(\overline{\Omega})$, and $|\alpha| = k$.

Bell's construction can be modified in the following way to obtain such an operator of order only $2s$. Let ψ be a smooth function equal to $|\nabla\rho|^{-1}$ in a neighborhood of $b\Omega$. Let Φ^0 denote the identity and define Φ^s recursively by the formula

$$\Phi^{s+1} = \Phi^s f - (s + 2)^{-1}(s + 1)^{-1}\Delta\big[(\Phi^s f)(\psi\rho)^2\big].$$

Then property (B.9) is obvious. Property (B.7) holds since if F is any smooth function then integration by parts implies that $\Delta(\rho^2 F)$ is orthogonal to the harmonic functions. To verify (B.8) observe that, modulo tangential derivatives, the principal part of the Laplacian at the boundary equals the second derivative in the normal direction.

I claim that Φ^s extends to be a bounded operator from $\mathrm{Har}\,W^r(\Omega)$ into $W^r(\Omega)$ when $r \geqslant 0$. The corresponding observation for the analogous operator on holomorphic functions is due to Bell [2], but in [3] he did not check this property for the harmonic Φ^s. It suffices by a density argument to prove the following a priori estimate.

LEMMA B.2. *For every nonnegative real number r and every positive integer s there is a constant C such that $\|\Phi^s u\|_r \leqslant C\|u\|_r$ for all harmonic functions u in $C^\infty(\overline{\Omega})$.*

By interpolation it suffices to prove the result for integral r. In view of property (B.9) of Φ^s it suffices to prove the following estimate.

LEMMA B.3. *For every nonnegative integer k there is a constant C such that $\|\rho^k u\|_k \leqslant C\|u\|_0$ for all harmonic functions u in $C^\infty(\overline{\Omega})$.*

PROOF. When $k = 0$ there is nothing to prove. When $k = 1$ integration by parts and the harmonicity of u give

$$\|\rho u\|_1^2 = \|\rho u\|_0^2 + \sum_{j=1}^N \int_\Omega D_j(\rho u) D_j(\rho\bar{u})$$

$$= \|\rho u\|_0^2 + \sum_{j=1}^N \int_\Omega (D_j\rho)^2 |u|^2 \leqslant C\|u\|_0^2.$$

Suppose now that the result is known for all nonnegative integers less than k, where $k \geqslant 2$. Then

$$\|\rho^k u\|_k \leqslant C\|\Delta(\rho^k u)\|_{k-2} \leqslant C\big(\|\rho^{k-1}u\|_{k-1} + \|\rho^{k-2}u\|_{k-2}\big) \leqslant C\|u\|_0.$$

Therefore the lemma holds by induction on k.

Both E. J. Straube and P. D. Smith have communicated to me alternative proofs and generalizations of Lemma B.3. Also E. Ligocka [38] has found proofs of Lemmas B.2, B.3, and B.4.

In [33, 34] Straube constructed for each nonnegative integer k an operator T^k such that $\mathbf{B}_{\mathrm{har}} T^k = \mathbf{B}_{\mathrm{har}}$ and such that T^k maps $W^r(\Omega)$ continuously into itself for every nonnegative r and maps $W^r(\Omega)$ continuously into $W_0^r(\Omega)$ when $0 \leqslant r \leqslant k$. In view of the previous discussion an alternative way to define T^k is $T^k = \Phi^k \mathbf{B}_{\mathrm{har}}$. The point of these ideas of Bell and Straube is the proof of (B.6), which I now restate for emphasis.

LEMMA B.4. *For every nonnegative real number r there exists a constant C such that*

$$\|u\|_{-r} \leqslant \|u\|_{-r}^* \leqslant C\|u\|_{-r}$$

for all harmonic functions u in $C^\infty(\bar{\Omega})$.

PROOF. The left-hand side is immediate from the definitions. To prove the right-hand side observe that if $u \in \mathrm{Har}\, C^\infty(\bar{\Omega})$ and $g \in C^\infty(\bar{\Omega})$ then choosing k to be an integer greater than r gives

$$|\langle u, g\rangle| = |\langle u, T^k g\rangle| \leqslant \|u\|_{-r} \|T^k g\|_r \leqslant C\|u\|_{-r}\|g\|_r.$$

Taking the supremum over g in the unit ball of $W^r(\Omega)$ gives the result.

With this special result for harmonic functions in hand it is easy to prove the more general Lemma B.1. It is useful to introduce the solution operator Q for the Dirichlet problem for the Laplacian with zero boundary condition. That is, Qf vanishes on $b\Omega$ and $\Delta Qf = f$. By the standard elliptic estimate (B.1),

$$\|Qf\|_r \leqslant C\|f\|_{r-2} \quad \text{if } r \geqslant 2.$$

On the other hand, if $r \geqslant 0$ and f and g are in $C^\infty(\bar{\Omega})$, then

$$|\langle Qf, g\rangle| = |\langle f, Qg\rangle| \leqslant C\|f\|_{-r-2}^* \|Qg\|_{r+2} \leqslant C\|f\|_{-r-2}^* \|g\|_r,$$

whence

$$\|Qf\|_{-r}^* \leqslant C\|f\|_{-r-2}^* \quad \text{if } r \geqslant 0.$$

By interpolation [26, p. 76] it follows that

(B.10) $$\|Qf\|_r^* \leqslant C\|f\|_{r-2}^* \quad \text{for all real } r.$$

PROOF OF LEMMA B.1. Write $f = (f - Q\Delta f) + Q\Delta f$. By (B.10),

$$\|Q\Delta f\|_r^* \leqslant C\|\Delta f\|_{r-2}^*,$$

and since $f - Q\Delta f$ is harmonic Lemma B.4 and estimate (B.2) imply

$$\|f - Q\Delta f\|_r^* \leqslant C\|\|f - Q\Delta f\|\|_{r-1/2} = C\|\|f\|\|_{r-1/2}.$$

Thus (B.3) holds. Moreover (B.2) and (B.10) imply

$$\|\|f\|\|_{r-1/2} = \|\|f - Q\Delta f\|\|_{r-1/2} \leqslant C\|f - Q\Delta f\|_r$$

$$\leqslant C\big(\|f\|_r + \|\Delta f\|_{r-2}^*\big),$$

so (B.4) holds. Finally (B.5) is a trivial consequence of (B.3) and (B.4).

Using Lemma B.1 it is not hard to localize the estimates (B.2) and (B.6) stating that various norms of harmonic functions are equivalent.

LEMMA B.5. *For all real numbers r and m there is a constant C such that for all standard cut-off functions ζ_1 and ζ_2*

$$
(B.11) \qquad \|\zeta_1 u\|_r^* \leqslant Cd^{-c}\|\zeta_1\|_c\big(\|\|\zeta_2 u\|\|_{r-1/2} + \|u\|_{-m}\big),
$$

$$
(B.12) \qquad \|\|\zeta_1 u\|\|_{r-1/2} \leqslant Cd^{-c}\|\zeta_1\|_c\big(\|\zeta_2 u\|_r + \|u\|_{-m}\big),
$$

$$
(B.13) \qquad \|\zeta_1 u\|_r^* \leqslant Cd^{-c}\|\zeta_1\|_c\big(\|\zeta_2 u\|_r + \|u\|_{-m}\big)
$$

for all harmonic functions u in $C^\infty(\overline{\Omega})$, where d is the distance between ζ_1 and ζ_2 (Definition 1.1).

PROOF. Put f equal to $\zeta_1 u$ in Lemma B.1. Then it remains only to estimate the error term $\Delta(\zeta_1 u)$. Since $\Delta u = 0$ it follows by iterative use of (B.5) that

$$
\big\|\Delta(\zeta_1 u)\big\|_{r-2}^* \leqslant Cd^{-c}\|\zeta_1\|_c\left(\|\zeta_2 u\|_r + \sum_{|\alpha|\leqslant q}\big\|\eta D^\alpha u\big\|_{r-2q}^*\right)
$$

where η is a cut-off function that is equal to 1 in a neighborhood of the support of ζ_1. Taking q so large that $r - q < -m$ and invoking the global estimate of Lemma B.4 gives

$$
\big\|\Delta(\zeta_1 u)\big\|_{r-2}^* \leqslant Cd^{-c}\|\zeta_1\|_c\big(\|\zeta_2 u\|_r + \|u\|_{-m}\big).
$$

This proves (B.12) and (B.13).

In view of (B.13) it is enough to prove (B.11) without the asterisk. To do this use the iteration trick of Appendix A with L and T equal to the identity, G equal to $\|\ \|_{-m}$, the norm M equal to $\|\|\ \|\|_{r-1/2}$, the open set U equal to $\mathbf{R}^N$, the number ε equal to $\frac{1}{2}$, and the subspace of $W^{-\infty}(\Omega)$ equal to $\text{Har}\,C^\infty(\overline{\Omega})$. By the elliptic estimate (B.3) and the estimate (B.13) already proved,

$$
\|\zeta_1 u\|_r \leqslant C\big(\|\|\zeta_1 u\|\|_{r-1/2} + \big\|\Delta(\zeta_1 u)\big\|_{r-2}^*\big)
$$

$$
\leqslant Cd^{-c}\|\zeta_1\|_c\big(\|\|\zeta_2 u\|\|_{r-1/2} + \|\zeta_2 u\|_{r-1/2} + \|u\|_{-m}\big).
$$

Hence (B.11) follows by the iteration trick.

Next consider derivatives of harmonic functions. Each derivative $(\partial/\partial x_j)$ maps $W^r(\Omega)$ continuously into $W^{r-1}(\Omega)$ except when $r = \frac{1}{2}$. The point of the next lemma is that restricting attention to harmonic functions eliminates the exception.

LEMMA B.6. *Let L be a linear differential operator of order k with $C^\infty(\overline{\Omega})$ coefficients. For all real numbers r and m there is a constant C such that for all standard cut-off functions ζ_1 and ζ_2*

$$
(B.14) \qquad \|\zeta_1 Lu\|_r \leqslant Cd^{-c}\|\zeta_1\|_c\big(\|\zeta_2 u\|_{r+k} + \|u\|_{-m}\big),
$$

$$
(B.15) \qquad \|\|\zeta_1 Lu\|\|_r \leqslant Cd^{-c}\|\zeta_1\|_c\big(\|\|\zeta_2 u\|\|_{r+k} + \|\|u\|\|_{-m}\big)
$$

for all harmonic functions u.

PROOF. By a density argument it is enough to consider u in $\operatorname{Har} C^\infty(\overline{\Omega})$. By induction it suffices to consider k equal to 1. By the remark preceding the lemma only the case of r equal to $-\frac{1}{2}$ in (B.14) is nontrivial; but since (B.14) and (B.15) are equivalent (by Lemma B.5), the result will be proved as soon as the case r equal -1 of (B.15) is verified.

If L is tangential at the boundary there is nothing to do, so the lemma reduces to showing that

$$\text{(B.16)} \qquad \||\zeta_1 D_\nu u\||_{-1} \leqslant Cd^{-c}\|\zeta_1\|_c \big(\||\zeta_2 u\||_0 + \||u\||_{-m}\big)$$

where D_ν is the directional derivative in the direction normal to the boundary. If $v \in \operatorname{Har} C^\infty(\overline{\Omega})$ then

$$\text{(B.17)} \qquad \int_{b\Omega} \bar{v}\zeta_1 D_\nu u = \int_{b\Omega} u D_\nu(\zeta_1 \bar{v}) - \int_\Omega u \Delta(\zeta_1 \bar{v})$$

by Green's theorem. Consider for a moment the special case of this identity with ζ_1 identically equal to 1 on Ω. It yields

$$\left| \int_{b\Omega} \bar{v} D_\nu u \right| \leqslant \||u\||_0 \||D_\nu v\||_0 \leqslant C\||u\||_0 \||v\||_1,$$

the second step following by the case r equals 0 of (B.15), which is equivalent to the known r equals $\frac{1}{2}$ case of (B.14). Taking the supremum over v in the unit ball of $\operatorname{Har} W^1(b\Omega)$ gives the global case of (B.16). Now return to (B.17) to obtain

$$\left| \int_{b\Omega} \bar{v}\zeta_1 D_\nu u \right| \leqslant Cd^{-c}\|\zeta_1\|_c \big(\||\zeta_2 u\||_0 + \||u\||_{-m}\big)\big(\||v\||_1 + \||D_\nu v\||_0\big).$$

Invoking the global result just proved and taking the supremum over v gives (B.16).

To compute the norm of a harmonic function it is enough to consider tangential derivatives, since the vanishing of the Laplacian makes it possible to convert normal derivatives to tangential ones. It is not quite so clear that the converse is true: it is enough to consider normal derivatives. That is the point of the next result.

LEMMA B.7. *For all real numbers r and m and every positive integer k there is a constant C such that for all standard cut-off functions ζ_1 and ζ_2*

$$\|\zeta_1 u\|_r \leqslant Cd^{-c}\|\zeta_1\|_c \big(\|\zeta_2 D_\rho^k u\|_{r-k} + \|u\|_{-m}\big)$$

for all harmonic functions u, where $D_\rho = \Sigma(\partial\rho/\partial x_j)(\partial/\partial x_j)$ and d is the distance between ζ_1 and ζ_2.

PROOF. Passing to the limit over interior approximating domains shows that there is no loss of generality in assuming that $u \in C^\infty(\overline{\Omega})$. It also suffices to show that

$$\text{(B.18)} \qquad \|\zeta_1 Lu\|_r \leqslant Cd^{-c}\|\zeta_1\|_c \big(\|\zeta_2 D_\rho Lu\|_{r-1} + \|\zeta_2 u\|_{r+k-2} + \|u\|_{-m}\big)$$

for an arbitrary linear differential operator L of order $k - 1$ with $C^\infty(\overline{\Omega})$ coefficients: the result then follows by the iteration trick of Appendix A and induction on k.

By the generalized elliptic estimate (B.3) and the previous two lemmas

(B.19) $\|\zeta_1 L u\|_r \leqslant C\left[\|\|\zeta_1 L u\|\|_{r-1/2} + d^{-c}\|\zeta_1\|_c\left(\|\zeta_2 u\|_{r+k-2} + \|u\|_{-m}\right)\right].$

Now for v in $\operatorname{Har} C^{\infty}(\overline{\Omega})$

$$\left|\int_{b\Omega} |\nabla\rho| \bar{v}\zeta_1 L u\right| = \left|\int_{\Omega} \bar{v}(\Delta\rho)\zeta_1 L u + \bar{v}\rho\Delta(\zeta_1 L u) + 2\bar{v}D_\rho(\zeta_1 L u)\right|$$

$$\leqslant Cd^{-c}\|\zeta_1\|_c\left(\|\zeta_2 u\|_{r+k-2} + \|u\|_{-m}\right)\left(\|v\|_{-r+1} + \|\rho v\|_{-r+2}^*\right)$$

$$+ C\|\zeta_1 D_\rho L u\|_{r-1}^*\|v\|_{-r+1}.$$

But by the general elliptic estimate

$$\|\rho v\|_{-r+2}^* \leqslant C\|\Delta(\rho v)\|_{-r}^* \leqslant C\|v\|_{-r+1}$$

and

$$\|\zeta_1 D_\rho L u\|_{r-1}^* \leqslant C\left(\|\zeta_1 D_\rho L u\|_{r-1} + \|\Delta(\zeta_1 D_\rho L u)\|_{r-3}^*\right)$$

$$\leqslant Cd^{-c}\|\zeta_1\|_c\left(\|\zeta_2 D_\rho L u\|_{r-1} + \|\zeta_2 u\|_{r+k-2} + \|u\|_{-m}\right),$$

so taking the supremum over v in the unit ball of $\operatorname{Har} W^{-r+1}(\Omega)$ gives

$$\|\|\zeta_1 L u\|\|_{r-1/2} \leqslant Cd^{-c}\|\zeta_1\|_c\left(\|\zeta_2 D_\rho L u\|_{r-1} + \|\zeta_2 u\|_{r+k-2} + \|u\|_{-m}\right).$$

Inserting this inequality in (B.19) gives the required estimate (B.18).

It is an easy corollary that the special operator $\mathbf{L}_\epsilon$ of (1.3) also determines the norm of a harmonic function. For simplicity of notation I omit the subscript on $\mathbf{L}$ when $\epsilon = 0$.

LEMMA B.8. *For all real numbers r and m there is a constant C such that*
(a) *for all sufficiently small positive ϵ,*

$$\|u\|_{W^r(\Omega_\epsilon)} \leqslant C\left(\|\mathbf{L}_\epsilon u\|_{W^{r-1}(\Omega_\epsilon)} + \|u\|_{W^{-m}(\Omega_\epsilon)}\right)$$

for all harmonic functions u in $C^{\infty}(\overline{\Omega}_\epsilon)$; and
(b) *for all standard cut-off functions ζ_1 and ζ_2*

$$\|\zeta_1 u\|_r \leqslant Cd^{-c}\|\zeta_1\|_c\left(\|\zeta_2 \mathbf{L} u\|_{r-1} + \|u\|_{-m}\right)$$

for all harmonic functions u in $C^{\infty}(\overline{\Omega})$.

PROOF. To prove (b), let η be a cut-off function intermediate between ζ_1 and ζ_2, constructed as in Appendix A. By the previous lemma

$$\|\zeta_1 u\|_r \leqslant Cd^{-c}\|\zeta_1\|_c\left(\|\eta D_\rho u\|_{r-1} + \|u\|_{-m}\right).$$

By the definition (1.3) of $\mathbf{L}$,

$$\|\eta D_\rho u\|_{r-1} \leqslant C\left(\|\eta \mathbf{L} u\|_{r-1} + \|u\|_{W^r(\omega)} + \|\eta\rho\nabla u \cdot \nabla\psi\|_{r-1} + \|\eta u\|_{r-1}\right),$$

where ω is a relatively compact subdomain of Ω containing the set where ψ vanishes. But $\|u\|_{W^r(\omega)} \leqslant C\|u\|_{-m}$ and by the general elliptic estimate (B.3) and Lemmas B.5 and B.6

$$\|\eta\rho\nabla u \cdot \nabla\psi\|_{r-1} \leqslant C\|\Delta(\eta\rho\nabla u \cdot \nabla\psi)\|_{r-3}^* \leqslant Cd^{-c}\left(\|\zeta_2 u\|_{r-1} + \|u\|_{-m}\right).$$

Combining these inequalities gives

$$\|\zeta_1 u\|_r \leqslant Cd^{-c}\|\zeta_1\|_c\big(\|\zeta_2 \mathbf{L}u\|_{r-1} + \|\zeta_2 u\|_{r-1} + \|u\|_{-m}\big),$$

so the result follows by the iteration trick of Appendix A.

Part (a) is a consequence. Indeed since the elliptic estimates hold uniformly on Ω_ε for small ε (see §8.1 of Chapter 2 of [26]) it is clear that the previous lemmas hold uniformly on Ω_ε. From the proof one sees that (b) also holds uniformly on interior approximating subdomains.

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