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A GLOBAL APPROACH TO THE RANKIN-SELBERG CONVOLUTION FOR GL(3, Z)

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ABSTRACT. We discuss the Rankin-Selberg convolution on GL(3, **Z**) in the 'classical' language of symmetric spaces and automorphic forms.

Introduction. Study of the Rankin-Selberg convolution of two automorphic forms on GL(2) has yielded many interesting number theoretic applications. Recently, this construction has been extended to automorphic forms on GL(n) by Jacquet, Piatetskii-Shapiro, and Shalika (see [**8**] for the local results). Their methods generalize those of Jacquet-Langlands; in particular, they make considerable use of the representation theory over local fields, as developed by Bernstein, Zelevinsky, and others. In this paper we shall give a discussion of their results in a classical, nonadelic, language in the simplest higher rank situation: that of two automorphic forms on GL(3, **Z**).

We must emphasize that most of the ideas in this paper are not really new, but simply restatements in this more classical language of those of Jacquet, Piatetskii-Shapiro, and Shalika; further, a classical sketch of the Rankin-Selberg method has been given by Jacquet [**7**]. However, we are able to sharpen these results at the archimedean place by giving the precise gamma factors at infinity, and also the behavior of the convolution at its poles (see also [**10**]). These are of particular interest because of two applications which require this precise version of the convolution.

The first, noted by Moreno-Shahidi [**30**] and Serre [**17**] (see also [**14**]), obtains the coefficient bound

$$|a(p)| < p^{1/5} + p^{-1/5}$$

for the size of the Fourier coefficient of a GL(2) Maass wave form which is also a Hecke eigenform (here the normalization is such that the Ramanujan-Petersson conjecture predicts $|a(p)| < 2$). This follows by combining the convolution with the Gelbart-Jacquet lifting [**3**] from GL(2) to GL(3) and a result of Chandrasekharan-Narasimhan [**2**] on the order of partial L series. The second is given in the thesis of

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G. Gilbert [4], who obtains multiplicity one theorems based on knowledge of the $a(p)$ for p in a Frobenius class of an extension; the explicit convolution allows him to give effective results in low rank cases (see also Moreno [12, 13]).

Now, let us describe the Rankin-Selberg Dirichlet series for GL(3). Let $\mathscr{Z}$ be the center of GL(3, **R**), and

$$H = \mathrm{GL}(3, \mathbf{R})/\mathscr{Z}O(3).$$

Then from the Iwasawa decomposition one sees that the symmetric space H may be regarded as the set of (cosets)

$$\tau = \begin{pmatrix} y_1 y_2 & y_1 x_2 & x_3 \\ & y_1 & x_1 \\ & & 1 \end{pmatrix} \tag{0.1}$$

with $x_1, x_2, x_3 \in \mathbf{R}$, $y_1, y_2 \in \mathbf{R}_{>0}$; we shall use this parametrization throughout. Left matrix multiplication induces an action of GL(3, **R**) on H, which we write as $\circ$.

An *automorphic form* on GL(3, **Z**) is a left-GL(3, **Z**) invariant function ϕ: $H \to \mathbf{C}$ satisfying certain differential equations and growth conditions. Such a form has a Fourier expansion; in the case of a cusp form, it is given by

$$\phi(\tau) = \sum_{g \in \Gamma^2_\infty \backslash \Gamma^2} \sum_{m,\, n=1}^{\infty} a_{m,n} (mn)^{-1} W\left(\begin{pmatrix} mn & & \\ & m & \\ & & 1 \end{pmatrix} g \circ \tau \right),$$

where

$$\Gamma^2 = \left\{ \begin{pmatrix} a & b & \\ c & d & \\ & & 1 \end{pmatrix} \middle| a, b, c, d \in \mathbf{Z},\ ad - bc = \pm 1 \right\},$$

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & b & e \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{SL}(3, \mathbf{Z}) \right\},$$

$$\Gamma^2_\infty = \Gamma^2 \cap \Gamma_\infty,$$

and $W(\tau)$ is a certain GL(3) Whittaker function, which depends only on the differential equations for ϕ (see §1 for details). The numbers $a_{m,n}$ are called the *Fourier coefficients* of ϕ; the factor $(mn)^{-1}$ is included for convenience.

Now let ϕ be as above, and φ be another GL(3, **Z**) form, with Fourier coefficients $b_{m,n}$. Then the Rankin-Selberg Dirichlet series is given by

$$D(s, \phi, \varphi) = \sum_{m,n=1}^{\infty} a_{m,n} \bar{b}_{m,n} m^{-2s} n^{-s}.$$

It converges absolutely for Re(s) sufficiently large.

The main properties of this series are described in Theorems 3.2, 3.4 (meromorphic continuation and functional equation under $s \to 1 - s$, $(m, n) \to (n, m)$), and 4.5 (Euler product) below. First, in §1, we review the basic information about GL(3, **Z**) automorphic forms which is needed in the sequel—Fourier expansions, Hecke operators, and the like—and also discuss briefly the standard minimal parabolic Eisenstein series. This section is based on the thesis of Daniel Bump, and I would like to thank him for providing me with prepublication access to this work

[1]. Then, in §2, the standard maximal parabolic Eisenstein series of type (2,1), $E(s,\tau)$, is introduced, and its properties are summarized. Next, in §3, we show how the Dirichlet series $D(s,\phi,\varphi)$ arises by integrating against $E(s,\tau)$, and obtain the basic properties of the Rankin-Selberg convolution from this. Finally, in §4 the Euler product for $D(s,\phi,\varphi)$ is derived from GL(3) Hecke theory, and used to explicitly evaluate the gamma factors 'at infinity.' This is accomplished modulo one hypothesis—that the Rankin-Selberg method extends to noncusp forms—which is not verified here.

Though we have confined ourselves to the case of $GL(3,\mathbf{Z})$ in this note, both for simplicity, and to take advantage of the results of [1], it is possible to extend this approach to the other cases considered in [7, 8].

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1. Automorphic forms on $GL(3,\mathbf{Z})$. Let $\Gamma = GL(3,\mathbf{Z}) = \{\gamma \in M(3,\mathbf{Z}) \mid \det\gamma = \pm 1\}$, $\Gamma_1 = SL(3,\mathbf{Z})$, and ν_1, ν_2 be complex numbers. Also, let $\mathscr{D}$ denote the center of the universal enveloping algebra of $GL(3,\mathbf{R})$, acting as an algebra of $GL(3,\mathbf{R})$-invariant differential operators on H.

DEFINITION 1.1. An *automorphic form on* Γ *of type* (ν_1,ν_2) is a function $\phi\colon H \to \mathbf{C}$ such that

(1) $\phi(\gamma\circ\tau) = \phi(\tau)$ for all $\gamma \in \Gamma$, $\tau \in H$,

(2) ϕ is an eigenfunction of $\mathscr{D}$ with the same eigenvalues as the function

$$I_{\nu_1,\nu_2}(\tau) = y_1^{2\nu_1+\nu_2} y_2^{\nu_1+2\nu_2},$$

(3) ϕ grows at most polynomially in y_1, y_2 as y_1, $y_2 \to \infty$, uniformly in x_1, x_2, and x_3.

DEFINITION 1.2. A *cusp form* is an automorphic form which satisfies

$$\int_0^1\int_0^1 \phi\left(\begin{pmatrix} 1 & & \xi_3 \\ & 1 & \xi_1 \\ & & 1 \end{pmatrix}\circ\tau\right) d\xi_1\, d\xi_3 = 0,$$

$$\int_0^1\int_0^1 \phi\left(\begin{pmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \\ & & 1 \end{pmatrix}\circ\tau\right) d\xi_2\, d\xi_3 = 0.$$

EXAMPLE 1.3. The *Eisenstein series* of type (ν_1,ν_2) associated to a minimal parabolic is given by

$$E_{\nu_1,\nu_2}(\tau) = \tfrac{1}{4}\zeta(3\nu_1)\zeta(3\nu_2)\zeta(3\nu_1+3\nu_2-1)\cdot \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} I_{\nu_1,\nu_2}(\gamma\circ\tau)$$

(where ζ is the Riemann zeta function) if $\mathrm{Re}(\nu_1)$, $\mathrm{Re}(\nu_2) > \frac{2}{3}$, and by analytic continuation in ν_1, ν_2 (based on the action of the Weyl group) for other values of ν_1, ν_2 (see [1] for details). It is an automorphic form for Γ of type (ν_1,ν_2), but not a cusp form. We shall give another example of a Γ-automorphic form, the maximal parabolic Eisenstein series, in §2 below.

Now set

$$\Gamma_1^2 = \Gamma^2 \cap \Gamma_1, \quad \mathbf{e}[x] = \exp(2\pi i x), \quad w_1 = \begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix},$$

and put

$$(1.1)\qquad W_{\nu_1,\nu_2}(\tau) = \int_{\mathbf{R}^3} I_{\nu_1,\nu_2}\left(w_1 \begin{pmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1 \end{pmatrix} \circ \tau \right) \mathbf{e}[-\xi_1 - \xi_2]\, d\xi_1\, d\xi_2\, d\xi_3$$

if $\operatorname{Re}(\nu_1), \operatorname{Re}(\nu_2) > \frac{1}{3}$. Explicitly,

$$(1.2)\qquad I_{\nu_1,\nu_2}(w_1 \circ \tau) = I_{\nu_1,\nu_2}(\tau)\left(x_3^2 + x_2^2 y_1^2 + y_1^2 y_2^2\right)^{-3\nu_1/2} \cdot \left((x_1x_2 - x_3)^2 + x_1^2 y_2^2 + y_1^2 y_2^2\right)^{-3\nu_2/2}.$$

As shown in [**1**], W_{ν_1,ν_2} too has a meromorphic continuation to all values of (ν_1, ν_2), which we again write W_{ν_1,ν_2}, based on the action of the Weyl group of GL(3) on (ν_1, ν_2) (see also [**20**]).

Proposition 1.4 (Shalika). *Let ϕ be an automorphic form of type (ν_1, ν_2) on Γ. Then ϕ has a Fourier expansion given by*

$$(1.3)\qquad \phi(\tau) = \sum_{n=-\infty}^{\infty} \phi_{0,n}(\tau) + \frac{1}{2} \sum_{\gamma \in \Gamma_\infty^2 \backslash \Gamma_1^2} \sum_{\substack{m,\, n=-\infty \\ m \neq 0}}^{\infty} \phi_{m,n}(\gamma \circ \tau),$$

where

$$(1.4)\qquad \phi_{m,n}(\tau) = \int_0^1 \int_0^1 \int_0^1 \phi\left(\begin{pmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1 \end{pmatrix} \circ \tau \right) \mathbf{e}[-m\xi_1 - n\xi_2]\, d\xi_1\, d\xi_2\, d\xi_3.$$

If ϕ is a cusp form, this may be simplified to

$$(1.5)\qquad \phi(\tau) = \sum_{\gamma \in \Gamma_\infty^2 \backslash \Gamma^2} \sum_{m,\, n=1}^{\infty} a_{m,n}(mn)^{-1} W_{\nu_1,\nu_2}\left(\begin{pmatrix} mn & & \\ & m & \\ & & 1 \end{pmatrix} \gamma \circ \tau \right).$$

The proof of this result, due to Shalika, may be found in Bump [**1**, Chapter 4] (we have also used his formula (3.16) and changed the notation slightly) and Shalika [**18**, Theorem 5.9]. A similar simplification may be given in the case of noncusp forms as well.

Given an automorphic form ϕ of type (ν_1, ν_2) on Γ, one can construct a form $\tilde{\phi}$ of type (ν_2, ν_1) as follows. Define an involution ι of $GL(3, \mathbf{R})$ by

$$^{\iota}g = w_1\, {}^{t}g^{-1} w_1;$$

ι induces an involution of H:

$$^{\iota}\tau = \begin{pmatrix} y_2 y_1 & -y_2 x_1 & x_1 x_2 - x_3 \\ & y_2 & -x_2 \\ & & 1 \end{pmatrix}.$$

Define the function $\tilde{\phi}$ by

$$\tilde{\phi}(\tau) = \phi({}^t\tau);$$

if ϕ is an automorphic form of type (ν_1, ν_2), with Fourier coefficients $a_{m,n}$, then one can verify that $\tilde{\phi}$ is an automorphic form of type (ν_2, ν_1), with Fourier coefficients $a_{n,m}$. The involution ~ will play a key role in the functional equation for the convolution.

Let us conclude this section with a brief discussion of L-series and GL(3) Hecke theory. For Re(s) sufficiently large, put

$$L(s, \phi) = \sum_{n=1}^{\infty} a_{1,n} n^{-s},$$

so that

$$L(s, \tilde{\phi}) = \sum_{m=1}^{\infty} a_{m,1} m^{-s}.$$

PROPOSITION 1.5. *Suppose that* ϕ *is an eigenfunction of the Hecke algebra, normalized so that* $a_{1,1} = 1$. *Then*

(1) $L(s, \phi)$, $L(s, \tilde{\phi})$ *have Euler products given by*

$$L(s, \phi) = \prod_p \left(1 - a_{1,p} p^{-s} + a_{p,1} p^{-2s} - p^{-3s}\right)^{-1},$$

$$L(s, \tilde{\phi}) = \prod_p \left(1 - a_{p,1} p^{-s} + a_{1,p} p^{-2s} - p^{-3s}\right)^{-1},$$

where $\prod_p$ *denotes the product over all primes* p.

(2)

$$a_{r,1} a_{m,n} = \sum_{\substack{tuv = r \\ u \mid m \\ t \mid n}} a_{mv/u, nu/t}.$$

For a proof, see [**1**]. Additional references for Hecke theory are [**23, 24, 25, 26**].

For example, in the case of the Eisenstein series E_{ν_1,ν_2} of Example 1.3 above, one has

PROPOSITION 1.6.

$$L\left(s, E_{\nu_1,\nu_2}\right) = \zeta(s + 1 - 2\nu_1 - \nu_2)\zeta(s + \nu_1 - \nu_2)\zeta(s - 1 + \nu_1 + 2\nu_2).$$

There is also a meromorphic continuation of $L(s, \phi)$ to the entire s plane, as well as a functional equation relating $L(s, \phi)$ to $L(1 - s, \tilde{\phi})$ (see [**1**] for details). Also, for an adelic, representation theoretic approach to automorphic forms on GL(3), see [**9**].

2. Maximal parabolic Eisenstein series. In this section we briefly give the properties of the Eisenstein series associated to the standard maximal parabolic of type $(2, 1)$, $E(s, \tau)$. Let $\hat{\Gamma}$ be the group

$$\hat{\Gamma} = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SL}(3, \mathbf{Z}) \right\}.$$

Note that for g in $\hat{\Gamma}$,

$$\det(g \circ \tau) = \det(\tau); \tag{2.1}$$

this is actually implied by the proof of Lemma 2.2 below, but is also easy to see directly. $E(s,\tau)$ is given by

$$E(s,\tau) = \sum_{\gamma \in \hat{\Gamma}\backslash\Gamma_1} [\det(\gamma \circ \tau)]^s;$$

by (2.1), this is well defined. It is clearly a Γ-automorphic form.

LEMMA 2.1. *The cosets of* $\hat{\Gamma} \setminus \Gamma_1$ *are in one-to-one correspondence with the relatively prime triples of integers via the map*

$$\hat{\Gamma}\gamma \to \textit{last row of } \gamma.$$

PROOF. The map is clearly well defined and injective, and is surjective since every relatively prime triple can be completed to a matrix in $\mathrm{SL}(3,\mathbf{Z})$. □

LEMMA 2.2. *Let*

$$\gamma = \begin{pmatrix} * & * & * \\ * & * & * \\ a & b & c \end{pmatrix}$$

be a representative for a coset of $\hat{\Gamma} \setminus \Gamma_1$. *Then*

$$\det(\gamma \circ \tau) = \frac{\det(\tau)}{\left[y_1^2|az_2 + b|^2 + (ax_3 + bx_1 + c)^2\right]^{3/2}},$$

where $z_2 = x_2 + iy_2$, *and one takes the positive square root.*

PROOF. Write

$$\gamma\tau = \tau' k(rI_3), \tag{2.2}$$

where τ' in $\mathrm{GL}(3,\mathbf{R})$ is of shape (0.1), k is in $O(3)$, r is real, and I_3 denotes the 3×3 identity. Then, by comparing the bottom rows of both sides of (2.2), and using $k \in O(3)$, one sees that

$$r^2 = (ay_1y_2)^2 + (ay_1x_2 + by_1)^2 + (ax_3 + bx_1 + c)^2.$$

But taking determinants in (2.2) gives

$$\det(\tau) = \det(\tau')|r|^3,$$

so the result follows. □

PROPOSITION 2.3. *The Eisenstein series* $E(s,\tau)$ *converges absolutely for* $\mathrm{Re}(s) > 1$.

PROOF. By Lemmas 2.1 and 2.2,

$$E(s,\tau) = \sum \frac{(y_1^2y_2)^s}{\left[y_1^2|az_2 + b|^2 + (ax_3 + bx_1 + c)^2\right]^{3s/2}}, \tag{2.3}$$

where the sum is over all relatively prime triples of integers (a,b,c). Application of the integral test then gives the result. □

For convenience, introduce the series

$$E^*(s,\tau) = \zeta(3s)E(s,\tau);$$

$E^*(s,\tau)$ is given by the right-hand side of (2.3), with the sum taken over *all* triples of integers $(a,b,c) \neq (0,0,0)$.

THEOREM 2.4. *The Eisenstein series* $E^*(s,\tau)$ $(\mathrm{Re}(s) > 1)$ *has a Fourier expansion given by* (1.3), *with*

$$\phi_{m,n}(\tau) = \begin{cases} \dfrac{4\left(y_1^{1/2}y_2\right)^s}{\Gamma(3s/2)} y_1^{1/2}\pi^{3s/2}|m|^{(3s-1)/2}\sigma_{1-3s}(|m|) \\ \qquad \cdot K_{(3s-1)/2}(2\pi|m|y_1)\mathbf{e}[mx_1], \qquad m \neq 0,\ n = 0, \\ \dfrac{4y_1^{1-s}y_2^{1-(s/2)}}{\Gamma(3s/2)}\pi^{3s/2}|n|^{(3s/2)-1}\sigma_{2-3s}(|n|) \\ \qquad \cdot K_{(3s/2)-1}(2\pi|n|y_2)\mathbf{e}[nx_2], \qquad m = 0,\ n \neq 0, \\ 2y_1^{2s}y_2^{s}\zeta(3s) + 2y_1^{1-s}y_2^{s}\zeta(3s-1)\Gamma\left(\dfrac{3s-1}{2}\right)\pi^{1/2}/\Gamma\left(\dfrac{3s}{2}\right) \\ \qquad + 2y_1^{1-s}y_2^{2-2s}\zeta(3s-2)\pi\Gamma\left(\dfrac{3s}{2}-1\right)/\Gamma\left(\dfrac{3s}{2}\right), \qquad m = n = 0, \end{cases}$$

and $\phi_{m,n}(\tau)$ *identically zero when* $mn \neq 0$ (*here* K *denotes the modified K-Bessel function of the third kind*,

$$K_s(y) = \frac{1}{2}\int_0^\infty e^{-y(t+1/t)/2}t^{s-1}\,dt \qquad (y > 0),$$

and σ *denotes the divisor function* $\sigma_s(n) = \sum_{0<d|n} d^s$).

PROOF. This follows by breaking the sum (2.3) for $E^*(s,\tau)$ into two pieces (the terms $a = 0$, $a \neq 0$), using the Fourier expansion of a GL(2) Eisenstein series to compute the $a = 0$ piece, and [**5**, formulas 3.276(2), 8.432(5)] and Poisson summation to evaluate the $a \neq 0$ terms (see Terras [**28**] for details). □

COROLLARY 2.5. *The Eisenstein series* $E(s,\tau)$ *can be meromorphically continued to the entire* s *plane, and satisfies the functional equation*

$$\pi^{-3s/2}\Gamma\left(\frac{3s}{2}\right)\zeta(3s)E(s,\tau) = \pi^{-3(1-s)/2}\Gamma\left(\frac{3(1-s)}{2}\right)\zeta(3(1-s))E(1-s,{}^{\iota}\tau).$$

Further,

$$\pi^{-3s/2}\Gamma\left(\frac{3s}{2}\right)\zeta(3s)E(s,\tau)$$

is holomorphic except for simple poles at $s = 0, 1$, *of residues* $-\frac{2}{3}$, $\frac{2}{3}$ *respectively.*

The analytic continuation and functional equation of Eisenstein series has been established in far greater generality by Selberg [**22**] and Langlands [**11**]. It can also be easily derived here since $E(s,\tau)$ is an Epstein zeta function (cf. Terras [**26, 27, 28**]). Additional references for the Fourier expansions of Eisenstein series include [**6, 21, 28, 29**].

3. The Rankin-Selberg integral. In this section we use the Rankin-Selberg method **[15, 16]** to study the properties of the Dirichlet series $D(s, \phi, \varphi)$. To do this, we first need

LEMMA 3.1. *The* GL(3, **R**)*-invariant (Haar) measure on H is given by*

$$d^H\tau = dx_1\, dx_2\, dx_3\, dy_1\, dy_2/(y_1 y_2)^3.$$

Now let ϕ and φ be two cusp forms for Γ, of types (ν_1, ν_2) and (μ_1, μ_2), and with Fourier coefficients $a_{m,n}$, $b_{m,n}$ respectively. By Lemma 3.1, the *Rankin-Selberg integral*

$$\int_{\Gamma \backslash H} \phi(\tau)\, \overline{\varphi(\tau)}\, E(s,\tau)\, d^H\tau$$

is well defined. For later use, let us examine the more general integral

$$I(s,\phi,\varphi,\xi_1,\xi_2) = \int_{\Gamma\backslash H} \phi(\tau)\overline{\varphi}\left(\tau\begin{pmatrix}1 & \xi_2 & \\ & 1 & \xi_1 \\ & & 1\end{pmatrix}\right) E(s,\tau)\, d^H\tau,$$

where ξ_1, ξ_2 are real. This integral is evaluated by

THEOREM 3.2. *For* $\operatorname{Re}(s)$ *sufficiently large*,

$$I(s,\phi,\varphi,\xi_1,\xi_2) = G(s,\nu_1,\nu_2,\mu_1,\mu_2,\xi_1,\xi_2) \sum_{m,\, n=1}^{\infty} a_{m,n}\overline{b}_{m,n} m^{-2s} n^{-s},$$

where

(3.1)

$$G(s,\nu_1,\nu_2,\mu_1,\mu_2,\xi_1,\xi_2) = \int_0^\infty \int_0^\infty W_{\nu_1,\nu_2}\left(\begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1\end{pmatrix}\right) \cdot \overline{W}_{\mu_1,\mu_2}\left(\begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1\end{pmatrix}\right) \mathbf{e}[-\xi_1 y_1 - \xi_2 y_2]\,(y_1^2 y_2)^s \frac{dy_1\, dy_2}{(y_1 y_2)^3}.$$

PROOF. First, since

$$E(s,\tau) = \frac{1}{2}\sum_{\gamma\in\hat{\Gamma}\backslash\Gamma} [\det(\gamma\circ\tau)]^s,$$

one sees by the usual 'unfolding' trick that

$$I(s,\phi,\varphi,\xi_1,\xi_2) = \frac{1}{2}\int_{\hat{\Gamma}\backslash H} \phi(\tau)\overline{\varphi}\left(\tau\begin{pmatrix}1 & \xi_2 & \\ & 1 & \xi_1 \\ & & 1\end{pmatrix}\right)(y_1^2 y_2)^s\, d^H\tau.$$

Let $\mathscr{F}$ be a fundamental domain for

$$\left\{\begin{pmatrix}1 & & e \\ & 1 & f \\ & & 1\end{pmatrix} \middle| e, f\in\mathbf{Z}\right\}\Big\backslash H.$$

Then we may identify $\hat{\Gamma}\backslash H$ with $\Gamma_1^2\backslash\mathscr{F}$. Further, the cosets of $\Gamma_1^2\backslash\mathscr{F}$ are in exactly two-to-one correspondence with the cosets of $\Gamma^2\backslash\mathscr{F}$. Since ϕ, φ, $[\det\tau]^s$, and $d^H\tau$ are all invariant under

$$\begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix},$$

one may rewrite the last line as

$$\int_{\Gamma^2\backslash\mathscr{F}} \phi(\tau)\overline{\varphi}\left(\tau\begin{pmatrix} 1 & \xi_2 & \\ & 1 & \xi_1 \\ & & 1 \end{pmatrix}\right)(y_1^2y_2)^s\, d^H\tau.$$

Substituting its Fourier expansion (1.5) for $\phi(\tau)$, and *again* using the unfolding trick, one sees that this in turn is equal to

$$\int_{\Gamma_\infty^2\backslash\mathscr{F}} \sum_{m,\,n=1}^{\infty} a_{m,n}(mn)^{-1}W_{\nu_1,\nu_2}\left(\begin{pmatrix} mn & & \\ & m & \\ & & 1 \end{pmatrix}\circ\tau\right)\overline{\varphi}\left(\tau\begin{pmatrix} 1 & \xi_2 & \\ & 1 & \xi_1 \\ & & 1 \end{pmatrix}\right)(y_1^2y_2)^s\, d^H\tau.$$

However,

$$\Gamma_\infty^2\backslash\mathscr{F}\cong\{\tau\in H\,|\,x_1,x_2,x_3\in(0,1), 0<y_1,y_2\in\mathbf{R}\}.$$

So, substituting in the Fourier expansion of φ, and switching the summations with the x_i-integrations ($i=1,2,3$) (justified by the absolute convergence of (1.5)), one obtains

$$\text{(3.2)}\qquad I(s,\phi,\varphi,\xi_1,\xi_2)=\int_0^\infty\int_0^\infty\sum_{m,\,n=1}^{\infty}\sum_{m',\,n'=1}^{\infty}a_{m,n}\bar{b}_{m',n'}(mnm'n')^{-1}$$
$$\cdot\sum_{\gamma\in\Gamma_\infty^2\backslash\Gamma^2}\int_0^1\int_0^1\int_0^1 W_{\nu_1,\nu_2}\left(\begin{pmatrix} mn & & \\ & m & \\ & & 1 \end{pmatrix}\circ\tau\right)$$
$$\cdot\overline{W}_{\mu_1,\mu_2}\left(\begin{pmatrix} m'n' & & \\ & m' & \\ & & 1 \end{pmatrix}\gamma\circ\tau\begin{pmatrix} 1 & \xi_2 & \\ & 1 & \xi_1 \\ & & 1 \end{pmatrix}\right)(y_1^2y_2)^s\, d^H\tau.$$

Now from its definition (1.1) and (1.2), one sees that

(3.3)

$$W\left(\begin{pmatrix} m'n' & & \\ & m' & \\ & & 1 \end{pmatrix}\circ\tau\begin{pmatrix} 1 & \xi_2 & \\ & 1 & \xi_1 \\ & & 1 \end{pmatrix}\right)$$
$$=\mathbf{e}[m'(x_1+\xi_1y_1)+n'(x_2+\xi_2y_2)]\,W\left(\begin{pmatrix} m'n' & & \\ & m' & \\ & & 1 \end{pmatrix}\circ\begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}\right).$$

Further, the action of

$$\gamma = \begin{pmatrix} * & * & \\ c & d & \\ & & 1 \end{pmatrix}$$

on τ sends x_1 to $cx_3 + dx_1$. But then (3.3) implies that the integrand in (3.2) corresponding to this choice of γ is $\mathbf{e}[m'cx_3]$ times a function *independent* of x_3. Integrating with respect to x_3, we see that a nonzero contribution occurs only when $c = 0$. Similar consideration of the x_1 and x_2 integrals reduces the sum to the case $\gamma = I_3$, $m = m'$, $n = n'$. So

(3.4)

$$I(s, \phi, \varphi, \xi_1, \xi_2) = \int_0^\infty \int_0^\infty \sum_{m,n=1}^\infty a_{m,n}\bar{b}_{m,n}(mn)^{-2} W_{\nu_1,\nu_2}\left(\begin{pmatrix} mny_1y_2 & & \\ & my_1 & \\ & & 1 \end{pmatrix}\right)$$

$$\cdot \overline{W}_{\mu_1,\mu_2}\left(\begin{pmatrix} mny_1y_2 & & \\ & my_1 & \\ & & 1 \end{pmatrix}\right)\mathbf{e}[-m\xi_1 y_1 - n\xi_2 y_2](y_1^2y_2)^s(y_1y_2)^{-3}\,dy_1\,dy_2.$$

Finally, it is easy to see that $D(s, \phi, \varphi)$, $G(s, \nu_1, \nu_2, \mu_1, \mu_2, \xi_1, \xi_2)$ converge absolutely for $\mathrm{Re}(s)$ sufficiently large. This allows one to perform the remaining interchanges of integration and summation in (3.4). Then making the change of variables $my_1 \to y_1$, $ny_2 \to y_2$ gives the result. □

REMARKS. (i) A similar argument shows that

$$\int_{\Gamma\backslash H} \phi(\tau)\bar{\varphi}\left(\tau\begin{pmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1 \end{pmatrix}\right) E(s, \tau)\, d^H\tau$$

is independent of ξ_3.

(ii) Note that by (1.1), (1.2), and (3.1), G is given for $\mathrm{Re}(s)$ sufficiently large as an integral over $(\mathbf{R}^+)^2 \times \mathbf{R}^6$. As we shall see below, G has a meromorphic continuation to the entire s plane, and is essentially a product of gamma functions.

(iii) Theorem 3.2 extends without change to the case where only one of ϕ and φ is a cusp form. Slightly modified, one should be able to extend it to the case where *neither* ϕ nor φ is a cusp form; the corresponding extension of the GL(2) Rankin-Selberg integral is given in Zagier [**19**] (the idea is to truncate the fundamental domain for $\Gamma \backslash H$). Since we do not give the details of this here, we assume it as hypothesis H.1 below.

For convenience in stating the next result, let

$$G^*(s, \nu_1, \nu_2, \mu_1, \mu_2, \xi_1, \xi_2) = \pi^{-3s/2}\Gamma\left(\frac{3s}{2}\right)G(s, \nu_1, \nu_2, \mu_1, \mu_2, \xi_1, \xi_2)$$

and

$$D^*(s, \phi, \varphi, \xi_1, \xi_2) = G^*(s, \nu_1, \nu_2, \mu_1, \mu_2, \xi_1, \xi_2)\zeta(3s)D(s, \phi, \varphi).$$

Also, let f be a function on $\mathbf{R}^2$ such that

$$(3.5)\quad \begin{aligned}&\text{(i)}\ f(\xi_1,\xi_2)=f(-\xi_2,-\xi_1),\\ &\text{(ii)}\ \hat{f}(y_1,y_2):=\int_{\mathbf{R}^2} f(\xi_1,\xi_2)\mathbf{e}[-\xi_1y_1-\xi_2y_2]\,d\xi_1\,d\xi_2\quad\text{is rapidly decreasing,}\end{aligned}$$

and set

$$G_f^*(s,\nu_1,\nu_2,\mu_1,\mu_2)=\int_{\mathbf{R}^2} G^*(s,\nu_1,\nu_2,\mu_1,\mu_2,\xi_1,\xi_2)f(\xi_1,\xi_2)\,d\xi_1\,d\xi_2,$$

$$D_f^*(s,\phi,\varphi)=G_f^*(s,\nu_1,\nu_2,\mu_1,\mu_2)\zeta(3s)D(s,\phi,\varphi).$$

Let us denote the GL(3)-Petersson inner product of two automorphic forms, at least one of which is a cusp form, by

$$\langle\phi,\varphi\rangle=\int_{\Gamma\backslash H}\phi(\tau)\overline{\varphi(\tau)}\,d^H\tau;$$

observe that $\langle\phi,\varphi\rangle=\langle\tilde{\phi},\tilde{\varphi}\rangle$. Then we have

PROPOSITION 3.3. *Let ϕ and φ be Γ-automorphic forms, of types (ν_1,ν_2), (μ_1,μ_2) respectively, at least one of which is a cusp form. Then*

(1) *$D^*(s,\phi,\varphi,\xi_1,\xi_2)$ has a meromorphic continuation to the entire s plane, and satisfies the functional equation*

$$D^*(s,\phi,\varphi,\xi_1,\xi_2)=D^*(1-s,\tilde{\phi},\tilde{\varphi},-\xi_2,-\xi_1).$$

(2) *$D^*(s,\phi,\varphi,0,0)$ is holomorphic when ϕ and φ are orthogonal, and otherwise is holomorphic except for simple poles at $s=0,1$, of residues $-\frac{2}{3}\langle\phi,\varphi\rangle$, $\frac{2}{3}\langle\phi,\varphi\rangle$, respectively.*

(3) *$D_f^*(s,\phi,\varphi)$ has a meromorphic continuation to the entire s plane, and satisfies the functional equation*

$$D_f^*(s,\phi,\varphi)=D_f^*(1-s,\tilde{\phi},\tilde{\varphi}).$$

(4) *There exists an f satisfying (3.5) such that $G_f^*(s,\nu_1,\nu_2,\mu_1,\mu_2)$ has an analytic continuation to the entire s plane which is never zero.*

PROOF. Observe that

$$\varphi\left({}^t\tau\begin{pmatrix}1&\xi_2&\\&1&\xi_1\\&&1\end{pmatrix}\right)=\tilde{\varphi}\left(\tau\begin{pmatrix}1&-\xi_1&\xi_1\xi_2\\&1&-\xi_2\\&&1\end{pmatrix}\right).$$

Hence (1), (2), and (3) follow by combining Corollary 2.5, Theorem 3.2, and Remark (i) above. As for (4), note that for $\mathrm{Re}(s)$ sufficiently large,

$$G_f^*(s,\nu_1,\nu_2,\mu_1,\mu_2)=\int_0^\infty\int_0^\infty \hat{f}(y_1,y_2)W_{\nu_1,\nu_2}\left(\begin{pmatrix}y_1y_2&&\\&y_1&\\&&1\end{pmatrix}\right)$$

$$\cdot\overline{W}_{\mu_1,\mu_2}\left(\begin{pmatrix}y_1y_2&&\\&y_1&\\&&1\end{pmatrix}\right)(y_1^2y_2)^s\frac{dy_1\,dy_2}{(y_1y_2)^3}.$$

Take a function f_1 whose Fourier transform is concentrated at a point in $[0, \infty] \times [0, \infty]$ where the Whittaker functions do not vanish (presumably they never do, but this is not needed). Symmetrize by setting

$$f(\xi_1, \xi_2) = f_1(\xi_1, \xi_2) + f_1(-\xi_2, -\xi_1).$$

Then it is clear that (4) holds for this f. □

The use of the extra variables ξ_1, ξ_2 was suggested to me by P. Sarnak.

Similar reasoning should give the meromorphic continuation and functional equation of $D^*(s, \phi, \varphi, \xi_1, \xi_2)$, $D_f^*(s, \phi, \varphi)$ even when neither ϕ nor φ is a cusp form (note, though, that the possibilities for the location and order of the poles are more diverse). Since we have not verified this here, we make it

HYPOTHESIS H.1. (1) *of Proposition* 3.3 *holds for* ϕ, φ *Eisenstein series.*

Note that (3) and (4) of Proposition 3.3 follow from Hypothesis H.1 in this case just as in the proof of Proposition 3.3. Making use of this, one can sharpen Proposition 3.3 by giving an explicit gamma factor. Namely, put

$$\begin{aligned}\Gamma^*(s, \nu_1, \nu_2, \mu_1, \mu_2) &= \pi^{-9s/2}\Gamma\left(\frac{s - 2\nu_1 - \nu_2 - 2\overline{\mu_1} - \overline{\mu_2} + 2}{2}\right)\\ &\cdot\Gamma\left(\frac{s - 2\nu_1 - \nu_2 - \overline{\mu_2} + \overline{\mu_1} + 1}{2}\right)\Gamma\left(\frac{s - 2\nu_1 - \nu_2 + \overline{\mu_2} + 2\overline{\mu_2}}{2}\right)\\ &\cdot\Gamma\left(\frac{s + \nu_1 - \nu_2 - 2\overline{\mu_1} - \overline{\mu_2} + 1}{2}\right)\Gamma\left(\frac{s + \nu_1 - \nu_2 - \overline{\mu_2} - \overline{\mu_1}}{2}\right)\\ &\cdot\Gamma\left(\frac{s + \nu_1 - \nu_2 + \overline{\mu_1} + 2\overline{\mu_2} - 1}{2}\right)\Gamma\left(\frac{s + \nu_1 + 2\nu_2 + 2\overline{\mu_1} - \overline{\mu_2}}{2}\right)\\ &\cdot\Gamma\left(\frac{s + \nu_1 + 2\nu_2 - \overline{\mu_2} + \overline{\mu_1} - 1}{2}\right)\Gamma\left(\frac{s + \nu_1 + 2\nu_2 + \overline{\mu_1} + 2\overline{\mu_2} - 2}{2}\right).\end{aligned}$$

We shall show

THEOREM 3.4. *Assume Hypothesis* H.1. *Let* ϕ *and* φ *be* Γ*-automorphic forms, of types* (ν_1, ν_2), (μ_1, μ_2) *respectively, at least one of which is a cusp form. Then*

$$\Gamma^*(s, \nu_1, \nu_2, \mu_1, \mu_2)\zeta(3s)D(s, \phi, \varphi)$$

has a meromorphic continuation to the entire s plane, which is holomorphic when ϕ *and* φ *are orthogonal, and otherwise is holomorphic except for simple poles at* $s = 0, 1$. *Further,* $\Gamma^*(s, \nu_1, \nu_2, \mu_1, \mu_2)\zeta(3s)D(s, \phi, \varphi)$ *satisfies the functional equation*

$$\begin{aligned}&\Gamma^*(s, \nu_1, \nu_2, \mu_1, \mu_2)\zeta(3s)D(s, \phi, \varphi)\\ &\quad = \Gamma^*(1 - s, \nu_2, \nu_1, \mu_2, \mu_1)\zeta(3(1 - s))D(1 - s, \tilde{\phi}, \tilde{\varphi}).\end{aligned}$$

Since the proof of Theorem 3.4 requires the Euler product for $D(s, \phi, \varphi)$, it is deferred to §4 below.

4. The Euler product for the convolution. Throughout this section assume that $a_{m,n}$, $b_{m,n}$ are the Fourier coefficients of normalized Hecke eigenforms ϕ, φ respectively (cf. §1). Also, write (m, n) for the greatest common divisor of m and n. The existence of an Euler product for $D(s, \phi, \varphi)$ is established by the following lemma.

LEMMA 4.1. *Let* $(mn, m'n') = 1$. *Then* $a_{mm',nn'} = a_{m,n}a_{m',n'}$.

PROOF. Induct on $mm'nn'$, using Proposition 1.5(2). □

COROLLARY 4.2. *Let*

$$D_p(s,\phi,\varphi) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{p^j,p^k}\bar{b}_{p^j,p^k}p^{-2js-ks}.$$

Then

$$D(s,\phi,\varphi) = \prod_p D_p(s,\phi,\varphi).$$

To simplify this, we have

LEMMA 4.3. $a_{p^j,p^k} = a_{p^j,1}a_{1,p^k} - a_{p^{j-1},1}a_{1,p^{k-1}}$.

PROOF. We again induct, this time on j. By Proposition 1.5(2), we have

$$a_{p^{j+1},p^k} = a_{p^{j+1},1}a_{1,p^k} - \sum_{t=1}^{\min(j+1,k)} a_{p^{j+1-t},p^{k-t}}.$$

Then, using the inductive hypothesis, one sees that the sum in the resulting expression telescopes. This gives the lemma. □

Combining Lemma 4.3 and Corollary 4.2, we see that the evaluation of the Euler product reduces to the computation of the two series

$$S_p(s,\phi,\varphi)_1 = \sum_{l=0}^{\infty} a_{1,p^l}\bar{b}_{1,p^l}p^{-ls},$$

$$S_p(s,\phi,\varphi)_2 = \sum_{l=0}^{\infty} a_{1,p^l}\bar{b}_{1,p^{l+1}}p^{-ls},$$

since one can obtain the remaining needed expressions from these by using the involution $\tilde{}$, and by reversing the roles of ϕ and φ. To give them, put

$$\text{(4.1)}\qquad \begin{aligned} 1 - a_{1,p}p^{-s} + a_{p,1}p^{-2s} - p^{-3s} &= \prod_{i=1}^{3}(1-\alpha_i p^{-s}),\\ 1 - b_{1,p}p^{-s} + b_{p,1}p^{-2s} - p^{-3s} &= \prod_{i=1}^{3}(1-\beta_i p^{-s}), \end{aligned}$$

and

$$L_p(s,\phi\otimes\varphi) = \prod_{i,j=1}^{3}\left(1-\alpha_i\bar{\beta}_j p^{-s}\right)^{-1}.$$

Then we have

PROPOSITION 4.4. (1)

$$S_p(s,\phi,\varphi)_1 = \Big[1 - a_{p,1}\bar{b}_{p,1}p^{-2s} + \left(a_{p,1}a_{1,p} + \bar{b}_{p,1}\bar{b}_{1,p} - 2\right)p^{-3s} - a_{1,p}\bar{b}_{1,p}p^{-4s} + p^{-6s}\Big]L_p(s,\phi\otimes\varphi).$$

(2)

$$S_p(s,\phi,\varphi)_2 = \Big[\bar{b}_{1,p} - a_{1,p}\bar{b}_{p,1}p^{-s} + \left(a_{1,p}^2 - a_{p,1}\right)p^{-2s} + \left(\bar{b}_{p,1}^2 - \bar{b}_{1,p}\right)p^{-3s} - a_{1,p}\bar{b}_{p,1}p^{-4s} + a_{p,1}p^{-5s}\Big]L_p(s,\phi\otimes\varphi).$$

PROOF. By Proposition 1.5, we have

$$a_{1,p^l} = \sum_{i+j+k=l} \alpha_1^i \alpha_2^j \alpha_3^k$$

and a similar expression for b_{1,p^l}. Thus

$$\begin{aligned} S_p(s,\phi,\varphi)_1 &= \sum_{l=0}^{\infty} \sum_{i+j+k=l} \alpha_1^i\alpha_2^j\alpha_3^k \sum_{i'+j'+k'=l} \bar\beta_1^{i'}\bar\beta_2^{j'}\bar\beta_3^{k'} p^{-ls} \\ &= \sum_{i,j,k=0}^{\infty} \sum_{i'=0}^{i+j+k} \sum_{j'=0}^{i+j+k-i'} \alpha_1^i\alpha_2^j\alpha_3^k\bar\beta_1^{i'}\bar\beta_2^{j'}\bar\beta_3^{i+j+k-i'-j'} p^{-(i+j+k)s}. \end{aligned}$$

Summing the geometric series here, simplifying, and making use of the relations (4.1), (1) follows. Part (2) is proved in a similiar way. □

We can now prove the main result of this section.

THEOREM 4.5. $\zeta(3s)D(s,\phi,\varphi) = \prod_p L_p(s,\phi\otimes\varphi)$.

PROOF. By Lemma 4.3 one sees that

(4.2)
$$\begin{aligned} D_p(s,\phi,\varphi) &= S_p(s,\phi,\varphi)_1 S_p(2s,\tilde\phi,\tilde\varphi)_1(1+p^{-3s}) \\ &\quad -p^{-3s}\left[S_p(s,\phi,\varphi)_2 S_p(2s,\tilde\phi,\tilde\varphi)_2 + \bar S_p(s,\varphi,\phi)_2 \bar S_p(2s,\tilde\varphi,\tilde\phi)_2\right]. \end{aligned}$$

Each of these terms is evaluated by the formulas of Proposition 4.4. After substituting these formulas into (4.2) and combining terms, one sees that $D_p(s,\phi,\varphi)$ is $L_p(s,\phi\otimes\varphi)L_p(2s,\tilde\phi\otimes\tilde\varphi)$ times a polynomial of degree 21 in p^{-s}. However, an explicit computation shows that this polynomial is exactly

$$(1-p^{-3s})L_p(2s,\tilde\phi\otimes\tilde\varphi)^{-1}.$$

Applying Corollary 4.2, the theorem is proved. □

REMARK. This computation can be further explained via a combinatorial identity involving certain Schur polynomials, since the Fourier coefficients a_{p^k,p^l} may be expressed by these polynomials [**1, 24**]; this allows its generalization to GL_n.

For example, in the case of the Eisenstein series E_{ν_1,ν_2}, E_{μ_1,μ_2} studied by Bump, we see from Proposition 1.6 that

(4.3)
$$\begin{aligned} \prod_p L_p(s, E_{\nu_1,\nu_2}\otimes E_{\mu_1,\mu_2}) &= \zeta(s-2\nu_1-\nu_2-2\bar\mu_1-\bar\mu_2+2) \\ &\cdot\zeta(s-2\nu_1-\nu_2-\bar\mu_2+\bar\mu_1+1)\zeta(s-2\nu_1-\nu_2+\bar\mu_1+2\bar\mu_2) \\ &\cdot\zeta(s+\nu_1-\nu_2-2\bar\mu_1-\bar\mu_2+1)\zeta(s+\nu_1-\nu_2-\bar\mu_2+\bar\mu_1) \\ &\cdot\zeta(s+\nu_1-\nu_2+\bar\mu_1+2\bar\mu_2-1)\zeta(s+\nu_1+2\nu_2-2\bar\mu_1-\bar\mu_2) \\ &\cdot\zeta(s+\nu_1+2\nu_2-\bar\mu_2+\bar\mu_1-1)\zeta(s+\nu_1+2\nu_2+\bar\mu_1+2\bar\mu_2-2). \end{aligned}$$

However, $G_f^*(s, \nu_1, \nu_2, \mu_1, \mu_2)$ depends only on $f, s, \nu_1, \nu_2, \mu_1$, and μ_2, and *not* on ϕ and φ. Hence (4.3), together with the analytic continuation of the Riemann zeta function, its functional equation, and Theorem 4.5, implies Theorem 3.4. Note that the ξ_1 and ξ_2 variables guarantee, by virtue of Proposition 3.3(4), that no extra poles are introduced when one replaces $G_f^*(s, \nu_1, \nu_2, \mu_1, \mu_2)$ by $\Gamma^*(s, \nu_1, \nu_2, \mu_1, \mu_2)$.

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