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WHITNEY CONTINUA OF CURVES

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ABSTRACT. In this paper, we prove several theorems relating shape properties of Whitney continua of curves. In particular, we investigate the fundamental dimension and the shape type of Whitney continua of curves.

1. Introduction. By a continuum, we mean a compact connected metric space. Let X be a continuum. Then the hyperspace $C(X)$ of subcontinua of X is metrized with the Hausdorff metric (e.g., see [**15**]). In [**27**], Whitney showed that for any continuum X there exists a map $\omega\colon C(X) \to [0, \omega(X)]$ satisfying

(1) $\omega(\{x\}) = 0$ for every $x \in X$, and

(2) if $A, B \in C(X)$ and $A \subsetneqq B$, then $\omega(A) < \omega(B)$.

Any such map is called a *Whitney map.* We may think of the map ω as measuring the size of a continuum. It is well known that every Whitney map ω is monotone, i.e., $\omega^{-1}(t)$ is a continuum for $0 \leq t \leq \omega(X)$. The continua $\omega^{-1}(t)$ $(0 \leq t < \omega(X))$ are called *Whitney continua.* A topological property P is called a *Whitney property* if whenever X has property P, so does every Whitney continua $\omega^{-1}(t)$ $(0 \leq t < \omega(X))$ for any Whitney map $\omega\colon C(X) \to [0, \omega(X)]$. It has been shown that several geometric properties are not Whitney properties for n-dimensional continua $(n \geq 2)$. For example, in [**17**], Petrus showed that the property of being (a) an AR or (b) an FAR is not a Whitney property, more precisely, there is a Whitney map $\omega\colon C(D) \to [0, \omega(D)]$ such that D is a disk and, for some $0 < t < \omega(D)$, $\omega^{-1}(t)$ is not an FAR, in fact, $\omega^{-1}(t)$ is not even an FANR (see [**8**, (1.12)]). But, for the case of curves (=1-dimensional continua) there are many geometric properties which are Whitney properties (see References). In partricular, Whitney continua of chainable continua and circle-like continua have been studied by many authors (see References). Krasinkiewicz [**10**] proved that the property of being (a) chainable or (b) proper circle-like is a Whitney property. Also, he proved that if X is a circle-like continuum, then $\operatorname{Sh} \omega^{-1}(t) = \operatorname{Sh} X$ for any Whitney map ω for $C(X)$ and $0 \leq t < \omega(X)$ (see [**11**]). In [**24**], Rogers showed that for any continuum X and any Whitney map ω for $C(X)$, there is an induced injection $r^*\colon H^1(\omega^{-1}(t)) \to H^1(X)$ for $0 \leq t \leq \omega(X)$. By using this result, he investigated Whitney continua of curves (see [**23, 24**]). In [**8**], the author proved that the property of being an FAR is a Whitney property for curves.

The aim of this paper is to investigate several shape properties of Whitney continua of curves. In §2, we study the fundamental properties of Whitney continua of graphs. In particular, we define an index $n(G)$ for a graph G and we prove

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that if $n(G) \leq m$, then $\operatorname{Fd} \omega^{-1}(t) \leq m-1$ for any Whitney map ω for $C(G)$ and $0 \leq t \leq \omega(G)$. In §3, we define natural shape morphisms between Whitney continua, and by using the shape morphisms we investigate shape properties of Whitney continua of curves. We prove that if X is a strongly winding curve, then $\operatorname{Sh} \omega^{-1}(t) = \operatorname{Sh} X$ for any Whitney map ω for $C(X)$ and $0 \leq t < \omega(X)$. Every tree-like continuum and every circle-like continuum are strongly winding curves. In §4, by using the technique of Rogers [**24**, Theorem 3], we prove that if X is a $\theta(m)$-curve and each proper subcontinuum of X is tree-like, then $\operatorname{Sh} \omega^{-1}(t) = \operatorname{Sh} X$ for any Whitney map ω for $C(X)$ and $0 \leq t < \omega(X)$.

We refer readers to [**9** and **15**] for hyperspace theory, and to [**1** and **13**] for shape theory.

2. Whitney continua of graphs. In [**3** and **4**], Duda described and analyzed polyhedral models for hyperspaces of graphs. We assume that the word *graph* means a finite connected 1-dimensional polyhedron. In this section, we investigate fundamental properties of Whitney continua of graphs. Let G be a graph with a triangulation T. For any points x, y of G, which belong to 1-simplex $\langle V_0, V_1 \rangle \in T$, define $d(x,y) = |t - t'|$, where $x = tV_0 + (1-t)V_1$, $y = t'V_0 + (1-t')V_1$. Let A be a subcontinuum of G. For $x, y \in A$ we define a metric d_A on A by

$$d_A(x,y) = \inf \left\{ \sum_{i=1}^{m} d(x_i, x_{i+1}) \mid x_0 = x,\ x_{m+1} = y,\ x_i \text{ and } x_{i+1}\ (0 \leq i \leq m) \right.$$
$$\text{belong to the 1-simplex } \sigma \text{ of } T \text{ such that the segment from}$$
$$\left. x_i \text{ to } x_{i+1} \text{ in } \sigma \text{ is contained in } A \text{ for } 0 \leq i \leq m \right\}.$$

Note that for $x_n, y_n \in A_n \in C(G)$ with $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, and $\lim_{n\to\infty} A_n = A$, we cannot conclude that $d_A(x,y) = \lim_{n\to\infty} d_{A_n}(x_n, y_n)$. But it is easily seen that if A contains no simple closed curve (= a tree), we can conclude that $d_A(x,y) = \lim_{n\to\infty} d_{A_n}(x_n, y_n)$. Then we have

(2.1) PROPOSITION. (1) *Let G be a graph which contains a simple closed curve and let ω be any Whitney map for $C(G)$. Let $t_0 = \min\{\omega(S) | S$ is a simple closed curve in $G\}$. Then there is a homotopy $H: \omega^{-1}([0,t_0)) \times I \to \omega^{-1}([0,t_0))$ such that*

(a) *$H(\{x\} \times I) = \{x\}$ for each $x \in G$,*

(b) *$H(A,0) = A$, $H(A,1) \in w^{-1}(0)$ for each $A \in \omega^{-1}([0,t_0))$, and*

(c) *if $s < s'$, then $H(A,s) \supsetneq H(A,s')$ for each $A \in \omega^{-1}((0,t_0))$.*

(2) [**6**, (2.17) *or* **17**, *Proposition* 12] *Let G be a tree and let ω be any Whitney map for $C(G)$. Then there is a homotopy $H: C(G) \times I \to C(G)$ such that*

(a′) *$H(\{x\} \times I) = \{x\}$ for each $x \in G$,*

(b′) *$H(A,0) = A$, $H(A,1) \in w^{-1}(0)$ for each $A \in C(G)$, and*

(c′) *if $s < s'$, then $H(A,s) \supsetneq H(A,s')$ for each $A \in \omega^{-1}((0,\omega(G)])$.*

PROOF. We shall prove (1). Since A contains no simple closed curve for each $A \in \omega^{-1}([0,t_0))$, we can choose an open covering $\mathfrak{U} = \{\langle U_{\lambda_1}, U_{\lambda_2}, \dots, U_{\lambda_k} \rangle | (\lambda_1, \lambda_2, \dots, \lambda_k) \in J\}$ of $\omega^{-1}([0,t_0))$ such that $\bigcup_{j=1}^{k} \overline{U}_{\lambda_j}$ is a subcontinuum of G and contains no simple closed curve for each $\langle U_{\lambda_1}, U_{\lambda_2}, \dots, U_{\lambda_k} \rangle \in \mathfrak{U}$, where U_{λ_j} is a connected open subset of G and

$$\langle U_{\lambda_1}, U_{\lambda_2}, \ldots, U_{\lambda_k}\rangle = \left\{ A \in \omega^{-1}([0,t_0)) | A \cap U_{\lambda_j} \neq \varnothing \ (j = 1, 2, \ldots, k) \text{ and } A \subset \bigcup_{j=1}^{k} U_{\lambda_j} \right\}.$$

Choose a locally finite closed covering $\mathfrak{V} = \{\mathfrak{A}\}$ of $\omega^{-1}([0,t_0))$ which is a refinement of $\mathfrak{U}$. Now, we shall prove that if $\mathfrak{A} \in \mathfrak{V}$ and if $\mathfrak{B}$ is a closed subset of $\mathfrak{A}$ and $f: \mathfrak{B} \to G$ is a map such that $f(A) \in A$ for each $A \in \mathfrak{B}$, then there exists an extension $\tilde{f}: \mathfrak{A} \to G$ of f such that $\tilde{f}(A) \in A$ for each $A \in \mathfrak{A}$. In fact, take a $\langle U_{\lambda_1}, U_{\lambda_2}, \ldots, U_{\lambda_k}\rangle \in \mathfrak{U}$ such that $\mathfrak{A} \subset \langle U_{\lambda_1}, U_{\lambda_2}, \ldots, U_{\lambda_k}\rangle$. Since $\bigcup_{j=1}^{k} \overline{U}_{\lambda_j}$ is an AR, there is an extension $f': \mathfrak{A} \to \bigcup_{j=1}^{k} \overline{U}_{\lambda_j}$ of f (note that $f(\mathfrak{B}) \subset \bigcup_{j=1}^{k} \overline{U}_{\lambda_j}$). Also, since $\bigcup_{j=1}^{k} \overline{U}_{\lambda_j}$ is a tree, for each $A \in \mathfrak{A}$, there is the unique point $\tilde{f}(A)$ of A such that

$$[f'(A), \tilde{f}(A)] \cap A = \{\tilde{f}(A)\},$$

where $[f'(A), \tilde{f}(A)]$ denotes the unique arc from $f'(A)$ to $\tilde{f}(A)$ in $\bigcup_{j=1}^{k} \overline{U}_{\lambda_j}$.

Clearly, the function $\tilde{f}$ is an extension of f and it is continuous such that $\tilde{f}(A) \in A$ for each $A \in \mathfrak{A}$. By using this fact, we can easily see that there is a map $h: \omega^{-1}([0,t_0)) \to G$ such that $h(A) \in A$ for each $A \in \omega^{-1}([0,t_0))$.

Next, we shall define a homotopy $H: \omega^{-1}([0,t_0)) \times I \to \omega^{-1}([0,t_0))$ satisfying the desired conditions. Let $A \in \omega^{-1}([0,t_0))$. Define a homotopy $H_A: A \times I \to A$ by $H_A(x,s) = y$ for $x \in A$, $s \in I$, where $y \in [x, h(A)]$ and $d_A(x,y) = s \cdot d_A(x, h(A))$. Then $H_A(x,0) = x$ and $H_A(x,1) = h(A)$ for $x \in A$. Also, define a function $H: \omega^{-1}([0,t_0)) \times I \to \omega^{-1}([0,t_0))$ by

$$H(A,s) = \{H_A(x,s) | x \in A\}.$$

It is easily seen that H is continuous and H has the desired conditions. This completes the proof.

(2.2) REMARK. In the statement of (1) of (2.1), there is no homotopy $H: \omega^{-1}([0,t_0]) \times I \to \omega^{-1}([0,t_0])$ satisfying the conditions (a), (b), and (c) of (1). Suppose, on the contrary, that such a homotopy H exists. Let S be a simple closed curve such that $\omega(S) = t_0$. By (c), $H(C(S) \times I) \subset C(S)$. By (a) and (b), $F_1(S) = \{\{x\} | x \in S\} \cong S$ is a strong deformation retract of $C(S)$. Since $C(S)$ is a disk, this implies a contradiction.

(2.3) PROPOSITION. *Let G be a graph and let ω be any Whitney map for $C(G)$. If G contains a simple closed curve, assume $t_0 = \min\{\omega(S) | S$ is a simple closed curve in $G\}$. If G is a tree, assume that $t_0 = w(G)$. Then $\omega^{-1}(t)$ $(0 \leq t < t_0)$ is homotopy equivalent to G, i.e., $\omega^{-1}(t) \simeq G$.*

PROOF. We shall prove only the case that G contains a simple closed curve. The case that G is a tree has been proved in [**17**]. Since G is a Peano continuum, there is a convex metric ρ on G. Define a homotopy $K: C(G) \times I \to C(G)$ by $K(A,s) = B(A, \alpha(A,s))$, where $\alpha(A,s)$ is the positive number such that $B(A, \alpha(A,s)) = \{y \in G | \rho(A,y) \leq \alpha(A,s)\}$ and $\omega(B(A, \alpha(A,s))) = (1-s) \cdot \omega(A) + s \cdot \omega(G)$. Let $0 \leq t < t_0$. Define a map $f_{0t}: G \to \omega^{-1}(t)$ by $f_{0t}(x) = K(\{x\}, \beta(x))$, where

$\omega(K(\{x\},\beta(x))) = t$. By using the homotopy H as in (2.1) and the homotopy K, we can easily see that f_{0t} is a homotopy equivalence (cf. [**7**, (2.5)]). Hence $\omega^{-1}(t) \simeq G$.

(2.4) PROPOSITION. *Let G be a graph. Then for any Whitney map ω for $C(G)$, $\omega^{-1}(t)$ is a polyhedron for each $0 \le t \le \omega(G)$.*

SKETCH OF THE PROOF. The proof is very similar to the proof of [**3**; **4**, (6.2) and (6.4)]. Let $A \subset B$ be a pair as in [**3**, p. 270] and let $\mathfrak{M}_{A\subset B}$ be the same family of subcontinua of X as in [**3**, p. 270]. Set $\mathfrak{M}_{A\subset B}(t) = \mathfrak{M}_{A\subset B} \cap \omega^{-1}(t)$. By similar arguments as in the proofs of [**3**, (5.2) and **4**, (6.2)], we obtain the following (see [**4**, (6.2)]): Let $A \subset B$ and $A' \subset B'$ be two distinct pairs of G. Then $\mathfrak{M}_{A\subset B}(t)$ and $\mathfrak{M}_{A'\subset B'}(t)$ are balls. Moreover, if $\mathfrak{M}_{A\subset B}(t)$ and $\mathfrak{M}_{A'\subset B'}(t)$ meet, then their common part either is equal to a ball or consists of finitely many disjoint balls, each lying on the surfaces of both $\mathfrak{M}_{A\subset B}(t)$ and $\mathfrak{M}_{A'\subset B'}(t)$. Hence we see that $\omega^{-1}(t)$ is a polyhedron.

Next, we need the following result of Lynch [**14**]. The author wishes to thank Professor D. Curtis for informing him of the result.

(2.5) (M. LYNCH [**14**]). *Let X be any continuum and $A \in C(X)$. Then for any Whitney map ω for $C(X)$, the set*

$$C_A(X,\omega,t) = \{B \in \omega^{-1}(t) | B \supset A\}$$

is an AR for $0 \le t \le \omega(X)$, where $\omega(A) \le t$.

The author wishes to thank Mr. K. Kawamura for pointing out the graph $G(m)$ as in the following proposition.

(2.6) PROPOSITION. *Let $G = G(m) = \bigcup_{i=1}^m A_i$ $(m \ge 2)$, where each A_i $(i = 1,2,\dots,m)$ is an arc from V_1 to V_2 and $A_i \ne A_j$ $(i \ne j)$. Assume that $A_i \cap A_j = \{V_1, V_2\}$ $(i \ne j)$. Let ω be any Whitney map for $C(G)$ and let*

$$t_0 = \max\{w(A_1 \cup A_2 \cup \cdots \cup A_{i-1} \cup A_{i+1} \cup \cdots \cup A_m) | i = 1,2,\dots,m\} < \omega(G).$$

Then $\omega^{-1}(t) \simeq S^{m-1}$ for $t_0 \le t < \omega(G)$, where S^{m-1} denotes the $(m-1)$-sphere.

PROOF. Let $t_0 \le t < \omega(G)$. Since $t \ge \omega(A_1 \cup A_2 \cup \cdots \cup A_{i-1} \cup A_{i+1} \cup \cdots \cup A_m)$ and $G - \{V_1, V_2\}$ is not connected, we have

(1) $\omega^{-1}(t) = C_{V_1}(G,\omega,t) \cup C_{V_2}(G,\omega,t)$, which is the union of two ARs (see (2.5)).

Also, we have

(2) $C_{V_1}(G,\omega,t) \cap C_{V_2}(G,\omega,t) = \bigcup_{i=1}^m C_{A_i}(G,\omega,t)$, which is the union of m ARs (see (2.5)).

Now, we shall prove that $C_{V_1}(G,\omega,t) \cap C_{V_2}(G,\omega,t) \simeq S^{m-2}$. Note that

(3) $\bigcap_{j\in J} C_{A_j}(G,\omega,t) = C_{(\bigcup_{j\in J} A_j)}(G,\omega,t)$ for each subset J of $\{1,2,\dots,m\}$.

By (3) and the definition of t_0, we have

(4) if $|J| \le m-1$, then $\bigcap_{j\in J} C_{A_j}(G,\omega,t)$ in an AR (nonempty) and $\bigcap_{j=1}^m C_{A_j}(G,\omega,t) = C_G(G,\omega,t) = \emptyset$.

Let Δ be the $(m-1)$-simplex with vertices $a_1, a_2, \dots, a_m$ and let $\partial\Delta$ denote the boundary of Δ. Consider the barycentric subdivision $\mathrm{Sd}\,\partial\Delta$ and the decomposition $\{\mathrm{St}(a_i; \mathrm{Sd}\,\partial\Delta) | i = 1,2,\dots,m\}$ of $\partial\Delta$ (see Figure 1).

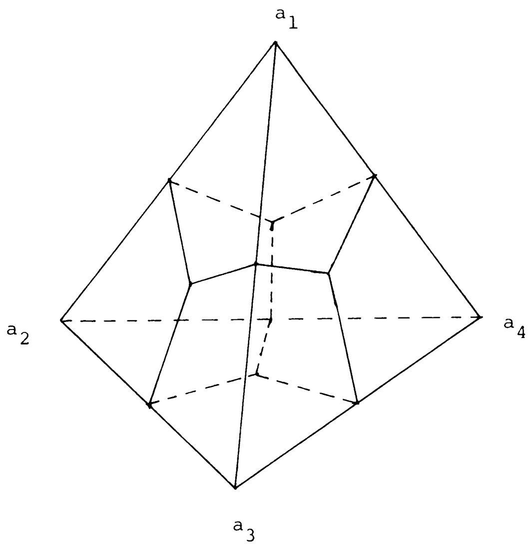

FIGURE 1

By (4), we can easily see that there exists a map $f: \partial\Delta \to C_{V_1}(G,\omega,t) \cap C_{V_2}(G,\omega,t)$ such that

(5) $f(\bigcap_{j\in J} \mathrm{St}(a_j; \mathrm{Sd}\,\partial\Delta)) \subset \bigcap_{j\in J} C_{A_j}(G,\omega,t)$ for each subset J of $\{1,2,\ldots,m\}$.

Also, we have a map $g: C_{V_1}(G,\omega,t) \cap C_{V_2}(G,\omega,t) \to \partial\Delta$ such that

(6) $g(\bigcap_{j\in J} C_{A_j}(G,\omega,t)) \subset \bigcap_{j\in J} \mathrm{St}(a_j; \mathrm{Sd}\,\partial\Delta)$ for each subset J of $\{1,2,\ldots,m\}$.

Then we can easily see that gf (resp. fg) is homotopic to the identity map on $\partial\Delta$ (resp. $C_{V_1}(G,\omega,t) \cap C_{V_2}(G,\omega,t)$), which implies that $C_{V_1}(G,\omega,t) \cap C_{V_2}(G,\omega,t) \simeq S^{m-2}$. Let $S^{m-1} = D_1 \cup D_2$, where D_i ($i = 1,2$) are $(m-1)$-balls such that $D_1 \cap D_2$ is homeomorphic to S^{m-2}. As before, we have a map $h: D_1 \cap D_2 \to C_{V_1}(G,\omega,t) \cap C_{V_2}(G,\omega,t)$ which is a homotopy equivalence. Since $C_{V_i}(G,\omega,t)$ is an AR, there is an extension $\tilde{h}: S^{m-1} \to C_{V_1}(G,\omega,t) \cup C_{V_2}(G,\omega,t) = \omega^{-1}(t)$ of h such that $\tilde{h}(D_i) \subset C_{V_i}(G,\omega,t)$ ($i = 1,2$). Then we can easily see that $\tilde{h}: S^{m-1} \to \omega^{-1}(t)$ is a homotopy equivalence. Hence $\omega^{-1}(t) \simeq S^{m-1}$.

(2.7) PROPOSITION. *Let G be a graph. If G contains a subgraph L which is homeomorphic to $G(m)$ as in (2.6), then there exist a Whitney map ω for $C(G)$ and a positive number $0 < t < \omega(G)$ such that $\mathrm{Fd}\,\omega^{-1}(t) \geq m-1$ (see [**1**, p. 227 or **13**] for the definition of $\mathrm{Fd}\,\omega^{-1}(t)$).*

PROOF. By induction, we can easily see that there is a metric ρ on G and a retraction $r: G \to L$ such that $\rho(x,y) \geq \rho(r(x),r(y))$ for $x,y \in G$. Let ω be the Whitney map for $C(G)$ as is defined by [**27**] and the metric ρ. Let $r^*: C(G) \to C(L)$ be the retraction of hyperspaces induced by r. Then we can easily see that $\omega(r^*(A)) \leq \omega(A)$ for each $A \in C(G)$. Hence $r^*|\omega^{-1}([0,t]): \omega^{-1}([0,t]) \to \omega_L^{-1}([0,t])$ is a retraction for any $0 \leq t \leq \omega(G)$, where $\omega_L = \omega|C(L)$. By (2.6), there is $t > 0$ such that $\omega_L^{-1}([0,t]) \simeq \omega_L^{-1}(t) \simeq S^{m-1}$. Hence $\mathrm{Fd}\,\omega^{-1}(t) = \mathrm{Fd}\,\omega^{-1}([0,t]) \geq m-1$. This completes the proof.

(2.8) EXAMPLE. Let $G = A_1 \cup A_2 \cup A_3$, where each A_i is an arc from V to W and $A_1 \cap A_2 \cap A_3 = A_i \cap A_j = \{V,W\}$ ($i \neq j$). Let ω be any Whitney map for $C(G)$. Set $t_1 = \min\{\omega(A_i)|i = 1,2,3\}$, $t_2 = \min\{\omega(A_i \cup A_j)|i \neq j\}$, and $t_3 = \max\{\omega(A_i \cup A_j)|i \neq j\}$. Then the Whitney continua $\omega^{-1}(t)$ are as follows:

(i) $t = 0$ (see Figure 2).

(ii) $0 < t < t_1$ (see Figure 3).

(iii) $t_1 \leq t < t_2$ (see Figure 4).

(iv) $t_2 \leq t < t_3$ (see Figure 5).

(v) $t_3 \leq t < \omega(G)$ (see Figure 6).

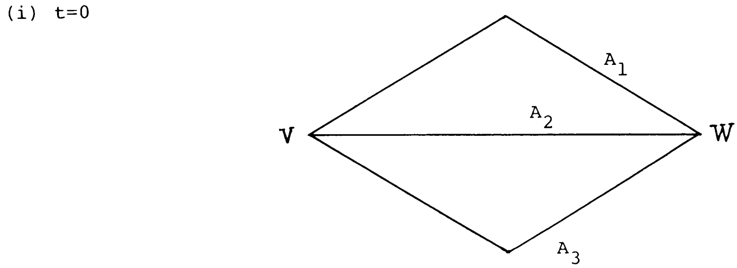

FIGURE 2

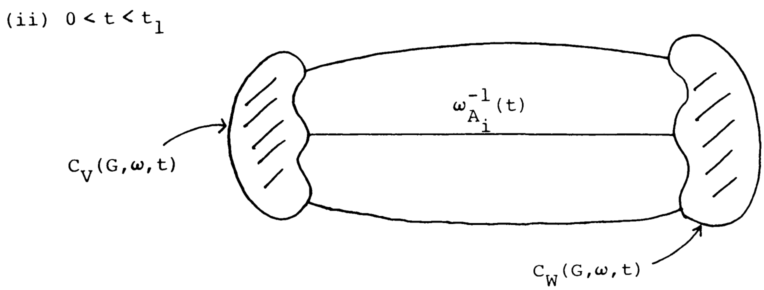

FIGURE 3

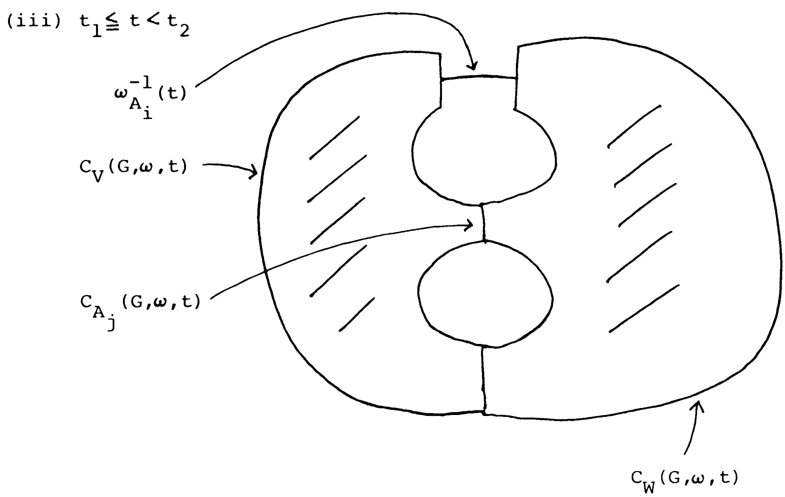

FIGURE 4

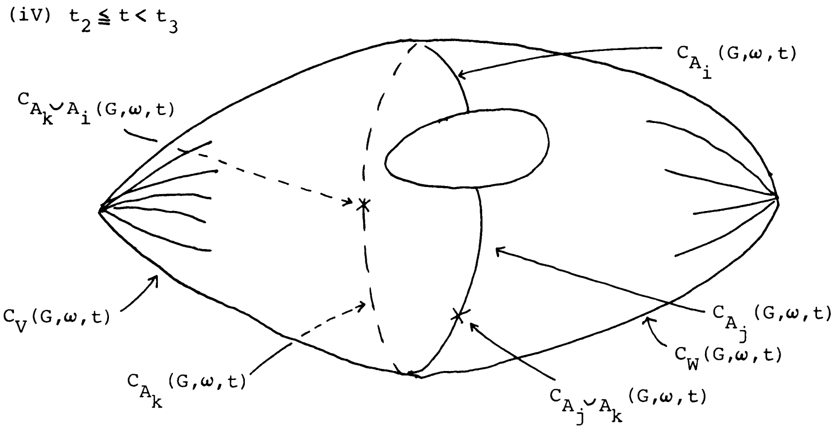

FIGURE 5

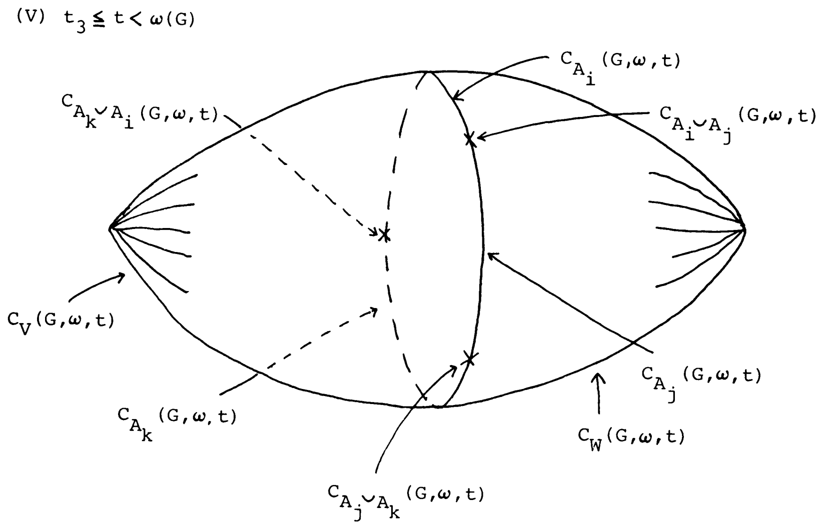

FIGURE 6

Let G be a graph and let $e = \langle V, W\rangle$ be an edge of G. Consider the family $A(e) = \{A_i\}$ of arcs from V to W in G. Note that $e \in A(e)$. Set $n(e) = |A(e)|$, where $|A(e)|$ denotes the cardinal number of $A(e)$. Also, set $n(G) = \max\{n(e) |\ e$ is an edge of $G\}$. Note that $n(G)$ is a topological invariant of G. Clearly, a graph G is a tree if and only if $n(G) \leq 1$. Then we have the following

(2.9) THEOREM. *Let G be a graph and let $m = 1, 2, \ldots$. If $n(G) \leq m$, then, for any Whitney map ω for $C(G)$, $\mathrm{Fd}\,\omega^{-1}(t) \leq m - 1$ for $0 \leq t \leq \omega(G)$.*

To prove (2.9), we need the following

(2.10) (S. NOWAK [**28**, (4.1)]). *Let X, Y be compacta. Then*

$$\mathrm{Fd}(X \cup Y) \leq \max\{\mathrm{Fd}(X), \mathrm{Fd}(Y), \mathrm{Fd}(X \cap Y) + 1\}.$$

PROOF OF (2.9). We shall prove the theorem by induction on the number i of edges of G. Let $0 < t < \omega(G)$. The statement is easily seen to be true for $i = 1$. Also, the case $m = 1$ is true (cf. (2.3)). We consider the case $m \geq 2$. Assume that it is true for $i \leq k$, and consider a graph G with $(k+1)$ edges. Let $e = \langle V, W\rangle$ be an edge of G such that $G - e$ is connected. Set $L = \overline{G - e}$. Note that $n(L) \leq m$. If $e \cap L = \{V\}$, then $\omega^{-1}(t) = \omega_L^{-1}(t) \cup \omega_e^{-1}(t) \cup C_V(G, \omega, t)$. By assumption, we can easily see that $\operatorname{Fd}\omega^{-1}(t) = \operatorname{Fd}\omega_L^{-1}(t) \leq m - 1$, because $\omega_e^{-1}(t)$ and $C_V(G, \omega, t)$ are ARs. Now we assume that $e \cap L = \{V, W\}$. Then we have

$$\omega^{-1}(t) = \omega_L^{-1}(t) \cap C_V(G, \omega, t) \cup C_W(G, \omega, t) \cup \omega_e^{-1}(t).$$

Consider the following cases (i) $\omega(e) > t$ and (ii) $\omega(e) \leq t$.

Case (i): $\omega(e) > t$. Note that $\omega_e^{-1}(t)$ is an arc. Then $C_V(G, \omega, t) \cap C_W(G, \omega, t) = \bigcup_{j=1}^{s} C_{A_i}(G, \omega, t)$, where A_i is an arc from V to W in L. Note that $s \leq m - 1$. If $\omega(L) \leq t$, then $\omega^{-1}(t) = C_V(G, \omega, t) \cup C_W(G, \omega, t) \cup \omega_e^{-1}(t)$. Note that $\bigcap_{j=1}^{s} C_{A_j}(G, \omega, t) = C_{\bigcup_{j\in J} A_j}(G, \omega, t)$ is an AR for each subset J of $\{1, 2, \ldots, s\}$. Hence $C_V(G, \omega, t) \cap C_W(G, \omega, t)$ is an AR. Thus we can easily see that $\omega^{-1}(t) \simeq S^1$, which implies that $\operatorname{Fd}\omega^{-1}(t) = 1 \leq m - 1$. Assume that $\omega(L) > t$. Then $\omega_L^{-1}(t)$ is a strong deformation retract of $\omega_L^{-1}(t) \cup C_V(G, \omega, t) \cup C_W(G, \omega, t)$. By assumption, we have

$$\operatorname{Fd}(\omega_L^{-1}(t) \cup C_V(G, \omega, t) \cup C_W(G, \omega, t)) = \operatorname{Fd}\omega_L^{-1}(t) \leq m - 1.$$

Since $(\omega_L^{-1}(t) \cup C_V(G, \omega, t) \cup C_W(G, \omega, t)) \cap \omega_e^{-1}(t)$ consists of two points, by (2.10) we have $\operatorname{Fd}\omega^{-1}(t) \leq m - 1$.

Case (ii): $\omega(e) \leq t$. Note that $\omega^{-1}(t) = \omega_L^{-1}(t) \cup C_V(G, \omega, t) \cup C_W(G, \omega, t)$. Since $C_V(G, \omega, t) \cap C_W(G, \omega, t) = \bigcup_{i=1}^{s} C_{A_i}(G, \omega, t) \cup C_e(G, \omega, t)$, by similar arguments as in the proof of (2.6) we can conclude that $C_V(G, \omega, t) \cap C_W(G, \omega, t)$ is homotopy equivalent to a $(m-2)$-dimensional polyhedron P. Also, $C_V(G, \omega, t) \cup C_W(G, \omega, t)$ is homotopy equivalent to ΣP, where ΣP denotes the suspension of P. Hence $\operatorname{Fd}(C_V(G, \omega, t) \cup C_W(G, \omega, t)) \leq m - 1$. Note that $\omega_L^{-1}(t) \cap (C_V(G, \omega, t) \cup C_W(G, \omega, t)) = C_V(L, \omega_L, t) \cup C_W(L, \omega_L, t)$. Since $C_V(L, \omega_L, t) \cap C_W(L, \omega_L, t) = \bigcup_{i=1}^{s} C_{A_i}(L, \omega_L, t)$ and $s \leq m - 1$, $C_V(L, \omega_L, t) \cup C_W(L, \omega_L, t)$ is homotopy equivalent to an $(m-2)$-dimensional polyhedron. Hence $\operatorname{Fd}(C_V(L, \omega_L, t) \cup C_W(L, \omega_L, t)) \leq m - 2$. By (2.10), we can conclude that $\operatorname{Fd}\omega^{-1}(t) \leq m - 1$.

(2.11) EXAMPLE. Let C be the Cantor set in the real line E, i.e., $C = \{x = \sum_{i=1}^{\infty} a_i/3^i \mid a_i = 0 \text{ or } 2\ (i = 1, 2, \ldots)\} \subset E = E \times \{0\} \subset E^2$. Consider the points $p = (0, 1)$, $q = (0, -1)$ of the plane E^2, and the sets $A_x = [p, x] \cup [q, x] \subset E^2$ for each $x \in C$, where $[p, x]$ denotes the segment from p to x in E^2. Set $X = \Sigma C$, where ΣC denotes the suspension of C with vertices p and q, i.e., $X = \bigcup_{x\in C} A_x$. Let ω be the Whitney map for $C(X)$ as defined by [**27**] and the Euclidean metric ρ. Let $A \in C(X)$. For each $n \geq 2$, let $F_n(A) = \{K \subset A \mid K \neq \emptyset$ and the cardinality of $K \leq n\}$, define $\lambda_n\colon F_n(A) \to [0, \infty)$ by letting $\lambda_n(\{a_1, a_2, \ldots, a_n\}) = \min\{\rho(a_i, a_j) \mid i \neq j\}$ for each $\{a_1, a_2, \ldots, a_n\} \in F_n(A)$, and let $\omega_n(A) = \sup \lambda_n(F_n(A))$. Then $\omega(A) = \sum_{n=2}^{\infty} \omega_n(A)/2^{n-1}$ (see [**27**]). Now, we shall show that $\operatorname{Fd}\omega^{-1}(t) = \infty$ for some t $(0 < t < \omega(X))$. First, we show that if $x_i \in C$ $(i = 1, 2, \ldots, k)$ and $x_i < x < y$ $(x, y \in C)$, then $\omega(\bigcup_{i=1}^{k} A_{x_i} \cup A_x) < \omega(\bigcup_{i=1}^{k} A_{x_i} \cup A_y)$. In fact, since there is a homeomorphism $h\colon \bigcup_{i=1}^{k} A_{x_i} \cup A_y \to \bigcup_{i=1}^{k} A_{x_i} \cup A_x$ such that $\rho(a, b) \geq \rho(h(a), h(b))$

for each $a, b \in \bigcup_{i=1}^k A_{x_i} \cup A_y$, then $\omega_n(\bigcup_{i=1}^k A_{x_i} \cup A_x) \leq \omega_n(\bigcup_{i=1}^k A_{x_i} \cup A_y)$. Also, note that $\omega_3(\bigcup_{i=1}^k A_{x_i} \cup A_x) = \rho(p,x) < \rho(p,y) = \omega_3(\bigcup_{i=1}^k A_{x_i} \cup A_y)$. Hence $\omega(\bigcup_{i=1}^k A_{x_i} \cup A_x) < \omega(\bigcup_{i=1}^k A_{x_i} \cup A_y)$.

Set $s_2 = \max\{\omega(A_{x_2(1)}), \omega(A_{x_2(2)})\}$ and $t_2 = \omega(A_{x_2(1)} \cup A_{x_2(2)})$, where $x_2(1) = 0$, $x_2(2) = 1$. Then $s_2 < t_2$. By induction, for each $n = 2, 3, \ldots$, choose points $x_n(1), x_n(2), \ldots, x_n(n)$ of C such that

(1) $x_n(1) < x_n(2) < \cdots < x_n(n)$, $x_{n+1}(1) = x_n(1), \ldots, x_{n+1}(n-1) = x_n(n-1)$, and $x_n(n-1) < x_{n+1}(n) < x_{n+1}(n+1) < x_n(n)$,

(2) $s_n = \max\{\omega(A_{x_n(1)} \cup \cdots \cup A_{x_n(i-1)} \cup A_{x_n(i+1)} \cup \cdots \cup A_{x_n(n)}) | i = 1, 2, \ldots, n\}$, $t_n = \omega(\bigcup_{i=1}^n A_{x_n(i)})$, and

(3) $s_{n-1} < s_n < t_n < t_{n-1}$ $(n = 3, 4, \ldots)$.

Note that $x_n(1) = 0$ for each $n = 2, 3, \ldots$. Let $t = \lim t_n$. We show that $\operatorname{Fd} \omega^{-1}(t) = \infty$. Since $s_n < t < t_n$ for each $n \geq 2$, by (2.6) $\operatorname{Fd} \omega_{Y_n}^{-1}(t) = n - 1$, where $Y_n = \sum(\{x_n(1), x_n(2), \ldots, x_n(n)\}) = \bigcup_{i=1}^n A_{X_n(i)}$. Since $\omega_{Y_n}^{-1}(t)$ is an ANR, for some $t' > t$ there exists a retraction $r: \omega_{Y_n}^{-1}([0,t']) \to \omega_{Y_n}^{-1}([0,t])$. Let $r_1: X \to Y_n$ be a retraction. Then we can choose a closed and open neighborhood H_n of $\{x_n(1), x_n(2), \ldots, x_n(n)\}$ in C such that $r_1^* \omega_{\sum H_n}^{-1}([0,t]) \subset \omega_{Y_n}^{-1}([0,t'])$. Also, choose a retraction $r_2: X \to \sum H_n$ such that if $x, y \in X$, then $\rho(r_2(x), r_2(y)) \leq \rho(x,y)$. Then $r_2^*(\omega^{-1}([0,t])) \subset \omega_{\sum H_n}^{-1}([0,t])$. Since $r \circ r_1^* \circ r_2^*: \omega^{-1}([0,t]) \to \omega_{Y_n}^{-1}([0,t])$ is a retraction, we can conclude that

$$\operatorname{Fd} \omega^{-1}(t) = \operatorname{Fd} \omega^{-1}([0,t]) \geq \operatorname{Fd} \omega_{Y_n}^{-1}([0,t]) = \operatorname{Fd} \omega_{Y_n}^{-1}(t) = n - 1$$

(see [**8**, (1.3) Theorem] or (3.3)). Hence $\operatorname{Fd} \omega^{-1}(t) = \infty$ (see Figure 7).

3. Shape of Whitney continua of curves. In this section, we define natural shape morphisms between Whitney continua by using inverse sequences. By using these shape morphisms, we investigate the shape of Whitney continua of curves.

In [**8**], the author defined shape morphisms between Whitney continua. In order to study Whitney continua of curves, we need another description of the shape morphisms by using inverse sequences of graphs, because the structures of Whitney continua of graphs are simple.

Let $\underline{X} = \{X_n, p_{nn+1}\}$ be an inverse sequence of continua and let $X = \varprojlim \underline{X}$. Then we have

(3.1) [**8**, (4.2)]. *If $\omega: C(X) \to [0, \omega(X)]$ is any Whitney map for $C(X)$, then there exist Whitney maps $\omega_n: C(X_n) \to [0, \omega_n(X_n)]$ satisfying the conditions; for any $\varepsilon > 0$, there is n_0 such that*

$$d(\omega, \omega_n p_n^*) = \sup\{|\omega(A) - \omega_n p_n^*(A)| \, | A \in C(X)\} < \varepsilon,$$

and

$$d(\omega_m p_{mn}^*, \omega_n) < \varepsilon \quad \textit{for each } n \geq m \geq n_0,$$

where $p_n: X \to X_n$ denotes the natural projection.

(3.2) PROPOSITION (CF. [**8**, (4.1)]). *Let ω_n be Whitney maps for $C(X_n)$ as in* (3.1). *Then there are positive numbers α_n $(n = 1, 2, \ldots)$ such that* $\lim_{n\to\infty} \alpha_n =$

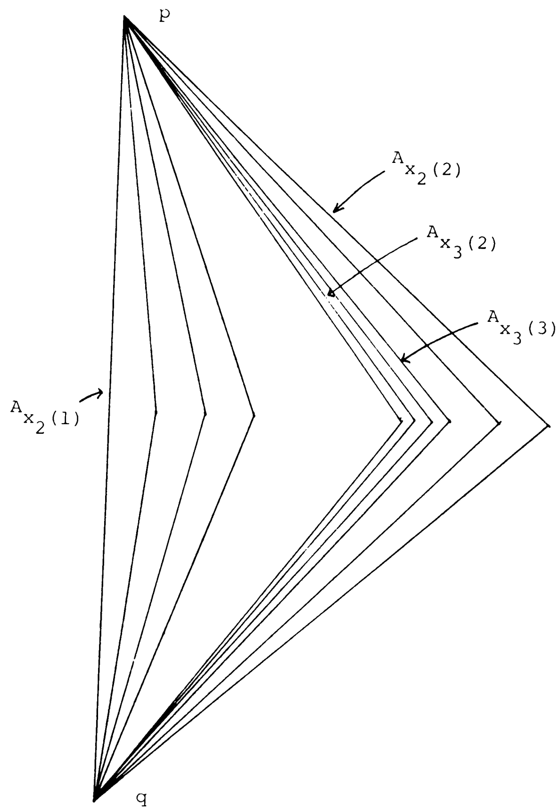

FIGURE 7

0 *and* $\omega^{-1}([s,t]) = \varprojlim \omega^{-1}([s,t])$ *for* $0 \le s \le t \le \omega(X)$, *where* $\underline{\omega^{-1}([s,t])}$ *is an inverse sequence as follows*:

$$\omega_{n_1}^{-1}([s-\alpha_1, t+\alpha_1]) \overset{p^*_{n_1n_2}}{\longleftarrow} \omega_{n_2}^{-1}([s-\alpha_2, t+\alpha_2]) \overset{p^*_{n_2n_3}}{\longleftarrow} \omega_{n_3}^{-1}([s-\alpha_3, t+\alpha_3]) \leftarrow \quad (n_1 < n_2 < n_3 < \cdots).$$

PROOF. Let $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \cdots$ be a decreasing sequence of positive numbers such that $\sum \varepsilon_i < \infty$. By (3.1), there are Whitney maps ω_n for $C(X_n)$ such that

(1) $d(\omega, \omega_n p_n^*) < \varepsilon_i$ and

(2) $d(\omega_m p^*_{mn}, \omega_n) < \varepsilon_i$ for each $n \ge m \ge n_i$, where $n_1 < n_2 < \cdots$. Set $\alpha_i = \sum_{j=i}^{\infty} \varepsilon_j$. By (2), we have

(3) $p^*_{n_i n_{i+1}}(\omega^{-1}_{n_{i+1}}([s-\alpha_{i+1}, t+\alpha_{i+1}])) \subset \omega^{-1}_{n_i}([s-\alpha_i, t+\alpha_i])$.

Hence, we can consider the following inverse sequence $\underline{\omega^{-1}([s,t])}$:

$$\omega_{n_1}^{-1}([s-\alpha_1, t+\alpha_1]) \overset{p^*_{n_1n_2}}{\longleftarrow} \omega_{n_2}^{-1}([s-\alpha_2, t+\alpha_2]) \overset{p^*_{n_2n_3}}{\longleftarrow} \omega_{n_3}^{-1}([s-\alpha_3, t+\alpha_3]) \leftarrow .$$

Let $A \in \omega^{-1}([s,t])$. By (1), we have

$$p^*_{n_i}(A) \subset \omega^{-1}_{n_i}([s-\varepsilon_i, t+\varepsilon_i]) \subset \omega^{-1}_{n_i}([s-\alpha_i, t+\alpha_i]).$$

Let $(A_{n_i}) \in \underleftarrow{\lim}\, \underline{\omega^{-1}([s,t])}$. Set $A = \underleftarrow{\lim}\{A_{n_i}, (p_{n_i n_{i+1}}|A_{n_{i+1}})\}$. Then $A \in C(X)$. Also, we have

$$s - \varepsilon_i - \alpha_i \leq \omega(A) \leq t + \varepsilon_i + \alpha_i \quad \text{for each } i = 1, 2, \ldots.$$

If $i \to \infty$, then $A \in \omega^{-1}([s,t])$. Thus $\omega^{-1}([s,t]) = \underleftarrow{\lim}\, \underline{\omega^{-1}([s,t])}$.

Now, by using (3.2), we shall define desired shape morphisms between Whitney continua. Let $\underline{X} = \{X_n, p_{nn+1}\}$ be an inverse sequence of Peano continua and let $X = \underleftarrow{\lim}\, \underline{X}$. Let ω be any Whitney map for $C(X)$. Then, by (3.2) there are Whitney maps ω_n for $C(X_n)$ and positive numbers α_n as in (3.2). Without loss of generality, we may assume that $n_1 = 1$, $n_2 = 2, \ldots$, $n_i = i, \ldots$ (see (3.2)). Since each X_n is a Peano continuum, there is a convex metric d_n on X_n. Define a homotopy $K_n: C(X_n) \times [0,\infty) \to C(X_n)$ (cf. the proof of (2.3)) by

$$K_n(A,u) = \{x \in X_n | d_n(A,x) \leq u\} \quad \text{for } A \in C(X_n) \text{ and } u \in [0,\infty).$$

Let $0 \leq s \leq t \leq \omega(X)$. Define a homotopy $F^n_{st}: \omega_n^{-1}([s-\alpha_n, t+\alpha_n]) \times I \to \omega_n^{-1}([s-\alpha_n, t+\alpha_n])$ $(n = 1,2,\ldots)$ by

$$F^n_{st}(A,u) = \begin{cases} A, & \text{if } \omega(A) \geq t - \alpha_n, \\ K_n(A, \beta(A,u)), & \text{if } \omega(A) \leq t - \alpha_n, \end{cases}$$

where $\omega_n(K_n(A,\beta(A,u))) = u \cdot (t - \alpha_n) + (1-u)\omega(A)$. Consider the maps $r^n_{st}: \omega_n^{-1}([s-\alpha_n, t+\alpha_n]) \to \omega_n^{-1}([t-\alpha_n, t+\alpha_n])$ defined by

$$r^n_{st} = F^n_{st}|\omega_n^{-1}([s-\alpha_n, t+\alpha_n]) \times \{1\}.$$

Then we have

(3.3) THEOREM. *$\underline{r}_{st} = \{r^n_{st}\}_{n=1,2,\ldots}: \omega^{-1}([s,t]) \to \omega^{-1}(t)$ is a shape equivalence such that $\underline{r}_{st}\underline{i} = \underline{1}_{\omega^{-1}(t)}$, where $\underline{i}: \omega^{-1}(t) \to \omega^{-1}([s,t])$ is the shape morphism induced by the inclusion map, and $\underline{1}_{\omega^{-1}(t)}$ denotes the identity morphism on $\omega^{-1}(t)$.*

PROOF. First, we shall prove that

$$(1) \qquad \begin{aligned} &r^n_{st}(p^*_{nn+1}|\omega^{-1}_{n+1}([s-\alpha_{n+1}, t+\alpha_{n+1}])) \\ &\quad \simeq (p^*_{nn+1}|\omega^{-1}_{n+1}([t-\alpha_{n+1}, t+\alpha_{n+1}]))r^{n+1}_{st}. \end{aligned}$$

Define a homotopy $R_n: \omega^{-1}_{n+1}([s-\alpha_{n+1}, t+\alpha_{n+1}]) \times I \to \omega_n^{-1}([t-\alpha_n, t+\alpha_n])$ by

$$(2) \qquad R_n(A,u) = \begin{cases} r^n_{st}F^n_{st}(p^*_{nn+1}(A), 1-2u), & \text{if } 0 \leq u \leq 1/2, \\ r^n_{st}p^*_{nn+1}F^{n+1}_{st}(A, 2u-1), & \text{if } 1/2 \leq u \leq 1. \end{cases}$$

It is easily checked that $R_n(A,0) = r^n_{st}p^*_{nn+1}(A)$ and $R_n(A,1) = p^*_{nn+1}r^{n+1}_{st}(A)$ for $A \in \omega^{-1}_{n+1}([s-\alpha_{n+1}, t+\alpha_{n+1}])$, which implies (1). Since each r^n_{st} is a homotopy equivalence, $\underline{r}_{st}$ is a shape equivalence. Also, since $r^n_{st}|\omega_n^{-1}([t-\alpha_n, t+\alpha_n]) =$ the identity map on $\omega_n^{-1}([t-\alpha_n, t+\alpha_n])$, $\underline{r}_{st}\underline{i} = \underline{1}_{\omega^{-1}(t)}$. This completes the proof.

Now, consider the shape morphisms $\underline{f}_{st} = \{f^n_{st}\}_{n=1,2,\ldots}: \omega^{-1}(s) \to \omega^{-1}(t)$ defined by $f^n_{st} = r^n_{st}|\omega^{-1}([s-\alpha_n, s+\alpha_n])$. Note that the shape morphisms $\underline{f}_{st}$ is the same as in [**8**, (1.13) and (1.14)]. By using $\underline{f}_{st}$, we investigate Whitney continua of curves.

A curve X is said to be a *winding curve* (resp. *strongly winding curve*) if there is an inverse sequence $\underline{X} = \{G_n, p_{nn+1}\}$ of graphs such that $X = \underleftarrow{\lim}\, \underline{X}$ and $\underline{X}$

satisfies the following conditions: $(*)$ If S is a simple closed curve in G_{n+1}, then $p_{nn+1}(S)$ contains a simple closed curve (resp. $(**)$ If S is a simple closed curve in G_{n+1}, then $p_{nn+1}(S) = G_n$). Clearly, every tree-like continuum and every circle-like continuum are strongly winding curves. Also, the Case-Chamberlin curve [**2**] is a strongly winding curve.

Then we have the following

(3.4) THEOREM. *Let X be a winding curve and let ω be any Whitney map for $C(X)$. Then there is a positive number t_0 ($t_0 < \omega(X)$) such that $\underline{f}_{0t}: X \to \omega^{-1}(t)$ is a shape equivalence for $0 \le t \le t_0$, i.e.,* $\operatorname{Sh}\omega^{-1}(t) = \operatorname{Sh} X$.

PROOF. Let $\underline{X} = \{G_n, p_{nn+1}\}$ be an inverse sequence of graphs such that $X = \varprojlim \underline{X}$ and $\underline{X}$ satisfies the conditions $(*)$. Consider the projection $p_1: X \to G_1$. Take a positive number t_0 such that if $A \in \omega^{-1}([0, t_0])$, $p_1^*(A)$ contains no simple closed curve. We shall prove that there is n_0 such that if $n \ge n_0$ for any $A \in \omega_n^{-1}([0, t_0 + \alpha_n])$, then A contains no simple closed curve in G_n, where ω_n is a Whitney map for $C(G_n)$ and α_n is a positive number as in (3.2) ($n = 1, 2, \ldots$). Choose a neighborhood $\mathfrak{U}$ of $p_1^*\omega^{-1}([0, t_0])$ in $C(G_1)$ such that for any $A \in \mathfrak{U}$, A contains no simple closed curve in G_1. By (3.2), $\omega^{-1}([0, t_0]) = \varprojlim \underline{\omega^{-1}([0, t_0])}$, where $\underline{\omega^{-1}([0, t_0])} = \{\omega_n^{-1}([0, t_0 + \alpha_n]), p_{nn+1}^*|\omega_{n+1}^{-1}([0, t_0 + \alpha_{n+1}])\}$ is the inverse sequence as in (3.2). Consider the Freudenthal space $\sigma\underline{\omega^{-1}([0, t_0])}$ of $\omega^{-1}([0, t_0])$, i.e., for an inverse sequence $\underline{Y} = \{Y_n, q_{nn+1}\}$, $\sigma\underline{Y}$ is the set $\varprojlim \underline{Y} \cup \bigcup_{n=1}^{\infty} Y_n$ with the topology defined by assuming the totality of the following sets; open subsets of the spaces Y_n, and sets of the form $q_m^{-1}(U) \cup \bigcup_{m \le n} q_{mn}^{-1}(U)$, where U is an open subset of Y_m. Define the map $p^*: \sigma\underline{\omega^{-1}([0, t_0])} \to C(G_1)$ by $p^*|\omega^{-1}([0, t_0]) = p_1^*|\omega^{-1}([0, t_0])$ and $p^*|\omega_n^{-1}([0, t_0 + \alpha_n]) = p_{1n}^*|\omega_n^{-1}([0, t_0 + \alpha_n])$ for $n \ge 1$. Note that $\lim_{n \to \infty} \omega_n^{-1}([0, t_0 + \alpha_n]) = \omega^{-1}([0, t_0])$ in $\sigma\underline{\omega^{-1}([0, t_0])}$. Hence there is n_0 such that if $n \ge n_0$, then $p_{1n}^*\omega_n^{-1}([0, t_0 + \alpha_n]) \subset \mathfrak{U}$. By $(*)$, we can easily see that n_0 is the desired positive integer. Let $0 \le t \le t_0$. Consider the shape morphism $\underline{f}_{0t}: X \to \omega^{-1}(t)$ as before. Since $\omega_n^{-1}(t + \alpha_n)$ contains no simple closed curve ($n \ge n_0$), $f_{0t}^n: G_n \to \omega_n^{-1}([t - \alpha_n, t + \alpha_n])$ is a homotopy equivalence (see the proof of (2.3)). Hence $\underline{f}_{0t}$ is a shape eqivalence, i.e., $\operatorname{Sh}\omega^{-1}(t) = \operatorname{Sh} X$. This completes the proof.

(3.5) THEOREM. *Let X be a strongly winding curve and let w be any Whitney map for $C(X)$. Then $f_{0t}: X \to \omega^{-1}(t)$ is a shape equivalence for $0 \le t < \omega(X)$, i.e.,* $\operatorname{Sh}\omega^{-1}(t) = \operatorname{Sh} X$.

PROOF. Let $\underline{X} = \{G_n, p_{nn+1}\}$ be an inverse sequence of graphs such that $X = \varprojlim \underline{X}$ and $\underline{X}$ satisfies the condition $(**)$. Let ω_n be the Whitney map for $C(G_n)$ and α_n be the positive number as in (3.2) ($n = 1, 2, \ldots$). Let $0 \le t < \omega(X)$. First, we shall prove that there is n_1 such that $p_{n_1}^*(A)$ contains no simple closed curve for each $A \in \omega^{-1}(t)$. In fact, for each $A \in \omega^{-1}(t)$, there is $n(A)$ such that $p_{n(A)}^*(A) \ne G_{n(A)}$. By $(**)$, $P_{n(A)+1}^*$ contains no simple closed curve. Choose a neighborhood $\mathfrak{U}(A)$ of A in $\omega^{-1}(t)$ such that $p_{n(A)+1}^*(B)$ contains no simple closed curve for each $B \in \mathfrak{U}(A)$. Since $\omega^{-1}(t)$ is compact, there are finite points

$A_1, A_2, \ldots, A_m$ of $\omega^{-1}(t)$ such that $\bigcup_{i=1}^m \mathfrak{U}(A_i) = \omega^{-1}(t)$. Set

$$n_1 = \max\{n(A_1)+1, n(A_2)+1, \ldots, n(A_m)+1\}.$$

Then $p_{n_1}^*(A)$ contains no simple closed curve for each $A \in \omega^{-1}(t)$. By the same argument as in the proof of (3.4), there is $n_0 \geq n_1$ such that if

$$A \in \omega_n^{-1}([t-\alpha_n, t+\alpha_n]) \qquad (n \geq n_0),$$

then A contains no simple closed curve. Hence $f_{0t}^n: X \to \omega_n^{-1}([t-\alpha_n, t+\alpha_n])$ $(n \geq n_0)$ are homotopy equivalence, which implies that $\underline{f}_{0t}: X \to \omega^{-1}(t)$ is a shape equivalence, i.e., $\operatorname{Sh}\omega^{-1}(t) = \operatorname{Sh} X$. This completes the proof.

As corollaries of (3.5), we have

(3.6) COROLLARY (J. KRASINKIEWICZ [**11**, (3.3)]). *Let X be a circle-like continuum and let ω be any Whitney map for $C(X)$. Then $\operatorname{Sh}\omega^{-1}(t) = \operatorname{Sh} X$ for $0 \leq t < \omega(X)$.*

(3.7) COROLLARY [**8**, (4.1)]. *Let X be a tree-like continuum and let ω be any Whitney map for $C(X)$. Then $\operatorname{Sh}\omega^{-1}(t)$ is trivial for $0 \leq t \leq \omega(X)$.*

(3.8) COROLLARY. *Let X be the Case-Chamberlin curve and let ω be any Whitney map for $C(X)$. Then $\operatorname{Sh}\omega^{-1}(t) = \operatorname{Sh} X$ for $0 \leq t < \omega(X)$.*

A curve X is said to be a *$\theta(m)$-curve* $(m = 1, 2, \ldots)$ provided that there is an inverse sequence $\underline{X} = \{G_i, p_{ii+1}\}$ of graphs such that $n(G_i) \leq m$ for each i. Note that a curve X is tree-like if and only if X is a $\theta(1)$-curve. By (2.9) and (3.2), we have

(3.9) PROPOSITION. *Let X be a curve and let ω be any Whitney map for $C(X)$. If X is a $\theta(m)$-curve, then $\operatorname{Fd}\omega^{-1}(t) \leq m-1$ for $0 \leq t \leq \omega(X)$.*

(3.10) PROPOSITION. *Let X be a curve and let ω be any Whitney map for $C(X)$. If X is a $\theta(2)$-curve and movable, then $\omega^{-1}(t)$ is also movable for $0 \leq t \leq \omega(X)$ (see [**1** or **13**] for the definition of the movability).*

PROOF. By [**8**], the property of being pointed 1-movable is a Whitney property. By (3.9), $\operatorname{Fd}\omega^{-1}(t) \leq 1$ for $0 \leq t \leq \omega(X)$. Hence $\omega^{-1}(t)$ is movable (see [**13**, p. 199]).

It is well known that if X is a continuum which is $\operatorname{Fd} X \leq 1$ and an FANR, then $\operatorname{Sh} X = \operatorname{Sh}\bigvee_{i=1}^n S_i$ for some $n < \infty$, where $\bigvee_{i=1}^n S_i$ denotes the one point union of n circles.

(3.11) THEOREM. *Let X be a curve which is a $\theta(2)$-curve, and let ω be any Whitney map for $C(X)$. If X is an FANR, $\omega^{-1}(t)$ is shape dominated by X, i.e., $\operatorname{Sh}\omega^{-1}(t) \leq \operatorname{Sh} X$. In particular, if $\operatorname{Sh} X = \operatorname{Sh}\bigvee_{i=1}^n S_i$, then $\operatorname{Sh}\omega^{-1}(t) = \operatorname{Sh}\bigvee_{i=1}^m S_i$ for some $m \leq n$. Moreover, there is a positive number $t_0 < \omega(X)$ such that $\operatorname{Sh}\omega^{-1}(t) = \operatorname{Sh} X$ for $0 \leq t \leq t_0$.*

PROOF. By (3.9) and (3.10), $\omega^{-1}(t)$ is movable and $\operatorname{Fd}\omega^{-1}(t) \leq 1$ for $0 \leq t \leq \omega(X)$. Note that $\operatorname{Sh}\omega^{-1}(t) = \operatorname{Sh}\bigvee_{i=1}^m S_i$ for some $m = 0, 1, 2, \ldots, \infty$. By [**8**, (1.13)], $H^1(\underline{f}_{0t}): H^1(\omega^{-1}(t)) \to H^1(X)$ is a monomorphism. Since $\operatorname{Sh} X = \operatorname{Sh}\bigvee_{i=1}^n S_i$ $(n < \infty)$, we conclude that $\operatorname{Sh}\omega^{-1}(t) = \operatorname{Sh}\bigvee_{i=1}^m S_i$ for some $m \leq n$.

Hence $\omega^{-1}(t)$ is shape dominated by X. By [**8**, (1.14)], there is a positive number $t_0 < \omega(X)$ such that $H^1(\underline{f}_{0t})$ is an isomorphism for $0 \le t \le t_0$. This implies that $\operatorname{Sh}\omega^{-1}(t) = \operatorname{Sh}X$.

(3.12) EXAMPLE. In the statement of (3.11), we cannot omit the condition that X is an FANR. Consider the set $X = \bigcup_{n=1}^{\infty} S_n$ in the plane E^2, where S_n is the circle in E^2 with the center $((n-1)/n, 0)$ and raidus $1/n$. Note that X is a $\theta(2)$-curve and not an FANR. Let ω be any Whitney map for $C(X)$ and let $0 < t < \omega(X)$. Then $\omega^{-1}(t)$ is an ANR such that $\omega^{-1}(t) \simeq \bigvee_{i=1}^{n} S_i$ for some $n < \infty$ (see [**14**]). Hence $\operatorname{Sh}X \ne \operatorname{Sh}\omega^{-1}(t)$ for $0 < t \le \omega(X)$.

4. Cell-like maps and Whitney continua. In §3, we proved that if X is a strongly winding curve, then $\operatorname{Sh}\omega^{-1}(t) = \operatorname{Sh}X$ for any Whitney map ω for $C(X)$ and $0 \le t < \omega(X)$. Note that if X is a strongly winding curve, then each proper nondegenerate subcontinuum of X is tree-like. Naturally, the following problem is raised: If X is a curve and each proper nondegenerate subcontinuum of X is tree-like, is $\operatorname{Sh}\omega^{-1}(t) = \operatorname{Sh}X$ for any Whitney map ω and $0 \le t < \omega(X)$? In this section, by using the technique of Rogers [**24**, Theorem 3] and a theorem on cell-like map, we give a partial answer to the above problem.

A map $f\colon X \to Y$ from a compactum X onto a compactum Y is a *cell-like map* if $f^{-1}(y)$ has trivial shape for each $y \in Y$. It is well known that there is a cell-like map $f\colon X \to Y$ such that $\operatorname{Sh}X \ne \operatorname{Sh}Y$, $\operatorname{Fd}X < \infty$, and $\operatorname{Fd}Y = \infty$. We need the following (see [**13**, pp. 284 and 286]).

(4.1) *Let $f\colon X \to Y$ be a cell-like map. If either* (a) $\operatorname{Fd}X < \infty$ *and* $\operatorname{Fd}Y < \infty$ *or* (b) $\dim Y < \infty$, *then f is a shape equivalence.*

(4.2) THEOREM. *Let X be a curve such that each proper nondegenerate subcontinuum of X is tree-like. Assume that ω is a Whitney map for $C(X)$ and* $\operatorname{Fd}\omega^{-1}(t) < \infty$ *for some t $(0 < t < \omega(X))$. Then the shape morphism $f_{0t}\colon X \to \omega^{-1}(t)$ is a shape equivalence, i.e.,* $\operatorname{Sh}\omega^{-1}(t) = \operatorname{Sh}X$.

By (3.9) and (4.2), we have

(4.3) COROLLARY. *Let X be a $\theta(m)$-curve* $(m < \infty)$. *If each proper subcontinuum of X is tree-like, then $f_{0t}\colon X \to \omega^{-1}(t)$ is a shape equivalence for any Whitney map ω for $C(X)$ and $0 \le t < \omega(X)$, i.e.,* $\operatorname{Sh}\omega^{-1}(t) = \operatorname{Sh}X$.

PROOF OF (4.2). The proof is essentially due to J. T. Rogers [**24**, Theorem 3]. Consider the subset $M = \{(x, A) | x \in X,\ A \in C_x(X, \omega, t)\}$ of $X \times \omega^{-1}(t)$. Note that M is a continuum. Let $p\colon M \to X$ and $q\colon M \to \omega^{-1}(t)$ be the projection maps. By (2.5) and the assumption, p and q are cell-like maps. Since $\dim X = 1$, by (4.1) p is a shape equivalence and $\operatorname{Fd}M \le 1$. Also, by (4.1), q is a shape equivalence. By the construction of $\underline{f}_{0t}$, we can easily see that there exists a shape morphism $\underline{g}_{0t}\colon X \to M$ such that $\underline{p} \cdot \underline{g}_{0t} = \underline{1}_X$ and $\underline{q} \cdot \underline{g}_{0t} = \underline{f}_{0t}$. Since $\underline{p}$ and $\underline{q}$ are shape equivalences, we conclude that $\underline{f}_{0t}\colon X \to \omega^{-1}(t)$ is a shape equivalence.

Finally, we give the following problem.

(4.4) PROBLEM. Let X be a curve. If each proper nondegenerate subcontinuum of X is tree-like, is $\operatorname{Fd}\omega^{-1}(t) \le 1$ for any Whitney map ω for $C(X)$ and $0 < t < \omega(X)$?

ADDED IN REMARK. Recently, the author proved that the problem (4.4) has an affirmative answer.

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