FREDHOLM, HODGE AND LIOUVILLE THEOREMS
ON NONCOMPACT MANIFOLDS

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ABSTRACT. Fredholm, Liouville, Hodge, and L^2-cohomology theorems are proved for Laplacians associated with a class of metrics defined on manifolds that have finitely many ends. The metrics are conformal to ones that are asymptotically translation invariant. They are not necessarily complete. The Fredholm results are, of necessity, with respect to weighted Sobolev spaces. Embedding and compact embedding theorems are also proved for these spaces.

Two of the most useful facts in analysis on a compact Riemannian manifold are that the Laplacian is Fredholm and its kernel consists of closed and coclosed forms that provide unique representatives for all the de Rham cohomology classes. Naturally one would like to extend these results to noncompact manifolds. The first such result in this direction is due to Kodaira [9, 12, p. 165]. It is that $L^2(\wedge^q M, g)$ is the orthogonal direct sum of $\overline{dC_0^\infty(\wedge^{q-1}M)}$, $\overline{d_g^*C_0^\infty(\wedge^{q+1}M)}$, and

$$\tilde{\mathfrak{H}}^2(\wedge^q M, g) = \left\{ \sigma \in L^2(\wedge^q M, g) \mid d\sigma = d_g^*\sigma = 0 \right\}.$$

Unfortunately, if no restriction is made on the manifold or the metric, one cannot improve on this. The Laplacian need not be Fredholm; L^2-harmonic forms need not be closed or coclosed; even if they are closed and coclosed, the space of harmonic L^2 forms need not be finite dimensional (see [7]); and even if the Laplacian is Fredholm and L^2-harmonic forms are closed and coclosed, those forms need not provide unique or total representation of de Rham cohomology.

Thus one of the main questions in analysis on noncompact manifolds is what conditions on M and g allow one to carry over the Fredholm and Hodge theorems for compact manifolds and, if they cannot be carried over completely, to what extent can they be? In the case of Hodge's theorem, i.e., properties of the space of L^2 harmonic forms, this question has been quite actively investigated recently (see [2–8, 11, and 13]).

For instance, one of the results of Atiyah, Patodi, and Singer in [2] is that if a manifold has cylindrical ends then $\tilde{\mathfrak{H}}^2(\wedge^g M, g)$ is naturally isomorphic to the image of $H_{comp}^q(M)$ in $H_{DR}^q(M)$ (see [2, Proposition 4.9]). In [11], Müller investigates the spectrum of the Laplacian on manifolds that outside a compact set are $\Omega \times \mathbf{R}^+$, with

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Ω compact, and are equipped with a metric that on $\Omega \times [1, \infty)$ is of the form $z^{-2}(dz^2 + ds_\Omega^2)$ with ds_Ω^2 a metric on Ω. A corollary of his Proposition 6.9 is that $\tilde{\mathfrak{H}}^2(\wedge^q M, g)$ is finite dimensional. In [4] Cheeger investigates, among other things, the L^2 cohomology of spaces of the form $M = \Omega \times (0, 1)$ equipped with the metric $g = dr^2 + r^{2\alpha}ds_\Omega^2$ with $\alpha \geqslant 1$. One of the consequences of Lemma 3.4 of [4] is that if Ω is compact then dim $\tilde{\mathfrak{H}}^2(\wedge^q M, g) \leqslant$ dim $H_{DR}^q(M)$. Using a separation of variables argument like that in [3] one can show that in fact dim $\tilde{\mathfrak{H}}^2(\wedge^q M, g) =$ dim $H_{DR}^q(M)$ when $2q < $ dim M.

In this paper we also examine the properties of L^2 harmonic forms on a class of manifolds equipped with certain metrics. However, unlike previous investigators who by and large treated Δ, d, and δ as unbounded operators on L^2, we consider the Fredholm properties of the Laplacian with respect to a particular family of weighted Sobolev spaces. There are three reasons for this. First of all, it allows for the consideration of a larger class of metrics. Secondly, a Fredholm theory for the Laplacian is as important to analysts as a Hodge theory. In particular, such a theory should be of use in nonlinear analysis on noncompact manifolds. It is for this reason that we do not restrict attention to weighted L^2 Sobolev spaces, but rather work with weighted L^p spaces for $1 < p < \infty$. It is also for this reason that we present several embedding theorems for the Sobolev spaces which we use. Thirdly, by using weights, we are able to prove Liouville type theorems for harmonic forms.

To be more precise, the noncompact manifolds we consider have finitely many ends. Thus M contains a compact manifold, M_0, with boundary; ∂M_0 has finitely many components, and $M_\infty = M - M_0 = \partial M_0 \times \mathbf{R}^+$. On such a manifold there is a natural $\mathbf{R}^+$-action on each end and so the notion of asymptotically translation invariant metric makes sense (see §2). If h is such a metric, then $g = e^{2\rho}h$ is said to be admissible if all the covariant derivatives of ρ have a limit at infinity (see 2.3.1).

Examples of admissible metrics abound. For instance, every metric that is asymptotically translation invariant is admissible. For h such a metric, the metric that on M_∞ is $g = e^{2z}h$ is admissible. As a special case of this, suppose $h = dz^2 + ds_{\partial M_0}^2$ on M_∞. Letting $r = e^z$ we see that $g = dr^2 + r^2 ds_{\partial M_0}^2$ on $(\omega, r) \in \partial M_0 \times (1, \infty) = M_\infty$. In particular if ∂M_0 is k-copies of S^{n-1}, then we have that every asymptotically Euclidean metric is admissible. The horns and cones of [4] that are of the form $g = dr^2 + r^{2\alpha}ds_{\partial M_0}^2$ on $(\omega, r) \in \partial M_0 \times (0, 1) = M_\infty$ and with $\alpha \geqslant 1$ are admissible in our sense. To see this, let $z = -\int_1^r t^{-\alpha} dt$. Then on $(\omega, z) \in \partial M_0 \times \mathbf{R}^+ = M_\infty$

$$(0.1) \quad g = \begin{cases} \left[\exp(-2(\alpha/(\alpha - 1))\ln((\alpha - 1)z + 1)) \right] \left(dz^2 + ds_{\partial M_0}^2 \right), & \alpha > 1, \\ e^{-2z} \left(dz^2 + ds_{\partial M_0}^2 \right), & \alpha = 1. \end{cases}$$

In both cases g is admissible. If $\rho = -\ln z$ on $\partial M_0 \times (1, \infty)$ then $g = e^{2\rho}h$ is admissible and so for instance the cuspidal metrics of [11] are admissible. For more examples see the end of the introduction.

It is for the Laplacians associated with admissible metrics that we prove Fredholm results. Since the manifolds under consideration are noncompact, weighted Sobolev spaces must be used. This is not due to some perversity of the author, but rather out

of necessity. It just is not the case that

$$(0.2) \qquad \Delta_g \colon L_2^p(\wedge^q M, g) \to L^p(\wedge^q M, g)$$

is always Fredholm. For instance, if g is asymptotically a product metric on M_∞, then (0.2) is Fredholm if and only if $H_{DR}^{q-1}(\partial M_0) = H_{DR}^q(\partial M_0) = 0$.

In the case of metrics asymptotically translation invariant on M_∞, the weights needed are $\exp(\tilde{\delta})$ with $\tilde{\delta} \in C^\infty(M; \mathbf{R})$ and $\tilde{\delta}(\omega, z) = \delta_l z$ on the lth end of M. What matters about $\tilde{\delta}$ is its values on M_∞, and so it may be identified with an element δ of $\mathbf{R}^L$, where L is the number of ends of M and $\tilde{\delta}(\omega, z) = \delta_l z$ on the lth end. Using this identification we write our weights as $e^{\delta z}$.

In the case of a general admissible metric, $g = e^{2\rho}h$, these weights must be modified. It turns out for $g = e^{2\rho}h$ the appropriate Sobolev spaces are the ones with the norms

$$(0.3) \qquad \|\sigma\|\big(W_{s,\delta,a}^p(\wedge^q M, g)\big) = \left(\sum_{t=0}^s \int_M \left\| e^{\delta z + (t+a)\rho} D_{(g)}^t \sigma \right\|_g^p dV_g \right)^{1/p}$$

with $\delta \in \mathbf{R}^L$, $1 < p < \infty$, $s \in \mathbf{N}$, and $a \in \mathbf{R}$. The associated Sobolev spaces are

$$(0.4) \qquad W_{s,\delta,a}^p(\wedge^q M, g) = \big\{ \sigma \in L_{s,\mathrm{loc}}^p(\wedge^q M) \mid \|\sigma\|\big(W_{s,\delta,a}^p(\wedge^q M, g)\big) < \infty \big\}.$$

Various properties of the spaces $W_{s,\delta,a}^p(\wedge^q M, g)$ are established in §§3 and 4. These include the density of $C_0^\infty(\wedge^q M)$, a weighted Sobolev embedding theorem, and a compact embedding theorem.

One of our principal Fredholm results is that

$$(0.5) \qquad \Delta_g \colon W_{s+2,\delta,a}^p(\wedge^q M, g) \to W_{s,\delta,a+2}^p(\wedge^q M, g)$$

is continuous for all δ and Fredholm for almost all $\delta \in \mathbf{R}^L$. Since $L^p(\wedge^q M, g) = W_{0,0,0}^p(\wedge^q M, g)$ and $W_{0,0,0}^p(\wedge^q M, g) \subset W_{0,\delta,0}^p(\wedge^q M, g)$ for all $\delta \in \mathbf{R}^L$ with $\delta_l \leqslant 0$, it follows from this Fredholm result and some regularity results for harmonic forms that the space of harmonic forms in $L^p(\wedge^q M, g)$ is finite dimensional. In fact more generally there is the Liouville theorem that for all $a \in \mathbf{R}$ and $\delta \in \mathbf{R}^L$ the space of harmonic forms in $W_{0,\delta,a}^p(\wedge^q M, g)$ is finite dimensional. This combined with the regularity results for harmonic forms in §5 implies that the space of harmonic forms satisfying $\sup_M \|e^{\delta z}\sigma\|_g < \infty$ is finite dimensional for each fixed $\delta \in \mathbf{R}^L$.

It is still not the case that

$$(0.6) \qquad \Delta_g \colon W_{2,0,-2}^p(\wedge^q M, g) \to W_{0,0,0}^p(\wedge^q M, g) = L^p(\wedge^q M, g)$$

is always Fredholm. However, because the kernel and the kernel of the adjoint of (0.6) are finite dimensional and because (0.5) with $a = -2$ is Fredholm for some $\delta \in \mathbf{R}^L$ with $\delta_l \leqslant 0$, it is possible to construct a Banach space $\tilde{W}_{2,0,-2}^p(\wedge^q M, g)$ such that

$$(0.7) \qquad \Delta_g \colon \tilde{W}_{2,0,-2}^p(\wedge^q M, g) \to L^p(\wedge^q M, g)$$

is Fredholm. If (0.6) is Fredholm, then $\tilde{W}_{2,0,-2}^p(\wedge^q M, g)$ turns out to be $W_{2,0,-2}^p(\wedge^q M, g)$ with an equivalent norm. In general $W_{2,0,-2}^p(\wedge^q M, g)$ is dense in $\tilde{W}_{2,0,-2}^p(\wedge^q M, g)$ and the latter space is contained in $\bigcap_{\eta < 0} W_{2,\eta,-2}^p(\wedge^q M, g)$. This along with the proof of the other Fredholm result is done in §5. The analogous results for $d + d_g^*$ are also proved there.

Because of these Fredholm results

$$(0.8) \qquad L^p(\wedge^q M, g) = \Delta_g\big(\tilde{W}^p_{2,0,-2}(\wedge^q M, g)\big) \oplus F$$

with dim F = dimension of the space of harmonic forms in $L^{p'}(\wedge^q M, g)$ $(1/p' + 1/p = 1)$. In §6 by using the weighted Sobolev imbedding theorem we provide sufficient conditions for F to consist of harmonic forms. In fact this is done for the general space $W^p_{s,\delta,a}(\wedge^q M, g)$. A consequence of these results is that on an orientable manifold if g is admissible and ϕ is a function such that $e^{\delta z}\phi \in L^p(M, g)$ for δ_l sufficiently negative for each l, then $\phi = d_g^*\sigma$ for some 1-form

$$\sigma \in \bigcap_{\tau < \delta} W^p_{0,\tau,-1}(\wedge^1 M, g).$$

In fact this is true even if g is only equivalent to an admissible metric; i.e., there is an admissible metric g_1 and $c \in \mathbf{R}^+$ such that $g/c \leqslant g_1 \leqslant cg$.

In §7 we turn our attention to the properties of the space of closed and coclosed forms in $L^2(\wedge^q M, g)$ for metrics equivalent to admissible metrics. This space is denoted $\tilde{\mathfrak{H}}^2(\wedge^q M, g)$. Note that in the case of complete metrics, $\tilde{\mathfrak{H}}^2(\wedge^q M, g)$ is the same as the space of L^2-harmonic forms.

Qne of the chief results of the section is that if a metric (not necessarily equivalent to an admissible one) is equivalent to one that is q-bounded above (see 7.3), then $\tilde{\mathfrak{H}}^2(\wedge^q M, g)$ contains no exact elements. Thus dim $\tilde{\mathfrak{H}}^2(\wedge^q M, g) \leqslant \dim H^q_{DR}(M)$ and each element of $\tilde{\mathfrak{H}}^2(\wedge^q M, g)$ is a unique closed and coclosed representative of a class in $H^q_{DR}(M)$.

Here and throughout the paper

$$(0.9) \qquad H^q_{DR}(M) = \big[\operatorname{Ker} d \colon C^\infty(\wedge^q M) \to C^\infty(\wedge^{q+1} M)\big]/dC^\infty(\wedge^{q-1} M).$$

There are many examples of metrics that are q-bounded above. For instance, if h is asymptotically translation invariant, then it is q-bounded above for all q. If $g = e^{2\phi}h$ is conformal to h (g not necessarily admissible) and $\phi(\omega, z)$ is increasing on M_∞ for each $\omega \in \partial M_0$, then g is q-bounded above for $q \geqslant n/2$; if ϕ is decreasing, then g is q-bounded above for $q \leqslant n/2$.

When g is equivalent to an admissible metric, the Fredholm results for $d + d_g^*$ mentioned above imply that every class of $H^q_{DR}(M)$ with a representative in $L^2(\wedge^q M, g)$ has a representative in $\tilde{\mathfrak{H}}^2(\wedge^q M, g)$. Thus the question arises as to which cohomology classes have representatives in $L^2(\wedge^q M, g)$. This is determined in (7.9).

For g a metric and $\Omega_1, \ldots, \Omega_L$ the components of ∂M_0 let

$$(0.10) \quad \mathscr{A} = \mathscr{A}(g; q) = \left\{ l \,\Big|\, \int_{\Omega_l \times \mathbf{R}^+} \|\phi\|_g^2 \, dV_g = \infty \text{ for some } \phi \in C^\infty(\wedge^q M)_{\mathbf{R}^+} \right\}$$

where $C^\infty(\wedge^q M)_{\mathbf{R}^+}$ is the space of C^∞ q-forms that are $\mathbf{R}^+$ invariant on M_∞. Also let $M^\infty_\infty(g; q) = \bigcup_{l \in \mathscr{A}} \Omega_l \times \mathbf{R}^+$. It is shown in (7.9) that if g is equivalent to an admissible metric, then the classes in $H^q_{DR}(M)$ with a representative in $\tilde{\mathfrak{H}}^2(\wedge^q M, g)$ are exactly those with compact support on $M^\infty_\infty(g; q)$; i.e. have a representative that is 0 on $\bigcup_{l \in \mathscr{A}} \Omega_l \times [R, \infty)$ for some $R > 0$. If in addition g is q-bounded above, then the representation is unique.

In the last section we look at some properties of the L^2-cohomology (see Definition (8.2)) of the Riemannian manifolds which are considered in this paper. In particular, if M is a manifold with finitely many ends, equipped with a metric g that is equivalent to an admissible one, and $H_2^q(M)$ is the L^2-cohomology of M, then we present some results concerning when the natural map

$$(0.11) \qquad\qquad i \colon \tilde{\mathfrak{H}}^2(\wedge^q M, g) \to H_2^q(M)$$

is 1-1, is onto, is an isomorphism. When it is an isomorphism we say that the Strong Hodge Theorem holds (see [4]). When (0.11) is not onto then we have the interesting fact that $H_2^q(M)$ is infinite dimensional.

For example, an easy result (Proposition (8.5)) is that if g is equivalent to a metric that is q-bounded above, then (0.11) is 1-1. A more difficult result (Theorem (8.7)) is that if g is equivalent to an admissible metric $g_1 = e^{2\rho}h$ and ρ is bounded below, then (0.11) is 1-1. Moreover, if $\limsup_{z \to \infty} \rho(\omega, z) = \infty$ on at least one end of M, then (0.11) is not onto for $q > 0$.

In Theorem (8.9) we treat the case in which g is equivalent to an admissible metric $g_1 = e^{2\rho}h$ and ρ is bounded above. One of the results of that theorem is that if $\rho \geq (\varepsilon - 1)\ln z$ on $\partial M_0 \times [1, \infty)$ for some $0 < \varepsilon < 1$ and ρ is slowly oscillating in the sense of (8.8) then (0.11) is onto if and only if $H_{DR}^{q-1}(\partial M_0) = 0$. For example, if ρ is bounded, then either the Strong Hodge Theorem holds or, when $H_{DR}^{q-1}(\partial M_0) \neq 0$, $H_2^q(M)$ is infinite dimensional (also see [13] for this result for bounded ρ).

The condition that $\rho \geq (\varepsilon - 1)\ln z$ for some $0 < \varepsilon < 1$ is in some sense sharp for the necessity of $H_{DR}^{q-1}(\partial M_0)$ being 0 for (0.11) to be onto. If g is q-bounded above and ze^ρ is bounded on $\partial M_0 \times \mathbf{R}^+$, then the Strong Hodge Theorem holds regardless of what $H_{DR}^q(\partial M_0)$ is (see (8.9)). Thus, for instance, this is true of the cuspidal metrics of [11].

Finally, if ρ is $O(z)$ but not $o(z)$ as $z \to \infty$ on $\partial M_0 \times \mathbf{R}^+$, then (0.11) is onto. This is the case, for example, for the metric cones in [4].

To illustrate the scope of these results, four examples will now be given. In all the examples there is the orthogonal decomposition

(0.12)
$$L^2(\wedge^q M, g) = d\tilde{W}_{1,0,-1}^2(\wedge^{q-1} M, g) \oplus d_g^*\tilde{W}_{1,0,-1}^2(\wedge^{q+1} M, g) \oplus \tilde{\mathfrak{H}}^2(\wedge^q M, g).$$

In the cases in which g is complete, $\tilde{\mathfrak{H}}^2(\wedge^q M, g)$ contains all harmonic, L^2 forms and so

$$(0.13) \quad \Delta_g\big(\tilde{W}_{2,0,-2}^2(\wedge^q M, g)\big) = d\tilde{W}_{1,0,-1}^2(\wedge^{q-1} M, g) \oplus d_g^* W_{1,0,-1}^2(\wedge^{q+1} M, g).$$

EXAMPLE (0.14). g is asymptotically translation invariant.

In this case g is q-bounded above for all q and admissible. Furthermore $M_\infty^\infty(g; q) = M_\infty$ for all q. Thus

$$(0.14.1) \qquad \tilde{\mathfrak{H}}^2(\wedge^q M, g) \approx \text{image of } H_{\text{comp}}^q(M) \text{ in } H_{DR}^q(M) \quad \text{for all } q$$

and provides unique representatives for that image. Also $H_2^q(M)$ is infinite dimensional if $H_{DR}^{q-1}(\partial M_0) \neq 0$. Otherwise, $H_2^q(M) \approx \tilde{\mathfrak{H}}^2(\wedge^q M, g)$.

EXAMPLE (0.15). h is asymptotically translation invariant, $g = e^{2\rho}h$ is admissible, $\rho(\omega, z)$ is increasing on M_∞ for each $\omega \in \partial M_0$, and $\rho(\omega, z) > [(1 + \varepsilon)/2]\ln z$ on $\partial M_0 \times [1, \infty)$ for some $\varepsilon > 0$.

In this case g is q-bounded above for $q \geqslant n/2$ and admissible. Furthermore $M_\infty^\infty(g, q) = M_\infty$ if $q \leqslant n/2$. However, if $q > n/2$ then $M(g, q)$ is empty. Thus

$$(0.15.1)\quad \tilde{\mathfrak{H}}^2(\wedge^q M, g) \approx \begin{cases} H_{DR}^q(M) & \text{for } q > n/2, \\ \text{image of } H_{\text{comp}}^q(M) \text{ in } H_{DR}^q(M) & \text{if } q = n/2 \end{cases}$$

and provides unique representatives for these spaces. For $q < n/2$ only the image of $H_{\text{comp}}^q(M)$ in $H_{DR}^q(M)$ has representatives in $\tilde{\mathfrak{H}}^2(\wedge^q M, g)$. Moreover the representatives need not be unique. For example, if M is orientable, then obviously $\tilde{\mathfrak{H}}^2(\wedge^q M, g) = *_g \tilde{\mathfrak{H}}^2(\wedge^{n-q} M, g)$ and so dim $\tilde{\mathfrak{H}}^2(\wedge^q M, g) = $ dim $H_{\text{comp}}^q(M)$ for $q < n/2$. Also dim $H_2^q(M) = \infty$ if $q > 0$.

EXAMPLE (0.16). h is asymptotically translation invariant, $g = e^{2\rho}h$ is admissible, $\rho(\omega, z)$ decreasing, and $\rho(\omega, z) < -[(1 + \varepsilon)/2]\ln z$ on $\partial M_0 \times [1, \infty)$ for some $\varepsilon > 0$.

This case is exactly the same as the previous one except that all the inequalities for q are reversed. Thus

$$(0.16.1)\quad \tilde{\mathfrak{H}}^2(\wedge^q M, g) \approx \begin{cases} H_{DR}^q(M) & \text{for } q < n/2, \\ \text{image of } H_{\text{comp}}^q(M) \text{ in } H_{DR}^q(M) & \text{if } q = n/2 \end{cases}$$

and provides unique representatives for these spaces. For $q > n/2$ only the image of $H_{\text{comp}}^q(M)$ in $H_{DR}^q(M)$ has representatives in $\tilde{\mathfrak{H}}^2(\wedge^q M, g)$. Moreover the representative need not be unique. If M is orientable, then dim $\tilde{\mathfrak{H}}^2(\wedge^q M, g) = $ dim $H_{\text{comp}}^q(M)$ for $q > n/2$. Also if $\varepsilon > 1$, then $H_2^q(M) \simeq \tilde{\mathfrak{H}}^2(\wedge^q M, g)$ for $q \leqslant n/2$. If moreover ρ is $O(z)$ but not $o(z)$ as $z \to \infty$ on $\partial M_0 \times \mathbf{R}^+$, then dim $H_2^q(M) \leqslant $ dim $\tilde{\mathfrak{H}}^2(\wedge^q M, g)$ for $q > n/2$.

EXAMPLE (0.17). h is asymptotically translation invariant, $g = e^{2\rho}h$ and $\rho(\omega, z)$ $= (\sin(\ln z))\ln z$ on $\partial M_0 \times [1, \infty)$.

Here g is admissible; however, it is q-bounded above only for $q = n/2$ (if n is even). Moreover $M_\infty^\infty(g; q) = M_\infty$ for all q. Thus only the image of $H_{\text{comp}}^q(M)$ in $H_{DR}^q(M)$ has representatives in $\tilde{\mathfrak{H}}^2(\wedge^q M, g)$. If $q = n/2$ these representatives are unique. For $q \neq n/2$ our techniques do not allow us to say anything more; except, of course, if $q = 0$ or n, which are trivial cases.

As we noted before, Example (0.15) contains all asymptotically Euclidean metrics on manifolds that outside M_0 are diffeomorphic to a finite number of copies of $\mathbf{R}^n$ minus a ball. Similarly Example (0.17) contains the metric cones and horns in [4] and cusps in [11]. Also note that if g is equivalent to any of the metrics in these examples, then everything said about the space of closed and coclosed L^2 forms for the metric that g is equivalent to is also true for $\tilde{\mathfrak{H}}^2(\wedge^q M, g)$. In fact one can even get a version of the decomposition (0.12) when M is oriented. More specifically, if g is equivalent to an admissible metric g_1, then

$$(0.18)\qquad L^2(\wedge^q M, g) = d\tilde{\tilde{W}}_{1,0,-1}^2(\wedge^{q-1} M, g_1) \oplus d_g^* B \oplus \tilde{\mathfrak{H}}^2(\wedge^q M, g)$$

where $B = *_g \tilde{\tilde{W}}_{1,0,-1}^2(\wedge^{n-q-1} M, g_1)$ and the norm on B is

$$\|\sigma\|_B = \left\| *_g \sigma \right\| \left(W_{1,0,-1}^2(\wedge^{n-q-1} M, g_1) \right).$$

This follows easily from (0.12) and Kodaira's theorem.

Finally there is the question as to whether one needs to use weighted Sobolev spaces in order to get Fredholm results for Δ_g and $d_g + d_g^*$ for admissible g. The answer is yes. For instance, suppose B is a Banach space of q-forms with the properties that $C_0^\infty(\wedge^q M)$ is dense in B and $\Delta_g \colon B \to L^p(\wedge^q M, g)$ is Fredholm. Since $\Delta_g \colon B \to L^p(\wedge^q M, g)$ is Fredholm, the norm on B is equivalent to $\|\Delta_g \sigma\|_p + \|\pi_B \sigma\|$ where π_B is a projection of B onto $\operatorname{Ker}\Delta_g \cap B = K_1$. Similarly the norm on $\tilde{W}_{2,0,-2}^p(\wedge^q M, g)$ is equivalent to $\|\Delta_g \sigma\|_p + \|\tilde{\pi}\sigma\|$ with $\tilde{\pi}$ a projection of $\tilde{W}_{2,0,-2}^p(\wedge^q M, g)$ onto

$$\operatorname{Ker}\Delta_g \cap \tilde{W}_{2,0,-2}^p(\wedge^q M, g) = K_2.$$

Thus if $F = B \cap \tilde{W}_{2,0,-2}^p(\wedge^q M, g)$ has norm $\|\sigma\|_F = \max(\|\sigma\|_B, \|\sigma\|(\tilde{W}_{2,0,-2}^p(\wedge^q M, g)))$, then $\| \ \|_F$ is equivalent to $\|\Delta_g \sigma\|_p + \|\pi\sigma\|$ where $\pi = (\pi_B, \pi)\colon F \to K_1 \oplus K_2$. It follows from this and the fact that $C_0^\infty(\wedge^q M) \subset F$ that $\Delta_g \colon F \to L^p(\wedge^q M, g)$ is Fredholm, which in turn means that as sets $B = \tilde{W}_{2,0,-2}^p(\wedge^q M, g)$ and that the norms are equivalent.

1. Notation. In this section we list some of the notation used that is not defined in the text. First of all $\mathbf{R}$ is the real numbers, $\mathbf{R}^+ = \{ z \in \mathbf{R} \mid z > 0 \}$, and $\mathbf{N}$ is the nonnegative integers.

As usual $T_r^q(M)$ is the bundle of (r, q) tensors over M, $\wedge^q M$ is the bundle of exterior q-forms (we often think of $\wedge^q M$ as being in $T_0^q M$), and $\wedge^* M = \oplus_{q=0}^n \wedge^q M$. If E is any of these bundles, then $\Gamma(E)$ is the space of all sections of E and $C^\infty(E)$ is the space of C^∞ sections; $C^\infty(M)$ is the C^∞ functions on M.

By a metric we always mean a C^∞, Riemannian metric, though many of the results hold for metrics with much less regularity. For a metric, g, we let $D_{(g)}$ be the covariant derivative associated with g by the Levi-Civita connection. If $t \in \mathbf{N}$, then $D_{(g)}^t = D_{(g)} \cdots D_{(g)}$ t times. On the other hand ∇_i is, as usual, the covariant derivative in the ith coordinate direction. In this regard $\nabla^i = g^{ij}\nabla_j$. Also, we take $\| \ \|_{g(x)}$ to be the norm on $T_r^q M \mid_x$, or $\wedge^q M \mid_x$, or $\wedge^* M \mid_x$ induced by $g(x)$.

For any metric, g, on M, $s \in \mathbf{N}$, and $1 \leqslant p < \infty$ and for $E = T_r^q M$, $\wedge^q M$, or $\wedge^* M$, we let

$$(1.1) \qquad L_s^p(E, g) = \left\{ \sigma \in \Gamma(E) \mid \left(\sum_{t=0}^s \int_M \left\| D_{(g)}^t \sigma \right\|_g^p dV_g \right)^{1/p} < \infty \right\}$$

where dV_g is the volume measure arising from g. Also we take

$$(1.2) \qquad L_{s,\mathrm{loc}}^p(E) = \left\{ \sigma \in \Gamma(E) \mid \phi\sigma \in L_s^p(E) \text{ for all } \phi \in C_0^\infty(M) \right\}.$$

Note that $L_{s,\mathrm{loc}}^p(E)$ is independent of the metric.

Lastly, if g is a metric, then d_g^* is the operator from $C^\infty(\wedge^q M)$ to $C^\infty(\wedge^{q-1} M)$ defined locally by

$$(1.3) \qquad \left(d_g^* \alpha \right)_{i, \ldots, i_{q-1}} = -\nabla^j \alpha_{j i_1 \cdots i_{q-1}}.$$

Thus it is the formal adjoint of d.

2. Admissible metrics on manifolds with finitely many ends. Let M be a manifold with finitely many ends. Recall this means that M contains a compact manifold, M_0, with boundary; that $M - M_0 = \partial M_0 \times (0, \infty)$, and that ∂M_0 has only finitely many components. There is an obvious $\mathbf{R}^+$-action on $M_\infty \equiv \partial M_0 \times \mathbf{R}^+$; namely $(\omega, z) \mapsto (\omega, z + z_0)$ for $z_0 \in \mathbf{R}^+$. We say that a metric on M is translation invariant, if on M_∞ it is invariant under this $\mathbf{R}^+$-action.

DEFINITION (2.1). Suppose h_∞ is a translation invariant metric, with covariant derivative D_∞. A metric, h, is asymptotic to h_∞ if for each $t \in \mathbf{N}$

$$(2.1.1) \qquad \lim_{z \to \infty} \sup_{\omega \in \partial M_0} \left\| D_\infty^t h(\omega, z) - D_\infty^t h_\infty(\omega) \right\|_{h_\infty} = 0.$$

If h is asymptotic to a translation invariant metric, then h is said to be asymptotically translation invariant.

CONVENTION (2.2). Henceforth h will always denote an asymptotically translation invariant metric.

DEFINITION (2.3). Let $g = e^{2\rho}h$ with $\rho \in C^\infty(M)$. The metric g is admissible if there is a C^∞, $\mathbf{R}^+$-invariant 1-form, θ, on M_∞ with the property

$$(2.3.1) \qquad \lim_{z \to \infty} \sup_{\omega \in \partial M_0} \left\| D_{(h)}^{t+1}\rho - D_{(h)}^t \theta \right\|_h = 0.$$

A few remarks about these definitions are in order. First of all note that all geometric quantities associated with a metric, h, that is asymptotic to a translation invariant metric, h_∞, asymptotically become those of h_∞. In fact one has the following

PROPOSITION (2.4). *Let h_∞ be translation invariant and h asymptotic to h_∞. Let A be the tensor field $\Gamma^i_{(h)jk} - \Gamma^i_{(h_\infty)jk}$. Then*

$$(2.4.1) \qquad \lim_{z \to \infty} \sup_{\omega \in \partial M_0} \left\| D_\infty^t A \right\|_{h_\infty} = 0$$

for all $t \in \mathbf{N}$. Furthermore if $X \in C^\infty(T_r^q M)$ satisfies

$$(2.4.2) \qquad \sup_M \left\| D_\infty^t X \right\|_{h_\infty} < \infty$$

for all t, then

$$(2.4.3) \qquad \lim_{z \to \infty} \sup_{\partial M_0} \left\| D_{(h)}^t X - D_\infty^t X \right\|_{h_\infty} = 0$$

for all t.

PROOF. Since

$$A^i_{jk} = \frac{1}{2}h^{il}\left[(D_\infty h)_{kil} + (D_\infty h)_{jlk} - (D_\infty h)_{ljk}\right],$$

(2.4.1) follows from (2.1.1). As for (2.4.3) we have for $l \geq 0$ that
(2.4.4)

$$\left\| D_{(h)}^{l+1}X - D_\infty^{l+1}X \right\|_{h_\infty} \leq \left\| D_{(h)}^l\left(D_{(h)}X - D_\infty X\right) - D_\infty^l\left(D_{(h)}X - D_\infty X\right) \right\|_{h_\infty}$$

$$+ \left\| D_\infty^l\left(D_{(h)}X - D_\infty X\right) \right\|_{h_\infty} + \left\| D_{(h)}^l D_\infty X - D_\infty^l D_\infty X \right\|_{h_\infty}.$$

Clearly if X satisfies (2.4.2), then so does $D_\infty X$. Furthermore because of (2.4.1) and the fact that $D_{(h)}X - D_\infty X$ is a sum of contractions of $\pm A \otimes X$, we have that $D_{(h)}X - D_\infty X$ also satisfies (2.4.2) when X does. Thus (2.4.3) holds for $t = l + 1$ if it holds for $t = l$. Since (2.4.3) is obviously true for $l = 0$, we are done. $\quad \square$

Another thing to note is that the norm and covariant derivative in (2.1.1) do not have to be those of h_∞.

PROPOSITION (2.5). *Let h, j, and k be asymptotically translation invariant metrics. If $X \in C^\infty(T_r^q M)$ satisfies*

$$(2.5.1) \qquad \lim_{z \to \infty} \sup_{\omega \in \partial M_0} \left\| D_{(h)}^t X \right\|_h = 0$$

for all t, then

$$(2.5.2) \qquad \lim_{z \to \infty} \sup_{\omega \in \partial M_0} \left\| D_{(j)}^t X \right\|_k = 0$$

for all t.

PROOF. Because h and k are asymptotically translation invariant, there is a constant, c, such that $c^{-1}\| \; \|_k \leqslant \| \; \|_h \leqslant c\| \; \|_k$. Thus (2.5.1) implies (2.5.2) for $t = 0$.

If $l \geqslant 0$, then

$$(2.5.3) \qquad \left\| D_{(j)}^{l+1} X \right\|_k \leqslant \left\| D_{(j)}^l D_{(h)} X \right\|_k + \sum_{i=1}^{3} \left\| D_{(j)}^l \left(D_{(i)} - D_{(i+1)} \right) X \right\|_k$$

with $D_{(1)} = D_{(j)}$, $D_{(2)} = D_{(j_\infty)}$, $D_{(3)} = D_{(h_\infty)}$, and $D_{(4)} = D_{(h)}$.

Let $B_{(i)} = D_{(i)} - D_{(i+1)}$. Then $D_{(i)} X - D_{(i+1)} X$ is a sum of contractions of $\pm B_{(i)} \otimes X$, and so by Leibnitz's rule

$$(2.5.4) \quad \left\| D_{(j)}^{l+1} X \right\|_k \leqslant \left\| D_{(j)}^l D_{(h)} X \right\|_k + c \sum_{s=0}^{l} \left\| D_{(j)}^s B_1 \right\|_k \left\| D_{(j)}^{l-s} X \right\|_k$$

$$+ c \sum_{s=0}^{l} \left\| D_{(j)}^s B_2 \right\|_k \left\| D_{(j)}^{l-s} X \right\|_k + \left\| D_{(j)}^l \left(D_{(h_\infty)} X - D_{(h)} X \right) \right\|_k.$$

It follows from the previous proposition that $D_{(h_\infty)}X - D_{(h)}X$ satisfies (2.5.1) for all t and that $\sup_M \| D_{(j)}^s B_1 \|_j < \infty$ for all s. Since $B_{(2)}$ is independent of z on $\partial M_0 \times \mathbf{R}^+$ it also follows from (2.4) that $\sup_M \| D_{(j)}^s B_2 \|_j < \infty$ for all s. Because $D_{(h)}X$ obviously satisfies (2.5.1) when X does and the j and k norms are equivalent, we have that (2.5.2) holds for $t = l + 1$ if it holds for $t = l$. Since it holds for $t = 0$ we are done. $\quad \square$

It is also useful to note that if $g = e^{2\rho}h$ is admissible, then ρ satisfies (2.3.1) for some θ.

PROPOSITION (2.6). *Suppose $g = e^{2\rho}h$ is admissible. Then ρ satisfies (2.3.1) for some θ.*

PROOF. By assumption there exists a metric, $\bar{h}$, asymptotic to a translation invariant metric, $\bar{h}_\infty$; a function, $\bar{\rho}$; and a translation invariant 1-form, $\bar{\theta}$, such that $g = e^{2(\bar{\rho})}\bar{h}$ and (2.3.1) holds for $\bar{\rho}$ and $\bar{\theta}$. Hence $h = e^{2(\bar{\rho}-\rho)}\bar{h}$. By (2.1.1) and the

previous proposition this means that

$$(2.6.1) \qquad \lim_{z \to \infty} \sup_{\omega \in \partial M_0} \left\| D_{(j)}^t \left(e^{2(\bar{\rho} - \rho)} \bar{h} - h_\infty \right) \right\|_j = 0$$

for all t and all asymptotically translation invariant metrics j. It follows that $\lim_{z \to \infty} (\bar{\rho}(\omega, z) - \rho(\omega, z))$ exists for each $\omega \in \partial M_0$, call it $\phi(\omega)$, and that $h_\infty = e^{2\phi} \bar{h}_\infty$. Thus ϕ is C^∞. Picking $j = \bar{h}_\infty$ in (2.6.1) we get

$$(2.6.2) \qquad \lim_{z \to \infty} \sup_{\omega \in \partial M_0} \left\| D_{(h_\infty)}^t \left(e^{2(\bar{\rho} - \rho)} \bar{h} - e^{2\phi} \bar{h}_\infty \right) \right\|_{\bar{h}_\infty} = 0.$$

It is easy to see from this that ρ satisfies (2.3.1) with $\theta = \bar{\theta} - d\phi$. □

Several examples of admissible metrics were given in the introduction. In these examples the exponent of the conformal factor, ρ, is a function of only z on M_∞ and is $O(z)$ as $z \to \infty$ on each component of M_∞. We now show that every admissible metric is of this form.

LEMMA (2.7). *Suppose ρ and θ are as in (2.3). If we express θ as $\theta = \psi(\omega) + f(\omega) \, dz$ with $\psi \in C^\infty(\Lambda^1 \partial M_0)$, then f is locally constant on ∂M_0.*

PROOF. Let $\rho = f(\omega) z + \sigma(\omega, z)$ on M_∞. Since

$$(2.7.1) \qquad d\rho = \left(f(\omega) + \partial_z \sigma(\omega, z) \right) dz + z \tilde{d} f + \tilde{d} \sigma,$$

where $\tilde{d}$ is the exterior derivative on ∂M_0, we have by (2.3.1) that

$$(2.7.2) \qquad \lim_{z \to \infty} \sup_{\omega \in \partial M_0} \left| \partial_z \sigma(\omega, z) \right| = 0.$$

Let $\gamma \colon [0, 1] \to \partial M_0$ be a path from ω_0 to ω_1. By Stokes's theorem and (2.7.2) we get

$$(2.7.3) \qquad \left| f(\omega_1) - f(\omega_0) \right| = \lim_{z \to \infty} \left| \int_\gamma \tilde{d} \partial_z \rho \right|.$$

Since h is asymptotically translation invariant, (2.7.3) combined with the Cauchy-Schwarz inequality gives

$$(2.7.4) \qquad \left| f(\omega_1) - f(\omega_0) \right| \leqslant c \lim_{z \to \infty} \sup_{\omega \in \partial M_0} \left\| \tilde{d} \partial_z \rho(\omega, z) \right\|_h.$$

However this last term is no greater than $\lim_{z \to \infty} \sup_{\omega \in \partial M_0} \left\| \partial_z d\rho(\omega, z) \right\|_h$ and by (2.3.1) this is 0. □

LEMMA (2.8). *Let $\theta = \psi(\omega) + f(\omega) \, dz$, as above. Then ψ is an exact 1-form on ∂M_0.*

PROOF. As in the previous lemma, let $\rho = f(\omega) z + \sigma(\omega, z)$ on M_∞. From that lemma we have that $d\rho = f(\omega) \, dz + d\sigma(\omega, z)$. Hence by condition (2.3.1)

$$(2.8.1) \qquad \lim_{z \to \infty} \sup_{\omega \in \partial M_0} \left\| d\sigma - \psi \right\|_h = 0.$$

If $\gamma \colon [0, 1] \to \partial M_0$ is any smooth loop, then

$$(2.8.2) \qquad \left| \int_\gamma \psi \right| = \left| \int_\gamma \psi - d\sigma(\omega, z) \right| \leqslant c \sup_{\partial M_0} \left\| \psi(\omega) - d\sigma(\omega, z) \right\|_h.$$

Since this is true for every z and (2.8.1) is true, $\int_\gamma \psi = 0$. □

THEOREM (2.9). *Suppose $g = e^{2\rho}h$ is admissible. There is a C^∞ function $\bar{\rho}$: $M \to \mathbf{R}$ and an asymptotically translation invariant metric $\bar{h}$ such that*

(2.9.1)   *$\bar{\rho}$ is a function of only z on M_∞;*

(2.9.2)   *if $\Omega_1, \ldots, \Omega_L$ are the components of ∂M_0, then there are constants $c_1, \ldots, c_L$ such that $\lim_{z \to \infty} \partial_z \bar{\rho}(z) = c_l$ uniformly on Ω_l;*

(2.9.3)   *$\displaystyle\lim_{z \to \infty} \partial_z^t \bar{\rho}(z) = 0$ uniformly on ∂M_0 for all $t > 1$;*

and

(2.9.4) $$g = e^{2\bar{\rho}}\bar{h}.$$

PROOF. For each $l = 1, \ldots, L$ let ω_l be a fixed point in Ω_l. We take $\bar{\rho}$ to be any C^∞ function on M that equals $\rho(\omega_l, z)$ on $\Omega_l \times \mathbf{R}^+$ and take $\bar{h}$ to be $e^{2(\rho - \bar{\rho})}h$.

If, as in (2.7) and (2.8), ρ is expressed on M_∞ as $\rho = f(\omega)z + \sigma(\omega, z)$, where $\theta = \psi(\omega) + f(\omega)\,dz$, then on $\Omega_l \times \mathbf{R}^+$ $\bar{\rho} = f(\omega_l)z + \sigma(\omega_l, z)$. That $\bar{\rho}$ satisfies (2.9.2) and (2.9.3) follows from condition (2.3.1).

To show that $\bar{h}$ is asymptotically translation invariant, first note that by Lemma (2.8) there is a function $k(\omega)$ such that $\psi = dk(\omega)$. Combining this with the fact that f is locally constant, we have that if γ: $[0, 1] \to \Omega_l$ is a path from ω_l to ω then

(2.9.5) $$\rho(\omega, z) - \rho(\omega_l, z) = \int_\gamma dk(\omega) + \int_\gamma [d\sigma(\omega, z) - dk(\omega)].$$

This means that

(2.9.6) $$\lim_{z \to \infty} \left| \rho(\omega, z) - \bar{\rho}(z) - \left(k(\omega) - k(\omega_l)\right) \right| \leqslant c \lim_{z \to \infty} \sup_{\omega \in \partial M_0} \|d\sigma - dk\|_h.$$

From condition (2.3.1) we have that for each $t \geqslant 0$

(2.9.7) $$\lim_{z \to \infty} \sup_{\omega \in \partial M_0} \left\| D_{(h)}^{t+1}\sigma - D_{(h)}^t\,dk \right\|_h = 0.$$

Thus the right-hand side of (2.9.6) is 0 and so

(2.9.8) $$\lim_{z \to \infty} \sup_{\omega \in \partial M_0} \left\| \rho(\omega, z) - \bar{\rho}(z) - \left(k(\omega) - k(\omega_l)\right) \right\|_h = 0.$$

Combining this with (2.9.7) and the fact that h is asymptotic to a translation invariant h_∞, we get that $\bar{h}$ is asymptotic to $e^{2(k(\omega) - j(\omega))}h_\infty$ where $j(\omega) = k(\omega_l)$ on Ω_l.

That $\bar{\rho}$ satisfies condition (2.3.1) for some $\bar{\theta}$ follows from Proposition 2.6. ($\bar{\theta}$ in fact is $f(\omega)\,dz$.) $\square$

We conclude this section with a lemma which will be useful in the next two sections.

LEMMA (2.10). *Let g be an admissible metric. If $T \in C^\infty(T_r^q M)$, then in terms of a coordinate system of the form $(\Omega_\nu \times \mathbf{R}^+, (\phi_\nu \times \mathrm{id}))$ on M_∞*

(2.10.1) $$\nabla_{i_1} \cdots \nabla_{i_l} T_{j_1 \cdots j_q}^{m_1 \cdots m_r} = \frac{\partial^l T_{j_1 \cdots j_q}^{m_1 \cdots m_r}}{\partial X_{i_1} \cdots \partial X_{i_l}} + L(T)$$

with $L = \sum_{|\alpha| < l} L_\alpha(\omega, z)\partial^\alpha$ *and* L_α *an* $(l + k)n \times kn$ *matrix. Moreover there exist translation invariant matrices* $L_\alpha^\infty(\omega)$ *such that for all* $\beta \in \mathbf{N}^n$

$$(2.10.2) \qquad \lim_{z \to \infty} \partial^\beta(L_\alpha(\omega, z) - L_\alpha^\infty(\omega)) = 0$$

uniformly on compact subsets of Ω_ν.

PROOF. This is a straightforward calculation that uses condition (2.3.1).   □

3. Weighted Sobolev spaces. As was mentioned in the introduction weighted Sobolev spaces need to be used on noncompact manifolds in order to get Fredholm results for elliptic operators. Moreover which weights should be used depends on the operator. In the case of elliptic operators that are asymptotically translation invariant, the proper weights are the functions $\exp(\tilde\delta)$ with $\tilde\delta \in C^\infty(M)$ and $\tilde\delta/z$ locally constant on M_∞. Since $\tilde\delta/z$ is locally constant on M_∞ and since we do not care what $\tilde\delta$ is on M_0, we may identify it with an element, δ, of $\mathbf{R}^L$, where L is the number of ends of M.

DEFINITION (3.1). If L = number of components of ∂M_0 and $\delta \in \mathbf{R}^L$, then δz is a C^∞ function on M that on the lth component of M_∞ equals $\delta_l z$.

NOTATION (3.2). If $\delta_1, \delta_2 \in \mathbf{R}^L$, then $\delta_1 \leqslant \delta_2$ means $\delta_{1l} \leqslant \delta_{2l}$ for $l = 1, \ldots, L$.

With this notation we are ready to define the weighted Sobolev spaces needed for Laplacians associated with asymptotically translation invariant metrics. In what follows we let E be either $T_r^q(M)$ or $\wedge^q(M)$ and, as usual, take h to be asymptotically translation invariant.

DEFINITION (3.3). Let $1 < p < \infty$, $\delta \in \mathbf{R}^L$, and $s \in \mathbf{N}$. For $\sigma \in L_{s,\mathrm{loc}}^p(E)$ set

$$(3.3.1) \qquad \|\sigma\|_{p,s,\delta} = \left(\sum_{t=0}^s \int_M \left\| e^{\delta z} D_{(h)}^t \sigma \right\|_h^p dV_h \right)^{1/p}.$$

DEFINITION (3.4). If $1 < p < \infty$, $\delta \in \mathbf{R}^L$, and $s \in \mathbf{N}$, then

$$(3.4.1) \qquad W_{s,\delta}^p(E) = \left\{ \sigma \in L_{s,\mathrm{loc}}^p(E) \mid \|\sigma\|_{p,s,\delta} < \infty \right\}.$$

If $1 < p < \infty$, $\delta \in \mathbf{R}^L$, and $-s \in \mathbf{N}$ then $W_{s,\delta}^p(E)$ is the dual space of $W_{-s,-\delta}^{p'}(E)$, where $1/p + 1/p' = 1$.

A few remarks about these definitions are in order. First of all, it is easy to show that these are Banach spaces. Secondly notice that by picking $\delta \in \mathbf{R}^L$ instead of in $\mathbf{R}$ we allow for different growth on the different ends of M. Thirdly note that the standard Sobolev space, $L_s^p(E, h)$, equals $W_{s,0}^p(E)$ and that if $\delta_1, \delta_2 \in \mathbf{R}^L$ with $\delta_1 \leqslant \delta_2$ and $s \geqslant 0$, then $W_{s,\delta_2}^p(E) \subset W_{s,\delta_1}^p(E)$. Thus we have embedded $L_s^p(E, h)$ into a family of weighted Sobolev spaces. Finally observe that any two asymptotically translation invariant metrics give rise to equivalent weighted Sobolev norms. Indeed it follows from Lemma (2.10) that the norm (3.3.1) is equivalent to the following metric independent one.

DEFINITION (3.5). Let $\{(\Omega_\nu, \tilde\phi_\nu)\}_{\nu=1}^N$ be an atlas for ∂M_0 and set $M_\nu = \Omega_\nu \times (0, \infty)$ and $\phi_\nu = (\tilde\phi_\nu, \mathrm{id})$. Then $\{(M_\nu, \phi_\nu)\}_{\nu=1}^N$ is an atlas for M_∞. Extend this to a finite atlas, $\{(M_\nu, \phi_\nu)\}_{\nu=1}^{N+J}$ of M and let $\{\psi_\nu\}_{\nu=1}^{N+J}$ be a subordinate partition of unity. If

$1 < p < \infty$, $\delta \in \mathbf{R}^L$, $s \in \mathbf{N}$, and $\sigma \in L^p_{s,\mathrm{loc}}(E)$, then

(3.5.1)

$$\|\sigma\|_{\tilde{p},s,\delta} = \left(\sum_{\nu=1}^{N+J} \sum_{|\alpha|=0}^{s} \sum_{\substack{i_1,\ldots,i_q=1 \\ j_1,\ldots,j_r=1}}^{n} \int_{\phi_\nu(M_\nu)} \left| \exp\left(\tilde{\delta} \circ \phi_\nu^{-1}\right) \partial^\alpha \left(\psi_\nu \sigma_{i_1\cdots i_q}^{j_1\cdots j_r} \right) \circ \phi_\nu^{-1} \right|^p dx \right)^{1/p}.$$

It is the norm (3.5.1) that is used in [10] and so the Fredholm results there hold for the spaces in (3.4). Before stating these results we must say what an asymptotically translation invariant operator is. Recall that if $A: C^\infty(E) \to C^\infty(E)$ is an mth order linear differential operator, then in terms of the covariant derivative of h we have $A = \sum_{t=0}^m a_t \cdot D^t_{(h)}$ with $a_t \in C^\infty(T^{q+r}_{q+r+t}M)$ and "·" meaning tensor product followed by contraction. A differential operator is said to be translation invariant if it is invariant under the $\mathbf{R}^+$-action on M_∞.

DEFINITION (3.6). Suppose that $A = \sum_{t=0}^m a_t \cdot D^t_{(h)}$ and $A_\infty = \sum_{t=0}^m a_t^\infty \cdot D^t_{(h)}$ are two mth order differential operators and that A_∞ is translation invariant. The operator A is asymptotic to A_∞, written $A \sim A_\infty$, if for every $l \in \mathbf{N}$ and $0 \leqslant t \leqslant m$

(3.6.1)
$$\lim_{z \to \infty} \sup_{\partial M_0} \left\| D^l_{(h)}\left(a_t - a_t^\infty\right) \right\|_h = 0.$$

If $A \sim A_\infty$ then A is said to be asymptotically translation invariant.

It is easy to see from Lemma (2.10) that this definition is independent of the choice of asymptotically translation invariant metric, h, used to express A and A_∞.

The Fredholm results we need are the following.

THEOREM (3.7). *Suppose A, A_∞: $C^\infty(E) \to C^\infty(E)$ are mth order, elliptic differential operators, A_∞ is translation invariant, and $A \sim A_\infty$. Then*

(3.7.1) *A is continuous from $W^p_{s+m,\delta}(E)$ to $W^p_{s,\delta}(E)$ for all $1 < p < \infty$, $s \in \mathbf{Z}$, and $\delta \in \mathbf{R}^L$.*

(3.7.2) *If $1 < p < \infty$, $s \in \mathbf{Z}$, and $\delta \in \mathbf{R}^L$, then for all $\sigma \in L^p_{s+m,\mathrm{loc}}(E)$*
$$\|\sigma\|_{p,s+m,\delta} \leqslant c\left(\|A\sigma\|_{p,s,\delta} + \|\sigma\|_{p,s,\delta}\right)$$
for some c independent of σ.

(3.7.3) *There is a subset $\mathscr{D}_A \subset \mathbf{R}^L$ such that if $1 < p < \infty$ and $s \in \mathbf{N}$, then A: $W^p_{s+m,\delta}(E) \to W^p_{s,\delta}(E)$ is Fredholm if and only if $\delta \in \mathbf{R}^L - \mathscr{D}_A$.*

PROOF. See inequality (2.4) in [10] for (3.7.2) and §7 in [10] for (3.7.3). □

Much can be said about the set $\mathscr{D}_A$. For instance $\mathscr{D}_A = \mathscr{D}_{A_\infty}$. Also it is of measure zero in $\mathbf{R}^L$. In fact it is of the form

(3.8) $\quad \mathscr{D}_A = \left(\mathscr{D}_{A_1} \times \mathbf{R}^{L-1} \right) \cup \left(\mathbf{R} \times \mathscr{D}_{A_2} \times \mathbf{R}^{L-2} \right) \cup \cdots \cup \left(\mathbf{R}^{L-1} \times \mathscr{D}_{A_L} \right)$

with each $\mathscr{D}_{A_i}$ a countable discrete set in $\mathbf{R}$. That this is so comes from the fact that $(\delta_1,\ldots,\delta_L)$ is in $\mathscr{D}_A$ if at least one of the δ_i is an eigenvalue of an associated generalized eigenvalue problem for A_∞ on $\partial M_0 \times \mathbf{R}$. (See [10] for details.) For example, suppose h is a metric that is the product metric $dz^2 + h_0$ on M_∞. Then on M_∞ a q-form σ equals $\psi(\omega, z) + \tau(\omega, z) \wedge dz$ for some $\psi \in \Gamma \wedge^q(\partial M_0)$ and $\tau \in \Gamma \wedge^{q-1}(\partial M_0)$ and

$$\Delta_h \sigma = -\frac{\partial^2 \psi}{\partial z^2} + \Delta_{h_0}\psi + (-1)^{q+1}\left(-\frac{\partial^2 \tau}{\partial z^2} + \Delta_{h_0}\tau \right) \wedge dz.$$

In this case $\delta = (\delta_1, \ldots, \delta_L)$ is in $\mathscr{D}_{\Delta_h}$ if for some l we have that δ_l is an eigenvalue for either Δ_{h_0}: $C^\infty(\wedge^q\Omega_l) \to C^\infty(\wedge^q\Omega_l)$ or Δ_{h_0}: $C^\infty(\wedge^{q-1}\Omega_l) \to C^\infty(\wedge^{q-1}\Omega_l)$ where Ω_l is the lth component of ∂M_0. Thus for instance

$$\Delta_h: L_2^p(\wedge^q M) = W_{2,0}^p(\wedge^q M) \to L^p(\wedge^q M) = W_{0,0}^p(\wedge^q M)$$

is Fredholm if and only if $H_{\mathcal{D}R}^{q-1}(\partial M_0) = H_{\mathcal{D}R}^q(\partial M_0) = 0$. Finally let us point out that even though $\mathscr{D}_A = \mathscr{D}_{A_\infty}$ it is not the case that $\mathrm{ind}_\delta A = \mathrm{ind}_\delta A_\infty$ if $\delta \in \mathbf{R}^L - \mathscr{D}_A$. However, if $\delta_1 < \delta_2$ and $\delta_1, \delta_2 \in \mathbf{R}^L - \mathscr{D}_A$, then it is true that $\mathrm{ind}_{\delta_1} A - \mathrm{ind}_{\delta_2} A = \mathrm{ind}_{\delta_1} A_\infty - \mathrm{ind}_{\delta_2} A_\infty$. (For details see [10].)

The reader may feel that the Fredholm result follows from the a priori inequality (3.7.2). However since M is not compact the imbedding $W_{s+m,\delta}^p(E) \to W_{s,\delta}^p(E)$ is not compact and so (3.7.2) does not imply Fredholm. We conclude this section by presenting various properties of the spaces $W_{s,\delta}^p(E)$, including a weighted compact imbedding theorem.

PROPOSITION (3.9). $C_0^\infty(E)$ is dense in $W_{s,\delta}^p(E)$.

PROOF. This follows from the fact that (3.5.1) gives an equivalent norm for these spaces and classical facts about Sobolev spaces in $\mathbf{R}^n$.

THEOREM (3.10) (WEIGHTED SOBOLEV EMBEDDING). There is a continuous embedding $W_{s,\delta}^p(E) \to W_{\bar{s},\bar{\delta}}^{\bar{p}}(E)$ if

(i) $s - \bar{s} \geqslant n/p - n/\bar{p}$ ($n = \dim M$),

(ii) $s \geqslant \bar{s} \geqslant 0$ and either

(iii) $1 < p \leqslant \bar{p} < \infty$ with $\bar{\delta} \leqslant \delta$ or

(iii′) $1 < \bar{p} < p < \infty$ with $\bar{\delta} < \delta$.

PROOF. See Lemma 7.2 in [10]. □

In proving the compact embedding theorem we need the following

LEMMA (3.11). For $\mathscr{F} \subset M$, a measurable set, define

$$(3.11.1) \qquad \|\sigma\|_{p,s,\delta}(\mathscr{F}) = \left(\sum_{t=0}^{s} \int_{\mathscr{F}} \left\| e^{\delta z} D_{(h)}^t \sigma \right\|_h^p dV_h \right)^{1/p}.$$

Also let $M_R = M_0 \cup (\partial M_0 \times (0, R))$. If s, $\bar{s}$, p, $\bar{p}$, δ and $\bar{\delta}$ satisfy the conditions in (3.10), $\sigma \in W_{s,\delta}^p(E)$, and $R > 1$, then

$$(3.11.2) \qquad \|\sigma\|_{\bar{p},\bar{s},\bar{\delta}}(M - M_{2R}) \leqslant c\|\sigma\|_{p,s,\delta}(M - M_R)$$

for some c independent of σ and R.

PROOF. For $\tilde{\phi} \in C_0^\infty([0,\infty))$ such that $\tilde{\phi} \equiv 1$ on $[0,1]$ and $\tilde{\phi} \equiv 0$ on $[2,\infty)$, take $\tilde{\phi}_R(z) = \tilde{\phi}(z/R)$. Define $\phi_R \in C_0^\infty(M)$ by

$$(3.11.3) \qquad \phi_R(x) = \begin{cases} 1 & \text{if } x \in M_0, \\ \tilde{\phi}_R(z) & \text{if } x = (\omega, z) \in M_\infty. \end{cases}$$

Clearly we have

$$(3.11.4) \qquad \|\sigma\|_{\bar{p},\bar{s},\bar{\delta}}(M - M_{2R}) \leqslant \|(1 - \phi_R)\sigma\|_{\bar{p},\bar{s},\bar{\delta}}.$$

By (3.10) we have

$$(3.11.5) \quad \left\| (1 - \phi_R)\sigma \right\|_{\bar{p},\bar{s},\bar{\delta}} \leqslant c \left\| (1 - \phi_R)\sigma \right\|_{p,s,\delta} = c \left\| (1 - \phi_R)\sigma \right\|_{p,s,\delta}(M - M_R).$$

However from the construction of ϕ_R we see that if $R > 1$, then

$$(3.11.6) \quad \left\| (1 - \phi_R)\sigma \right\|_{p,s,\delta}(M - M_R) \leqslant c \left\| \sigma \right\|_{p,s,\delta}(M - M_R)$$

with c independent of σ and R.

THEOREM (3.12) (COMPACT EMBEDDING). *If*
  (i) $s - \bar{s} > n/p - n/\bar{p}$,
  (ii) $s > \bar{s} \geqslant 0$, *and*
  (iii) $\bar{\delta} < \delta$,
then the embedding $W^p_{s,\delta}(E) \to W^{\bar{p}}_{\bar{s},\delta}(E)$ is compact.

PROOF. Suppose $\{\sigma_j\}$ is a bounded sequence in $W^p_{s,\delta}(E)$. Using the notation in (3.11) and Minkowski's inequality we get

$$(3.12.1) \quad \left\| \sigma_j \right\|_{\bar{p},\bar{s},\bar{\delta}} \leqslant c \left(\left\| \sigma_j \right\|_{\bar{p},\bar{s},\bar{\delta}}(M_{2R}) + \left\| \sigma_j \right\|_{\bar{p},\bar{s},\bar{\delta}}(M - M_{2R}) \right).$$

Let $\hat{\delta}$ be defined by $\hat{\delta}_l = (\delta_l + \bar{\delta}_l)/2$. Then $\bar{\delta} < \hat{\delta} < \delta$ and so by (3.11.2)

$$(3.12.2) \quad \left\| \sigma_j \right\|_{\bar{p},\bar{s},\bar{\delta}}(M - M_{2R}) \leqslant c \left\| \sigma_j \right\|_{p,s,\hat{\delta}}(M - M_R).$$

For all j we have

$$(3.12.3) \quad \left\| \sigma_j \right\|_{p,s,\hat{\delta}}(M - M_R) = \left(\sum_{t=0}^{s} \int_{\partial M_0 \times (R,\infty)} \left\| e^{(\hat{\delta} - \delta)z} e^{\delta z} D^t_{(h)} \sigma_j \right\|_h^p dV_h \right)^{1/p}$$

$$\leqslant e^{-bR} \left\| \sigma_j \right\|_{p,s,\delta} \leqslant c e^{-bR}$$

with $0 < b = \min\{\delta_l - \hat{\delta}_l\}$ and $c = \sup_j\{\left\| \sigma_j \right\|_{p,s,\delta}\}$. On the other hand, it follows from the classical Rellich theorem that there is a subsequence of $\{\sigma_j\}$ that converges in the $\| \ \|_{\bar{p},\bar{s},\bar{\delta}}(M_{2R})$ norm. Thus if $N \in \mathbf{N}$, we may pick R large enough so that there is a subsequence, $\{\tilde{\sigma}_j\}$ of $\{\sigma_j\}$ that satisfies

$$(3.12.4) \quad \left\| \tilde{\sigma}_j - \tilde{\sigma}_m \right\|_{\bar{p},\bar{s},\bar{\delta}} \leqslant 2^{-N}$$

for all j and m.

Let $\{\sigma_{j1}\}$ be a subsequence of $\{\sigma_j\}$ such that $\left\| \sigma_{j1} - \sigma_{m1} \right\|_{\bar{p},\bar{s},\bar{\delta}} \leqslant 2^{-1}$ for all j and m. For $N = 2, 3, \ldots$ pick $\{\sigma_{jN}\}$ to be a subsequence of $\{\sigma_{jN-1}\}$ satisfying $\left\| \sigma_{jN} - \sigma_{mN} \right\|_{\bar{p},\bar{s},\bar{\delta}} \leqslant 2^{-N}$ for all j and m. Finally let $\hat{\sigma}_j = \sigma_{jj}$. Since $\left\| \hat{\sigma}_j - \hat{\sigma}_m \right\|_{\bar{p},\bar{s},\bar{\delta}} \leqslant 2^{-K}$ for $j, m > K$, the sequence $\{\hat{\sigma}_j\}$ converges in $W^{\bar{p}}_{\bar{s},\delta}(E)$. □

4. Weighted Sobolev spaces for admissible metrics. For the Laplacians associated with a general admissible metric, $g = e^{2\rho}h$, the Sobolev spaces introduced in the last section are not the right ones to use, unless ρ is bounded. In this section we introduce the correct spaces and give some of their properties. As before E is either $T^q_r M$ or $\wedge^q M$ and h is an asymptotically translation invariant metric.

DEFINITION (4.1). Let $1 < p < \infty$, $\delta \in \mathbf{R}^L$, $a \in \mathbf{R}$, and $s \in \mathbf{N}$. Suppose $g = e^{2\rho}h$ is admissible. If $\sigma \in L^p_{s,\text{loc}}(E)$, then

$$(4.1.1) \quad \left\| \sigma \right\| (W^p_{s,\delta,a}(E,g)) = \left(\sum_{t=0}^{s} \int_M \left\| e^{\delta z + (t+a)\rho} D^t_{(g)} \sigma \right\|_g^p dV_g \right)^{1/p}.$$

DEFINITION (4.2). *If* $1 < p < \infty$, $\delta \in \mathbf{R}^L$, $a \in \mathbf{R}$, *and* $s \in \mathbf{N}$, *then*

$$(4.2.1) \qquad W^p_{s,\delta,a}(E, g) = \left\{ \sigma \in L^p_{s,\mathrm{loc}}(E) \mid \|\sigma\|(W^p_{s,\delta,a}(E, g)) < \infty \right\}.$$

If $1 < p < \infty$, $\delta \in \mathbf{R}^L$, $a \in \mathbf{R}$, and $-s \in \mathbf{N}$, then $W^p_{s,\delta,a}(E, g)$ is the dual space of $W^{p'}_{-s,-\delta,-a}(E, g)$, where $1/p + 1/p' = 1$.

Once again it is straightforward to show that these are Banach spaces. Also notice that if ρ is bounded, and so g is asymptotically translation invariant, then $W^p_{s,\delta,a}(E, g) = W^p_{s,\delta}(E)$, equipped with an equivalent norm. Note too that $W^p_{0,0,0}(E, g) = L^p(E, g)$. However, if ρ is not bounded, then $W^p_{s,0,0}(E, g)$ is not the classical Sobolev space, $L^p_s(E, g)$.

The need for the term $\exp((t + a)\rho)$ in (4.1.1) may seem strange. It is best explained by Proposition (4.4). Before giving that proposition let us mention the following useful representation for elements of $W^p_{s,\delta,a}(T^q_r M, g)$ for $s < 0$.

PROPOSITION (4.3). *For every* $\sigma \in W^p_{s,\delta,a}(T^q_r M, g)$ *with* $s < 0$, *there is a unique* $(1 - s)$-*tuple,* $(\sigma_0, \ldots, \sigma_{-s}) \in \bigoplus_{t=0}^{-s} W^p_{0,\delta,a-t}(T^{q+t}_r M, g)$, *such that*

$$(4.3.1) \qquad \sigma[\phi] = \int_M \sum_{t=0}^{-s} \langle \sigma_t, D^t_{(g)}\phi \rangle_g \, dV_g$$

for $\phi \in W^{p'}_{-s,-\delta,-a}(T^q_r M, g)$, *and*

$$(4.3.2) \quad \|\sigma\|(W^p_{s,\delta,a}(T^q_r M, g)) = \left(\sum_{t=0}^{-s} \left[\|\sigma_t\|(W^p_{0,\delta,a-t}(T^{q+t}_r M, g)) \right]^p \right)^{1/p}.$$

PROOF. The standard proof, as in [1], can be used here too. □

PROPOSITION AND DEFINITION (4.4). *For* $g = e^{2\rho}h$ *admissible define the operator* $K_{a,p}$ *on* $W^p_{s,\delta,a}(T^q_r M, g)$ *as follows: if* $s \in \mathbf{N}$, *then*

$$(4.4.1) \qquad\qquad K_{a,p}\sigma = e^{(a+r-q+n/p)\rho}\sigma.$$

If $-s \in \mathbf{N}$ *and*

$$(\sigma_0, \ldots, \sigma_{-s}) \in \bigoplus_{t=0}^{-s} W^p_{0,\delta,a-t}(T^{q+t}_r M, g)$$

represents $\sigma \in W^p_{s,\delta,a}(T^q_r M, g)$ *as in* (4.3), *then*

$$(4.4.2) \quad (K_{a,p}\sigma)[\psi] = \int_M \sum_{t=0}^{-s} \left\langle \exp\left(\left(a + r - q - t + \frac{n}{p}\right)\rho\right)\sigma_t, D^t_{(h)}\phi \right\rangle_h dV_h$$

for $\phi \in W^{p'}_{-s,-\delta,0}(T^q_r M, h)$.

For all $s \in \mathbf{Z}$

$$(4.4.3) \qquad K_{a,p} : W^p_{s,\delta,a}(T^q_r M, g) \to W^p_{s,\delta,0}(T^q_r M, h) = W^p_{s,\delta}(T^q_r M)$$

is a Banach space isomorphism onto.

PROOF. If $s \geqslant 0$, then

$$(4.4.4) \quad \left\| e^{(a+r-q+n/p)\rho}\sigma \right\|_{p,s,\delta}$$

$$\leqslant \left(\sum_{t=0}^{s} \sum_{b \leqslant t} c_{bt} \int_M \left\| e^{\delta z + (a+r-q+n/p)\rho} D^{t-b}_{(h)}\rho D^b_{(h)}\sigma \right\|^p_h dV_h \right)^{1/p}.$$

It follows from condition (2.3.1) on ρ that this is

$$(4.4.5) \qquad \leqslant c \left(\sum_{b=0}^{s} \int_M \left\| e^{\delta z + (a - q + r + n/p)\rho} D_{(h)}^b \sigma \right\|_h^p dV_h \right)^{1/p}$$

$$= c \left(\sum_{b=0}^{s} \int_M \left\| e^{\delta z + (a + b)\rho} D_{(h)}^b \sigma \right\|_g^p dV_g \right)^{1/p}.$$

Using Lemma (2.10) we get that this in turn is $\leqslant c\|\sigma\|(W_{s,\delta,a}^p(T_r^q M, g))$.

Thus (4.4.3) is continuous for $s \in \mathbf{N}$. In exactly the same way we can show that $(K_{a,p})^{-1}$ defined by

$$(4.4.6) \qquad \left(K_{a,p} \right)^{-1} \gamma = e^{(q - r - a - n/p)\rho} \gamma$$

is continuous. This proves the proposition for $s \geqslant 0$. The case of $s < 0$ follows easily from this. $\square$

We shall also use $K_{a,p}$ on $W_{s,\delta,a}^p(\wedge^q M, g)$ by considering $\wedge^q M$ to be in $T_0^q M$. The proposition is true then in this case too.

COROLLARY (4.5). *If g is admissible, then $C_0^\infty(0)$ is dense in $W_{s,\delta,a}^p(E, g)$.*

PROOF. The operator $K_{a,p}$ obviously preserves supports and so this follows from Proposition (3.9). $\square$

PROPOSITION (4.6). *For admissible g*

$$(4.6.1) \qquad D_{(g)}: W_{s,\delta,a}^p(E, g) \to W_{s-1,\delta,a+1}^p\left(T_r^{q+1} M, g \right)$$

is continuous.

PROOF. For $s \geqslant 1$ this follows easily from Definition (4.1). For $s < 1$ we have by the density of $C_0^\infty(T_r^{q+1})$ in $W_{1-s,-\delta,-a-1}^{p'}(T_r^{q+1} M, g)$ that

$$(4.6.2) \qquad \left(D_{(g)} \sigma \right)[\psi] = \sigma \left[D_{(g)}^* \psi \right]$$

where $\psi \in W_{1-s,-\delta,-a-1}^{p'}(T_r^{q+1} M, g)$ and $D_{(g)}^*$ is the formal adjoint of $D_{(g)}$. Thus (4.6.1) is continuous for $s \leqslant 0$ provided

$$(4.6.3) \qquad D_{(g)}^*: W_{1-s,-\delta,-a-1}^{p'}\left(T_r^{q+1} M, g \right) \to W_{-s,-\delta,-a}^{p'}\left(T_r^q M, g \right)$$

is continuous. But (4.6.3) is continuous because

$$(4.6.4) \qquad \left(D_{(g)}^* \psi \right)_{i_1 \cdots i_q}^{j_1 \cdots j_r} = e^{-2\rho} h^{lk} \nabla_l \psi_{k i_1 \cdots i_q}^{j_1 \cdots j_r}$$

with ∇ the covariant derivative for g. $\square$

It is apparent from the proof of (4.4) that $K_{a,p}$ would not be bounded were the term $e^{t\rho}$ not in the norm (4.1.1). However once one has introduced $e^{t\rho}$ then $e^{a\rho}$ is needed too in order for Proposition (4.6) to be true.

An immediate corollary of (4.6) is

COROLLARY (4.7). *For g admissible, $s \in \mathbf{Z}$, $\delta \in \mathbf{R}^L$, $a \in \mathbf{R}$ and $1 < p < \infty$ the following operators are continuous:*

$$(4.7.1) \qquad d: W_{s,\delta,a}^p\left(\wedge^q M, g \right) \to W_{s-1,\delta,a+1}^p\left(\wedge^{q+1} M, g \right),$$

$$(4.7.2) \qquad d_g^*: W_{s,\delta,a}^p\left(\wedge^q M, g \right) \to W_{s-1,\delta,a+1}^p\left(\wedge^{q-1} M, g \right),$$

(4.7.3) $$\Delta_g \colon W^p_{s,\delta,a}(\wedge^q M, g) \to W^p_{s-2,\delta,a+2}(\wedge^q M, g),$$

(4.7.4) $$d + d^*_g \colon \bigoplus_{q=0}^{n} W^p_{s,\delta,a}(\wedge^q M, g) \to \bigoplus_{q=0}^{n} W^p_{s-1,\delta,a+1}(\wedge^q M, g). \quad \square$$

Proposition (4.4) allows us to carry over the properties of the spaces $W^p_{s,\delta}(E)$ in §3 to the spaces $W^p_{s,\delta,a}(E, g)$.

THEOREM (4.8) (WEIGHTED SOBOLEV EMBEDDING). *Suppose g is admissible. There is a continuous embedding*

(4.8.1) $$W^p_{s,\delta,a}(E, g) \to W^{\bar{p}}_{\bar{s},\bar{\delta},a+n(1/p-1/p')}(E, g)$$

if

  (i) $s - \bar{s} \geqslant n(1/p - 1/\bar{p})$,
  (ii) $s \geqslant \bar{s} \geqslant 0$ *and either*
  (iii) $1 < p \leqslant \bar{p} < \infty$ *with* $\bar{\delta} \leqslant \delta$ *or*
  (iii′) $1 < \bar{p} < p < \infty$ *with* $\bar{\delta} < \delta$.

PROOF. Let $g = e^{2\rho}h$. By (4.4) and (3.10) we have the sequence of continuous maps:

(4.8.2) $$W^p_{s,\delta,a}(E, g) \overset{K_{a,p}}{\underset{\cong}{\to}} W^p_{s,\delta}(E) \to W^{\bar{p}}_{\bar{s},\bar{\delta}}(E)$$

$$\overset{(K_{a+n(1/p-1/\bar{p}),\bar{p}})^{-1}}{\underset{\cong}{\to}} (\bar{p})^{-1} W^{\bar{p}}_{\bar{s},\bar{\delta},a+n(1/p-1/\bar{p})}(E, g).$$

Furthermore by (4.4.1) and (4.4.6) if $\sigma \in W^p_{s,\delta,a}(E, g)$, then in (4.8.2) it is mapped to

(4.8.3) $$e^{(q-r-a-n/p+n/\bar{p}-n/\bar{p})\rho} \cdot e^{(a+r-q+n/p)\rho}\sigma = \sigma. \quad \square$$

THEOREM 4.9 (COMPACT EMBEDDING). *The embedding* (4.8.1) *is compact when*
  (i) $s - \bar{s} > n/p - n/\bar{p}$,
  (ii) $s > \bar{s} \geqslant 0$ *and*
  (iii) $\bar{\delta} < \delta$.

PROOF. Theorem (3.11) implies the middle map in (4.8.2) is compact when (i)–(iii) hold. $\square$

5. Fredholm results. In this section the Fredholm properties of Laplacians associated with admissible metrics are established. Throughout the section h is, as usual, an asymptotically translation invariant metric. We start with a straightforward lemma.

LEMMA (5.1). *If $g = e^{2\rho}h$ is admissible, then $D^t_{(g)}$ is an asymptotically translation invariant operator for every $t \in \mathbf{N}$.*

PROOF. It is clear from Definition (3.6) that the composition of asymptotically translation invariant operators is again asymptotically translation invariant. Hence we need only consider $t = 1$. In that case $D_{(g)} = D_{(h)} + L$ with L a 0th order operator with coefficients that depend only on h and $D_{(h)}\rho$. The lemma, then, is a consequence of condition (2.3.1) on ρ.

THEOREM (5.2). *Let $g = e^{2\rho}h$ be admissible. For every $1 < p < \infty$, $s \in \mathbf{N}$, and $a \in \mathbf{R}$ there is a subset $\mathscr{D}_\Delta = \mathscr{D}_\Delta(g, p, q, a)$ of $\mathbf{R}^L$ with the properties*

$$(5.2.1) \quad \mathscr{D}_\Delta = \left(\mathscr{D}_1 \times \mathbf{R}^{L-1} \right) \cup \left(\mathbf{R} \times \mathscr{D}_2 \times \mathbf{R}^{L-2} \right) \cup \cdots \cup \left(\mathbf{R}^{L-1} \times \mathscr{D}_L \right)$$

with each $\mathscr{D}_l$ countable and discrete.

(5.2.2) *$\mathscr{D}_\Delta$ depends on p, q, a, the metric g, and $n = \dim M$, but not $s \in \mathbf{N}$.*

$$(5.2.3) \qquad \Delta_g \colon W^p_{s+2,\delta,a}(\wedge^q M, g) \to W^p_{s,\delta,a+2}(\wedge^q M, g)$$

is Fredholm if and only if $\delta \in \mathbf{R}^L - \mathscr{D}_\Delta$.

PROOF. For $b \in \mathbf{R}$, define $A(b)$ to be the operator

$$(5.2.4) \qquad A(b) = e^{(2+b)\rho}\Delta_g e^{-b\rho}.$$

Letting $\overline{\nabla}$ be the covariant derivative for g, we have by a standard identity that
(5.2.5)

$$\left(A(b)\sigma \right)_{i_1 \cdots i_q} = e^{(2+b)\rho}\left[e^{-b\rho}\Delta_g\sigma_{i_1 \cdots i_q} - 2\left(\overline{\nabla}^k e^{-b\rho}\overline{\nabla}_k\sigma_{i_1 \cdots i_q} \right) - \left(\Delta_g e^{-b\rho} \right)\sigma_{i_1 \cdots i_q} \right].$$

Computing the second and third terms on the right-hand side of (5.2.5) and using the definition of Δ_g we get
(5.2.6)

$$\left(A(b)\sigma \right)_{i_1 \cdots i_q} = e^{2\rho}\left[-\overline{\nabla}^k\overline{\nabla}_k\sigma_{i_1 \cdots i_q} + \sum_{\nu=1}^{q} (-1)^{\nu}\left(\overline{\nabla}_{i_\nu}\overline{\nabla}^k - \overline{\nabla}^k\overline{\nabla}_{i_\nu} \right)\sigma_{k i_1 \cdots \hat{i}_\nu \cdots i_q} \right.$$

$$\left. + 2b\overline{\nabla}_\rho^k\overline{\nabla}_k\sigma_{i_1 \cdots i_q} - \left(b^2\|\overline{\nabla}\rho\|_g^2 - b\Delta_g\rho \right)\sigma_{i_1 \cdots i_q} \right].$$

Since $g = e^{2\rho}h$, it follows from (5.2.6) that
(5.2.7)

$$\left(A(b)\sigma \right)_{i_1 \cdots i_q}$$

$$= h^{tk}\left[-\overline{\nabla}_t\overline{\nabla}_k\sigma_{i_1 \cdots i_q} + \sum_{\nu=1}^{q} (-1)^{\nu}\left(\overline{\nabla}_{i_\nu}\overline{\nabla}_t - \overline{\nabla}_t\overline{\nabla}_{i_\nu} \right)\sigma_{k i_1 \cdots \hat{i}_\nu \cdots i_q} + 2b\left(\overline{\nabla}_t\rho\overline{\nabla}_k\sigma_{i_1 \cdots i_q} \right) \right]$$

$$+ \left(b\Delta_h\rho - b(b-n+2)\|\overline{\nabla}\rho\|_h \right)\sigma_{i_1 \cdots i_q}.$$

Thus as a result of Lemma (5.1) and condition (2.3.1) on ρ, we see that $A(b)$ is an asymptotically translation invariant operator. Hence by Theorem (3.7) there is a subset $\mathscr{D}_{A(b)}$ of $\mathbf{R}^L$ of the form described in (5.2.1) with the property that

$$(5.2.8) \qquad A(b) \colon W^p_{s+2,\delta}(\wedge^q M) \to W^p_{s,\delta}(\wedge^q M)$$

is Fredholm if and only if $\delta \in \mathbf{R}^L - \mathscr{D}_{A(b)}$. In particular this is true for $b = a - q + n/p$. However, in this case we also have the commutative diagram:

$$(5.2.9) \qquad \begin{array}{ccc} W^p_{s+2,\delta}(\wedge^q M) & \xrightarrow{A(a-q+n/p)} & W^p_{s,\delta}(\wedge^q M) \\[2mm] \downarrow (K_{a,p})^{-1} & & \uparrow K_{a+2,P} \\[2mm] W^p_{s+2,\delta,a}(\wedge^q M, g) & \xrightarrow{\Delta_g} & W^p_{s,\delta,a+2}(\wedge^q M, g) \end{array}$$

Because of the fact that $(K_{a,p})^{-1}$ and $K_{a+2,p}$ are Banach space isomorphisms, the operator (5.2.3) is Fredholm exactly when (5.2.8) with $b = a - q + n/p$ is Fredholm. In fact we see from (5.2.9) that the horizontal maps in that diagram have isomorphic kernels, ranges, and cokernels. In particular when they are Fredholm, they have the same indices. □

Using the ideas in the proof of this theorem, we may establish the next fact about harmonic forms.

PROPOSITION (5.3). *If $s \in \mathbf{Z}$ and $\sigma \in W^p_{s,\delta,a}(\wedge^q M, g)$ is harmonic, then $\sigma \in W^p_{t,\delta,a}(\wedge^q M, g)$ for all $t \in \mathbf{Z}$.*

PROOF. Letting $A(a - q + n/p)$ be as above, we see from (5.2.9) that $K_{a,p}\sigma \in \text{Ker } A(a - q + n/p) \cap W^p_{s,\delta}(\wedge^q M)$. Since Δ_g is formally selfadjoint, elliptic, and has C^∞ coefficients, σ is in $C^\infty(\wedge^q M)$ and so $K_{a,p}\sigma$ is also in $L^p_{k,\text{loc}}(\wedge^q M)$ for all k. Thus if we use the a priori inequality (3.7.2) for $A = A(a - q + n/p)$ and a standard bootstrap argument, we get that $K_{a,p}\sigma \in W^p_{t,\delta}(\wedge^q M)$ for all t. But this means $\sigma \in W^p_{t,\delta,a}(\wedge^q M, g)$ for all t. □

Because of this proposition the next definition is independent of s.

DEFINITION (5.4). For g admissible

$$(5.4.1) \qquad \mathfrak{H}^p_{\delta,a}(\wedge^q M, g) = \left\{ \sigma \in W^p_{s,\delta,a}(\wedge^q M, g) \mid \Delta_g \sigma = 0 \right\}.$$

COROLLARY (5.5). *For all $0 \leqslant q \leqslant n$, $\delta \in \mathbf{R}^L$, and $a \in \mathbf{R}$ the space $\mathfrak{H}^p_{\delta,a}(\wedge^q M, g)$ is finite dimensional for admissible g.*

PROOF. For $\delta \in \mathbf{R}^L - \mathscr{D}_\Delta$ this is merely a property of Δ_g being Fredholm. For $\delta \in \mathscr{D}_\Delta$ it follows from the fact that $\mathscr{D}_\Delta$ has the form (5.2.1) that there is a $\bar{\delta} \in \mathbf{R}^L - \mathscr{D}_\Delta$ such that $\bar{\delta} < \delta$. The assertion is then a result of the obvious fact that $\mathfrak{H}^p_{\bar{\delta},a}(\wedge^q M, g) \subset \mathfrak{H}^p_{\delta,a}(\wedge^q M, g)$ for $\bar{\delta} < \delta$. □

COROLLARY (5.6). *Suppose g is admissible. For each fixed $\delta \in \mathbf{R}^L$, the space of harmonic σ satisfying*

$$(5.6.1) \qquad\qquad\qquad \sup_M \|e^{\delta z}\sigma\|_g < \infty$$

is finite dimensional.

PROOF. If σ satisfies (5.6.1), then $\sigma \in W^p_{0,\bar{\delta},0}(\wedge^q M, g)$ for any $\bar{\delta}$ satisfying $\bar{\delta}_l - \delta_l < -n/p$ for $l = 1, \ldots, L$. □

We see from Corollary (5.5) that the kernel of

$$(5.7) \qquad\qquad \Delta_g: W^p_{s+2,\delta,a}(\wedge^q M, g) \to W^p_{s,\delta,a+2}(\wedge^q M, g)$$

is always finite dimensional. The kernel of the adjoint of (5.7) is also always finite dimensional; for it is $\mathfrak{H}^{p'}_{-\delta,-a-2}(\wedge^q M, g)$, where $1/p + 1/p' = 1$. Hence if (5.7) is not Fredholm, it is because it does not have a closed range. Nevertheless we shall see in Theorem (5.10) that there is always a Banach space $\tilde{W}^p_{s+2,\delta,a}(\wedge^q M, g)$ such that

$$(5.8) \qquad\qquad \Delta_g: \tilde{W}^p_{s+2,\delta,a}(\wedge^q M, g) \to W^p_{s,\delta,a+2}(\wedge^q M, g)$$

is Fredholm. The key is the next technical lemma.

LEMMA (5.9). *Let B_0, B_1, R_0, and R_1 be Banach spaces. Suppose $B_0 \subset B_1$ and $R_0 \subset R_1$ as sets and the inclusions are continuous. Also suppose A is a linear operator that is continuous from B_i to R_i for $i = 0, 1$. If $A: B_1 \to R_1$ is Fredholm and $A^*: R_0^* \to B_0^*$ has a finite dimensional kernel, then there is a Banach space, $\tilde{B}$, with the properties:*

(5.9.1) $B_0 \subset \tilde{B} \subset B_1$ and the inclusions are continuous,

(5.9.2) B_0 is dense in $\tilde{B}$,

(5.9.3) $A: \tilde{B} \to R_0$ is defined, continuous, and Fredholm,

(5.9.4)   *the range of $A: \tilde{B} \to R_0$ is the closure of the range of $A: B_0 \to R_0$.*

PROOF. Let $B = \{ u \in B_1 \mid Au \in R_0 \}$ and equip B with the norm $\|u\|_B = \|u\|_{B_1} + \|Au\|_{R_0}$. It is straightforward to show that B is a Banach space, that $B_0 \subset B \subset B_1$, as sets, that the inclusions are continuous, and that $A: B \to R_0$ is continuous. In fact $A: B \to R_0$ is Fredholm.

To see this let K_B be the kernel of A in B and W be the closure of $A(B)$ in R_0. Since $B \subset B_1$ and $A: B_1 \to R_1$ is Fredholm, K_B is finite dimensional. Also if $w \in W$ then $w \in R_1$ and in fact W is in the closure of $A(B_1)$ in R_1. Thus there is a $u \in B_1$ such that $Au = w$. However $u \in B_1$ and $Au \in R_0$ means $u \in B$. Hence $A(B) = W$. Finally W contains the closure of $A(B_0)$ in R_0 and so W is finite codimensional, for $A^*: R_0^* \to B_0^*$ has a finite dimensional kernel.

Unfortunately, B might not satisfy (5.9.2) and (5.9.4). For this reason we take $\tilde{B}$ to be the closure of B_0 in B with respect to $\|\ \|_B$ and equip it with the norm $\|\ \|_B$. Clearly $A: \tilde{B} \to R_0$ is continuous and has a finite dimensional kernel. Also, since $A(B_0) \subset A(\tilde{B})$, the closure of $A(\tilde{B})$ in R_0 is finite codimensional. Thus to finish we only have to show that $A(\tilde{B})$ is closed in R_0.

Recalling that K_B is the kernel of A in B, we have from the finite dimensionality of K_B that $E = K_B + \tilde{B}$ is closed in B. Hence $C_B E$, the set complement of E in B, is open. It follows from the fact that $A: B \to A(B)$ is an open, onto map that $A(C_B E)$ is open in $A(B)$ and that $A(B) = A(E) \cup A(C_B E)$. Since $A(B)$ is closed in R_0 we are done if we show that $A(E) \cap A(C_B E)$ is empty. But this is clear; for if $A(g) = A(e)$ for some $g \in C_B E$ and $e \in E$ then $g = e + k$ with $k \in K_B$, i.e. $g \in E$.   □

THEOREM (5.10). *Suppose g is an admissible metric, $\delta \in \mathbf{R}^L$, $a \in \mathbf{R}$, $s \in \mathbf{N}$, and $1 < p < \infty$. There is a Banach space, $\tilde{W}^p_{s+2,\delta,a}(\wedge^q M, g)$, with the properties:*

(5.10.1) $\Delta_g: \tilde{W}^p_{s+2,\delta,a}(\wedge^q M, g) \to W^p_{s,\delta,a+2}(\wedge^q M, g)$

is continuous and Fredholm; $\tilde{W}^p_{s+2,\delta,a}(\wedge^q M, g) = W^p_{s+2,\delta,a}(\wedge^q M, g)$, equipped with an equivalent norm, if $\delta \in \mathbf{R}^L - \mathscr{D}_\Delta$; in general

(5.10.2) $\tilde{W}^p_{s+2,\delta,a}(\wedge^q M, g) \subset \bigcap_{\eta < \delta} W^p_{s+2,\eta,a}(\wedge^q M, g)$;

and the range of (5.10.1) is the closure of the range of

(5.10.3) $\Delta_g: W^p_{s+2,\delta,a}(\wedge^q M, g) \to W^p_{s,\delta,a+2}(\wedge^q M, g).$

PROOF. Since $\mathscr{D}_\Delta$ has the form (5.2.1), almost all $\tau \leqslant \delta$ are in $\mathbf{R}^L - \mathscr{D}_\Delta$. For each such τ let $\tilde{B}_\tau$ be the closure of $W_{s+2,\delta,a}^p(\wedge^q M, g)$ in

$$(5.10.4) \qquad B_\tau = \left\{ \sigma \in W_{s+2,\tau,a}^p(\wedge^q M, g) \mid \Delta_g \sigma \in W_{s,\delta,a+2}^p(\wedge^q M, g) \right\}$$

with respect to the norm

$$(5.10.5) \qquad \|\sigma\|_{B_\tau} = \|\sigma\|\left(W_{s+2,\tau,a}^p(\wedge^q M, g)\right) + \|\Delta_g \sigma\|\left(W_{s,\delta,a+2}^p(\wedge^q M, g)\right)$$

and equip $\tilde{B}_\tau$ with the norm $\|\ \|_{B_\tau}$. By the previous lemma

$$(5.10.6) \qquad \Delta_g \colon \tilde{B}_\tau \to W_{s,\delta,a+2}^p(\wedge^q M, g)$$

is continuous and Fredholm with range equal to the closure of the range of (5.10.3).

We claim that all the $\tilde{B}_\tau$ are equal as sets and have equivalent norms. To see this first suppose $\tau_1 < \tau_2 \leqslant \delta$. Then there is a continuous inclusion $i \colon \tilde{B}_{\tau_2} \to \tilde{B}_{\tau_1}$, and so we have the commutative diagram

$$(5.10.7) \qquad \begin{array}{ccc} \tilde{B}_{\tau_2} & \overset{\Delta_g}{\to} & W_{s,\delta,a+2}^p(\wedge^q M, g) \\[4pt] i \downarrow & \nearrow \Delta_g & \\[4pt] \tilde{B}_{\tau_1} & & \end{array}$$

Because $\Delta_g \colon \tilde{B}_{\tau_j} \to W_{s,\delta,a+2}^p(\wedge^q M, g)$ is Fredholm, $i \colon \tilde{B}_{\tau_2} \to \tilde{B}_{\tau_1}$ also is. This means $\tilde{B}_{\tau_2}$ is closed in $\tilde{B}_{\tau_1}$. But $W_{s+2,\delta,a}^p(\wedge^q M, g)$ is dense in both. Hence $\tilde{B}_{\tau_1}$ and $\tilde{B}_{\tau_2}$ are equal as sets and i is an isomorphism.

For general, $\tau_1, \tau_2 < \delta$ and in $\mathbf{R}^L - \mathscr{D}_\Delta$ there is a $\tau_3 \in \mathbf{R}^L - \mathscr{D}_\Delta$ such that τ_1, $\tau_2 < \tau_3 \leqslant \delta$. Thus $\tilde{B}_{\tau_2} = \tilde{B}_{\tau_3} = \tilde{B}_{\tau_1}$ and they all have equivalent norms. Because of this we may take $\tilde{W}_{s+2,\delta,a}^p(\wedge^q M, g)$ to be any one of the $\tilde{B}_\tau$.

To see that (5.10.2) holds for $\tilde{W}_{s+2,\delta,a}^p(\wedge^q M, g)$ all we have to do is note that for $\eta < \delta$ there is a $\tau \in \mathbf{R}^L - \mathscr{D}_\Delta$ such that $\eta < \tau \leqslant \delta$. For such τ it is clear that $\tilde{B}_\tau \subset W_{s+2,\eta,a}^p(\wedge^q M, g)$.

Finally if $\delta \in \mathbf{R}^L - \mathscr{D}_\Delta$, then we have $\tilde{B}_\delta$. Obviously $\tilde{B}_\delta$ is $W_{s+2,\delta,a}^p(\wedge^q M, g)$ equipped with an equivalent norm. $\quad\square$

Observe that while it is true that the range of (5.10.1) is the closure of the range of (5.10.3), the kernel of (5.10.1) may be strictly larger than that of (5.10.3). For instance if g is a product metric on $\partial M_0 \times \mathbf{R}^+$, then $\sigma(x) \equiv 1$ is in $\tilde{W}_{2,0}^p(\wedge^0 M, g)$ though not in $W_{2,0}^p(\wedge^0 M)$.

To finish this section we shall prove an analogous result for $d + d_g^*$. This result will be useful in §7. In stating it we let

$$W_{s,\delta,a}^p(\wedge^* M, g) = \overset{q=0}{\underset{n}{\bigoplus}} W_{s,\delta,a}^p(\wedge^q M, g).$$

THEOREM (5.11). *Suppose g is admissible, $\delta \in \mathbf{R}^L$, $a \in \mathbf{R}$, $s \in \mathbf{N}$, and $1 < p < \infty$. There is a Banach space $\tilde{W}_{s+1,\delta,a}^p(\wedge^* M, g)$, with the properties:*

$$(5.11.1) \qquad d + d_g^* \colon \tilde{W}_{s+1,\delta,a}^p(\wedge^* M, g) \to W_{s,\delta,a+1}^p(\wedge^* M, g)$$

is continuous and Fredholm; if

$$(5.11.2) \qquad d + d_g^* \colon W_{s+1,\delta,a}^p(\wedge^* M, g) \to W_{s,\delta,a+1}^p(\wedge^* M, g)$$

is Fredholm, then $\tilde{W}^p_{s+1,\delta,a}(\wedge^*M, g) = W^p_{s+1,\delta,a}(\wedge^*M, g)$, *equipped with an equivalent norm; in general*

$$(5.11.3) \qquad \tilde{W}^p_{s+1,\delta,a}(\wedge^*M, g) \subset \bigcap_{\eta < \delta} W^p_{s+1,\eta,a}(\wedge^*M, g);$$

and the range of (5.11.1) *is the closure of the range of*

$$(5.11.4) \qquad d + d^*_g : W^p_{s+1,\delta,a}(\wedge^*M, g) \to W^p_{s,\delta,a+1}(\wedge^*M, g).$$

PROOF. Let $\Gamma = \bigcap_{q=0}^n \mathscr{D}_\Delta(g, p, q, a - 1)$ and let $\delta_0 \in \Gamma$. Then

$$(5.11.5) \qquad \Delta_g : W^p_{s+2,\delta_0,a-1}(\wedge^*M, g) \to W^p_{s,\delta_0,a+1}(\wedge^*M, g)$$

is Fredholm. Moreover it factors as

$$(5.11.6) \qquad W^p_{s+2,\delta_0,a-1}(\wedge^*M, g) \xrightarrow{d+d^*_g} W^p_{s+1,\delta_0,a}(\wedge^*M, g)$$

$$\xrightarrow{d+d^*_g} W^p_{s,\delta_0,a+1}(\wedge^*M, g).$$

Since (5.11.5) has a closed, finite codimensional range, so does

$$(5.11.7) \qquad d + d^*_g : W^p_{s+1,\delta_0,a}(\wedge^*M, g) \to W^p_{s,\delta_0,a+1}(\wedge^*M, g).$$

Because the kernel of (5.11.7) is contained in $\mathfrak{H}^p_{\delta_0,a}$, it is also finite dimensional. Thus (5.11.7) is Fredholm for $\delta_0 \in \Gamma$.

For any $\delta \in \mathbf{R}^L$ there is a $\delta_0 \in \Gamma$ such that $\delta_0 \leqslant \delta$. Hence we may proceed as in the previous theorem. $\square$

6. Hodge decomposition. In the last section we showed for admissible metrics that $W^p_{s,\delta,a}(\wedge^q M, g) = \Delta_g \tilde{W}^p_{s+2,\delta,a-2}(\wedge^q M, g) \oplus F$ with F a finite dimensional space. Unlike the situation for a compact manifold, F cannot always be picked to be $\mathfrak{H}^p_{\delta,a}(\wedge^q M, g)$. In fact the dimensions usually are not the same, for the dimension of F is the same as that of the kernel of the adjoint of $\Delta_g : W^p_{s+2,\delta,a-2}(\wedge^q M, g) \to W^p_{s,\delta,a}(\wedge^q M, g)$, and so equals dim $\mathfrak{H}^{p'}_{-\delta,-a}(\wedge^q M, g)$ where $1/p + 1/p' = 1$. Since any $\sigma \in \mathfrak{H}^{p'}_{-\delta,-a}(\wedge^q M, g)$ that is also in $W^p_{s,\delta,a}(\wedge^q M, g)$ may be put in F, this suggests that F can be chosen to be $\mathfrak{H}^{p'}_{-\delta,-a}(\wedge^q M, g)$. However, that can only be done if $\mathfrak{H}^{p'}_{-\delta,-a}(\wedge^q M, g) \subset W^p_{s,\delta,a}(\wedge^q M, g)$. In the next theorem sufficient conditions are provided for this to be true.

THEOREM (6.1). *For* $1 < p < \infty$, $\delta \in \mathbf{R}^L$, $a \in \mathbf{R}$, $s \in \mathbf{N}$, *and* g *admissible* $\mathfrak{H}^{p'}_{-\delta,-a}(\wedge^q M, g) \subset W^p_{s,\delta,a}(\wedge^q M, g)$ *if any of the following is true:*

$$(6.1.1) \qquad p \geqslant 2 \text{ and } \delta z + (a + n/2 - n/p')\rho \text{ is bounded above},$$

$$(6.1.2) \qquad \begin{array}{l} p < 2 \text{ and } (\delta + \varepsilon)z + (a + n/2 - n/p')\rho \text{ is bounded above} \\ \text{for some } \varepsilon > 0, \end{array}$$

$$(6.1.3) \qquad p \leqslant 2, \delta \leqslant 0, a = 0, \text{ and } \int_M dV_g < \infty.$$

PROOF. All we need to do is show that $\mathfrak{H}^{p'}_{-\delta,-a}(\wedge^q M, g) \subset W^p_{0,\delta,a}(\wedge^q M, g)$, for then it is in $\mathfrak{H}^p_{\delta,a}(\wedge^q M, g)$, which by (5.3) is in $W^p_{s,\delta,a}(\wedge^q M, g)$ for all s. In the case of (6.13) this is a result of the fact that $W^{p'}_{0,-\delta,0}(\wedge^q M, g) \subset W^p_{0,\delta,0}(\wedge^q M, g)$, which follows easily from Hölder's inequality.

As for the first two cases, note that according to Theorem (4.7) we can, for large enough t, continuously imbed $W_{t,-\delta,-a}^{p'}(\wedge^q M, g)$ into $W_{0,\hat\delta,-a+n/p'-n/p}^{p}(\wedge^q M, g)$ provided either

(i) $p \geqslant 2$ and $\hat\delta \leqslant -\delta$ or

(ii) $p < 2$ and $\hat\delta < -\delta$.

Hence $\mathfrak{H}_{-\delta,-a}^{p'}(\wedge^q M, g) \subset W_{0,\hat\delta,-a+n/p'-n/p}^{p}(\wedge^q M, g)$ when either of these conditions hold.

Since

(6.1.4)

$$\delta z + a\rho = (\delta - \hat\delta)z + 2(a + n/2 - n/p')\rho + \hat\delta z + (-a + n/p' - n/p)\rho,$$

Hölder's inequality implies that $W_{0,\hat\delta,-a+n/p'-n/p}^{p}(\wedge^q M, g) \subset W_{0,\delta,a}^{p}(\wedge^q M, g)$ if $(\delta - \hat\delta)z + 2(a + n/2 - n/p')\rho$ is bounded above. For $p \geqslant 2$ we may pick $\hat\delta = -\delta$, but for $p < 2$ we must pick $\hat\delta = -\delta - \varepsilon$ for some $\varepsilon > 0$. Hence we have conditions (6.1.1) and (6.1.2). $\square$

COROLLARY (6.2). *Given* $a \in \mathbf{R}$, $1 < p < \infty$, *and an admissible metric g there is a* $\delta_0 \in \mathbf{R}^L$ *with the property that*

(6.2.1) $W_{s,\delta,a}^{p}(\wedge^q M, g) = \Delta_g(\tilde W_{s+2,\delta,a-2}^{p}(\wedge^q M, g)) \oplus \mathfrak{H}_{-\delta,-a}^{p'}(\wedge^q M, g)$

for all $\delta < \delta_0$.

PROOF. A consequence of Theorem (2.9) is that $|\rho(z)| \leqslant cz$ on M_∞ for some c. Thus (6.11) and (6.12) can be satisfied by picking δ sufficiently negative on each end. $\square$

COROLLARY (6.3). *For admissible g the decomposition*

(6.3.1) $L^p(\wedge^q M, g) = \Delta_g(\tilde W_{2,0-2}^{p}(\wedge^q M, g)) \oplus \mathfrak{H}_{0,0}^{p'}(\wedge^q M, g)$

holds if any of the following are true:

(6.3.2) $p = 2,$

(6.3.3) $p \geqslant 2$ *and* ρ *is bounded below,*

(6.3.4) $p < 2$ *and* $\displaystyle\int_M dV_g < \infty.$ $\square$

If g is admissible but not complete, then $\mathfrak{H}_{0,0}^{p'}(\wedge^q M, g)$ can have elements that are not closed or coclosed. However, δ can be picked so that every element of $\mathfrak{H}_{-\delta,-a}^{p'}(\wedge^q M, g)$ is closed and coclosed.

PROPOSITION (6.4). *For $a \in \mathbf{R}$ and g admissible, there is a* $\hat\delta \in \mathbf{R}^L$ *such that every element of* $\mathfrak{H}_{-\delta,-a}^{p'}(\wedge^q M, g)$ *is closed and coclosed if $\delta < \hat\delta$.*

PROOF. As a result of Theorem (2.9) there is a $\hat\delta \in \mathbf{R}^L$ such that

$$\hat\delta z + ((a - 1) + n/2 - n/p')\rho$$

is bounded above. Thus by the proof of (6.1) we have, for large enough t, a continuous inclusion

(6.4.1) $W_{t,-\delta,-a+1}^{p'}(\wedge^r M, g) \subset W_{0,\delta,a-1}^{p}(\wedge^r M, g)$

for all r and all $\delta < \hat\delta$.

Let $\sigma \in \mathfrak{H}_{-\delta,-a}^{p'}(\wedge^q M, g)$. Since $C_0^\infty(\wedge^q M)$ is dense in $W_{t+1,-\delta,-a}^{p'}(\wedge^q M, g)$, there is a sequence $\{\phi_i\} \subset C_0^\infty(\wedge^q M)$ that converges to σ in $W_{t+1,-\delta,-a}^{p'}(\wedge^q M, g)$. From the continuity of d and d_g^*, we get that $d\phi_i \to d\sigma$ in $W_{t,-\delta,-a+1}^{p'}(\wedge^{q+1} M, g)$ and $d_g^*\phi_i \to d_g^*\sigma$ in $W_{t,-\delta,-a+1}^{p'}(\wedge^{q-1} M, g)$. Hence by (6.4.1) $d\phi_i \to d\sigma$ in $W_{0,\delta,a-1}^{p}(\wedge^q M, g)$ and $d_g^*\phi_i \to d_g^*\sigma$ in $W_{0,\delta,a-1}^{p}(\wedge^{q-1} M, g)$. It follows from this and the duality of the spaces $W_{0,-\delta,-a+1}^{p'}(\wedge^r M, g)$ and $W_{0,\delta,a-1}^{p}(\wedge^r M, g)$ that

$$\int_M \langle d\sigma, d\sigma \rangle_g \, dV_g \quad \text{and} \quad \int_M \langle d_g^*\sigma, d_g^*\sigma \rangle_g \, dV_g$$

are finite and that

(6.4.2) $$\int_M \langle d\sigma, d\sigma \rangle_g \, dV_g = \lim_{i \to \infty} \int_M \langle d\sigma, d\phi_i \rangle_g \, dV_g$$

and

(6.4.3) $$\int_M \langle d_g^*\sigma, d_g^*\sigma \rangle_g \, dV_g = \lim_{i \to \infty} \int_M \langle d_g^*\sigma, d_g^*\phi_i \rangle_g \, dV_g.$$

But Stokes's theorem implies that the integrals on the right-hand side of (6.4.2) and (6.4.3) are zero for $\sigma \in \mathfrak{H}_{-\delta,-a}^{p'}(\wedge^q M, g)$. □

In the next theorem we deal with a larger class of metrics than the admissible ones; namely those equivalent to admissible ones in the following sense.

DEFINITION (6.5). Two metrics, g_0 and g_1, are equivalent if there is a constant $c > 0$ such that $g_0/c \leqslant g_1 \leqslant cg_0$. This relation is denoted $g_0 \approx g_1$.

Notice that if $g_0 \approx g_1$, then $L^p(\wedge^q M, f dV_{g_0})$ and $L^p(\wedge^q M, f dV_{g_1})$ are equal as vector spaces and have equivalent norms for any continuous $f: M \to \mathbf{R}^+$. Also note that equivalent metrics can have quite different curvature properties. For instance on $S^1 \times \mathbf{R}$ the metrics $d\theta^2 + dx^2$ and $(2 + \sin(e^{x^2}))(d\theta^2 + dx^2)$ are equivalent.

THEOREM (6.6). *In addition to the conditions already assumed for M, suppose M is oriented. Let g_1 be a metric on M that is equivalent to an admissible metric g. There is a $\bar{\delta} \in \mathbf{R}^L$ such that if $\delta < \bar{\delta}$ and $\phi \in W_{0,\delta,0}^p(\wedge^0 M, g)$, then $\phi = d_{g_1}^*\sigma$ for some $\sigma \in \bigcap_{\tau < \delta} W_{0,\tau,-1}^p(\wedge^1 M, g)$.*

PROOF. By (6.2) and (6.4) there is a $\hat{\delta}$ such that if $\delta < \hat{\delta}$ then

(6.6.1) $$W_{0,\delta,0}^p(\wedge^0 M, g) = d_g^*d\big(\tilde{W}_{2,\delta,-2}^p(\wedge^0 M, g)\big) \oplus \mathfrak{H}_{-\delta,0}^{p'}(\wedge^0 M, g)$$

with $\mathfrak{H}_{-\delta,0}^{p'}$ containing only constants. Picking $\bar{\delta} \leqslant \hat{\delta}$ so that $\int_M e^{-p\bar{\delta}z} \, dV_g = \infty$, we get

(6.6.2) $$W_{0,\delta,0}^p(\wedge^0 M, g) = d_g^*d\big(\tilde{W}_{2,\delta,-2}^p(\wedge^0 M, g)\big)$$

for $\delta < \bar{\delta}$.

Since g_1 is equivalent to g, the isomorphism

(6.6.3) $$*_{g_1}*_g : \wedge^q M \to \wedge^q M$$

extends to an isomorphism

(6.6.4) $$*_{g_1}*_g : W_{0,\tau,a}^p(\wedge^q M, g) \to W_{0,\tau,a}^p(\wedge^q M, g).$$

Hence the solution to $\phi = d_{g_1}^*\sigma$ we seek is $\sigma = *_{g_1}*_g\tilde{\sigma}$ with $\tilde{\sigma}$ the solution to $(-1)^{n-1}*_g*_{g_1}\phi = d_g^*\tilde{\sigma}$. Since $\tilde{\sigma}$ can be picked to be dF for some $F \in \tilde{W}_{2,\delta,-2}^p(\wedge^0 M, g)$, it follows from Theorem (5.10) that $\tilde{\sigma}$ has all the properties we want. Hence by (6.6.4) σ also does. $\square$

7. Hodge representation. We now turn our attention to determining the dimension of the space of closed and coclosed forms in L^2 and determining to what extent its elements represent de Rham cohomology. The key result is Theorem (7.4), which holds for many admissible and nonadmissible metrics. In the theorem we do not assume g is admissible.

NOTATION (7.1). Let g_0 be a metric on M that is a product metric on M_∞. If g is a metric on M, then $F_g = (\det g/\det g_0)^{1/2}$. Thus $dV_g = F_g dV_{g_0}$.

NOTATION (7.2). For $0 \leqslant t$ the function $\mathcal{M}_t: \partial M_0 \times \overline{\mathbf{R}}^+ \to \partial M_0 \times \overline{\mathbf{R}}^+$ is given by $(\omega, z) \to (\omega, tz)$.

DEFINITION (7.3). A metric g on M is q-bounded above if it is conformal to an asymptotically translation invariant one and if for every $0 \leqslant t \leqslant 1$, every $(\omega, z) \in M_\infty$, and every $\nu \in \wedge^q M_{(\omega, zt)}$ the inequality

$$(7.3.1) \quad \left\| dz \wedge \left(\mathcal{M}_t^*\nu\right)(\omega, z) \right\|_{g(\omega, z)}^2 F_g(\omega, z) \leqslant c \|(dz \wedge \nu)(\omega, tz)\|_{g(\omega, tz)}^2 F_g(\omega, tz)$$

holds for some c independent of ν, t, and (ω, z).

THEOREM (7.4). *Suppose g is q-bounded above. If $\sigma \in L^2(\wedge^q M, g) \cap dC^\infty(\wedge^{q-1} M)$, then $\sigma \in \overline{dC_0^\infty(\wedge^{q-1} M)}$, the closure of $dC_0^\infty(\wedge^{q-1} M)$ in $L^2(\wedge^q M, g)$.*

PROOF. Let $\sigma = d\psi$ for some $\psi \in C^\infty(\wedge^{q-1} M)$. With ϕ_R as in Lemma (3.11), we have $\sigma = d\phi_1\psi + d(1 - \phi_1)\psi$. Since $d\phi_1\psi \in dC_0^\infty(\wedge^{q-1} M)$, all we need to do is show that $\sigma_\infty = d(1 - \phi_1)\psi$ is in $\overline{dC_0^\infty(\wedge^{q-1} M)}$. Now $(1 - \phi_1)$ has its support in $\partial M_0 \times [1, \infty)$ and there $\sigma_\infty = \eta + \nu \wedge dz$ with $\eta(\omega, z) \in C^\infty(\wedge^q \partial M_0)$ and $\nu \in C^\infty(\wedge^{q-1} \partial M_0)$. Define $\mu(\omega, z)$ by

$$(7.4.1) \qquad \mu(\omega, z) = \begin{cases} (-1)^{q-1} \int_0^1 z\left(\mathcal{M}_t^*\nu\right)(\omega, z)\, dt & \text{on } M_\infty, \\ 0 & \text{on } M_0. \end{cases}$$

Then $\mu \in C^\infty(\wedge^{q-1} M)$, and, as an easy calculation shows, $d\mu = \sigma_\infty$.

Setting $\mu_R = \phi_R\mu$, we get

$$(7.4.2) \qquad d\mu_R = \frac{(-1)^{q+1}}{R}\tilde{\phi}'\left(\frac{z}{R}\right)\int_0^1 z\, dz \wedge \left(\mathcal{M}_t^*\nu\right)(\omega, z)\, dt + \phi_R\sigma_\infty$$

with $\tilde{\phi}$ as in (3.11). Clearly $\phi_R\sigma_\infty \to \sigma_\infty$ in $L^2(\wedge^q M, g)$ as $R \to \infty$. Therefore, to finish the proof we must show the convergence to 0 in $L^2(\wedge^q M, g)$ as $R \to \infty$ of

$$(7.4.3) \qquad \frac{1}{R}\tilde{\phi}'\left(\frac{z}{R}\right)\int_0^1 z\, dz \wedge \left(\mathcal{M}_t^*\nu\right)(\omega, z)\, dt = \lambda_R(\omega, z).$$

Because of the fact that $\operatorname{supp} \tilde{\phi}'(z/R) \subset \partial M_0 \times [R, 2R] \equiv M_R$,

$$(7.4.4) \quad \|\lambda_R\|_2 = \left(\int_{M_R} \frac{z^2}{R^2} \left(\tilde{\phi}'\left(\frac{z}{R} \right) \right)^2 \left\| \int_0^1 dz \wedge (\mathcal{M}_t^* \nu)(\omega, z) \, dt \right\|_{g(\omega, z)}^2 \, dV_g \right)^{1/2}$$

$$\leqslant c \left(\int_{M_R} \left\| \int_0^1 dz \wedge (\mathcal{M}_t^* \nu)(\omega, z) \, dt \right\|_{g(\omega, z)}^2 \, dV_g \right)^{1/2}.$$

In turn, Minkowski's inequality applied to the last integral in (7.4.4) gives

$$(7.4.5) \quad \|\lambda_R\|_2 \leqslant c \int_0^1 \left(\int_{\partial M_0} \int_R^{2R} \| dz \wedge (\mathcal{M}_t^* \nu)(\omega, z) \|_{g(\omega, z)}^2 F_g(\omega, z) \, dz \, d\omega \right)^{1/2} \, dt.$$

Since g is q-bounded above, it follows from (7.4.5) that

$$(7.4.6) \quad \|\lambda_R\|_2 \leqslant c \int_0^1 \left(\int_{\partial M_0} \int_R^{2R} \| dz \wedge \nu \|_{g(\omega, tz)}^2 F_g(\omega, tz) \, dz \, d\omega \right)^{1/2} \, dt.$$

The change of variable, $\xi = zt$, in (7.4.6) then yields

$$(7.4.7) \quad \|\lambda_R\|_2 \leqslant c \int_0^1 t^{-1/2} \left(\int_{\partial M_0} \int_{tR}^{2tR} \| dz \wedge \nu \|_g^2 F_g \, d\xi \, d\omega \right)^{1/2} \, dt$$

$$\leqslant c \int_0^{R^{-1/2}} t^{-1/2} \| \nu \wedge dz \|_2 \, dt$$

$$+ c \int_{R^{-1/2}}^1 t^{-1/2} \left(\int_{\partial M_0 \times [\sqrt{R}, 2R]} \| \nu \wedge dz \|_g^2 \, dV_g \right)^{1/2} \, dt$$

$$= c R^{-1/4} \| \nu \wedge dz \|_2 + c \left(\int_{\partial M_0 \times [\sqrt{R}, 2R]} \| \nu \wedge dz \|_g^2 \, dV_g \right)^{1/2}.$$

This last quantity obviously goes to 0 as $R \to \infty$. □

COROLLARY (7.5). *Let*

$$(7.5.1) \quad \tilde{\mathfrak{H}}^2(\wedge^q M, g) = \left\{ \sigma \in L^2(\wedge^q M, g) \mid d\sigma = d_g^* \sigma = 0 \right\}.$$

If g is q-bounded above, then

$$(7.5.2) \quad \dim \tilde{\mathfrak{H}}^2(\wedge^q M, g) \leqslant \dim H_{DR}^q(M).$$

PROOF. By the theorem of Kodaira mentioned in the introduction, the space of closed forms in $L^2(\wedge^q M, g)$ is the orthogonal direct sum

$$(7.5.3) \quad \overline{dC_0^\infty(\wedge^{q-1} M)} \oplus \tilde{\mathfrak{H}}^2(\wedge^q M, g).$$

Thus by Theorem (7.4) there are no exact forms in $\tilde{\mathfrak{H}}^2(\wedge^q M, g)$. □

Examples of q-bounded above metrics were given in the introduction. In the case that g is equivalent to a q-bounded above metric that is also admissible, we may use the Fredholm results of the previous sections to obtain a precise determination of $\tilde{\mathfrak{H}}^2(\wedge^q M, g)$.

THEOREM (7.6). *Suppose that g_1 is both q-bounded above and admissible and that g is equivalent to g_1. Every $v \in H^q_{DR}(M)$ with a representative in $L^2(\wedge^q M, g)$ has a unique representative in $\tilde{\mathfrak{H}}^2(\wedge^q M, g)$.*

PROOF. Once again the equivalence of g and g_1 means $L^2(\wedge^q M, g) = L^2(\wedge^q M, g_1)$ with equivalent norms. Taking $\tilde{W}^2_{1,0,-1}(\wedge^* M, g_1)$ to be as in Theorem (5.11) and $\tilde{W}^2_{1,0,-1}(\wedge^{q-1} M, g_1)$ to be the $q-1$ forms in $\tilde{W}^2_{1,0,-1}(\wedge^* M, g_1)$, we have from that theorem that $d\tilde{W}^2_{1,0,-1}(\wedge^{q-1} M, g_1)$ is closed in $L^2(\wedge^q M, g)$. Since $C^\infty_0(\wedge^{q-1} M) \subset \tilde{W}^2_{1,0,-1}(\wedge^{q-1} M, g_1)$ this means that

$$(7.6.1) \qquad \overline{dC^\infty_0(\wedge^{q-1} M)} = d\tilde{W}^2_{1,0,-1}(\wedge^{q-1} M, g_1).$$

In other words every element of $\overline{dC^\infty_0(\wedge^{q-1} M)}$ is exact.

The theorem then follows from Kodaira's theorem and the fact that no element of $\tilde{\mathfrak{H}}^2(\wedge^q M, g_1)$ is exact. $\square$

Naturally, the question arises as to which cohomology classes have representatives in $L^2(\wedge^q M, g)$. We answer this in the next theorem. But first some terminology needs to be established.

DEFINITION (7.7). For Ω_l a component of ∂M_0, a cohomology class $v \in H^q_{DR}(M)$ is said to have compact support on $\Omega_l \times \mathbf{R}^+$ if there is a representative, σ, of v such that $\sigma \equiv 0$ on $\Omega_l \times [R, \infty)$ for some $R \geqslant 0$.

DEFINITION (7.8). Let $\Omega_1, \ldots, \Omega_L$ be the components of ∂M_0. Also take $C^\infty(\wedge^q M)_{\mathbf{R}^+}$ to be the set of $\phi \in C^\infty(\wedge^q M)$ that are $\mathbf{R}^+$ invariant on M_∞. For a metric g

$$(7.8.1) \quad \mathscr{A}(g; q) = \left\{ l \,\Big|\, \int_{\Omega_l \times \mathbf{R}^+} \|\phi\|^2_g \, dV_g = \infty \text{ for some } \phi \in C^\infty(\wedge^q M)_{\mathbf{R}^+} \right\}.$$

Observe that in the case $g = e^{2\rho}h$ is admissible

$$\mathscr{A}(g, q) = \left\{ l \,\big|\, e^{-2q\rho} \notin L^1(\Omega_l \times \mathbf{R}^+, dV_g) \right\}.$$

THEOREM (7.9). *Suppose $g_1 = e^{2\rho}h$ is admissible and $g \approx g_1$. With $\mathscr{A}$ and Ω_l as in (7.8), set $M^\infty_\infty(g; q) = \bigcup_{l \in \mathscr{A}} \Omega_l \times \mathbf{R}^+$. The cohomology classes in $H^q_{DR}(M)$ with representatives in $\tilde{\mathfrak{H}}^2(\wedge^q M, g)$ are exactly those that have compact support on $M^\infty_\infty(g; q)$.*

PROOF. As in Theorem (7.6), the equivalence of g and g_1, combined with Kodaira's theorem and Theorem (5.11) implies that the classes in $H^q_{DR}(M)$ with representatives in $\tilde{\mathfrak{H}}^2(\wedge^q M, g)$ are the same as those with representatives in $\tilde{\mathfrak{H}}^2(\wedge^q M, g_1)$. Hence we only need to prove the theorem for g_1.

According to Theorem (2.9) there is a $\bar{\rho} \in C^\infty(M)$ that is a function only of z on M_∞ and an asymptotically translation invariant metric $\bar{h}$ such that $g = e^{2\bar{\rho}}\bar{h}$. By Propositions (2.5) and (2.6) $\bar{\rho}$ satisfies condition (2.3.1) for some θ and any asymptotically translation invariant metric. In particular if h_0 is any metric that on M_∞ is the product metric $ds^2_{\partial M_0} + dz^2$, then $e^{2\bar{\rho}}h_0$ is admissible. Since g_1 is equivalent to $e^{2\bar{\rho}}h_0$, we can use the argument in the preceding paragraph to see that we only have to prove the theorem for metrics of the form $e^{2\bar{\rho}}h_0$.

Thus without loss of generality suppose g is an admissible metric and that $g = e^{2\rho}h_0$ with ρ a function only of z on M_∞ and $h_0 = ds^2_{\partial M_0} + dz^2$ on M_∞. Let $\tilde{\Delta}$ be the Laplacian on ∂M_0 associated with $ds^2_{\partial M_0}$, $\tilde{d}$ the exterior derivative, and $\tilde{d}*$ the coderivative. Take $\{\eta_{-1i}\} \cup \{\eta_{0i}\} \cup \{\eta_{1i}\} \cup \cdots \cup \{\eta_{Li}\}$ to be an orthonormal basis of $L^2(\wedge^{q-1}\partial M_0, ds^2_{\partial M_0})$ consisting of eigenforms of $\tilde{\Delta}$ with the η_{-1i} closed but not coclosed, the η_{0i} coclosed but not closed, and the η_{li} harmonic and equal to 0 on Ω_j for $j \neq l$. Say $\tilde{\Delta}\eta_{01} = \lambda^2_{0i}\eta_{0i}$. If $\{\psi_{-1i}\}$ is an orthonormal basis for the coclosed nonharmonic forms in $L^2(\wedge^q\partial M_0, ds^2_{\partial M_0})$ and $\{\psi_{li}\}$ is an orthonormal basis for the harmonic q-forms on ∂M_0 such that $\psi_{li} \equiv 0$ on Ω_j for $j \neq l$, then $\{\psi_{-1l}\} \cup \{\lambda^{-1}_{0i}\tilde{d}\eta_{0i}\} \cup \{\psi_{1i}\} \cup \cdots \cup \{\psi_{Li}\}$ is an orthonormal basis for $L^2(\wedge^q\partial M_0, ds^2_{\partial M_0})$.

Every $\sigma \in L^2(\wedge^q M_\infty, g)$ has an orthogonal series expansion of the form

$$(7.9.1) \qquad \sigma = \sum_{l=-1}^{L} \sum_i f_{li}(z)\, dz \wedge \eta_{li} + \sum_i h_{-1i}(z)\psi_{-1i}$$

$$+ \sum_i h_{0i}(z)\lambda^{-1}_{0i}\tilde{d}\eta_{0i} + \sum_{l=1}^{L} \sum_i h_{li}(z)\psi_{li}.$$

From this we see that

$$(7.9.2) \quad d\sigma = \sum_i \left(h'_{0i}(z)\lambda^{-1}_{0i} - f_{0i}(z)\right) dz \wedge \tilde{d}\eta_{0i} + \sum_i h'_{-1i}(z)\, dz \wedge \psi_{-1i}$$

$$+ \sum_i h_{-1i}(z)\, \tilde{d}\psi_{-1i} + \sum_{l=1}^{L} \sum_i h'_{li}(z)\, dz \wedge \psi_{li}.$$

If $d\sigma = 0$, then it follows that

$$(7.9.3) \qquad h_{-1i} \equiv 0, \quad h'_{0i} = \lambda_{0i}f_{0i}, \quad \text{and} \quad h_{li} = a_{li} \in \mathbf{R} \quad \text{for } l > 0.$$

Similarly we have

$$(7.9.4) \quad d*\sigma = -\sum_i \left[f_{-1i}(z)e^{(n-2q)\rho}\right]' e^{(2q-n)\rho}\eta_{-1i} - \sum_i f_{-1i}(z)\, dz \wedge \tilde{d}*\eta_{-1i}$$

$$+ \sum_i \left(h_{0i}(z)\lambda_{0i}e^{(n-2q)\rho} - \left[f_{0i}(z)e^{(n-2q)\rho}\right]'\right)e^{(2q-n)\rho}\eta_{0i}$$

$$- \sum_{l=1}^{L} \sum_i \left[f_{li}(z)e^{(n-2q)\rho}\right]' e^{(2q-n)\rho}\eta_{li}.$$

If $d*\sigma = 0$, then it follows that

$$(7.9.5) \quad f_{-1i} \equiv 0, \quad \left[f_{0i}(z)e^{(n-2q)\rho}\right]' = \lambda_{0i}h_{0i}(z)e^{(n-2q)\rho},$$

$$\text{and } f_{li} = b_{li}e^{(2q-n)\rho} \quad \text{for } l > 0 \text{ and } b_{li} \in \mathbf{R}.$$

As a result of (7.9.3) and (7.9.5) we have that if $\sigma \in \tilde{\mathfrak{H}}^2(\wedge^q M, g)$, then on M_∞

$$(7.9.6) \quad \sigma = d\left[\sum_i h_{0i}(z)\lambda^{-1}_{0i}\eta_{01} + \sum_{l=1}^{L} \sum_i b_{li}\int_0^z e^{(2q-n)\rho(\zeta)}\, d\zeta\eta_{li}\right] + \sum_{l=1}^{L} \sum_i a_{li}\psi_{li}$$

with h_{0i} satisfying $h''_{0i} + (n - 2q)\rho'h'_{01} - \lambda^2_{0i}h_{0i} = 0$.

In order for σ to be in $L^2(\wedge^q M, g)$ each term in (7.9.1) must be in $L^2(\wedge^q M_\infty, g)$. Therefore the a_{li} in (7.9.6) must be 0 if $l \in \mathscr{A}(g; q)$ and may be anything if $l \notin \mathscr{A}(g; q)$. Hence letting

$$(7.9.7) \qquad \tilde\beta = \sum_i h_{0i}(z)\lambda_{0i}^{-1}\eta_{0i} + \sum_{l=1}^{L} \sum_i b_{li} \int_0^z e^{(2q-n)\rho(\zeta)}\,d\zeta\,\eta_{li}$$

on $\partial M_0 \times [1, \infty)$ and letting β be a C^∞ extension of $\tilde\beta$ to all of M we get that on $\partial M_0 \times [1, \infty)$

$$(7.9.8) \qquad \sigma - d\beta = \sum_{l \notin \mathscr{A}} \sum_i a_{li}\psi_{li}.$$

This means that the only classes in $H^q_{DR}(M)$ with representatives in $\tilde{\mathfrak{H}}^2(\wedge^q M, g)$ are the ones that are compactly supported on $M_\infty^\infty(g; q)$. On the other hand if $\nu \in H^q_{DR}(M)$ is such a class and ϕ is a representative of ν, then by de Rham's theorem on M_∞

$$(7.9.9) \qquad \phi = \sum_{l \notin \mathscr{A}} \sum_i a_{li}\psi_{li} + d\gamma_0.$$

Choosing γ to be a C^∞ extension to all of M of the restriction of γ_0 to $\partial M_0 \times \{1, \infty\}$, we see that ν has a representative that is $\sum_{l \notin \mathscr{A}} \sum_i a_{li}\psi_{li}$ on M_∞, namely $\phi - d\gamma$. Thus ν has a representative in $L^2(\wedge^q M, g)$. It follows, therefore, from Kodaira's theorem and Theorem (5.11) that ν has a representative in $\tilde{\mathfrak{H}}^2(\wedge^q M, g)$. $\square$

COROLLARY (7.10). *If $g \approx g_1$ and g_1 is admissible and q-bounded above, then the classes of $H^q_{DR}(M)$ with representatives in $\tilde{\mathfrak{H}}^2(\wedge^q M, g)$ are exactly those compactly supported on $M_\infty^\infty(g; q)$. Moreover the representation is unique.* $\square$

8. L^2-cohomology. In this section we give a few applications of the results of the preceding sections to L^2-cohomology. Recently this cohomology has received quite a bit of attention; in part, because of its use in the study of singular algebraic varieties (see [4, 5 and 13]). We begin by defining various subspaces of L^2.

DEFINITION 8.1. For an arbitrary Riemannian manifold X

$$(8.1.1) \qquad L_e^2(\wedge^q X) = \text{the closure of } dC_0^\infty(\wedge^{q-1}X) \text{ in } L^2(\wedge^q X),$$

$$(8.1.2) \qquad L_{ce}^2(\wedge^q X) = \text{the closure of } d_g^* C_0^\infty(\wedge^{q+1}X) \text{ in } L^2(\wedge^q X),$$

$$(8.1.3) \qquad \mathfrak{Q}^q(X) = \left\{ \sigma \in L^2(\wedge^q X) \mid d\sigma \in L^2(\wedge^{q+1}X) \right\},$$

$$(8.1.4) \qquad \mathfrak{Q}_{ce}^q(X) = \mathfrak{Q}^q(X) \cap L_{ce}^2(\wedge^q X).$$

We equip $L_e^2(\wedge^q X)$ and $L_{ce}^2(\wedge^q X)$ with the L^2 norm and equip $\mathfrak{Q}^q(X)$ and $\mathfrak{Q}_{ce}^q(X)$ with the norm $\|\sigma\|_{\mathfrak{Q}} = (\|\sigma\|_2^2 + \|d\sigma\|_2^2)^{1/2}$. So equipped each space is a sub-Hilbert space of L^2.

DEFINITION (8.2). The qth-L^2-cohomology group of X is

$$(8.2.1) \qquad H_2^q(X) = \left[\operatorname{Ker} d\colon \mathfrak{Q}^q(X) \to \mathfrak{Q}^{q+1}(X)\right]/d\mathfrak{Q}^{q-1}(X).$$

One of the interesting questions about $H_2^q(X)$ is whether or not the map

$$(8.3) \qquad i\colon \tilde{\mathfrak{H}}^2(\wedge^q X, g) \to H_2^q(X)$$

is an isomorphism. If it is, then one says that the Strong Hodge Theorem holds (see [**4** and **5**]). It follows from Kodaira's theorem that

$$(8.4) \qquad H_2^q(X) = \frac{L_e^2(\wedge^q X) \oplus \tilde{\mathfrak{H}}^2(\wedge^q X, g)}{d\mathfrak{L}_{ce}^{q-1}(X)}.$$

Hence (8.3) is 1-1 if $d\mathfrak{L}_{ce}^{q-1}(X) \subset L_e^2(\wedge^q X)$ and is onto if $d\mathfrak{L}_{ce}^{q-1}(X)$ is closed in $L^2(\wedge^q X, g)$.

For the remainder of this section we restrict our attention to the manifolds and metrics considered in the preceding sections. Thus M has finitely many ends and g is equivalent to $g_1 = e^{2\rho}h$ with h asymptotically translation invariant. An easy consequence of Theorem (7.4) is

PROPOSITION (8.5). *If g is equivalent to a metric that is q-bounded above, then (8.3) is 1-1.*

LEMMA (8.6). *Suppose $g = e^{2\rho}h$ is an admissible metric on M and that ρ is bounded below. Then $\mathfrak{L}_{ce}^q(M) \subset W_{1,0,-1}^2(\wedge^q M, g)$.*

PROOF. Let $\sigma \in \mathfrak{L}_{ce}^q(M)$. Then $(d + d_g^*)\sigma \in L^2(\wedge^{q+1}M, g)$ and so $\Delta_g\sigma \in W_{-1,0,1}^2(\wedge^q M, g)$. It follows from Proposition (4.4) that $K_{1,2}\Delta_g\sigma \in W_{-1,0,0}^2(\wedge^q M, h) = W_{-1,0}^2(\wedge^q M)$. However from the definition of the asymptotically translation invariant operator $A(-1 + q + n/2)$ in Theorem (5.2) we see that

$$K_{1,2}\Delta_g\sigma = A(-1 + q + n/2)K_{-1,2}\sigma.$$

Thus $A(-1 + q + n/2)K_{-1,2}\sigma$ is in $W_{-1,0}^2(\wedge^q M)$. Hence using the a priori inequality (3.7.2) we get

$$(8.6.1) \quad \|K_{-1,2}\sigma\|_{2,1,0} \leqslant c\left(\left\|A\left(-1 + q + \frac{n}{2}\right)K_{-1,2}\sigma\right\|_{2,-1,0} + \|K_{-1,2}\sigma\|_{2,-1,0}\right).$$

Since ρ is bounded below and $\sigma \in L^2(\wedge^q M, g)$, we have that $\sigma \in W_{0,0,-1}^2(\wedge^q M, g)$. This means that $K_{-1,2}\sigma \in W_{0,0}^2(\wedge^q M, g)$ and so in $W_{-1,0}^2(\wedge^q M, g)$. Thus by (8.6.1) we have that $K_{-1,2}\sigma \in W_{1,0}^2(\wedge^q M, g)$ which by Proposition (4.4) means that $\sigma \in W_{1,0,-1}^2(\wedge^q M, g)$.

THEOREM (8.7). *Suppose g is equivalent to an admissible metric, $g_1 = e^{2\rho}h$, and that ρ is bounded below. Then (8.3) is 1-1. Moreover, if $\limsup_{z\to\infty}\rho(\omega, z) = \infty$ on at least one end, then (8.3) is not onto for $q > 0$; and so the Strong Hodge Theorem does not hold in this case.*

PROOF. As usual we only need to prove this for g_1. Furthermore, we may assume that ρ is a function of just z on M_∞ and that h is a product metric on M_∞. The first part of the theorem follows trivially from Lemma (8.6) and the fact that $dW_{1,0,-1}^2(\wedge^{q-1}M, g) \subset L_e^2(\wedge^q M)$.

For the second part of the theorem, let

$$(8.7.1) \qquad K = \text{Ker } d_g^*: W_{1,0,-1}^2(\wedge^{q-1}M, g) \to L^2(\wedge^{q-2}M, g).$$

It follows from Theorem (5.11) that $K = d_g^* \tilde{W}_{2,0,-2}^2(\wedge^q M, g) + \tilde{\mathfrak{H}}_{1,0,-1}^2(\wedge^{q-1} M)$ where $\tilde{\mathfrak{H}}_{1,0,-1}^2(\wedge^{q-1} M)$ is the finite dimensional space of closed and coclosed forms in $W_{1,0,-1}^2(\wedge^{q-1} M, g)$. Note that since $d_g^* \tilde{W}_{2,0,-2}^2(\wedge^q M, g)$ is finite codimensional in K and is the image of a continuous operator, it is closed in K and so also in $W_{1,0,-1}^2(\wedge^{q-1} M, g)$. Also note that $\mathfrak{L}_{ce}^{q-1}(M) \subset K$.

Suppose $d\colon K \to L_{\acute{e}}^2(\wedge^q M)$ is onto; if it is not then a fortiori $d\colon \mathfrak{L}_{ce}^{q-1}(M) \to L_{\acute{e}}^2(\wedge^q M)$ is not. In this case an equivalent norm on K is

$$(8.7.2) \qquad \|\sigma\|_K = \|\sigma\|\big(W_{0,0,-1}^2(\wedge^{q-1} M, g)\big) + \|d\sigma\|_2.$$

It is easy to see that we then have the commutative diagram of continuous maps

$$(8.7.3) \qquad
\begin{array}{ccc}
\mathfrak{L}_{ce}^{q-1}(M) & \xrightarrow{d} & L_{\acute{e}}^2(\wedge^q M) \\[2mm]
\downarrow i & \nearrow d & \\[2mm]
K & &
\end{array}$$

where i is the inclusion. If $d\mathfrak{L}_{ce}^{q-1} = L_{\acute{e}}^2(\wedge^q M)$, then i must be Fredholm and so $\mathfrak{L}_{ce}^{q-1}(M)$ is closed in K. Now $C_0^\infty(\wedge^q M)$ is dense in $\tilde{W}_{2,0,-2}^2(\wedge^q M, g)$ and so $d_g^* C_0^\infty(\wedge^q M)$ is dense in $d_g^* \tilde{W}_{2,0,-2}^2(\wedge^q M, g)$. Since $d_g^* C_0^\infty(\wedge^q M) \subset \mathfrak{L}_{ce}^{q-1}(M)$ and $\mathfrak{L}_{ce}^{q-1}(M)$ is closed in K, it follows that $d_g^* \tilde{W}_{2,0,-2}^2(\wedge^q M, g) \subset \mathfrak{L}_{ce}^{q-1}(M)$. In fact, since $\mathfrak{L}_{ce}^{q-1}(M)$ contains no closed forms and $dK = d\mathfrak{L}_{ce}^{q-1}(M)$, we have that $d_g^* \tilde{W}_{2,0,-2}^2(\wedge^q M, g) = \mathfrak{L}_{ce}^{q-1}(M)$.

However this is impossible if $\limsup_{z \to \infty} \rho(z) = \infty$ on an end of M. To see this note that it follows from (2.3) that there is an increasing sequence $\{z_n\}$ such that $z_1 > 4$, $z_{n+1} - z_n \geqslant 4$, and $\rho(z) > n$ on $(z_n - 1, z_n + 1)$. Now let $\nu \in C^\infty(\wedge^q M)$ be such that ν is translation invariant on M_∞ and $\nu(\omega, z) \in \wedge^q \partial M_0$ for $(\omega, z) \in M_\infty$. Let $\phi(z) \in C_0^\infty(\mathbf{R})$ be such that $\operatorname{supp} \phi \subset [-1, 1]$ and $\phi \equiv 1$ on $[-\frac{1}{2}, \frac{1}{2}]$.

Finally take $f \in C^\infty(M)$ to be such that $f(z) = \sum_{n=1}^\infty n^{-1}\phi(z - z_n)$ on the end of M on which $\limsup_{z \to \infty} \rho(z) = \infty$ and $f \equiv 0$ elsewhere. If $\tau = e^{(q+2-n/2)\rho} f \nu$, then a calculation shows that $\tau \in W_{2,0,-2}^2(\wedge^q M, g)$ but $d_g^* \tau \notin L^2(\wedge^{q-1} M, g)$.

DEFINITION (8.8). Let G be a compact manifold. A function $\rho\colon G \times \mathbf{R}^+ \to \mathbf{R}$ is slowly oscillating if

$$(8.8.1) \qquad \sup_{t \in [z/2, z]} |\rho(\omega, z) - \rho(\omega, t)| < c \quad \text{for some } c \text{ independent of } \omega \text{ and } z.$$

THEOREM (8.9). *Suppose g is equivalent to an admissible metric $g_1 = e^{2\rho} h$ and that ρ is bounded above. For $i = 1, 2, 3$ let G_i be a, possibly empty, union of components of ∂M_0 satisfying (i) $G_i \cap G_j = \varnothing$ for $1 \leqslant i, j \leqslant 3$ and $i \neq j$ and (ii) $G_1 \cup G_2 \cup G_3 = \partial M_0$. Suppose ρ is slowly oscillating on $G_1 \times \mathbf{R}^+$ and that $\rho(\omega, z) \geqslant (\varepsilon - 1)\ln z$ on $G_1 \times [1, \infty)$ for some $0 < \varepsilon < 1$. Also assume that on $G_2 \times [1, \infty)$ both g_1 is q-bounded above and ze^ρ is bounded. Finally assume that ρ is $O(z)$ but not $o(z)$ as $z \to \infty$ on $G_3 \times [1, \infty)$. Then (8.3) is onto if and only if $H_{DR}^{q-1}(G_1) = 0$.*

PROOF. Suppose $\sigma = d\Psi$ for some $\Psi \in \tilde{W}_{1,0,-1}^2(\wedge^{q-1} M, g_1)$. Let $\{\phi_0, \phi_1, \phi_2, \phi_3\}$ be a partition of unity subordinate to the cover $M_0 \cup (\partial M_0 \times [0, 2))$, $G_1 \times (1, \infty)$, $G_2 \times (1, \infty)$, $G_3 \times (1, \infty)$, with ϕ_i omitted if G_i is empty. Then $\sigma = \sum d(\phi_i \Psi)$ and $\phi_0 \Psi$ is clearly in $\mathfrak{L}^{q-1}(M)$. Also if G_3 is not empty, then it follows from Definition (2.3) and the construction of $\tilde{W}_{1,0,-1}^2(\wedge^* M, g_1)$ that $\phi_3 \Psi \in \mathfrak{L}^{q-1}(M)$.

If $G_3 = \partial M_0$, then we are done. Therefore assume that $G_1 \cup G_2 = \bigcup_{l=1}^K \Omega_l$ where ($K \leqslant L$ and $\Omega_1, \ldots, \Omega_L$) are the components of ∂M_0. Also, without loss of generality, assume that ρ just depends on z and h is a product metric on $(G_1 \cup G_2) \times \mathbf{R}^+$.

Let $\sigma_\infty = d(\phi_1 \Psi) + d(\phi_2 \Psi)$. Using the notation of Theorem (7.9) we have that

$$(8.9.1) \qquad \sigma_\infty = \sum_{l=-1}^K \sum_i f_{li}(z)\,dz \wedge \eta_{li} + \sum_i h_{-1i}(z)\Psi_{-1i}$$

$$+ \sum_i h_{0i}(z)\lambda_{0i}^{-1}\tilde{d}\eta_{0i} + \sum_{l=1}^K \sum_i h_{li}(z)\Psi_{li}.$$

Since σ_∞ is exact it follows that $h_{-1i} \equiv 0$; $h'_{0i} = \lambda_{0i} f_{0i}$; and $h_{li} \equiv 0$ for $l > 0$. Thus $\sigma_\infty = d\tau$ where

$$(8.9.2) \quad \tau = \sum_i h_{0i}(z)\lambda_{0i}^{-1}\eta_{0i} + \sum_i f_{-1i}(z)\,dz \wedge \gamma_{-1i} + \sum_{l=1}^K \sum_i \int_0^1 z f_{li}(zt)\,dt\,\eta_{li},$$

with $d\gamma_{-1i} = \eta_{-1i}$.

Since ρ is bounded above we get from the fact that $\sigma_\infty \in L^2(\wedge^q M, g_1)$ that

$$(8.9.3) \quad \tilde{\tau} = \sum_i h_{0i}(z)\lambda_{0i}^{-1}\eta_{0i} + \sum_i f_{-1i}(z)\,dz \wedge \gamma_{-1i} \quad \text{is in } L^2(\wedge^{q-1}M, g_1).$$

Furthermore, if $\Omega_l \subset G_2$, then $\int_0^1 z f_{li}(zt)\,dt\,\eta_{li}$ also is in $L^2(\wedge^{q-1}M, g_1)$. This follows from the chain of inequalities

$$(8.9.4) \quad \left\| \int_0^1 z f_{li}(zt)\,dt\,\eta_{li} \right\| \left(L^2(\wedge^{q-1}M, g_1) \right)$$

$$= \left(\int_0^\infty \int_{\Omega_l} e^{(n-2q+2)\rho} \left\| \int_0^1 z f_{li}(zt)\,dt\,\eta_{li} \right\|_h^2 \, d\omega\, dz \right)^{1/2}$$

$$\leqslant c \left(\int_0^\infty e^{(n-2q)\rho} \left| \int_0^1 f_{li}(zt)\,dt \right|^2 dz \right)^{1/2}$$

$$\leqslant c \int_0^1 \left(\int_0^\infty e^{(n-2q)\rho(z)} \left| f_{li}(zt) \right|^2 dz \right)^{1/2} dt$$

$$\leqslant c \int_0^1 \left(\int_0^\infty e^{(n-2q)\rho(zt)} \left| f_{li}(zt) \right|^2 dz \right)^{1/2} dt < \infty$$

where the next to last inequality is Minkowski's and the last is q-bounded above. Thus if $H_{\tilde{D}R}^{q-1}(G_1) = 0$, we have that $\tau \in L^2(\wedge^{q-1}M, g_1)$ and so $\sigma_\infty \in d\mathfrak{L}^{q-1}(M)$. But this means that $d\tilde{W}_{1,0,-1}^2(\wedge^{q-1}M, g_1) = d\mathfrak{L}^{q-1}(M)$ and hence that (8.3) is onto.

If, however, there is an $\Omega_l \subset G_1$ and $H_{\tilde{D}R}^{q-1}(\Omega_l) \neq 0$, then we can construct an infinite dimensional space of f_{li} such that $f_{li}(z)\,dz \wedge \eta_{li} \in L^2(\wedge^q M, g_1)$, but $\int_0^z f_{li}(t)\,dt + k$ is not in $L^2(\wedge^{q-1}M, g_1)$ for any k. It follows from Kodaira's theorem that in this case $d\mathfrak{L}^{q-1}(M)$ is not closed and so (8.3) is not onto.

More specifically for $-1/2 - \varepsilon/2 < \alpha < -1/2$ let $f_\alpha \in C^\infty(M)$ be such that $f_\alpha \equiv 0$ on $M - (\Omega_l \times [1, \infty))$ and $f_\alpha(\omega, z) = e^{(q - n/2)\rho}z^\alpha$ on $\Omega_l \times [2, \infty)$. If η_{l1} is a harmonic representative of a nonzero class in $H_{\tilde{D}R}^{q-1}(\Omega_l)$, then $\sigma_\alpha = f_\alpha\,dz \wedge \eta_{l1}$ is exact and in $L^2(\wedge^q M, g_1)$. Moreover if $\sigma_\alpha = d\tau_\alpha$, then on $\Omega_l \times [1, \infty)$ we have $\tau_\alpha = (\int_2^z f_\alpha(t)\,dt + k)\eta_{l1}$. However $\tau_\alpha \notin L^2(\wedge^{q-1}M, g_1)$ for any value of k.

To see this note that

$$(8.9.5) \quad \| \tau_\alpha \| L^2 \big(\textstyle\bigwedge^{q-1} M, g_1 \big)$$

$$\geq c \left(\int_4^\infty e^{(n-2q+2)\rho(z)} \left| \int_2^z e^{(q-n/2)\rho(t)} t^\alpha \, dt + k \right|^2 dz \right)^{1/2}.$$

First suppose that $\int_2^\infty e^{(q-n/2)\rho(t)} t^\alpha \, dt = \infty$. Then there is a $z_0 \geq 4$, depending on k, such that for $z \geq z_0$, we have

$$(8.9.6) \qquad \int_2^z e^{(q-n/2)\rho(t)} t^\alpha \, dt + k \geq \frac{1}{2} \int_2^z e^{(q-n/2)\rho(t)} t^\alpha \, dt.$$

Thus in this case the right-hand side of (8.9.5) is

$$(8.9.7) \qquad \geq c \left(\int_{z_0}^\infty e^{(n-2q+2)\rho(z)} \left| \int_2^z e^{(q-n/2)\rho(t)} t^\alpha \, dt \right|^2 dz \right)^{1/2}$$

$$\geq c \left(\int_{z_0}^\infty z^{2\varepsilon-2} \left| \int_{z/2}^z e^{(n/2-q)(\rho(z)-\rho(t))} t^\alpha \, dt \right|^2 dz \right)^{1/2}.$$

From (8.8.1) we get that this last expression is

$$(8.9.8) \qquad \geq c \left(\int_{z_0}^\infty z^{2\varepsilon-2} \left| z^{\alpha+1} \right|^2 dz \right)^{1/2}$$

$$= c \left(\int_{z_0}^\infty z^{2\varepsilon+2\alpha} \, dz \right)^{1/2} \geq c \left(\int_{z_0}^\infty z^{\varepsilon-1} \, dz \right)^{1/2} = \infty.$$

Finally, suppose that $\int_2^\infty e^{(q-n/2)\rho(t)} t^\alpha \, dt < \infty$. Since $\alpha > -1$, this means that $q > n/2$ and so $\int_4^\infty e^{(n-2q+2)\rho(t)} \, dt = \infty$. Hence the only way the right-hand side of (8.9.5) could be finite is if $k = -\int_2^\infty e^{(q-n/2)\rho(t)} t^\alpha \, dt$. In this case we have that the right-hand side of (8.9.5) is

$$(8.9.9) \qquad = c \left(\int_4^\infty e^{(n-2q+2)\rho(z)} \left| \int_z^\infty e^{(q-n/2)\rho(t)} t^\alpha \, dt \right|^2 dz \right)^{1/2}$$

$$\geq c \left(\int_4^\infty z^{2\varepsilon-2} \left| \int_z^{2z} e^{(n/2-q)(\rho(z)-\rho(t))} t^\alpha \, dt \right|^2 dz \right)^{1/2}.$$

Again by (8.8.1) we get that this last expression is

$$(8.9.10) \qquad \geq c \left(\int_4^\infty z^{2\varepsilon-2} \left| z^{\alpha+1} \right|^2 dz \right)^{1/2} = \infty.$$

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