ON ROOT INVARIANTS OF PERIODIC CLASSES 
IN Ext\(_A(\mathbb{Z}/2, \mathbb{Z}/2)\)

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ABSTRACT. We prove that if a class in the cohomology of the mod 2 Steenrod algebra is \(v_r\)-periodic in the sense of [10], then its root invariant must be \(v_{n+1}\)-periodic, where \(v_n\) denotes the \(n\)th generator of \(\pi_*(\text{BP})\).

1. Introduction and statement of results. This work is motivated by a desire to understand the relationship between two ideas of recent interest in stable homotopy theory. The first concept is that of \(v_r\)-periodicity in stable homotopy, which has been extensively studied in the setting of the Novikov spectral sequence [11, 12]. In [2], a start was made toward studying this phenomenon in the setting of the classical Adams spectral sequence (clASS). This study was continued in [10], where complete definitions were given of the notion of \(v_r\)-periodicity in \(\text{Ext}_A(\mathbb{Z}/2, \mathbb{Z}/2)\). The second idea is that of the root invariant. This invariant is defined using W.-H. Lin’s theorem, which relates the stable homotopy of spheres with that of projective spaces. Complete definitions can be found in [9 and 12].

To state our results, we need to review the definitions and earlier results involved. The following theorems and definitions can be found in [10]. We recall that a class \(x \in R\), a commutative ring, is said to be a nonzero divisor if \(rx^n \neq 0\) for all nonzero \(r \in R\), and all \(n \in \mathbb{N}\). Here \(A\) denotes the Steenrod algebra at the prime 2, and \(A_i\) denotes the Hopf subalgebra generated by \(\{Sq^0, Sq^1, Sq^2, \ldots, Sq^{2^i}\}\). Let \(Q_i\) denote the \(i\)th Milnor generator and \(E(Q_i)\) be the exterior algebra over \(\mathbb{Z}/2\).

**Theorem (1.1).** For each \(i \geq 1\), there exists a unique nonzero divisor \(w_i \in \text{Ext}_{A_i}^{2i+1, 2i+1}(\mathbb{Z}/2, \mathbb{Z}/2)\) such that \(w_i\) restricts nontrivially to \(\text{Ext}_{E(Q_i)}(\mathbb{Z}/2, \mathbb{Z}/2)\) and corresponds to the class \(v_r\) \(i\) \(\in \pi_*(\text{BP})\).

We hereafter use the notation \(v_i^{2i+1} \in \text{Ext}_{A_i}(\mathbb{Z}/2, \mathbb{Z}/2)\). We henceforth suppress the second module in \(\text{Ext}(M, N)\) whenever it is \(\mathbb{Z}/2\).

For \(k > i\), there is also some power of \(v_i^{2i+1}\) present in \(\text{Ext}_{A_i}(\mathbb{Z}/2)\). In fact, we have the following result.

**Theorem (1.2).** For \(k\) any positive integer, there exist positive integers \(N_1, N_2, \ldots, N_k\) such that

\[
\mathbb{Z}/2\left[h_0, v_1^{4N_1}, v_2^{6N_2}, \ldots, v_i^{2i+1N_i}, \ldots, v_k^{2k+1N_k}\right] \subset \text{Ext}_{A_i}(\mathbb{Z}/2).
\]
Note that the integer \( N_i \) also depends upon \( k \). Note also that \( N_k \) can be chosen to be 1 by Theorem 1.1. It should be mentioned that although \( v_i^{2i+1} \) and its multiples are classes in \( \text{Ext}_A(\mathbb{Z}/2) \), \( v_i^{2i+1}N_i \) is a coset in \( \text{Ext}_A(\mathbb{Z}/2) \), for \( k > i \). Theorem (1.2) follows easily from a theorem of Lin [6]. In particular, Theorem (1.2) implies that for all \( k \geq i \), \( \mathbb{Z}/2[v_i^{2i+1}N_i] \subset \text{Ext}_A(\mathbb{Z}/2) \). For each \( k > i \), we localize \( \text{Ext}_A(\mathbb{Z}/2) \) with respect to \( v_i \). Since \( \text{Ext}_A(\mathbb{Z}/2) = \lim_{\leftarrow k} \text{Ext}_A(\mathbb{Z}/2) \) this gives a map

\[
f_i: \text{Ext}_A^i(\mathbb{Z}/2) \to \lim_{\leftarrow k} \left[ \text{Ext}_A^i(\mathbb{Z}/2)(v_i^{-1}) \right],
\]

which enables us to define the following concept.

**Definition (1.3).** A class \( x \in \text{Ext}_A(\mathbb{Z}/2) \) is \( v_i \)-periodic if \( f_i(x) \neq 0 \), and is \( v_i \)-torsion otherwise.

Notice that this definition is equivalent to the following: if \( q_k^*: \text{Ext}_A(\mathbb{Z}/2) \to \text{Ext}_A(\mathbb{Z}/2) \) denotes the usual restriction then \( x \in \text{Ext}_A(\mathbb{Z}/2) \) is \( v_i \)-periodic if there exists a \( K > 0 \) such that \( q_k^*(x)(v_i^{2i+1}N_k)^s \neq 0 \) for all \( s \geq 0 \) and \( k \geq K \).

The main result of [10] is

**Theorem (1.4).** If \( x \in \text{Ext}_A(\mathbb{Z}/2) \) is \( v_n \)-periodic, then \( x \) is also \( v_{n+k} \)-periodic for all \( k \geq 0 \).

Equivalently, if \( x \in \text{Ext}_A(\mathbb{Z}/2) \) is \( v_n \)-torsion, then \( x \) is also \( v_k \)-torsion, for all \( k \) such that \( 0 \leq k \leq n \).

In the setting of \( BP_\bullet BP \)-comodules, this result is due to Johnson and Yosimura [3]. Theorem (1.4) allows one to prove

**Corollary (1.5).** There is a filtration, which we call the chromatic filtration,

\[
\text{Ext}_A(\mathbb{Z}/2) = F_{-1} \subset F_0 \subset F_1 \subset \cdots \subset F_i \subset \cdots
\]

such that \( F_i - F_{i+1} \) is the set of classes that are \( v_{i+1} \)-periodic but \( v_k \)-torsion for all \( k \leq i \).

The second major concept that we deal with here is the root invariant, first defined in [8]. This invariant is constructed using W.-H. Lin’s theorem [4]. To state this, let \( RP_{-k} \) denote the Thom spectrum of \( -k \) times the canonical line bundle over \( RP^\infty \). Let \( P \) denote the direct limit \( \lim_{\leftarrow j} H^*(RP_j; \mathbb{Z}/2) \). Here, \( P \) is isomorphic to the ring of Laurent series \( \mathbb{Z}/2[x, x^{-1}] \), where \( |x| = 1 \). The Steenrod algebra action is given by \( Sq^ixi = (i)x_{i+1}^i \).

**Theorem (1.6) (Lin’s Theorem).** The inverse limit \( \lim_{\leftarrow j} H^*(RP_j) \equiv \pi_*^i(S^1) \). Also, \( \text{Ext}_A(P) \equiv \text{Ext}_A(S^{-1}Z/2) \).

We use the following ideas to define the root invariant. Let \( P_m \) denote the \( A \)-module \( H^*(RP_m; \mathbb{Z}/2) \), where \( m \) is any integer. We have a map of \( A \)-modules \( j_m: P_m \to \Sigma^m Z/2 \), induced from the map generating \( \tau_m \). There is also a map \( k_m: P_m \to P \), given by the system of maps \( RP_{m-k} \to RP_m \) which collapse the bottom \( k \) cells of \( RP_{m-k} \). With these conventions, we define the root invariant as follows: for \( a \in \text{Ext}_A^i(\mathbb{Z}/2) \), we may regard \( a \) as living in \( \text{Ext}_A^{i-1}(P) \). There exists a maximal
integer $N$ such that $k_N^*(a) \neq 0$ in $\text{Ext}^s_{\mathcal{A}^t}(P_N)$. We then define the root invariant of $a$, $R(a)$, to be the coset given by

$$R(a) = \{ y \in \text{Ext}^s_{\mathcal{A}^t}(\Sigma^N \mathbb{Z}/2) : j_N^*(y) = k_N^*(a) \}.$$ 

Note that $N$ will always be negative, for $s > 0$, by the proof of the algebraic Kahn-Priddy theorem [5], so that $R$ will preserve the $s$-filtration and raise the $(t - s)$-filtration of a class. The diagram one should have in mind is

$$\begin{array}{ccc}
\text{Ext}^s_{\mathcal{A}^t}(\mathbb{Z}/2) & \xrightarrow{=} & \text{Ext}^s_{\mathcal{A}^t}(\mathbb{P}) \\
\downarrow R & & \\ \\
\text{Ext}^s_{\mathcal{A}^t-N-1}(\mathbb{Z}/2) & \xrightarrow{j_N^*} & \text{Ext}^s_{\mathcal{A}^t-N}(P_N) \\
\end{array}$$

For $a \in \pi^s_{2n}(S^0)$, we define the geometric root invariant of $a$, $R_C(a)$, in a similar manner. This geometric root invariant appears as the “Mahowald invariant” in [13]. Calculations of $R(a)$ for $a \in \text{Ext}^s_{\mathcal{A}^t}(\mathbb{Z}/2)$, $t - s < 16$, have appeared in [9].

The goal of this paper is to prove the following result, which links these concepts of $v_i$-periodicity and root invariants.

**Theorem A.** Let $a \in \text{Ext}^s_{\mathcal{A}^t}(\mathbb{Z}/2)$ be $v_i$-periodic in the sense of Definition (1.3). Then the root invariant of $a$, $R(a)$, is $v_{i+1}$-periodic.

The geometric version of this result was conjectured by Mahowald and Ravenel, and has been attacked by Hopkins and Wegmann using techniques from the proof of the Nilpotence Theorem. Also, this result seems closely tied into the notion of smooth linear $\mathbb{Z}/p$ actions on exotic spheres, as the work of Schultz and Stolz in [14] and [16] points out.

The proof of this theorem uses the machinery of Koszul-type resolutions, presented in [1], together with the techniques used in the proof of Lin’s theorem, found in [7]. The major concept in [7] is the following splitting of $\mathcal{A}$-modules, due to Davis and Mahowald:

$$(\text{Davis-Mahowald splitting}): \gamma_i: A \otimes_{A_i} \mathbb{P}/F_m \xrightarrow{=} \bigoplus_{k \geq m} \Sigma^k \mathbb{Z}/2 \rightarrow (A \otimes_{A_{i-1}} \mathbb{Z}/2)$$

where $F_m$ is the $A_r$-submodule generated by $\{ x^j \in \mathbb{P} : j < m \}$. This gives a splitting in $\text{Ext}$, after the change of rings isomorphism and taking the limit as $m$ goes to minus infinity:

$$\gamma_i^*: \bigoplus_{k \in \mathbb{Z}} \Sigma^k \mathbb{Z}/2 \rightarrow \text{Ext}_{A_{i-1}}(\mathbb{Z}/2) \xrightarrow{=} \text{Ext}_{A_i}(\mathbb{P}).$$

We use this to define the $i$th root invariant

$$R_i: \text{Ext}_{A_{i-1}}(\mathbb{Z}/2) \rightarrow \text{Ext}_{A_i}(\mathbb{Z}/2)$$
by including Ext}_{A_{i-1}}(\mathbb{Z}/2) in as the \(-1\) summand in Ext}_{A_i}(P). By taking inverse limits, we can now analyze the root invariant

\[ R: \text{Ext}^s_{A_{i-1}}(\mathbb{Z}/2) \rightarrow \text{Ext}^s_{A_{i-1}}(\mathbb{Z}/2) \]

in terms of these \(R_i\)’s. Davis and Mahowald have completely calculated

\[ R_2: \text{Ext}_{A_{i-1}}(\mathbb{Z}/2) \rightarrow \text{Ext}_{A_i}(\mathbb{Z}/2) \]

in [1]. Recalling that for all \(i > 0\), we have \(v_i^{2i+1} \in \text{Ext}_{A_{i-1}}(\mathbb{Z}/2)\), with all of its powers nonzero, we prove the following result.

**Theorem B.** For the class \(v_i^{2i+1} \in \text{Ext}_{A_{i-1}}(\mathbb{Z}/2)\), we have \(R_i(v_i^{2i+1}) = v_i^{2i+1} \in \text{Ext}_{A_i}(\mathbb{Z}/2)\). Also, \(R_1(h^4) = v_1^4 \in \text{Ext}_{A_i}(\mathbb{Z}/2)\).

This is proved using the Koszul spectral sequence, a tool first presented in [1]. A brief summary is given in §2.

The paper is organized as follows: in §2, we analyze the Koszul spectral sequence used to calculate \(\text{Ext}_{A_i}(P)\). In §3 we prove Theorem B. Finally we prove Theorem A in §4. Throughout the paper, we use homology and cohomology with \(\mathbb{Z}/2\) coefficients. By “spectrum”, we mean a connective spectrum localized at the prime 2. The author would like to thank Stewart Priddy, Don Davis, and especially Mark Mahowald for many helpful conversations that helped to produce this paper, and also the referee for his helpful comments. The main results of this paper form part of the author’s Ph.D. thesis, completed at Northwestern University in 1984 under the direction of Mark Mahowald [15].

### 2. Calculation of the Koszul spectral sequence for \(\text{Ext}_{A_i}(P, \mathbb{Z}/2)\)

In this section, we explicitly calculate the Koszul spectral sequence for \(\text{Ext}_{A_i}(P, \mathbb{Z}/2)\) in terms of the components given by the Davis-Mahowald splitting (1.9). We first briefly review the construction of the Koszul spectral sequence, which first appeared in [1]. For complete details, see [10].

The Koszul spectral sequence (hereafter abbreviated KSS) is a tool which can be used to calculate \(\text{Ext}_{A_i}(M)\), where \(M\) is any \(A\)-module, in terms of \(\text{Ext}_{A_{i-1}}(\_, \mathbb{Z}/2)\) of certain modules. We construct an exact complex of \(A\)-modules and apply the functor \(\text{Ext}_{A_{i-1}}(\_, \mathbb{Z}/2)\) to it.

To construct this complex, we exploit the following fact about the mod 2 Steenrod algebra: \((A_i \otimes A_{i-1}, \mathbb{Z}/2)^* \cong E(\xi_1^{2^i}, \xi_2^{2^i-1}, \ldots, \xi_{i+1}^1)\), both as algebras and as left \(A\)-modules, where \(E(\) \()\) denotes an exterior algebra over the field \(\mathbb{F}_2\). Here \(\xi_k\) is \(x_k(\xi_k)\), the conjugate of the \(k\)th Milnor generator. The Steenrod algebra action is given on the right by \((\xi_k^{2^i-1})Pu^2 = \xi_k^{2^i+1}\) and \((\xi_k^1)Pu^2 = 1\), extended by the Cartan formula. For convenience, we denote \((A_i \otimes A_{i-1}, \mathbb{Z}/2)^* \cong (A_i//A_{i-1})^*\) by \(E(i)\). \(E(i)\) is an \(A\)-module but not an \(A\)-module. We decompose \(E(i)\) as an \(F_2\) vector space into a direct sum \(E(i) \cong \bigoplus_{\sigma \geq 0} E_\sigma(i)\), where \(E_\sigma(i)\) is the \(F_2\) vector space spanned by the monomials of length \(\sigma\) in \(\{\xi_1^{2^i}, \xi_2^{2^i-1}, \ldots, \xi_{i+1}^1\}\). Each of these \(E_\sigma(i)\)’s is closed under the \(A_{i-1}\)-action inherited from \(E(i)\), so the decomposition holds as an \(A_{i-1}\)-module.

We resolve this exterior algebra by using pieces of a polynomial algebra. Let \(R(i) = \mathbb{Z}/2[\xi_1^{2^i}, \xi_2^{2^i-1}, \ldots, \xi_{i+1}^1]\). This is a right \(A\)-module, with the same action as \(E(i)\). We can decompose \(R(i)\) as an \(A_{i-1}\)-module \(R(i) \cong \bigoplus_{\sigma \geq 0} R_\sigma(i)\), where \(R_\sigma(i)\)
is the $\mathbb{F}_2$ vector space spanned by monomials of length $s$ in $\{\xi_1^{2^i}, \xi_2^{2^i-1}, \ldots, \xi_{i+1}\}$. Each of the $R_\sigma$'s is an $A$-module.

To construct the resolution, form the tensor product $E_r(i) \otimes_{\mathbb{Z}/2} \mathbb{Z}_2 R_s(i)$, a right $A$-module with the action given by the Cartan formula. (We will abbreviate this by $E_r \otimes R_s$.) Define $k_{r,s}: E_r \otimes R_s \rightarrow E_{r-1} \otimes R_{s+1}$ by

$$k_{r,s}(x_1 x_2 \cdots x_r \otimes p) = \sum_{j=1}^{r} x_1 x_2 \cdots \hat{x}_j \cdots x_r \otimes x_j p,$$

for all $r \geq 1, s \geq 0$, where each $x_k$ is an element of $\{\xi_1^{2^i}, \xi_2^{2^i-1}, \ldots, \xi_{i+1}\}$ and $p$ is a polynomial of these. Each $k_{r,s}$ is an $A_i$-module map. Composing these we get an exact sequence

$$0 \rightarrow E_{i+1} \otimes R_s \rightarrow E_i \otimes R_{s+1} \rightarrow \cdots \rightarrow E_0 \otimes R_{s+i+1} \rightarrow 0.$$

Summing these sequences over a constant $s$, we obtain

$$0 \rightarrow \mathbb{Z}/2 \rightarrow E(i) \otimes R_0(i) \rightarrow E(i) \otimes R_1(i) \rightarrow \cdots$$

which is an exact sequence. The differential is given by

$$d_\sigma[(x_1 x_2 \cdots x_r) \otimes p] = \sum_{j=1}^{r} (x_1 x_2 \cdots \hat{x}_j \cdots x_r) \otimes x_j p.$$

Denote the dual of $R_\sigma(i)$ by $N_\sigma(i)$. Then, dualizing the exact sequence above, we have

**Theorem (2.1).** For $i > 0$ there exists a family of $A$-modules, $N_\sigma(i)$, $\sigma \geq 0$, defined above, and $A_i$-module maps $\delta_\sigma: A_i \otimes_{A_{i-1}} N_{\sigma+1}(i) \rightarrow A_i \otimes_{A_{i-1}} N_\sigma(i)$, such that the sequence

$$0 \leftarrow \mathbb{Z}/2 \leftarrow A_i \otimes_{A_{i-1}} N_0(i) \leftarrow A_i \otimes_{A_{i-1}} N_1(i) \leftarrow \cdots \leftarrow A_i \otimes_{A_{i-1}} N_\sigma(i) \leftarrow \cdots$$

is exact as a sequence of $A_i$-modules.

We refer to this as the Koszul-type resolution of $\mathbb{Z}/2$ over $A_i$ (KR$_i(\mathbb{Z}/2)$ or just KR if $i$ is understood).

Applying the functor $\text{Ext}^{\sigma, \cdot, i}(\cdot)$ to the complex, we obtain

**Theorem (2.2).** For $i$ any positive integer, there is a family of $A$-modules, $N_\sigma(i)$, $\sigma \geq 0$, defined above, such that for any $A_i$-module $M$ there is a trigraded spectral sequence converging to $\text{Ext}^{\sigma, \cdot, i}(M)$, with $E_1^{\sigma, \cdot, i} \equiv \text{Ext}^{\sigma, \cdot, i}(N_\sigma(i) \otimes M)$.

This is called the Koszul spectral sequence for $M$ over $A_i$ (KSS$(M)$). Note that a trigraded spectral sequence is a family of spectral sequences, one for each positive integer $i$.

We use this KSS to calculate part of $\text{Ext}_{A_i}(\mathbb{P})$. We recall that the Davis-Mahowald splitting of $A_i \otimes_{A_{i-1}} \mathbb{P}/F$ (1.8) yields, after change of rings and limits,

$$\gamma_*: \bigoplus_{k \in \mathbb{Z}} \Sigma^{k2^i+1-1} \text{Ext}_{A_{i-1}}(\mathbb{Z}/2) \rightarrow \text{Ext}_{A_i}(\mathbb{P}).$$

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There are KSS's converging to both sides of this isomorphism. To the right-hand side, we have
\[ E_1^{s,s,t} = \text{Ext}^{s-\sigma,t}(A_i \otimes A_{i-1} N_\sigma(i) \otimes P) = \text{Ext}^{s-\sigma,t}(N_\sigma(i) \otimes P), \]
converging to \( E_0^0 \text{Ext}^{s,t}(P) \). On the left-hand side we have
\[ E_1^{s,s,t} = \text{Ext}^{s-\sigma,t}(A_{i-1} \otimes A_{i-2} N_\sigma(i-1) \otimes \left( \bigoplus_{k \in \mathbb{Z}} \Sigma^{k2^{i-1}-1} \mathbb{Z}/2 \right)) \]
\[ \cong \text{Ext}^{s-\sigma,t}(\bigoplus_{k \in \mathbb{Z}} \Sigma^{k2^{i-1}-1} N_\sigma(i-1)), \]
converging to \( E_0^0 \text{Ext}^{s,t}(\bigoplus_{k \in \mathbb{Z}} \Sigma^{k2^{i-1}-1} \mathbb{Z}/2) \).

We now explicitly relate these two \( E_1 \) terms, using the following two lemmas.

**Lemma (2.5).** As an \( A_{i-2} \)-module, \( N_\sigma(i) \) splits as a direct sum
\[ N_\sigma(i) \cong \bigoplus_{k=0}^{\sigma} \Sigma^{2^i} N_k(i-1). \]

**Proof.** Assume that \( N_{\sigma-1}(i) \) splits in this fashion. We recall that the dual of \( N_{\sigma}(i) \), \( R_{\sigma}(i) \), is given as the vector space spanned by monomials of length \( \sigma \) in \( \{ \xi_1^{2^i}, \xi_2^{2^i-1}, \ldots, \xi_{i+1} \} \). Now \( R_{\sigma}(i) \) automatically contains a copy of \( \xi_1^{2^i} \cdot R_{\sigma-1}(i) \), since \( \xi_1^{2^i} \cdot m \) is of length \( \sigma \) if \( m \) is of length \( \sigma - 1 \). Further, considered as an \( A_{i-2} \)-module, this copy of \( \xi_1^{2^i} \cdot R_{\sigma-1}(i) \) splits off as a direct sum, since no \( \xi_1^{2^i} \cdot m \) can be a target of \( \text{Sq}^{2^j} \) for \( j \leq i - 1 \). This shows that \( R_{\sigma}(i) \cong \xi_1^{2^i} \cdot R_{\sigma-1}(i) \oplus M \), where \( M \) is given by all monomials having no factor of \( \xi_1^{2^i} \) in them. Thus \( M \) is given \( R_{\sigma}(i-1) \), under the doubling homomorphism \( \xi_n \rightarrow \xi_{n+1} \). This raises dimension by \( 2^i \cdot \sigma \), so that \( R_{\sigma}(i) \cong \Sigma^{2^i} R_{\sigma}(i-1) \oplus \Sigma^{2^i} R_{\sigma-1}(i) \). By our inductive hypothesis, we have the result. The \( i = 1 \) case that initiates the induction can be readily computed by hand (see [15]).

**Lemma (2.6).** Let \( M \) be any finite \( A \)-module. Then
\[ \text{Ext}_{A_{i-1}}(\bigoplus_{k \in \mathbb{Z}} \Sigma^{k2^{i-1}-1} M) \cong \text{Ext}_{A_i}(M \otimes \mathbb{Z}/2 P). \]

**Proof.** If we tensor the Davis-Mahowald splitting (1.8) with \( M \) we get
\[ \bigoplus_{k \geq m} \Sigma^{k2^{i-1}-1} (A \otimes A_{i-1} \mathbb{Z}/2) \otimes M \cong A \otimes A_i \mathbb{P} / F_m \otimes M. \]
Applying \( \text{Ext}_{A}(-) \), the change of rings theorem gives us the result. Note that it is necessary that \( M \) be a \( A \)-module, not just an \( A_i \)-module. This is the case for the \( N_\sigma(i) \)'s which we shall use.

We now examine the relationship between the two \( E_1 \) terms given earlier. On the RHS:
\[ E_1^{s,s,t} \cong \text{Ext}^{s-\sigma,t}(\bigoplus_{m \in \mathbb{Z}} \Sigma^{m2^{i-1}} N_\sigma(i)) \]
\[ \cong \text{Ext}^{s-\sigma,t}(\bigoplus_{m \in \mathbb{Z}} \Sigma^{m2^{i-1}}\left[ \bigoplus_{j=0}^{\sigma} \Sigma^{2^j} N_j(i-1) \right]) \]
by Lemma (2.6).

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So the $E_1$ term for the $\text{Ext}^t_{A_i}(\mathbb{P})$ contains all of the $\tilde{E}_1$ term for 
$\text{Ext}^t_{A_i-1}(\sum k^{2^i-1} \mathbb{Z}/2)$, plus a considerable amount of excess. The diagram
that one should have in mind is

\begin{equation}
E_1 = \bigoplus_{\sigma} \text{Ext}^{t-\sigma,i}_{A_i-1}(A_i \otimes_{A_i-1} N_\sigma(i) \otimes \mathbb{P}) \xrightarrow{\text{KSS}} \text{Ext}^{t,i}_{A_i}(\mathbb{P})
\end{equation}

and

\begin{equation}
E_1 = \bigoplus_{\sigma} \text{Ext}^{t-\sigma,i}_{A_i-1}(A_i \otimes_{A_i-1} N_\sigma(i-1)) \otimes [k^{2^{i+1}-1} \mathbb{Z}/2] \xrightarrow{\text{KSS}} \text{Ext}^{t,i}_{A_i-1}(\sum k^{2^{i+1}-1} \mathbb{Z}/2).
\end{equation}

In particular, by Lemma (2.5),
$N_{2i+1}(i) \equiv \sum 2^{2i+1} N_{2i+1}(i-1) \oplus (\text{other terms})$
where $\xi_{2i+1}$ corresponds to $\sum 2^{2i+1} \xi_{2i+1}$, as the top cell in each module. Let $g_k$ denote
the class in $\text{Ext}^0_{A_i}(\mathbb{P})$, so that $g_k$ is nonzero for $k \equiv -1 \pmod{2^{i+1}}$. Let $\iota_k$ denote
the analogous class in $\text{Ext}^0_{A_i-1}(\sum \mathbb{Z}/2)$, also nonzero for $k \equiv -1 \pmod{2^{i+1}}$. Then
in diagram (2.8) one observes that $i(\iota_{2i+1} \cdot \iota_{2i+1}) = \xi_{2i+1} \cdot g_{-1}$ in Ext$^0$.

3. Calculation of $R_i(v_{2i+1}^j)$. We recall the definition of the $i$th root invariant, $R_i$. It
uses the Davis-Mahowald splitting (1.9)

\begin{equation}
\gamma_*: \bigoplus_{k \in \mathbb{Z}} \sum k^{2^{i+1}-1} \text{Ext}_{A_i-1}(\mathbb{Z}/2) \xrightarrow{\text{KSS}} \text{Ext}_{A_i}(\mathbb{P}).
\end{equation}

This splitting commutes with the natural projections [7, (1.5)]. We use this, together
with the maps $j_m$ and $k_m$ defined in §1 (now thought of as $A_i$-module maps), to
define $R_i$ as follows:

\begin{equation}
\begin{align}
\text{Ext}^{t,i}_{A_i-1}(\Sigma^{-1} \mathbb{Z}/2) & \rightarrow \text{Ext}^{t,i}_{A_i}(\mathbb{P}) \\
\downarrow & \downarrow \Gamma_i \\
\text{Ext}^{t-1,i}_{A_i}(\Sigma N \mathbb{Z}/2) & \rightarrow \text{Ext}^{t,i}_{A_i}(\mathbb{P}_N)
\end{align}
\end{equation}

Thus $\Gamma_i: \text{Ext}_{A_i}(\mathbb{Z}/2) \rightarrow \text{Ext}_{A_i}(\mathbb{Z}/2)$ is given by $\Gamma_i(a) = \{ b \in \text{Ext}_{A_i}(\Sigma N \mathbb{Z}/2):$
$j_N^*(b) = k_N^*(a) \}$, where $N$ is the maximal integer s.t. $k_N^*(a) \neq 0$.

To calculate $R_i(v_{2i+1}^j)$, we need to recall the construction of the classes $v_{2i+1}^j \in$
$\text{Ext}_{A_i}(\mathbb{Z}/2)$, as given in [10, §2]. In the Koszul-type resolution (2.2), the top class
$(\xi_{2i+1})^* \in N_\sigma(i)$, for $\sigma = 2^{i+1}$, can be split off by $A_i$-module maps

$\Sigma' \mathbb{Z}/2 \rightarrow N_\sigma(i) \rightarrow \Sigma' \mathbb{Z}/2$, where $i = 2^{i+1}(2^{i+1} - 1)$.
This leads to a splitting of complexes (3.2)
\[ A_i \otimes_{A,_{i-1}} N_0(i) \leftarrow A_i \otimes_{A,_{i-1}} N_1(i) \leftarrow \cdots \]
\[ 0 \leftarrow \mathbb{Z}/2 \leftarrow A_i \otimes_{A,_{i-1}} N_0(i) \leftarrow A_i \otimes_{A,_{i-1}} N_1(i) \leftarrow \cdots \]
\[ \cdots \leftarrow A_i \otimes_{A,_{i-1}} N_0(i) \leftarrow A_i \otimes_{A,_{i-1}} N_1(i) \leftarrow \cdots \]

The map \( \hat{g} \) corresponds to a class \( g' \in \text{Hom}_{A_i}(A_i \otimes_{A,_{i-1}} N_0(i), \mathbb{Z}/2) = \text{Ext}_0^{i,t}_{A,_{i-1}}(N_0(i)) \), where \( \sigma = 2^i+1 \) and \( t = 2^i+1(2^i+1 - 1) \). This is the \( E_1^{2^i+1,i} \) term of the KSS. In [10], it is shown that this class is a nonbounding cycle in the KSS and projects to the class \( w_i \in \text{Ext}^{2^i+1,2^i+1(2^i+1 - 1)}_{A,_{i-1}}(\mathbb{Z}/2) \) of Theorem (1.1).

Recall that
\[ g_{k 2^i+1 - 1} \in \text{Ext}^{0,2^i+1-1}_{A,_{i}}(P), \quad t_{k 2^i+1 - 1} \in \bigoplus_{k \in \mathbb{Z}} \text{Ext}^{0,2^i+1-1}_{A,_{i-1}}(\Sigma^{k 2^i+1-1}\mathbb{Z}/2) \]
denote the appropriate nonzero classes. There is a Yoneda product in \( \text{Ext}_{A,_{i}}(P) \) given by the pairing \( \text{Ext}^{i,t}_{A,_{i}}(\mathbb{Z}/2) \otimes \text{Ext}^{j,t'}_{A,_{i}}(P) \rightarrow \text{Ext}^{i+j+t+t'}_{A,_{i}}(P) \). In particular, there is a class given by \( v_i^{2^i+1} \otimes g_{-1} \) which we will denote by \( v_i^{2^i+1} g_{-1} \in \text{Ext}^{2^i+1,i-1}_{A,_{i}}(P) \) where \( t \) is as above. This class is nonzero. In fact this class can be constructed by tensoring the above diagram (3.2) with the \( A_{i-1} \)-module \( P \).

Thus, the nonzero class in \( \text{Ext}^{0,-1}_{A,_{i-1}}(P) \) corresponds to a nonzero class in \( \text{Ext}^{0,-1}_{A,_{i-1}}(N_0(i) \otimes P) \),

where \( \sigma = 2^i+1 \) as before. This class must be a nonbounding cycle in the KSS, by the proof of Theorem A in [10]. This nontrivial class is exactly \( v_i^{2^i+1} g_{-1} \).

**Theorem (3.4).** For \( i \geq 1 \), \( \gamma_i(v_i^{2^i+1} g_{-1}) = v_i^{2^i+1} t_{2^i+1,i-1} \).

**Proof.** \( v_i^{2^i+1} \in \text{Ext}^{i,-1}_{A,_{i-1}}(\mathbb{Z}/2) \) is obtained by splitting off the top cell \( (\xi^{2^i+1})^* \) in \( N_{2^i+1,i-1} \). We now calculate \( \gamma_i(v_i^{2^i+1} g_{-1}) \) on the \( E_1 \) level of the KSS’s. Again, let \( t = 2^i+1(2^i+1 - 1) \) and \( \sigma = 2^i+1 \). \( v_i^{2^i+1} g_{-1} \) corresponds to a class \( \{u\} \in \text{Ext}^{0,i-1}_{A,i}(N_0(i) \otimes P) \), given as the top class in \( N_0(i) \), tensored with \( P \). But
\[
\text{Ext}^{0,i-1}_{A,i}(N_0(i) \otimes P) \cong \text{Ext}^{0,i-1}_{A,i-2} \left( \bigoplus_{m \in \mathbb{Z}} \sum_{j=0}^{\sigma} \Sigma^{m 2^i-1} \left[ \bigoplus_{j=0}^{\sigma} \Sigma^{m 2^i-1} N_j(i-1) \right] \right)
\]
by Lemmas (2.5) and (2.6). Now \{u\} lies completely in the \(m = 0\) summand of \(\text{Ext}_{A_{-i}}(\bigoplus_{m \in \mathbb{Z}} \Sigma^{m^2 - 1}[\bigoplus_{j=0}^{m/2} \Sigma^{2j + 1}N_j(i - 1)])\) for dimensional reasons. Since the isomorphism of Lemma (2.6) is given here by \(\gamma \otimes \text{id}_{N_i}\), we conclude that \{u\} corresponds to the top cell in \(\Sigma^{2i+1}N_o(i - 1)\), the top summand in \(N_o(i)\). By the observation at the end of §2, this class \{u\} corresponds precisely to \(\Sigma^{2i+1}t_{2i+1}N(i - 1)\), which yields \(v_{2i+1}^{2i+1}t_{2i+1}N(i - 1)\) in \(\text{Ext}_{A_{-i}}(\bigoplus \Sigma \mathbb{Z}/2)\). Since this class \{u\} yields \(v_{2i+1}^{2i+1}g_{-1}\) in \(\text{Ext}_{A}(\mathbb{P})\) in one KSS and \(v_{2i+1}^{2i+1}t_{2i+1}N(i - 1)\) in the other, we conclude that \(\gamma^* (v_{2i+1}^{2i+1}g_{-1}) = v_{2i+1}^{2i+1}t_{2i+1}N(i - 1)\). It should be pointed out that we are dealing with classes in \(\text{Ext}_{A_i}\) and \(\text{Ext}_{A_{-i}}\), not cosets, so that this is actually an equality here. The easiest way to view the calculation is in the following diagram.

\[
\begin{align*}
\text{Ext}_{A_{-i}}^0(N_o(i) \otimes \mathbb{P}) & \Rightarrow \text{Ext}_{A_{-i}}^2(N_o(i) \otimes \mathbb{P}) \\
\text{Ext}_{A_{-i}}^0(\bigoplus_{m \in \mathbb{Z}} \Sigma^{m^2 - 1}[\bigoplus_{j=0}^{m/2} \Sigma^{2j + 1}N_j(i - 1)]) & \Rightarrow \text{Ext}_{A_{-i}}^2(\bigoplus_{m \in \mathbb{Z}} \Sigma^{m^2 - 1}N_o(i - 1)) \\
\text{Ext}_{A_{-i}}^0(N_o(i) \otimes \mathbb{P}) & \Rightarrow \text{Ext}_{A_{-i}}^2(N_o(i) \otimes \mathbb{P}) \\
\end{align*}
\]

(3.5)

From this theorem, we can deduce the following

**Theorem B.** For the class \(v_{2i+1}^{2i+1} \in \text{Ext}_{A_{-i}}(\mathbb{Z}/2)\), we have \(R_i(v_{2i+1}^{2i+1}) = v_{2i+1}^{2i+1} \in \text{Ext}_{A_i}(\mathbb{Z}/2)\). Also, \(R_i(h_0^+)^* = v_i^+ \in \text{Ext}_{A_i}(\mathbb{Z}/2)\).

**Proof.** The class \(v_{2i+1}^{2i+1}g_{-1} \in \text{Ext}_{A_i}(\mathbb{Z}/2)\) survives \(k^+\) but not \(k^-\), by construction. Further, \(\gamma^* (v_{2i+1}^{2i+1}g_{-1}) = v_{2i+1}^{2i+1}t_{2i+1}N(i - 1)\), so that we have

\[
\begin{align*}
\text{Ext}_{A_{-i}}^{2i+1,t-1}(\Sigma^{2i+1}t_{2i+1}N(i - 1)) & \Rightarrow \text{Ext}_{A_{-i}}^{2i+1,t-1}(\Sigma^{2i+1}t_{2i+1}N(i - 1)) \\
\text{Ext}_{A_{-i}}^{2i+1,t-1}(\Sigma^{2i+1}t_{2i+1}N(i - 1)) & \Rightarrow \text{Ext}_{A_{-i}}^{2i+1,t-1}(\Sigma^{2i+1}t_{2i+1}N(i - 1)) \\
\end{align*}
\]

(3.6)

Here the map \(Q = \Sigma^{2i+1}t_{2i+1}R_i\). Desuspending the entire diagram \(2i+1\) times, together with the fact that \(\text{Ext}_{A_i}(\mathbb{Z}/2)\) has only one nonzero class, completes the proof of Theorem B.

**4. On the root invariant of a \(v_i\)-periodic class.** In this section, we prove the main theorem.

**Theorem A.** Let \(a \in \text{Ext}_{A_i}^i(\mathbb{Z}/2)\) be \(v_i\)-periodic in the sense of Definition (1.3). Then the root invariant of \(a\), \(R(a)\), is \(v_{i+1}\)-periodic.

We recall that \(a \in \text{Ext}_{A_i}(\mathbb{Z}/2)\) is \(v_i\)-periodic if \(f_i(a) \neq 0\), where \(f_i: \text{Ext}_{A_i}^i(\mathbb{Z}/2) \rightarrow \lim_k [\text{Ext}_{A_i}^i(\mathbb{Z}/2)(v_i^{k-1})]\). We will show that \(f_{i+1}(R(a)) \neq 0\).

We also recall that for \(k > i\), there are cosets \(v_i^N\) and \(v_i^M\) in \(\text{Ext}_{A_i}(\mathbb{Z}/2)\).

**Lemma (4.1).** For the \(k\)th root invariant, \(R_k: \text{Ext}_{A_{-i}}^k(\mathbb{Z}/2) \rightarrow \text{Ext}_{A_{-i}}^k(\mathbb{Z}/2)\), we have \(v_i^N \subset R_k(v_i^{N-1})\), whenever both \(v_i^{N-1}\) and \(v_i^N\) are nonzero in \(\text{Ext}_{A_{-i}}^k(\mathbb{Z}/2)\) and \(\text{Ext}_{A_{-i}}^k(\mathbb{Z}/2)\) respectively.
Proof. For $k = i$, this is an easy consequence of the proof of Theorem B (the proof works for any power of $v_i$, not just for the $2^{i+1}$st power). For $k > i$, we note that the cosets in question here are of $\sigma$-filtration zero in the KSS’s. Thus, the calculations actually take place in the KSS’s on the $\text{Ext}_{A_k-2}$ and $\text{Ext}_{A_k-1}$ levels. An easy induction starting at $R_i$ completes the proof.

We can now complete the proof of Theorem A by using the $A$-module structure preserved by the Davis-Mahowald splitting (1.8). We recall that the splitting is given by (1.8)

$$(\text{Davis-Mahowald Splitting}): \gamma_k: A \otimes_{A_k} P/F_m \cong \bigoplus_{j \geq m} \Sigma^{2^{i+1}-1}(A \otimes_{A_{k-1}} Z/2)$$

where $F_m$ is the $A_k$-submodule generated by $\{x^j \in P: j < m\}$. This gives a splitting in $\text{Ext}$, after the change of rings isomorphism and taking the limit as $m$ goes to minus infinity:

$$(1.9) \quad \gamma_k^*: \bigoplus_{j \in \mathbb{Z}} \Sigma^{2^{k+1}-1} \text{Ext}_{A_{k-1}}(Z/2) \cong \text{Ext}_{A_k}(P).$$

Since (1.8) is an isomorphism of $A$-modules, the induced map in $\text{Ext}_A(\_)$ must respect Yoneda products with classes from $\text{Ext}_A(Z/2)$. After change of rings, the induced map in $\text{Ext}$ (1.9) must therefore respect Yoneda products with classes $q_k^*(a)$ for $a \in \text{Ext}_A(Z/2)$. With this in mind, we can prove Theorem A.

Proof of Theorem A. Let $a \in \text{Ext}(Z/2)$ be $v_i$-periodic. Let $k$ be large enough so that $q_k^*(a)$ and $q_k^{*-1}(a)$ are nonzero in $\text{Ext}_{A_k}(Z/2)$ and $\text{Ext}_{A_k-1}(Z/2)$, respectively. For ease of notation, denote $q_k^*(a)$ by $a' \in \text{Ext}_{A_k}(Z/2)$. Now, since $a$ is $v_i$-periodic in $\text{Ext}_A(Z/2)$, we have $v_i^2a' \neq 0$ in $\text{Ext}_{A_k-1}(Z/2)$, for all $s$ where $v_i^s \neq 0$ there. Consider the action of the map $\gamma_k^*$ on this class. Since $a'$ is the projection of a class from $\text{Ext}_A(Z/2)$, $\gamma_k^*(a' \cdot b_{-1}) = a' \cdot \gamma_k^*(b)g_{-1}$, because the map $\gamma_k$ is an $A$-module map.

Now $\gamma_k^*(a' \cdot b_{-1}) = a'g_{-1}$, and if we consider the map $k_N^*$ on this class we have $k_N^*(a'g_{-1}) = R(a')g_N$ in $\text{Ext}_{A_k}(P_N)$, by the definition of root invariant, where $N$ is the maximal $N$ such that $k_N^*\gamma_k^*(a') \neq 0$.

Recall that $\gamma_k^*(v_{i}^{2m+1}g_{-1}) = v_{i+1}^{2m}g_{q_{-1}}$, where $q = 2^{m+i+1} + 1$, by Theorem (3.4) and the fact that $\gamma_k$ commutes with the natural projections. (Note that for $m$ sufficiently large, $q \equiv -1 \pmod{2^{k+1}}$.) Thus, $\gamma_k^*(v^{2m \cdot a_{i-1}'}g_{q_{-1}}) = v_{i+1}^{2m}a'g_{q_{-1}}$, and if we consider $k_N^*$ on this class, we have $q_N^*(v^{2m \cdot a_{i-1}'}g_{q_{-1}}) = v^{2m}_{i+1}R(a')g_{q_{-1}}$ in $\text{Ext}_{A_k}(P_{q_{-1}})$, where $q$ and $N$ are as above. Thus $R(a')$ is $v_{i+1}$-periodic in $\text{Ext}_{A_k}(Z/2)$, completing the proof of Theorem A.

References


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