

GLOBAL SOLVABILITY ON COMPACT NILMANIFOLDS OF THREE OR MORE STEPS

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ABSTRACT. We apply the methods of representation theory of nilpotent Lie groups to study the convergence of Fourier series of smooth global solutions to first order invariant partial differential equations $Df = g$ in C^∞ of a compact nilmanifold of three or more steps. We investigate which algebraically well-defined conditions on D in the complexified Lie algebra imply that smooth infinite-dimensional irreducible solutions, when they exist, satisfy estimates strong enough to guarantee uniform convergence of the irreducible (or primary) Fourier series to a smooth global solution. This extends and improves the results of an earlier two step paper.

0. Introduction and preliminaries. Let N be a nilpotent Lie group, $\Gamma \backslash N$ a compact nilmanifold, and D a differential operator, left-invariant on N , and viewed on $\Gamma \backslash N$. If $g \in C^\infty(\Gamma \backslash N)$ and if g_π is the orthogonal component of g in some irreducible subspace corresponding to the irreducible unitary representation π , then $g_\pi \in C^\infty(\Gamma \backslash N)$ too [2]. Modulo unitary equivalence, we may think of g_π as being a C^∞ -vector in any concrete realization, or model, of π . We will determine algebraically well-defined conditions on first order and suitable higher order D under which the global solvability of $Df = g$ in $C^\infty(\Gamma \backslash N)$ is equivalent to the solvability of $\pi(D)f_\pi = g_\pi$ in the C^∞ -vectors for each π in the spectrum of $\Gamma \backslash N$. In one sense, we will be presenting algebraic conditions on D for the reduction of a global (geometrical) problem on $\Gamma \backslash N$ to a collection of purely group (representation) theoretic problems, none of which needs to be regarded as living on the manifold $\Gamma \backslash N$. Operators D admitting such a reduction are called globally regular (Definition (0.1)). In effect, we will prove global regularity for suitable operators by showing that if the smooth solutions f_π of $\pi(D)f_\pi = g_\pi$ exist for each π , then they can be summed uniformly to a smooth global solution of $Df = g$. (Globally regular operators are usually *neither locally solvable nor onto* $C^\infty(\Gamma \backslash N)$! [12].) In order to make the necessary estimates on f_π , we construct a suitable Schrödinger model of π , which, for convenience, is not in the Hilbert space $L^2(\Gamma \backslash N)$.

In this paper, we extend to nilmanifolds of three or more steps (and correct) some of the results for two step nilmanifolds presented in [7].

In order to describe the main results, it will be helpful to review the classical situation on a torus T^2 of two dimensions (the situation being similar for T^n with $n > 2$). Let $D = \alpha\partial/\partial x + \beta\partial/\partial y$ and suppose, for simplicity, that α and β are real.

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Then D is globally regular if and only if β/α is not a (transcendental) Liouville number. The problem with Liouville numbers is that, in solving for the Fourier transform of the solution function, *small divisors* occur. Now, every nilmanifold $\Gamma \backslash N$ contains the structure of a torus, $\Gamma[N, N] \backslash N$, although this torus does not reflect any of the nonabelian structure of N . The only representations in $(\Gamma \backslash N)^\wedge$ which are not infinite dimensional are the one-dimensional characters of $\Gamma[N, N] \backslash N$. Since the presence of this torus is inescapable, we denote, for each $g \in C^\infty(\Gamma \backslash N)$, the sum of the one-dimensional components of g by g_0 . Then global regularity is informally taken to mean that the solvability of $Df = g$ is equivalent to the solvability of $\pi(D)f_\pi = g_\pi$ in C^∞ -vectors for each infinite-dimensional $\pi \in (\Gamma \backslash N)^\wedge$, modulo the solvability of $Df_0 = g_0$ in C^∞ of the torus $\Gamma[N, N] \backslash N$. In order to state the formal definitions and summarize the new results, we need a few preliminaries.

Let \mathfrak{N} denote a finite-dimensional real nilpotent Lie algebra, and $N = \exp \mathfrak{N}$ the corresponding Lie group. Then Malcev proved there exists a discrete subgroup $\Gamma \subset N$ which is cocompact, meaning that the nilmanifold $\Gamma \backslash N$ is a compact homogeneous space, of cosets of the form Γn , if and only if \mathfrak{N} has rational structure constants with respect to some suitable basis. Such a Γ is never normal when N is nonabelian. If Γ exists, the rational basis of \mathfrak{N} can be selected from $\log \Gamma$. If $\Gamma \backslash N$ is a compact nilmanifold, a subspace $V \subset \mathfrak{N}$, is called rational if and only if it is spanned by vectors which are finite linear combinations with rational coefficients of elements of a rational basis of \mathfrak{N} . A subgroup $M \subset N$ is called rational if and only if $\mathfrak{M} = \log M$ is a rational subspace of \mathfrak{N} , and this is equivalent to $\Gamma \cap M \backslash M$ being compact [13].

If $\Gamma \backslash N$ is a compact nilmanifold and M is a normal rational subgroup of N , then Malcev proved the existence of *one-parameter coordinate subgroups* $d_1(t), \dots, d_k(t)$, where $k = \dim(M \backslash N)$, with the following properties. If $N_k = M \rtimes d_k(\mathbf{R})$, a semidirect product with M normal, and if $N_i = N_{i+1} \rtimes d_i(\mathbf{R})$, then $N_1 = N$. Also, $d_i(n) \in \Gamma$ for each $n \in \mathbf{Z}$, the integers [13].

Let \hat{N} denote the space of equivalence classes of all irreducible unitary representations on N . Then Kirillov proved that the elements of \hat{N} are in one-to-one correspondence with the so-called Kirillov orbits of $\text{Ad}^* N$ acting in \mathfrak{N}^* , the linear dual of \mathfrak{N} . If $\pi \in \hat{N}$ corresponds to an orbit $\mathcal{O}_N(\pi) = (\text{Ad}^* N)\Lambda$, we may write $\pi = \pi_\Lambda$, and we may speak of $\Lambda \in \hat{N}$, for convenience. If $\Lambda \in \mathcal{O}_N(\pi)$, there is a subalgebra $\mathfrak{M} \subset \mathfrak{N}$, of maximal dimension so as to be subordinate to Λ , in the sense that $\Lambda([\mathfrak{M}, \mathfrak{M}]) = 0$. Then Λ determines a character of $M = \exp \mathfrak{M}$ by $\chi_\Lambda(m) = \exp[i\Lambda(\log m)]$, and χ_Λ induces π , in the sense of Mackey. The *Mackey induced representation space* is

$$\{f: N \rightarrow C \mid f(mn) = \chi_\Lambda(m)f(n),$$

$$\text{for all } m \in M, n \in N, \text{ and } |f| \in L^2(M \backslash N)\},$$

and $\pi_\Lambda(n)$ acts by right translation on this space.

The following definition, theorem, and proof are contained in [18, §3]. They are included here since we will refer later to certain details of the proof.

DEFINITION 0. Let \mathfrak{N} be any nonabelian nilpotent Lie algebra and $\Lambda \in \mathfrak{N}^* \sim \{0\}$. A *maximal subordinate subalgebra* \mathfrak{M}_Λ for Λ is called *special* if there is a decomposition $\mathfrak{N} = \mathbf{R}X_1 \oplus \dots \oplus \mathbf{R}X_k \oplus \mathfrak{M}_\Lambda$, where the vectors X_1, \dots, X_k , called the *external vectors*, have the following properties.

(i) $\mathfrak{N}_j = \mathbf{R}X_j \oplus \cdots \oplus \mathbf{R}X_k \oplus \mathfrak{M}_\Lambda$ is a subalgebra of \mathfrak{N} for each $j = 1, \dots, k + 1$, where by \mathfrak{N}_{k+1} we mean \mathfrak{M}_Λ , and $\mathfrak{N}_1 = \mathfrak{N}$.

(ii) For each $j = 1, \dots, k$, there is an ideal \mathfrak{W}_j of \mathfrak{N}_j with $\Lambda(\mathfrak{W}_j) = \{0\}$, such that $\mathfrak{N}_j/\mathfrak{W}_j$ has 1-dimensional center, on which the image of Λ is nontrivial.

(iii) For each $j = 1, \dots, k$ there is a vector $\bar{Y}_j \in \mathfrak{N}_j/\mathfrak{W}_j$ such that $[\bar{X}_j, \bar{Y}_j]$ spans the center of $\mathfrak{N}_j/\mathfrak{W}_j$, and the image $\overline{\mathfrak{M}}_\Lambda$ of \mathfrak{M}_Λ lies in the centralizer $\mathfrak{Z}(\bar{Y}_j)$, which equals \mathfrak{N}_{j+1} . If $j = k$, we have $\overline{\mathfrak{M}}_\Lambda = \mathfrak{Z}(\bar{Y}_k)$.

THEOREM 0. *Let \mathfrak{N} be any nilpotent Lie algebra and $\Lambda \in \mathfrak{N}^* \sim \{0\}$. Then there exists a special maximal subordinate subalgebra \mathfrak{M}_Λ for Λ . If there exists a cocompact discrete subgroup $\Gamma \subset N = \exp \mathfrak{N}$, and if $\Lambda(\log \Gamma) \subset \mathbf{Q}$, the rational numbers, then \mathfrak{M}_Λ , the external vectors, and Y_1, \dots, Y_k (preimages of $\bar{Y}_1, \dots, \bar{Y}_k$) can all be taken to be rational.*

PROOF. Let $\mathfrak{Z}(\mathfrak{A})$, where \mathfrak{A} is an algebra, denote the center of \mathfrak{A} , whereas $\mathfrak{Z}(Y)$, where Y is a vector, denotes the centralizer of Y . If $\mathfrak{N}/\ker(\Lambda|\mathfrak{Z}(\mathfrak{N}))$ has 1-dimensional center, then we let $\mathfrak{W}_1 = \ker(\Lambda|\mathfrak{Z}(\mathfrak{N}))$. Otherwise, we factor out $\ker(\Lambda|\mathfrak{Z}(\mathfrak{N}/\ker(\Lambda|\mathfrak{Z}(\mathfrak{N}))))$, etc., until we obtain an ideal $\mathfrak{W}_1 \subset \mathfrak{N}_1 = \mathfrak{N}$ such that $\mathfrak{N}_1/\mathfrak{W}_1$ has 1-dimensional center, on which Λ is nontrivial. The existence of \mathfrak{W}_1 is guaranteed by the fact that $\Lambda \neq 0$, and because $\dim(\mathfrak{N}) < \infty$. If Λ is rational, so is \mathfrak{W}_1 .

By Kirillov’s structure theorem [12], for nilpotent Lie algebras with 1-dimensional center, there are vectors \bar{X}_1 and \bar{Y}_1 in $\mathfrak{N}_1/\mathfrak{W}_1$ with \bar{Y}_1 in the next to the last stage of the lower central series of $\mathfrak{N}_1/\mathfrak{W}_1$, for which $\Lambda([\bar{X}_1, \bar{Y}_1]) \neq 0$ (by abuse of notation, we regard Λ as acting on $\mathfrak{N}_1/\mathfrak{W}_1$). By the Malcev structure theory [13], in the case of Λ rational, we can pick the preimages X_1 and Y_1 rational as well. In either case, we can decompose $\overline{\mathfrak{N}}_1 = \mathbf{R}\bar{X}_1 \oplus \mathfrak{Z}(\bar{Y}_1)$, as in Kirillov [12]. We denote $\overline{\mathfrak{N}}_2 = \mathfrak{Z}(\bar{Y}_1)$. If $\overline{\mathfrak{N}}_2$ is abelian, then $\mathfrak{N}_2 = \mathfrak{M}_\Lambda$ the special maximal subordinate subalgebra. If not, we proceed as follows, noting that \mathfrak{N}_2 is automatically rational if Λ and Y_1 are rational.

Next we find an ideal \mathfrak{W}_2 in $\mathfrak{N}_2, \mathfrak{W}_2 \supset \mathfrak{W}_1$, just as we found $\mathfrak{W}_1 \subset \mathfrak{N}_1$ above. In $\mathfrak{N}_2/\mathfrak{W}_2$, we find \bar{X}_2 and \bar{Y}_2 just as we found \bar{X}_1 and \bar{Y}_1 above. Thus we have $\mathfrak{N}_2/\mathfrak{W}_2 = \mathbf{R}\bar{X}_2 \oplus \mathfrak{Z}(\bar{Y}_2)$ and $\mathfrak{N}_2 = \mathbf{R}X_2 \oplus \mathfrak{N}_3$, where the preimages X_2 and Y_2 can again be taken to be rational in the case of rational Λ . If $\overline{\mathfrak{N}}_3$ is abelian, then $\mathfrak{N}_3 = \mathfrak{M}_\Lambda$. If not, we continue this process, which must terminate in finitely many steps, since $\dim(\mathfrak{N}) < \infty$. Now $\mathfrak{N} = \mathbf{R}X_1 \oplus \cdots \oplus \mathbf{R}X_k \oplus \mathfrak{M}_\Lambda$, where \mathfrak{M}_Λ is clearly subordinate to Λ . It remains only to show that \mathfrak{M}_Λ is of maximal dimension so as to be subordinate to Λ .

If we denote $B_\Lambda(X, Y) = \Lambda([X, Y])$, then the dimension of a maximal subordinate subalgebra is always the dimension of the algebra minus $\frac{1}{2} \text{rank}(B_\Lambda)$, and it is also always the dimension of the algebra minus $\frac{1}{2}$ the dimension of the coadjoint orbit. The maximality of $\overline{\mathfrak{M}}_\Lambda$ in $\overline{\mathfrak{N}}_k$ implies the maximality of \mathfrak{M}_Λ in \mathfrak{N}_k , since $\text{ad}^* \mathfrak{W}_k$ does not enlarge the orbit. But now \mathfrak{M}_Λ is maximal in \mathfrak{N}_{k-1} also, since the dimension of the algebra is increased by 1 and the rank of B_Λ is increased by 2. Hence, by induction, we conclude that \mathfrak{M}_Λ is a special maximal subordinate subalgebra for Λ . This proves Theorem 0.

We will denote by $(\Gamma \setminus N)^\wedge$ the subspace of \hat{N} occurring in the discrete direct sum decomposition of $L^2(\Gamma \setminus N)$. By the combined results of C. C. Moore [15] and the

second author [18], $\pi \in (\Gamma \setminus N)^\wedge$ if and only if there is a rational $\Lambda \in \mathcal{O}_N(\pi)$, and a rational special maximal subordinate subalgebra \mathfrak{M}_Λ , such that $\Lambda: \log(\Gamma \cap M_\Lambda) \rightarrow \mathbf{Z}$. Henceforth, we consider only such rational Λ . In particular, $\Lambda: \log(\Gamma \cap \mathfrak{Z}(N)) \rightarrow \mathbf{Z}$. ($\mathfrak{Z}(N)$ is rational, by [13].) The multiplicity $m(\pi)$ with which π occurs in the π -primary summand $\mathfrak{H}_\pi \subset L^2(\Gamma \setminus N)$ is known explicitly, but here we need only the fact that $m(\pi) < \infty$ [18].

If $\mathfrak{H}_\pi = H_{\pi,1} \oplus \cdots \oplus H_{\pi,m(\pi)}$ is an irreducible decomposition, the spaces $H_{\pi,j}$ are not canonical. Nevertheless, we will use irreducible decompositions because of the convenience of their Schrödinger models, which we will construct. If $g \in L^2(\Gamma \setminus N)$, we will write $g = \sum_{\pi \in (\Gamma \setminus N)^\wedge} g_\pi$, the primary Fourier decomposition into components, and $g_\pi = \sum_{q=1}^{m(\pi)} g_{\pi,q}$ for some particular irreducible decomposition. Auslander and Brezin proved that, if $g \in C^\infty(\Gamma \setminus N)$, then each $g_{\pi,q} \in C^\infty(\Gamma \setminus N)$, which implies in turn that $g_\pi \in C^\infty(\Gamma \setminus N)$. Furthermore, the sums $g = \sum g_\pi = \sum g_{\pi,q}$ both converge uniformly, if $g \in C^\infty(\Gamma \setminus N)$. Furthermore, if we begin with some functions $g_{\pi,q} \in H_{\pi,q}^\infty = H_{\pi,q} \cap C^\infty(\Gamma \setminus N)$, then $\sum_{\pi,q} g_{\pi,q} = g \in C^\infty(\Gamma \setminus N)$ if, and only if, $\sum_{\pi,q} \|Ug_{\pi,q}\|_2^2 < \infty$, for each fixed $U \in \mathcal{U}(\mathfrak{N})$, the universal enveloping algebra of \mathfrak{N} [2]. If D is a left-invariant differential operator on N , viewed on $\Gamma \setminus N$, it follows that $Df_\pi = g_\pi$ can be solved in \mathfrak{H}_π^∞ if and only if $Df_{\pi,q} = g_{\pi,q}$ can be solved in $H_{\pi,q}^\infty$ for each irreducible component $g_{\pi,q}$ of the π -primary function g_π .

The following lemma is already known, but we include a proof for completeness.

LEMMA 1. *Let $\mathfrak{N} \supset \mathfrak{N}_1 \supset \cdots \supset \mathfrak{N}_k \supset \{0\}$ be the lower central series of a nilpotent Lie algebra \mathfrak{N} of step 3 or more ($k \geq 2$). Let $\Lambda \in \mathfrak{N}^*$, $\Lambda|_{\mathfrak{N}_k} \neq 0$. Then there exists a maximal subordinate subalgebra \mathfrak{M} for Λ which contains \mathfrak{N}_{k-1} .*

PROOF. Let \mathfrak{M} be any maximal subordinate subalgebra for Λ . Let $Y \in \mathfrak{N}_{k-1} \sim \mathfrak{M}$. There is $X \in \mathfrak{M}$ such that $[X, Y] \notin \ker(\Lambda)$, and by the Jacobi identity $X \notin \mathfrak{N}_1 \supseteq \mathfrak{N}_{k-1}$ (if $k \geq 2$). Since \mathfrak{M} contains the center, the span $\langle \mathfrak{M}, Y \rangle$ is a subalgebra of \mathfrak{N} . Let $\mathfrak{M}' = \mathfrak{Z}(Y) =$ centralizer of Y in $\langle \mathfrak{M}, Y \rangle$ (modulo $\ker(\Lambda|_{\mathfrak{N}_k})$). Then \mathfrak{M}' has codim = 1 in $\langle \mathfrak{M}, Y \rangle$, so $\dim(\mathfrak{M}') = \dim(\mathfrak{M})$, and \mathfrak{M}' is subordinate to Λ . If there is still a $Y' \in \mathfrak{N}_{k-1} \sim \mathfrak{M}$, we note by Jacobi that $[\mathfrak{N}_{k-1}, Y'] = 0$, so Y' may also be added to \mathfrak{M}' . Eventually, $\mathfrak{M}' \supset \mathfrak{N}_{k-1}$.

LEMMA 1'. *Under the hypothesis of Lemma 1 there exists a special maximal subordinate subalgebra \mathfrak{M} for Λ which contains \mathfrak{N}_{k-1} .*

PROOF. We will retrace the proof of Theorem 0, showing how to guarantee that $\mathfrak{M}_\Lambda \supset \mathfrak{N}_{k-1}$. Since $\Lambda|_{\mathfrak{N}_k} \neq 0$, $\mathfrak{N}/\mathfrak{W}_1$ will still have a k -stage lower central series, where \mathfrak{W}_1 is as in the proof of Theorem 0. (Caution: We are no longer using the symbol \mathfrak{N}_j as in Theorem 0, however.) Hence Y_1 , chosen as in the proof of Theorem 0, will also lie in \mathfrak{N}_{k-1} . Let $Z_1 = [X_1, Y_1]$. By Jacobi's identity, $\mathfrak{Z}(\overline{Y}_1) \supset \overline{\mathfrak{N}}_{k-1}$. If $\overline{\mathfrak{N}}_{k-1} \neq \mathbf{R}\overline{Y}_1$, then there exists a $(k-1)$ -fold bracket product $\overline{A} \in \mathfrak{Z}(\overline{Y}_1) \sim \mathbf{R}\overline{Y}_1$. If \overline{A} is in the center of $\mathfrak{Z}(\overline{Y}_1)$, then $A \in \mathbf{R}Z_1 \oplus \mathfrak{W}_2$ constructed as in the proof of Theorem 0. Otherwise, we simply choose $\overline{Y}_2 = \overline{A}$, in the proof of Theorem 0. Since $[\overline{\mathfrak{M}}, \overline{A}]$ is central in $\overline{\mathfrak{M}}$ and hence in $\mathfrak{Z}(\overline{Y}_1)$ too, Kirillov's Structure Theorem provides that $\mathfrak{Z}(\overline{Y}_1) = \mathbf{R}\overline{X}_2 + \mathfrak{Z}(\overline{A})$, where $\mathfrak{Z}(\overline{A})$ is the centralizer in $\mathfrak{Z}(\overline{Y}_1)$. At the j th stage of the construction, any remaining central elements of $\overline{\mathfrak{N}}_{k-1}$ will be in the corresponding $Z_j + \mathfrak{W}_j$, and any noncentral $\overline{A} \in \overline{\mathfrak{N}}_{k-1} \sim \mathbf{R}\overline{Y}_1 \oplus \cdots \oplus \mathbf{R}\overline{Y}_{j-1}$ can be chosen to be \overline{Y}_j . This process cannot terminate until some l th stage at

which $\overline{\mathfrak{N}}_{k-1} \subset \mathbf{R}\overline{Y}_1 \oplus \cdots \oplus \mathbf{R}\overline{Y}_l \oplus \mathfrak{W}_l \oplus \mathbf{R}\overline{Z}_1 \oplus \cdots \oplus \mathbf{R}\overline{Z}_l$. Since \mathfrak{M}_Λ contains $Y_1, \dots, Y_k, Z_1, \dots, Z_k, \mathfrak{W}_1, \dots, \mathfrak{W}_k$, as in the proof of Theorem 0, we have $\mathfrak{M}_\Lambda \supset \mathfrak{N}_{k-1}$. This proves Lemma 1'.

COROLLARY. *If \mathfrak{N} is of step 3, i.e. $\mathfrak{N} \supset \mathfrak{N}_1 \supset \mathfrak{N}_2 \supset \{0\}$, then there exists a (special) maximal subordinate \mathfrak{M} for Λ which contains $\mathfrak{N}_1 = [\mathfrak{N}, \mathfrak{N}]$ —the commutator of \mathfrak{N} —so that \mathfrak{M} is an ideal in \mathfrak{N} .*

Now let $\Gamma \setminus N$ be a compact nilmanifold and let $g \in C^\infty(\Gamma \setminus N)$. Let g_0 denote the sum of all the 1-dimensional primary components of g , so that g_0 really lives on the torus $\Gamma[N, N] \setminus N$. (On this torus, the phenomenon of small divisors is inescapable, but none of the nonabelian structure of N is present here.) Let $\pi \in (\Gamma \setminus N)^\wedge$ be infinite dimensional, and let $\mathfrak{X}_\pi = H_{\pi,1} \oplus \cdots \oplus H_{\pi,m(\pi)}$ be an irreducible decomposition of the π -primary subspace of $L^2(\Gamma \setminus N)$. We write $g_\pi = g_{\pi,1} + \cdots + g_{\pi,m(\pi)}$ for the corresponding components of g , all of which are in

$$C^\infty(\Gamma \setminus N) \cap \mathfrak{X}_\pi = \mathfrak{X}_\pi^\infty = H_{\pi,1}^\infty \oplus \cdots \oplus H_{\pi,m(\pi)}^\infty.$$

(0.1). **DEFINITION.** A left-invariant differential operator D is called *globally regular* on $\Gamma \setminus N$ if the following three conditions imply that the $f_{\pi,q}$'s can be chosen, if necessary, in such a way that $\sum_{\pi \in (\Gamma \setminus N)^\wedge} \sum_{q=1}^{m(\pi)} f_{\pi,q}$ converges uniformly to a function $f \in C^\infty(\Gamma \setminus N)$ such that $Df = g$:

- (i) $g \in C^\infty(\Gamma \setminus N)$.
- (ii) For each infinite-dimensional $\pi \in (\Gamma \setminus N)^\wedge$ and $j \in \{1, \dots, m(\pi)\}$ there exists $f_{\pi,q} \in H_{\pi,q}^\infty$ such that $Df_{\pi,q} = g_{\pi,q}$.
- (iii) There exists $f_0 \in C^\infty(\Gamma[N, N] \setminus N)$ such that $Df_0 = g_0$.

Note that the function $g_{\pi,q} \in H_{\pi,q}$ can just as well be regarded as a C^∞ -vector $\tilde{g}_{\pi,q}$ in any realization of the irreducible representation π . From this viewpoint, if D is globally regular, then the solution in $C^\infty(\Gamma \setminus N)$ of $Df = g \in C^\infty(\Gamma \setminus N)$ is reduced (except for the problem on the torus $\Gamma[N, N] \setminus N$) to the solution of $\pi(D)\tilde{f}_{\pi,q} = \tilde{g}_{\pi,q}$ in the C^∞ -vectors of the chosen realization of π . Thus the global geometrical problem on $\Gamma \setminus N$ is replaced by a purely group (representation) theoretic problem (together with a well-understood classical problem on a torus). This viewpoint is also critical to the proofs we present here since we will obtain our estimates on the $f_{\pi,q}$ by solving $\pi(D)\tilde{f}_{\pi,q} = \tilde{g}_{\pi,q}$ in a suitable Schrödinger model of π . The freedom of choice of this model grants us the flexibility needed to obtain our estimates.

Note that global regularity carries no implication that D is onto $C^\infty(\Gamma \setminus N)$. (See [19].) Global regularity means only that if $\pi(D)\tilde{f}_{\pi,q} = \tilde{g}_{\pi,q}$ can be solved in the C^∞ -vectors for π , for each component $g_{\pi,q}$ of $g \in C^\infty(\Gamma \setminus N)$, and if $Df_0 = g_0$ can be solved in C^∞ of the torus, then $Df = g$ can be solved in $C^\infty(\Gamma \setminus N)$.

It follows from the definition of global regularity that suitable compositions of finitely many globally regular operators are again globally regular.

PROPOSITION 1. *Let $D = D_n \circ D_{n-1} \circ \cdots \circ D_1$ be a composition of n globally regular operators having the property that $\ker(D_{i+1}) \subseteq \text{range}(D_i \circ D_{i-1} \circ \cdots \circ D_1)$, $i = 1, \dots, n - 1$. Then D is globally regular too.*

PROOF. It will suffice to give the proof for $n = 2$. Suppose, for each $\Lambda \in (\Gamma \setminus N)^\wedge$, there is a smooth solution f_Λ to $Df_\Lambda = g_\Lambda$, where $\sum_\Lambda g_\Lambda = g \in C^\infty(\Gamma \setminus N)$.

Then, for each Λ , $D_2(D_1 f_\Lambda) = g_\Lambda$ and, by hypothesis, there exists $h_\Lambda \in \ker(D_2)$ such that $D_2(h_\Lambda + D_1 f_\Lambda) = g_\Lambda$ and $\sum_\Lambda (h_\Lambda + D_1 f_\Lambda) = h \in C^\infty(\Gamma \setminus N)$. Then there exists $k_\Lambda \in C^\infty(\Gamma \setminus N)$ such that $D_1 k_\Lambda = h_\Lambda + D_1 f_\Lambda$, and, since D_1 is globally regular, there exists $k'_\Lambda \in C^\infty$ such that $\sum_\Lambda k'_\Lambda = k \in C^\infty(\Gamma \setminus N)$ and $Dk_\Lambda = D_2(h_\Lambda + D_1 f_\Lambda) = g_\Lambda$ for each Λ . This proves the proposition.

We remark that the range of each D_i seems to be very large—nearly all of C^∞ , as indicated by the examples in [19].

Our theorems about global regularity will be stated for $D = X + iY$ in the complexified Lie algebra $\mathfrak{N}_\mathbb{C}$. Then many compositions of globally regular first order operators will again be globally regular.

Let V be a real (finite-dimensional) vector space and $\mathbf{E} = \{e_1, \dots, e_n\}$ a basis of V . Then $\alpha = \alpha_1 e_1 + \dots + \alpha_n e_n$ is called a non-Liouville vector in V with respect to \mathbf{E} if there exist positive constants d and C such that

$$|k \cdot \alpha|^{-1} = |k_1 \alpha_1 + \dots + k_n \alpha_n|^{-1} \leq C(k_1^2 + \dots + k_n^2)^d = C|k|^d$$

for all integers k_1, \dots, k_n such that $k_1 \alpha_1 + \dots + k_n \alpha_n \neq 0$.

REMARK 1. If $\alpha \in V$ is a non-Liouville vector with respect to a basis \mathbf{E} of V , then α is also a non-Liouville vector with respect to any basis \mathbf{E}' of V rationally equivalent to \mathbf{E} .

PROOF. Let $\mathbf{E} = \tilde{Q}\mathbf{E}' = q^{-1}Q\mathbf{E}'$, where \tilde{Q} is a rational matrix and Q is an integer matrix such that $q\tilde{Q} = Q$ for some integer q . Then $|Q^T \alpha \cdot k| = |\alpha \cdot Qk| \geq C|Qk|^{-d} \geq C'|k|^{-d}$.

If a compact nilmanifold $\Gamma \setminus N$ has been specified, we call a vector in \mathfrak{N} non-Liouville if it is non-Liouville with respect to any (every) rational basis selected from $\log \Gamma$.

As in [21, p. 18] one can show that the set of all Liouville vectors (= the complement of the set of all non-Liouville vectors) has Lebesgue measure zero in $\mathfrak{N} \simeq \mathbf{R}^n$.

We call a vector in \mathfrak{N} algebraic if and only if it can be expressed as a linear combination of vectors in $\log \Gamma$ with coefficients which are algebraic numbers.

Of course, every algebraic vector is non-Liouville (see e.g. [3, p. 79]). (Note that rational vectors are non-Liouville under our definition above, since we require $|k \cdot \alpha|^{-1}$ to be polynomially bounded only when $k \cdot \alpha \neq 0$.)

(0.2). DEFINITION. $X \in \mathfrak{N}$ is said to have the non-Liouville (resp. algebraic, rational) supplementation property (relative to Γ) if for each stage \mathfrak{N}_j of the lower central series of \mathfrak{N} , $\text{ad}_X \mathfrak{N}_j$ contains a finite set Z_1^j, \dots, Z_m^j of non-Liouville (resp. algebraic, rational) vectors relative to $\log \Gamma$ such that every rational subspace V of \mathfrak{N}_{j+1} of codimension 1 in \mathfrak{N}_{j+1} can be complemented to \mathfrak{N}_{j+1} by some Z_i^j , $1 \leq i \leq m$.

REMARK 2. The rational supplementation property simply means that ad_X maps each stage of the lower central series of \mathfrak{N} onto the next.

Clearly, there are Liouville vectors that have the rational supplementation property. On the other hand, we have the following

PROPOSITION 2. Let X_1, \dots, X_n Lie-generate a rational nilpotent Lie algebra \mathfrak{N} . If $X = \alpha_1 X_1 + \dots + \alpha_n X_n$ is a non-Liouville vector (relative to the basis \mathbf{B} of \mathfrak{N} consisting of X_1, \dots, X_n , and a sufficient set of their commutators) such that $\{\alpha_1, \dots, \alpha_n\}$ is linearly independent over \mathbf{Q} , then X has the non-Liouville supplementation property (relative to the discrete subgroup generated by $\exp B$).

PROOF. Fix $r \leq k =$ length of the lower central series. Let the r -fold bracket products, $\Xi_l = [X_{i_l}[X_{j_l}[X_{k_l} \cdots]]$, $l = 1, \dots, m$, be a basis of \mathfrak{N}_r . Let $V =$ (linear) span of $\{\sum_{l=1}^m \xi_{l_s} \Xi_l, 1 \leq s \leq m - 1, \text{ all } \xi_{l_s} \in \mathbf{Q}\}$ be a rational subspace of \mathfrak{N}_r of codimension 1. Then the $m - 1$ by m matrix (ξ_{l_s}) is of rank $m - 1$. Let s_0 be the number of a column the algebraic complement of which in (ξ_{l_s}) is of full rank. Let $Z_{s_0} \in \mathfrak{N}_r$ be defined by replacing $X_{i_{s_0}}$ in the definition of Ξ_{s_0} by the vector X :

$$(0.3) \quad Z_{s_0} = [X[X_{j_{s_0}}[X_{k_{s_0}} \cdots]] = \sum_{i=1}^n \alpha_i [X_i[X_{j_{s_0}}[X_{k_{s_0}} \cdots]] = \sum_{l=1}^m \beta_l \Xi_l.$$

We notice that $\beta_{s_0} = \alpha_{i_{s_0}} +$ a rational linear combination of the other α_i 's; hence $\beta_{s_0} \neq 0$. Also, none of the formulas for β_l , $l \neq s_0$, resulting from (0.3), contains any term with $\alpha_{i_{s_0}}$. In order to prove that $\mathbf{R}Z_{s_0} \oplus V = \mathfrak{N}_r$, we will show that the m by m matrix obtained by augmenting (ξ_{l_s}) by a row consisting of β_1, \dots, β_m is of rank m . Indeed, the determinant of this matrix, when expanded along the row of β_j 's, is of the form

$$\begin{aligned} & (-1)^{j_0+1} \det[(\xi_{l_s})_{s \neq s_0}] \beta_{s_0} + \sum_{p \neq s_0} (-1)^{p+1} \det[(\xi_{l_s})_{s \neq p}] \beta_j \\ & = (-1)^{j_0+1} \det[(\xi_{l_s})_{s \neq s_0}] \alpha_{i_{s_0}} \\ & \quad + \text{a rational linear combination of the } \alpha_i \text{'s with } i \neq i_{s_0}. \end{aligned}$$

This is nonzero, since the α_j 's are linearly independent over \mathbf{Q} , and the coefficient of $\alpha_{i_{s_0}}$ is $\neq 0$. As in the proof of Remark 1 we can show now that Z_{s_0} is a non-Liouville vector. Thus Z_{s_0} , $s_0 = 1, \dots, m$, as defined by (0.3) is the finite set of non-Liouville vectors of Definition (0.2) complementing every rational subspace V of codim 1 in \mathfrak{N}_r .

Now we are ready to summarize the new results in this paper. In Theorem 1 (§1) we prove that every real field with the (non-Liouville or rational) supplementation property (Definition (0.2)) is globally regular on every compact nilmanifold. This confirms an informal conjecture made by Roger Howe. In Theorem 2 (§1) we show that if X is a non-Liouville vector field in the next to the last step of the lower central series of \mathfrak{N} , then $D = X$ is globally regular. The hypothesis that X be non-Liouville cannot be relaxed as Example 1 shows. Example 2 illustrates some of the obstructions that one encounters in dealing with an arbitrary non-Liouville vector field X . All the other theorems, in §§2-4, deal with complex vector fields $D = X + iY$ with $X, Y \in \mathfrak{N}$, $[X, Y]$ central (or 0). Thus in Theorem 1_C (§2) we prove that D is globally regular if X and Y commute and have property (P_k) , $k = 1, \dots, n$ (see (2.6)), which is a stronger hypothesis than the requirement that both X and Y have (non-Liouville) supplementation. Example 4 of §2 explains that such a stronger requirement is in fact needed. Theorem 2_C (§2) states that D is globally regular if both X and Y are non-Liouville vectors and are in the next to the last step of the lower central series of a metabelian algebra \mathfrak{N} . For 3-step (and lower) compact nilmanifolds $\Gamma \setminus N$, we show in Theorems 3 and 4 (§§3 and 4) that D is globally regular if X has the (non-Liouville) supplementation property, Y is a non-Liouville vector and is in the next to the last step of the lower central series of \mathfrak{N} (which is the commutator $[\mathfrak{N}, \mathfrak{N}]$ if \mathfrak{N} is of step 3, and \mathfrak{N} itself if \mathfrak{N} is of step 2) provided $[X, Y]$ (which then has to be central) is a non-Liouville vector

and some commutativity relations between X and \mathfrak{N} and Y and \mathfrak{N} hold. In the case of a step 3 algebra \mathfrak{N} and a noncentral $[X, Y]$ there is an example, suggested to us by L. Corwin, in which D is not globally regular, even though X has the supplementation property and Y is an algebraic vector $\notin [\mathfrak{N}, \mathfrak{N}]$. The key thing there is that X and Y generate a step 3 subalgebra of \mathfrak{N} .

We elaborate upon that example in §5. It would be interesting to determine, in that context, what happens if X and Y generate a step 2 subalgebra of \mathfrak{N} . We are unable to settle this question. Example 2(c) of §5 indicates that $\pi_\Lambda(D)$ then is closely related to the not everywhere locally solvable operator $\partial/\partial x + ix\partial/\partial y$.

If \mathfrak{N} is of step 2, the above mentioned theorems yield the following three conditions, each of which alone is sufficient to imply that $D = X + iY$ is globally regular on every step 2 compact nilmanifold $\Gamma \setminus N$.

1°. X and Y satisfy (P_1) (Corollary 2 of §4).

2°. X has the non-Liouville supplementation property, and both Y and $[X, Y]$ are non-Liouville vectors (Theorem 4, §4).

3°. All the 3 vectors X, Y and $[X, Y]$ are non-Liouville or, in particular, both X and Y are algebraic vectors (Corollary 1 of §4).

The relevance of the above step 2 theorems to what has been stated in our step 2 paper [7] is as follows: Statements 1° and 2° correct Theorems (3.5) and (3.13); statement 3° is new. To be more specific, statement 2° is precisely what is called Case I of Theorem (3.5) there (p. 542) when $Y = 0$. We obtain the same result here by different method. Case II of Theorem (3.5) when Y is central is still statement 2° and then the assumption, necessary as Example 1 of §1 here demonstrates, that “ Y is algebraic (or non-Liouville)” is missing there. In Case III, $X, Y \notin [\mathfrak{N}, \mathfrak{N}]$ the assumption of Theorems (3.5) and (3.13) there that “ $D \in \mathfrak{N}_\mathbb{C}$ has the supplementation property” meaning $[X, \mathfrak{N}] + [Y, \mathfrak{N}] = [\mathfrak{N}, \mathfrak{N}]$ requires the additional hypothesis that X be algebraic (or non-Liouville) in case $[X, \mathfrak{N}] \neq [\mathfrak{N}, \mathfrak{N}]$ and that Y be algebraic (or non-Liouville) too in case $[Y, \mathfrak{N}] \neq [\mathfrak{N}, \mathfrak{N}]$, as can be seen from Example 1 again. Also our proofs here correct an error in Cases II and III of Theorem (3.5). There the exponential function in the integrand did not behave as claimed in all cases. Thus if $[X, \mathfrak{N}] \neq [\mathfrak{N}, \mathfrak{N}]$ and $[Y, \mathfrak{N}] \neq [\mathfrak{N}, \mathfrak{N}]$ we need the assumption that both X and Y are algebraic (or non-Liouville) and then we do not need any supplementation! However, we still need $[X, Y]$ to be algebraic or non-Liouville (statement 3°). If, on the other hand, $[X, \mathfrak{N}] = [\mathfrak{N}, \mathfrak{N}]$ and $[Y, \mathfrak{N}] \neq [\mathfrak{N}, \mathfrak{N}]$ (or resp. $[Y, \mathfrak{N}] = [\mathfrak{N}, \mathfrak{N}]$ and $[X, \mathfrak{N}] \neq [\mathfrak{N}, \mathfrak{N}]$) we need Y (or resp. X) to be non-Liouville, statement 2°, and then again we need to assume that $[X, Y]$ is algebraic (or non-Liouville). Finally, if both $[X, \mathfrak{N}] = [\mathfrak{N}, \mathfrak{N}]$ and $[Y, \mathfrak{N}] = [\mathfrak{N}, \mathfrak{N}]$ we do not need either X or Y to be algebraic, but we still need a stronger hypothesis to prove that D is globally regular (see Example 2 of §2). The stronger hypothesis we assume is that X and Y satisfy (P_1) , and of course that $[X, Y]$ is non-Liouville—statement 1°.

As for the methods of proofs we use throughout the paper, roughly speaking, the idea is this. “ X having the supplementation property” implies $\pi_\Lambda(X) = \partial/\partial x$ in a suitably chosen Schrödinger model for $\pi_\Lambda \in (\Gamma \setminus N)^\wedge$. This differentiation can of course be converted into a multiplication by $i\xi$ by means of a partial Fourier transform. “ X being in the next to the last step of the lower central series of \mathfrak{N} ”

implies $\pi_\Lambda(X) = i\lambda x$ (again in a suitable model for π_Λ), that is we embed in \mathfrak{N} a 3-dim Heisenberg algebra spanned by $\{X_1, Y_1, Z_1\}$ with either X identified with X_1 or with Y identified with Y_1 . If now $D = X + iY$ is a complex vector field, $[X, Y] = 0$, and some extra commutativity conditions on \mathfrak{N} hold, the two variables x of $\pi_\Lambda(X)$ and y of $\pi_\Lambda(Y)$ are in fact separated, i.e. a 5-dim Heisenberg algebra spanned by $\{X_1, X_2, Y_1, Y_2, Z\}$ can be embedded in \mathfrak{N} with either X identified with X_1 and Y identified with Y_2 (Theorem 3) or with X identified with X_1 and Y identified with X_2 (Theorem 1_C) or with X identified with Y_1 and Y identified with Y_2 (Theorem 2_C), and $\pi_\Lambda(D)$, acting on $\mathbf{H}_\Lambda \simeq L^2(\mathbf{R}^p)$ can be viewed as a multiplication by a polynomial in two variables. In our problems of global regularity of D we deal with the estimates in terms of the Sobolev norms of some Schwartz functions divided by polynomials. The way those estimates depend on $\Lambda \in (\Gamma \setminus N)^\wedge$ turns out to be essential for our problem. If on the other hand $[X, Y] \neq 0$ is central, $\pi_\Lambda(D)$ can be reduced to the form $\partial/\partial x - \lambda x$ rather than a multiplication by a polynomial, which again corresponds to embedding in \mathfrak{N} a 3-dim Heisenberg algebra spanned by X, Y and $[X, Y]$. Using appropriate integral estimates adapted from [7] we can show global regularity of D .

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1. Real fields on n -step nilmanifolds.

THEOREM 1. *Let N be any 1-connected nilpotent Lie group possessing a co-compact discrete subgroup Γ [13]. If $X \in \mathfrak{N}$ has the non-Liouville (or rational) supplementation property (relative to Γ), then $D = X$ is globally regular on $\Gamma \setminus N$.*

PROOF. Let $\pi \in (\Gamma \setminus N)^\wedge$, the set of irreducible unitary representations of N in the spectrum of $\Gamma \setminus N$. The proof is accomplished by establishing inductively that, independent of global solvability questions, $\pi(D)$ satisfies certain Sobolev-type estimates for every infinite-dimensional $\pi \in (\Gamma \setminus N)^\wedge$, in a suitable concrete model of π . It is the formula for the estimates that is treated inductively, not the global regularity. Let Λ be any member of the Kirillov orbit of π which is rational on $\log \Gamma$ and integral on the center [13]. (The theorem is already known for 2-step groups—Case I of Theorem (3.5) of [7].) If Λ were trivial on the center \mathfrak{Z} , then we could factor \mathfrak{Z} out—it is easy to see that $\pi(D)$ would not be affected by such a factoring process. So we can assume without loss of generality that Λ is nontrivial on $\mathfrak{Z} \cap \mathfrak{N}_k$. Let Z_1, \dots, Z_m be the set of non-Liouville vectors in $\text{ad}_X \mathfrak{N}_{k-1}$ described in Definition (0.2). Then there is some $1 \leq j \leq m$ such that $\Lambda(Z_j) = \lambda_j \neq 0$. We remark that $Z_j = \alpha_1 e_1 + \dots + \alpha_n e_n$ for a fixed rational basis e_1, \dots, e_n of \mathfrak{N}_k , and there are constants $C > 0$ and (integer) $d > 0$ (independent of Λ) such that

$$(1.1) \quad |\lambda_j|^{-1} = |\Lambda(Z_j)|^{-1} \leq C(\Lambda(e_1)^2 + \dots + \Lambda(e_n)^2)^d$$

for all (integral on \mathfrak{N}_k) $\Lambda \in (\Gamma \setminus N)^\wedge$, $\Lambda(Z_j) \neq 0$. (If D simply had rational supplementation, we would have picked a rational basis Z_1, \dots, Z_r of \mathfrak{N}_k rather than the set Z_1, \dots, Z_m of non-Liouville vectors. Then we could replace (1.1) by

$$(1.1') \quad |\lambda_j|^{-1} \leq M_j,$$

for some suitable constant M_j , using the fact that the set $\{\Lambda(Z_j)|\pi_\Lambda \in (\Gamma \setminus N)^\wedge, \Lambda$ integral on \mathfrak{Z} , and $\Lambda(Z_j) \neq 0\}$ is bounded away from 0.)

By hypothesis, there is a vector Y_j in the next to the last stage of the lower central series of \mathfrak{N} such that $[X, Y_j] = Z_j$. Let $\overline{\mathfrak{N}} = \mathfrak{N}/\ker(\Lambda|\mathfrak{Z})$ if it has 1-dim center, or $\mathfrak{N}/\mathfrak{W}_1$ as in the beginning of the proof of Theorem 0, §0, otherwise; so $[\overline{X}, \overline{Y}_j] = \overline{Z}_j$, and $\overline{\mathfrak{N}} = \mathbf{R}\overline{X} \oplus \mathfrak{Z}(\overline{Y}_j)$, where $\mathfrak{Z}(\overline{Y}_j)$ is the centralizer of \overline{Y}_j . Without loss of generality, we can take a maximal subordinate subalgebra \mathfrak{M}_Λ for Λ such that $\overline{\mathfrak{M}}_\Lambda = \mathbf{R}\overline{Z}_j \oplus \mathbf{R}\overline{Y}_j \oplus \mathbf{R}\overline{X}_1 \oplus \dots \oplus \mathbf{R}\overline{X}_i$, and $\mathfrak{Z}(\overline{Y}_j) = \overline{\mathfrak{M}}_\Lambda \oplus \mathbf{R}\overline{X}_{i+1} \oplus \dots \oplus \mathbf{R}\overline{X}_n$, where the basis vectors X_1, \dots, X_n are chosen at our convenience and fixed. Pass one-parameter coordinate subgroups through $\exp \overline{Z}_j$, $\exp \overline{Y}_j$, $\exp \overline{X}$, and $\exp \overline{X}_i$, $i = 1, \dots, n$, and we may write

$$\overline{N} = \overline{M}_\Lambda \rtimes \exp(\mathbf{R}\overline{X}_{i+1}) \rtimes \dots \rtimes \exp(\mathbf{R}\overline{X}_n) \rtimes \exp(\mathbf{R}\overline{X}),$$

a sequence of nested semidirect products [18]. In the corresponding Schrödinger model of π in $L^2(\mathbf{R}^{n+1-i})$, we have $\pi(\overline{D}) = \pi(\overline{X}) = \partial/\partial t$, $\pi(\overline{Y}_j) = i\lambda_j t$, and $\pi(\overline{Z}_j) = i\lambda_j$, where t denotes the coordinate in $\exp(\mathbf{R}\overline{X})$. Taking the Euclidean partial Fourier transform $\hat{\cdot}$ with respect to the variable t we have

$$\hat{X} = \hat{\pi}(\overline{X}) = i\xi \quad \text{and} \quad \hat{Y}_j = \hat{\pi}(\overline{Y}_j) = -\lambda_j \frac{\partial}{\partial \xi}$$

where ξ denotes the dual variable of t .

Let the component of $g \in C^\infty(\overline{\Gamma} \setminus \overline{N})$ in the q th irreducible π -subspace (noncanonical) be denoted $g_{\Lambda,q} \in C^\infty(\overline{\Gamma} \setminus \overline{N})$. Let $\tilde{g}_{\Lambda,q}$ denote the corresponding Schwartz function in $L^2(\mathbf{R}^{n+1-i})$, and $\hat{g}_{\Lambda,q}$ the (Euclidean) partial Fourier transform of the Schwartz function $\tilde{g}_{\Lambda,q}$ [21]. If there is an $f_{\Lambda,q} \in C^\infty(\Gamma \setminus N)$ such that $Df_{\Lambda,q} = g_{\Lambda,q}$, or $\tilde{D}\tilde{f}_{\Lambda,q} = \tilde{g}_{\Lambda,q}$, then

$$(1.2) \quad \hat{D}f_{\Lambda,q} = \hat{g}_{\Lambda,q}$$

and

$$(1.3) \quad \hat{f}_{\Lambda,q}(s_\Lambda, \xi) = \hat{g}_{\Lambda,q}(s_\Lambda, \xi)(i\xi)^{-1}$$

is the unique Schwartz solution of (1.2), where the $(n-i)$ -tuple s_Λ denotes the coordinates of the second kind with respect to $\overline{X}_{i+1}, \dots, \overline{X}_n$.

Next, observe that since $\tilde{f}_{\Lambda,q} \in S(\mathbf{R}^{n+1-i})$ it must be that $\hat{g}_{\Lambda,q}(s_\Lambda, 0) = 0$, and by Taylor's formula

$$(1.4) \quad |\hat{f}_{\Lambda,q}(s_\Lambda, \xi)| = \left| \xi \left(\frac{\partial}{\partial \xi} \right) \hat{g}_\Lambda(s_\Lambda, \xi_0) \right| |\xi^{-1}| = |\hat{Y}_j \hat{g}_{\Lambda,q}(s_\Lambda, \xi_0)| |\lambda_j|^{-1}$$

for some ξ_0 between 0 and ξ . To estimate the $L^2(\Gamma \setminus N)$ norm of $f_{\Lambda,q}$ or, which is the same, the $L^2(\mathbf{R}^{n+1-i})$ norm $\|\cdot\|$ of $\hat{f}_{\Lambda,q}$ we notice that by (1.4)

$$(1.5) \quad \begin{aligned} \|\hat{f}_{\Lambda,q}\|^2 &= \int ds_\Lambda \left\{ \int_{-1}^1 + \int_{|\xi| \geq 1} \right\} |\hat{g}_{\Lambda,q}(s_\Lambda, \xi)|^2 \xi^{-2} d\xi \\ &\leq \int 2 \sup_\xi |\hat{Y}_j \hat{g}_{\Lambda,q}(s_\Lambda, \xi)|^2 \lambda_j^{-2} ds_\Lambda + \|\hat{g}_{\Lambda,q}\|^2. \end{aligned}$$

Applying the \mathbf{R}^1 -Sobolev inequality

$$(1.6) \quad \sup_{\xi} |h(\xi)|^2 \leq 2 \left(\|h\|^2 + \left\| \frac{dh}{d\xi} \right\|^2 \right)$$

to $h(\xi) = \hat{Y}_j \hat{g}_{\Lambda,q}(s_{\Lambda}, \xi)$ in (1.5) and replacing $\partial/\partial\xi$ of (1.6) by $-\lambda_j^{-1} \hat{Y}_j$ we obtain

$$(1.7) \quad \|\hat{f}_{\lambda,q}\|^2 \leq 4\lambda_j^{-2} (\|\hat{Y}_j \hat{g}_{\Lambda,q}\|^2 + \lambda_j^{-2} \|\hat{Y}_j^2 g_{\Lambda,q}\|^2) + \|\hat{g}_{\Lambda,q}\|^2.$$

To get estimates on $Uf_{\Lambda,q}$, for each $U \in \mathcal{U}(\mathfrak{N})$, we begin by writing $\hat{D}\hat{U}\hat{f}_{\Lambda,q} = \hat{U}\hat{g}_{\Lambda,q} + [\hat{D}, \hat{U}]\hat{f}_{\Lambda,q}$. Now, if it happens that $[D, [D, U]] = 0$, then we substitute $\hat{f}_{\Lambda,q} = \hat{g}_{\Lambda,q}/P_{\Lambda}$ in the right side above and we obtain

$$\hat{U}\hat{f}_{\Lambda,q} = (\hat{U}\hat{g}_{\Lambda,q} + [\hat{D}, \hat{U}]\hat{g}_{\Lambda,q}/P_{\Lambda})/P_{\Lambda},$$

and so

$$\hat{U}\hat{f}_{\Lambda,q} = (\hat{D}\hat{U}\hat{g}_{\Lambda,q} + [\hat{D}, \hat{U}]\hat{g}_{\Lambda,q})P_{\Lambda}^{-2}.$$

But, if $(\text{ad } D)^2 U \neq 0$, then we apply the same reasoning to $[\hat{D}, \hat{U}]\hat{f}_{\Lambda,q}$ as we used previously for $\hat{U}\hat{f}_{\Lambda,q}$. Hence

$$\hat{U}\hat{f}_{\Lambda,q} = (\hat{U}\hat{g}_{\Lambda,q} + \{[\hat{D}, \hat{U}]\hat{g}_{\Lambda,q} + [\hat{D}[\hat{D}, \hat{U}]]\hat{f}_{\Lambda,q}\}P_{\Lambda}^{-1})P_{\Lambda}^{-1}.$$

If $(\text{ad } D)^3 U = 0$, then we substitute $\hat{f}_{\Lambda,q} = \hat{g}_{\Lambda,q}/P_{\Lambda}$ and obtain

$$\hat{U}\hat{f}_{\Lambda,q} = (\hat{D}^2 \hat{U}\hat{g}_{\Lambda,q} + \hat{D}[\hat{D}, \hat{U}]\hat{g}_{\Lambda,q} + [\hat{D}[\hat{D}, \hat{U}]]\hat{g}_{\Lambda,q})P_{\Lambda}^{-3}.$$

Proceeding inductively, we see that, if $m \geq 1$ is such that $(\text{ad } D)^m U = 0$, then

$$(1.8) \quad \hat{U}\hat{f}_{\Lambda,q} = \hat{h}_m P_{\Lambda}^{-m},$$

where

$$\begin{aligned} \hat{h}_m &= [D[D \cdots [D, U] \cdots]]g_{\Lambda,q} + D[D \cdots [D, U] \cdots]g_{\Lambda,q} \\ &\quad + \cdots + D^{m-2}[D, U]g_{\Lambda,q} + D^{m-1}Ug_{\Lambda,q} \in C^{\infty}(\Gamma \setminus N), \end{aligned}$$

and $P_{\Lambda} = i\xi$. Here the first bracket involves $m - 1$ D 's with the number of D 's inside the brackets decreasing by 1 in each successive summand.

Also we notice that if $P_{\Lambda}(\xi_0) = 0$ then all the (partial) derivatives of order less than m of the Schwartz function $\hat{h}_m = P_{\Lambda}^m \hat{U}\hat{f}_{\Lambda,q}$ vanish at ξ_0 . (Of course, here $P_{\Lambda}(\xi_0) = 0$ if and only if $\xi_0 = 0$, but later we will refer to this argument in the context of a more general P_{Λ} .) Hence all the nonzero terms in Taylor's expansion of \hat{h}_m at ξ_0 contain (partial) derivatives of \hat{h}_m of order at least m and we have the estimate

$$|\hat{h}_m(s_{\Lambda}, \xi)| \leq \sum_{|\alpha|=m} (\alpha!)^{-1} D_{\xi}^{\alpha} \hat{h}_m(s_{\Lambda}, \xi_{\alpha}) |(\xi - \xi_0)^{\alpha}|,$$

for suitable ξ_{α} .

Thus dividing the above inequality by P_{Λ}^m , for $P_{\Lambda} = i\xi$, $\xi_0 = 0$, $\alpha = m$, and $D_{\xi} = -\lambda_j^{-1} \hat{Y}_j$ we get

$$(1.9) \quad |\hat{h}_m P_{\Lambda}^{-m}(s_{\Lambda}, \xi)| \leq (m!)^{-1} \sup_{\xi} |\hat{Y}_j^m \hat{h}_m(s_{\Lambda}, \xi)| |\lambda_j|^{-m}.$$

Now we estimate the $L^2(\mathbf{R}^{n-i})$ norm of $\hat{h}_m P_\Lambda^{-m}$ in exactly the same manner as we estimated $\|\hat{f}_{\Lambda,q}\|^2$ in (1.5) but using now (1.9) instead of (1.4) and we get

$$(1.10) \quad \|\hat{U}\hat{f}_{\Lambda,q}\|^2 \leq 4(m!)^{-1}\lambda_j^{-2m}(\|\hat{Y}_j^m \hat{h}_m\|^2 + \lambda_j^{-2}\|\hat{Y}_j^{m+1} \hat{h}_m\|^2) + \|\hat{h}_m\|^2.$$

(If $m = 1$ and $U = I$ then $\hat{h}_1 = \hat{g}_{\lambda,q}$ and (1.10) is just (1.7).)

If X has the rational supplementation property, (1.1') "eliminates" the λ_j 's from (1.10). If it has the non-Liouville supplementation, we apply (1.1) to get

$$\begin{aligned} \lambda_j^{-2m}\|\hat{Y}_j^m \hat{h}_m\|^2 &\leq C^2\|(\Lambda(e_1)^2 + \dots + \Lambda(e_n)^2)^{md}\hat{Y}_j^m \hat{h}_m\|^2 \\ &= c^2\|\phi_{j,\Lambda,q}\|^2, \end{aligned}$$

and

$$\begin{aligned} \lambda_j^{-2(m+1)}\|\hat{Y}_j^{m+1} \hat{h}_m\|^2 &\leq C^2\|(\Lambda(e_1)^2 + \dots + \Lambda(e_n)^2)^{(m+1)d}\hat{Y}_j^{m+1} \hat{h}_m\|^2 \\ &= C^2\|\psi_{j,\Lambda,q}\|^2, \end{aligned}$$

where $\phi_{j,\Lambda,q}$ and $\psi_{j,\Lambda,q}$ are defined by the equations above, and where $\phi_j = \sum_{\Lambda,q} \phi_{\Lambda,q}$ and $\psi_j = \sum_{\Lambda,q} \psi_{\Lambda,q}$ are in $C^\infty(\Gamma \setminus N)$. Thus, denoting by $||| \cdot |||$ the norm in $L^2(\Gamma \setminus N)$, we have (recalling the j which was fixed early in the proof)

$$\begin{aligned} |||Uf|||^2 &= \sum_{\Lambda,q} \|\hat{U}f_{\Lambda,q}\|^2 \\ &\leq \sum_{j=1}^m \sum_{\{\Lambda,q:\Lambda(Z_j) \neq 0\}} C_1^2(\|\hat{\phi}_{j,\Lambda,q}\|^2 + \|\hat{\psi}_{j,\Lambda,q}\|^2) + 2\|\hat{h}_m\|^2 \\ &\leq \sum_j C_1^2(|||\phi_j|||^2 + |||\psi_j|||^2) + 2m|||h_m|||^2, \end{aligned}$$

i.e. $f = \sum_{\Lambda,q} f_{\Lambda,q} \in C^\infty(\Gamma \setminus N)$ and D is globally regular.

REMARK 1. Suppose $D = X + iZ$ with $X \in \mathfrak{N}$ satisfying the hypothesis of Theorem 1 and $Z \in \mathfrak{Z}$. Then, in the notation of Theorem 1, $\hat{D} = i\xi - \Lambda(\bar{Z})$. We could replace (1.3) by

$$(1.3') \quad \hat{f}_{\Lambda,q}(s_\Lambda, \xi) = \hat{g}_{\Lambda,q}(s_\Lambda, \xi)(i\xi - \Lambda(\bar{Z}))^{-1},$$

and then (1.3') would yield the estimate $|\hat{f}_{\Lambda,q}(s_\Lambda, \xi)| \leq |\hat{g}_{\Lambda,q}(s_\Lambda, \xi)| |\Lambda(\bar{Z})|^{-1}$ or, more generally, by (1.8), $|\hat{U}\hat{f}_{\Lambda,q}| \leq |\hat{h}_m| |\Lambda(\bar{Z})|^{-m}$, provided $\Lambda(Z) \neq 0$. If now Z were a non-Liouville vector, as in (1.1) we would have

$$\begin{aligned} |\Lambda(\bar{Z})|^{-m} &= |\Lambda(Z)|^{-m} \leq C^m(\Lambda(e_1)^2 + \dots + \Lambda(e_n)^2)^{md} \\ &= C^m(\Lambda(\bar{e}_1)^2 + \dots + \Lambda(\bar{e}_n)^2)^{md} \end{aligned}$$

with some rational basis e_1, \dots, e_n of \mathfrak{Z} . Hence

$$(1.11) \quad |||Uf|||^2 \leq C^m |||(\Lambda(e_1)^2 + \dots + \Lambda(e_n)^2)^{md} h_m|||^2 + \sum_{\{\Lambda,q:\Lambda(Z)=0\}} \|\hat{U}\hat{f}_{\Lambda,q}\|^2,$$

while the sum over those Λ such that $\Lambda(Z) = 0$, is finite by the estimates of the proof of Theorem 1. What we have just shown can be stated as the following slightly modified version of Theorem 1.

THEOREM 1'. *Let N be any 1-connected nilpotent Lie group possessing a cocompact discrete subgroup Γ . If $X \in \mathfrak{N}$ has the non-Liouville supplementation property and $Z \in \mathfrak{Z}(\mathfrak{N})$ (center of \mathfrak{N}) is a non-Liouville vector then $D = X + iZ$ is globally regular on $\Gamma \backslash N$.*

The hypothesis of Theorem 1' that Z be a non-Liouville vector cannot be relaxed, even if \mathfrak{N} is only two-step, as the following example shows.

EXAMPLE 1. Let \mathfrak{N} be spanned by $\{X, Y_1, Y_2, Z_1, Z_2\}$, where all nontrivial bracket products are generated by $[X, Y_j] = Z_j, j = 1, 2$. Let $D = X + iZ$, where $Z = Z_1 + \alpha Z_2$ with some fixed real number α . Let $N = \exp \mathfrak{N}$ and let Γ be the discrete cocompact subgroup of N generated by $\exp X, \exp Y_1$ and $\exp Y_2$. Let $\pi \in (\Gamma \backslash N)^\wedge$ be infinite dimensional, and $\Lambda \in \hat{\mathcal{O}}_N(\pi)$ with $\Lambda(Z_j) = \lambda_j \in \mathbf{Z}, j = 1, 2$. Then there is a Schrödinger model of π in $L^2(\mathbf{R})$ derived from the Mackey induced representation space using the maximal subordinate subalgebra \mathfrak{M} spanned by $\{Y_1, Y_2, Z_1, Z_2\}$. In this model $\pi(X) = d/dt, \pi(Y_j) = i\lambda_j t$, and $\pi(Z_j) = i\lambda_j, j = 1, 2$. Taking the (Euclidean) Fourier transform $\hat{\cdot}$ we obtain $\hat{\pi}(X) = i\xi, \hat{\pi}(Y_j) = -\lambda_j(\partial/\partial\xi), \hat{\pi}(Z_j) = i\lambda_j$, and $\hat{\pi}(D) = i\xi - (\lambda_1 + \alpha\lambda_2)$. Hence $\hat{f}_\Lambda(\xi) = \hat{g}_\Lambda(\xi)(i\xi - \lambda_1 - \alpha\lambda_2)^{-1}$ is the unique Schwartz solution of $\hat{\pi}(D)\hat{f}_\Lambda = \hat{g}_\Lambda$. If α is a Liouville number, which corresponds to Z being a Liouville vector in the center \mathfrak{Z} of \mathfrak{N} , we can find, for each integer $p > 0$, a pair $(\lambda_{1p}, \lambda_{2p})$ such that $|\lambda_{1p} + \alpha\lambda_{2p}| < (|\lambda_{1p}| + |\lambda_{2p}|)^{-p}$. Let $\Lambda_p \in (\Gamma \backslash N)^\wedge$ be such that $\Lambda_p(Z_j) = \lambda_{jp}, j = 1, 2$. Define $g = \sum_p g_p$ where $g_p \in H_{\Lambda_p}, p = 1, 2, \dots$, in such a way that $\hat{g}_p(\xi) = (|\lambda_{1p}| + |\lambda_{2p}|)^{-p/2}\phi(\xi)$ with some fixed $\phi \in C^\infty(\mathbf{R}), \phi(\xi) = 1$ on a neighborhood of $\xi = 0$ and $\phi(\xi) = 0$ outside a compact neighborhood of zero. Then $g \in C^\infty(\Gamma \backslash N)$, but $\sum_p \|\hat{f}_p(0)\|^2$ diverges. By the Sobolev inequality (1.6)

$$\begin{aligned} \infty &= \sum_p \|\hat{f}_p(0)\|^2 \leq \sum_p \sup_\xi \|\hat{f}_p(\xi)\|^2 \leq \sum_p 2(\|\hat{f}_p\|^2 + \lambda_j^{-2}\|\hat{Y}_j \hat{f}_p\|) \\ &\leq 2\|f\|^2 + 2\|Y_j f\|^2. \end{aligned}$$

Thus $f = \sum_p f_p \notin C^\infty(\Gamma \backslash N)$ and D is not globally regular on $\Gamma \backslash N$.

The hypothesis of Theorem 1 (and also that of Theorem 1') that X has the non-Liouville supplementation property can be replaced by a number theoretic condition, as illustrated by Example 2. The example also shows what obstructions one encounters in attempting to prove a general theorem of this sort.

EXAMPLE 2. Let the Lie algebra \mathfrak{N}_n be spanned by the vectors $X, Y_1, \dots, Y_{n-1}, Z$, where Z is central. Let the only nontrivial bracket products be generated by $[X, Y_j] = Y_{j+1}, 1 \leq j < n-1$, and $[X, Y_{n-1}] = Z$. We call $N_n = \exp \mathfrak{N}_n$ the n -step chain group. Let Γ be a cocompact discrete subgroup of N_n such that X and Y_1 , the generators of \mathfrak{N}_n , are in $\log \Gamma$. If \mathfrak{M} is spanned by $\{Y_1, \dots, Y_{n-1}, Z\}$, and if $\Lambda(Z) \neq 0$, then \mathfrak{M} is a Γ -rational subalgebra of \mathfrak{N}_n and is maximal subordinate to Λ . If we now let $n = 3$, and $D = Y_1 + \beta Y_2 + \gamma Z (\in \mathfrak{M})$, we will show that D is globally regular on $\Gamma \backslash N_3$ if $\beta^2/2 - \gamma$ is not a Liouville number. (In particular, if both β and γ are algebraic numbers, then $\beta^2/2 - \gamma$ is non-Liouville.) Here $[D, \mathfrak{N}_1] = \{0\} \subsetneq \mathbf{R}Z = \mathfrak{N}_2$, and consequently D does not have any form of the supplementation property.

All the nonequivalent unitary irreducible representations in the spectrum of $\Gamma \setminus N_3$ which are nontrivial on the center can be parametrized by pairs of integers (a, c) with $a \neq 0$ that correspond to integral $\Lambda \in \mathfrak{N}_3^*/\text{Ad } N_3$ with $\Lambda(Z) = a$, $\Lambda(Y_1) = c$, $\Lambda(Y_2) = 0$. All those representations, say π_{ac} , can be realized on $L^2(\mathbf{R})$ in such a way that (cf. [4, p. 119]) $\pi_{ac}(X) = d/dx$, $\pi_{ac}(Y_1) = 2\pi i(c + ax^2/2)$, $\pi_{ac}(Y_2) = 2\pi iax$, and $\pi_{ac}(Z) = 2\pi ia$. Thus

$$\pi_{ac}(D) = 2\pi i(c + ax^2/2) + \beta ax + \gamma a = a\pi i(x - x_+)(x - x_-),$$

with $x_{\pm} = -\beta \pm (\beta^2 - 2\gamma - 2c/a)^{1/2}$. Therefore

$$\tilde{f}_{ac}(x) = \tilde{g}_{ac}(x)[\pi ia(x - x_+)(x - x_-)]^{-1}$$

is the unique Schwartz solution of the equation $\pi_{ac}(D)\tilde{f}_{ac} = \tilde{g}_{ac}$ and we can write

$$\begin{aligned} \tilde{f}_{ac}(x) &= [\pi ia(x_+ - x_-)]^{-1}[\tilde{g}_{ac}(x)(x - x_+)^{-1} - \tilde{g}_{ac}(x)(x - x_-)^{-1}], \\ &\quad \text{if } x_+ - x_- = 2(\beta^2 - 2\gamma - 2c/a)^{1/2} \neq 0, \end{aligned}$$

or

$$\tilde{f}_{ac}(x) = \tilde{g}_{ac}(x)(x - \beta)^2/\pi ia \quad \text{otherwise.}$$

Now, if $\beta^2/2 - \gamma$ is not a Liouville number, there exist a positive constant K and a positive integer p such that $|\beta^2 - 2\gamma - 2c/a| \geq K^{-2}(2\pi a)^{-2(p+1)}$ for all integers c and a , $a \neq 0$. Hence

$$\begin{aligned} |\tilde{f}_{ac}(x)| &\leq K|\tilde{g}_{ac}(x)(2\pi a)^p/(x - x_+)| + K|\tilde{g}_{ac}(x)(2\pi a)^p/(x - x_-)| \\ &= K|\tilde{Z}^p\tilde{g}_{ac}(x)|/|x - x_+| + K|\tilde{Z}^p\tilde{g}_{ac}(x)|/|x - x_-|. \end{aligned}$$

If x_+ and x_- are complex roots, then for all real x ,

$$|x - x_{\pm}| \geq (\beta^2 - 2\gamma - 2c/a)^{1/2} \geq K^{-1}(2\pi a)^{-p-1},$$

and we have the estimate $|\tilde{Z}^p\tilde{g}_{ac}(x)|/|x - x_{\pm}| \leq K|\tilde{Z}^{2p+1}\tilde{g}_{ac}(x)|$ which implies the corresponding norm estimate $\|\tilde{Z}^p\tilde{g}_{ac}/(x - x_{\pm})\| \leq K\|\tilde{Z}^{2p+1}\tilde{g}_{ac}\|$. If x_+ and x_- are real, we notice that $\tilde{g}_{ac}(x_{\pm}) = 0$ and we use Taylor's expansion of $\tilde{Z}^p\tilde{g}_{ac}(x)$ at $x = x_{\pm}$ to get to estimate

$$(1.12) \quad |\tilde{Z}^p\tilde{g}_{ac}(x)|/|x - x_{\pm}| \leq \sup_x |\tilde{X}\tilde{Z}^p\tilde{g}_{ac}(x)|.$$

To estimate $\|\tilde{f}_{ac}\|$, as in (1.5) we break the integral $\int \cdots dx$ over \mathbf{R} into two integrals: one over the interval $|x - x_{\pm}| \leq 1$ and the other one over its complement $|x - x_{\pm}| > 1$, we apply (1.12) to the integrand of the first integral and then use Sobolev's inequality (1.6) with $h(x) = \tilde{X}\tilde{Z}^p\tilde{g}_{ac}(x)$. The resulting estimate is

$$\|\tilde{f}_{ac}\|^2 \leq 8K^2(\|\tilde{X}\tilde{Z}^p\tilde{g}_{ac}\|^2 + \|\tilde{X}^2\tilde{Z}^p\tilde{g}_{ac}\|^2) + 2K^2\|\tilde{Z}^p\tilde{g}_{ac}\|^2.$$

Next we proceed as we did in (1.8) to obtain the estimates on $\tilde{U}\tilde{f}_{ac}$, $U \in \mathcal{U}(\mathfrak{N}_3)$, a , c integers, $a \neq 0$. The remaining ∞ -dimensional representations of N_3 which are in the spectrum of $\Gamma \setminus N_3$ correspond to integral $\Lambda \in \mathfrak{N}_3^*/\text{Ad } N_3$ which are 0 on the center $\mathbf{R}Z$ of \mathfrak{N}_3 , and therefore can be viewed as representations of the quotient group $\bar{N}_3 = N_3/\exp \mathbf{R}Z$ isomorphic to the 3-dimensional Heisenberg group. On this quotient group $\bar{N}_3, \bar{D} = \bar{Y}_1 + \beta\bar{Y}_2$ already has the rational supplementation property: $[\bar{D}, \bar{\mathfrak{N}}_3] = \mathbf{R}\bar{Y}_2$, and is thus globally regular by Theorem 1.

Conversely, if $\beta^2/2 - \gamma$ is a Liouville number that can be rapidly approximated by rationals from the right, i.e. if there exists an infinite sequence of rational numbers p_n/q_n , $(p_n, q_n) = 1$, such that $0 < p_n/q_n - \beta^2/2 + \gamma < q_n^{-n}$, then $D = Y_1 + \beta Y_2 + \gamma Z$ is not globally regular. (Clearly there exist such Liouville numbers; that they do not comprise the whole set of Liouville numbers follows from the properties of convergents of continued fractions.) To show this we may actually assume that there is a sequence $\{p_n/q_n\}$ of rational numbers such that $0 < p_n/q_n - \beta^2/2 + \gamma < (q_n + |p_n|)^{-n}$. Let $g = \sum_{n=1}^\infty g_n \in C^\infty(\Gamma \setminus N)$ be such that each g_n is in an (irreducible) representation space $H_\pi \subset L^2(\Gamma \setminus N)$ of $\pi_{q_n p_n}$ and corresponds to $\tilde{g}_n(x) = (q_n + |p_n|)^{-n/2} \psi(x)$ in the $L^2(\mathbf{R})$ realization of H_π , as described above ($\psi(x) \in C^\infty(\mathbf{R})$, $\psi(-\beta) = 1$, $\psi \geq 0$, compactly supported). We have the following estimates on the solutions \tilde{f}_n of $\pi_{q_n p_n}(D)\tilde{f}_n = \tilde{g}_n$:

$$\begin{aligned} \sup |\tilde{f}_n(x)| &= \sup |\tilde{g}_n(x)/[\pi i q_n(x - x_+)(x - x_-)]| \\ &\geq |\tilde{g}_n(-\beta)/\pi q_n(\beta^2 - 2\gamma - 2p_n/q_n)| \geq (q_n + |p_n|)^{n-n/2-1}, \end{aligned}$$

where $x_\pm = -\beta \pm i(p_n/q_n - \beta^2 + 2\gamma)^{1/2}$, and the sup is taken over all $x \in \mathbf{R}$. As in Example 1, this shows that $f = \sum f_n \notin C^\infty(\Gamma \setminus N)$. (Note that the right sidedness of the Liouville number was used to insure that neither x_+ nor x_- is real.)

On the chain group N_n , any D without the supplementation property acts in the representation space of π_Λ realized on $L^2(\mathbf{R})$ as a multiplication by a polynomial, say $P_\Lambda(x)$. Example 2 exhibits that the crucial thing for the global regularity of such D then is how rapidly the distances from the real line of the complex roots of $P_\Lambda(x)$ vary with $\Lambda \in \mathfrak{N}_3^*/\text{Ad}^* N_3$ “as $\Lambda \rightarrow \infty$ ”. We will finish this section with the following

THEOREM 2. *Let N be any 1-connected nilpotent Lie group possessing a co-compact discrete subgroup Γ . Let $D = Y \in \mathfrak{N}_{k-1}$ —the next to the last step of the lower central series of \mathfrak{N} , the Lie algebra of N . Suppose Y is a non-Liouville vector (relative to Γ). Then D is globally regular on $\Gamma \setminus N$.*

PROOF. The hypothesis in Theorem 1 that X has supplementation was to assure the existence of \bar{Y}_j in $\bar{\mathfrak{N}}_{k-1}$ such that $[\bar{X}, \bar{Y}_j] = \bar{Z}_j$, so that we could build a Schrödinger model for the representation π_Λ in which $\pi_\Lambda(\bar{X}) = \partial/\partial x$, $\pi_\Lambda(\bar{Y}_j) = i\Lambda(\bar{Z}_j)x$, and $\pi_\Lambda(\bar{Z}_j) = i\Lambda(Z_j)$. We can now reverse the roles of X and Y_j in this picture. That is, we start with $D = Y \in \mathfrak{N}_{k-1}$ and a rational basis Z_1, \dots, Z_n of \mathfrak{N}_k . Then for given Λ , if $\Lambda(Z_j) \neq 0$, either \bar{Y} is in the center of $\bar{\mathfrak{N}}$ (a quotient of \mathfrak{N} with one-dimensional center spanned by \bar{Z}_j), in which case we get the estimates on $\pi_\Lambda(\bar{D})$ using (1.1) as we did in (1.11), with $m = 1$, and noting that the passage from \mathfrak{N} to $\bar{\mathfrak{N}}$ does not affect these estimates, or else $\bar{Y} \in \bar{\mathfrak{N}}_{k-1} \sim$ center of $\bar{\mathfrak{N}}$. It is in the former case that we need non-Liouville, while in the latter case, there exists $\bar{X}_j \in \bar{\mathfrak{N}}$ such that $[\bar{X}_j, \bar{Y}] = \bar{Z}_j$. We again build a Schrödinger model for π_Λ in which $\pi_\Lambda(\bar{X}_j) = \partial/\partial x$, $\pi_\Lambda(\bar{Y}) = i\Lambda(Z_j)x$, and $\pi_\Lambda(\bar{Z}_j) = i\Lambda(Z_j)$. Again, $\tilde{f}_\Lambda(x) = \tilde{g}_\Lambda(x)[i\Lambda(Z_j)x]^{-1}$ is the only Schwartz solution of $\pi_\Lambda(D)\tilde{D}_\Lambda = \tilde{g}_\Lambda$, and we can use $\pi_\Lambda(X_j) = \partial/\partial x$ in Taylor’s formula and Sobolev’s inequality that are needed for the estimates on $\tilde{U}\tilde{f}_\Lambda$, $U \in \mathcal{U}(\mathfrak{N})$, as in the proof of Theorem 1, noticing that $|\Lambda(Z_j)|^{-1} \leq M_j$ for all integral on \mathfrak{M} such that $\Lambda(Z_j) \neq 0$.

We mention also the following modification of Theorem 2.

THEOREM 2'. *Let N be any 1-connected nilpotent Lie group possessing a co-compact discrete subgroup Γ . Let $D = Y + iZ$ with $Y \in \mathfrak{N}_{k-1}$ and $Z \in$ the center of \mathfrak{N} . Suppose that both Y and Z are non-Liouville vectors (relative to Γ). Then D is globally regular on $\Gamma \setminus N$.*

PROOF. Let $\Lambda \in (\Gamma \setminus N)^\wedge$. If, after reducing \mathfrak{N} to $\overline{\mathfrak{N}}$ with one-dimensional center $\overline{\mathfrak{Z}}$, $\overline{Y} \in \overline{\mathfrak{Z}}$, then $\pi_\Lambda(D) = i\Lambda(Y) - \Lambda(Z)$, $\tilde{f}_\Lambda = \tilde{g}_\Lambda(i\Lambda(Y) - \Lambda(Z))^{-1}$, and we use either the estimate $|\tilde{f}_\Lambda| \leq |\tilde{g}_\Lambda| |\Lambda(Z)|^{-1}$ if $\Lambda(Z) \neq 0$, or we use $|\tilde{f}_\Lambda| \leq |\tilde{g}_\Lambda| |\Lambda(Y)|^{-1}$ if $\Lambda(Z) = 0$, and the proof goes now as in Theorem 1' using (1.1). If, on the other hand, $\overline{Y} \notin \overline{\mathfrak{Z}}$, then $\pi_\Lambda(D) = i\Lambda(Z_j)x - \Lambda(Z)$, again we proceed as in the proof of Theorem 1' when $\Lambda(Z) \neq 0$, while if $\Lambda(Z) = 0$, we are in the situation of Theorem 2; cf. the comment at the beginning of the proof of Theorem 1, §1.

2. n -step nilmanifolds—complex fields. We can “complexify” Theorem 2 as follows. Suppose $D = Y_1 + iY_2$ with $Y_1, Y_2 \in \mathfrak{N}_{k-1}$, both non-Liouville vectors. If \mathfrak{N} is of step 3 or more, then $[Y_1, Y_2] = 0$. If \mathfrak{N} is of step 2, $\mathfrak{N}_{k-1} = \mathfrak{N}$, and we simply assume that $[Y_1, Y_2] = 0$. Let Z_1, \dots, Z_n be a fixed rational basis of \mathfrak{N}_k , $\Lambda \in \mathfrak{N}^*$, rational (as remarked after Theorem 0), $\Lambda(Z_j) \neq 0$ for some j , be given. After reducing to one-dimensional center, we may assume that $\overline{Y}_1, \overline{Y}_2 \in \overline{\mathfrak{N}}_{k-1} \sim \overline{\mathfrak{Z}}$. (Otherwise we would use the estimates from the proof of Theorem 2', noting that these estimates are independent of the passage from N to \overline{N} .) Therefore, there exists $X_j \in \mathfrak{N}$ such that $[\overline{X}_j, \overline{Y}_1] = \overline{Z}_j$. We construct a Schrödinger model for π_Λ choosing X_j to be the first external vector. Since $Y_2 \in \mathfrak{N}_{k-1}$, we have $[\overline{X}_j, \overline{Y}_2] \in \mathfrak{N}_k \subseteq \text{center}(\mathfrak{N}) = \mathbf{R}\overline{Z}_j$, so $[\overline{X}_j, \overline{Y}_2] = \alpha_j \overline{Z}_j$ for some $\alpha_j \in \mathbf{R}$. If $\overline{Y}_2 \in \text{center of } \mathfrak{Z}(\overline{Y}_1)$, then in our model

$$(2.1) \quad \begin{aligned} \pi_\Lambda(Y_1) &= i\Lambda(Z_j)x, & \pi_\Lambda(X_j) &= \frac{\partial}{\partial x}, \\ \pi_\Lambda(Y_2) &= i\Lambda([X_j, Y_2])x, & \text{and } \pi_\Lambda(Z_j) &= i\Lambda(Z_j). \end{aligned}$$

Thus $\pi_\Lambda(D) = \{i\Lambda(Z_j) - \Lambda([X_j, Y_2])\}x$ and we can use the same estimates on \tilde{f}_Λ , unique Schwartz solution of $\pi_\Lambda(D)\tilde{f}_\Lambda = \tilde{g}_\Lambda$, as we did in the proof of Theorem 2 (we notice that $|i\Lambda(Z_j) - \Lambda([X_j, Y_2])|^{-1} \leq |\Lambda(Z_j)|^{-1}$). If on the other hand, $\overline{Y}_2 \notin \text{center of } \mathfrak{Z}(\overline{Y}_1)$, then there exists $\overline{V}_j \in \mathfrak{Z}(\overline{Y}_1)$ such that $[\overline{V}_j, \overline{Y}_2] = \overline{Z}_j$ in $\mathfrak{Z}(\overline{Y}_1)$, and we pick V_j to be the second external vector in our model. In this model $\pi_\Lambda(Y_1) = i\Lambda(Z_j)x$,

$$(2.2) \quad \pi_\Lambda(X_j) = \frac{\partial}{\partial x}, \quad \pi_\Lambda(Y_2) = i\Lambda(Z_j)v + i\Lambda([X_j, Y_2])x,$$

and

$$\pi_\Lambda(V_j) = \frac{\partial}{\partial v} + i\Lambda([X_j, V_j])x$$

provided $[X_j, V_j]$ is central. Hence

$$\pi_\Lambda(D) = i\Lambda(Z_j)x - \{\Lambda(Z_j)v + \Lambda([X_j, Y_2])x\} \equiv P_\Lambda(x, v),$$

and $|P_\Lambda(x, v)|^2 = \langle A(x, v), (x, v) \rangle$, where the real symmetric matrix $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, and $a = \Lambda(Z_j)^2 + \Lambda([X_j, Y_2])^2$, $b = \Lambda(Z_j)\Lambda([X_j, Y_2])$, and $c = \Lambda(Z_j)^2$. Then the

minimum eigenvalue of A is positive and is readily calculated to be

$$(2.3) \quad \begin{aligned} c_{\Lambda_j} &= 2(ac - b^2)\{a + c + ((a - c)^2 + 4b^2)^{1/2}\}^{-1} \\ &= 2\Lambda(Z_j)^4\{2\Lambda(Z_j)^2 + \Lambda([X_j, Y_2])^2 \\ &\quad + |\Lambda([X_j, Y_2])|(\Lambda([X_j, Y_2])^2 + 4\Lambda(Z_j)^2)^{1/2}\}^{-1}. \end{aligned}$$

Then

$$|P_\Lambda(x, v)|^2 \geq c_{\Lambda_j}(x^2 + v^2), \quad \text{for all real } x \text{ and } v.$$

In order to obtain L^2 -estimates on \tilde{f}_Λ , the unique Schwartz solution of $\pi_\Lambda(D)\tilde{f}_\Lambda = \tilde{g}_\Lambda$, we proceed as in the proof of Theorem 1, with small modifications. First, $P_\Lambda(x, v) = 0$ iff $x = v = 0$ implies that $\tilde{g}_\Lambda(s_\Lambda, 0, 0) = 0$. Instead of (1.4) we now apply Taylor's formula on \mathbf{R}^2 combined with the expressions (2.2) for the partial derivatives $\partial/\partial x$ and $\partial/\partial v$. Thus by elementary inequalities we have

$$(2.4) \quad \begin{aligned} |\tilde{f}_{\Lambda,q}(s_\Lambda, v, x)|^2 &\leq \left(\left| x \left(\frac{\partial \tilde{g}_\Lambda}{\partial x} \right) (s_\Lambda, v_0, x_0) \right| + \left| v \left(\frac{\partial \tilde{g}_\Lambda}{\partial v} \right) (s_\Lambda, v_1, x_1) \right| \right)^2 \cdot P_\Lambda(v, x)^{-2} \\ &\leq c_{\Lambda_j}^{-1} \left(\sup |\tilde{X}_j \tilde{g}_\Lambda(s_\Lambda, v, x)|^2 + \sup |(\tilde{V}_j + i[\tilde{X}_j, \tilde{V}_j] \tilde{Y}_1 \Lambda(Z_j)^{-1}) \tilde{g}_\Lambda(s_\Lambda, v, x)|^2 \right) \\ &\leq c_{\Lambda_j}^{-1} \left(\sup |\tilde{X}_j \tilde{g}_\Lambda|^2 + 2 \sup |\tilde{V}_j \tilde{g}_\Lambda|^2 + 2 \sup |[\tilde{X}_j, \tilde{V}_j] \tilde{Y}_1 \tilde{g}_\Lambda|^2 M_j^2 \right) \end{aligned}$$

where $M_j = \text{Sup}\{|\Lambda(Z_j)|^{-1}: \Lambda \in (\Gamma \setminus N)^\wedge, \Lambda(Z_j) \neq 0\}$ as in (1.1'), and all the suprema are taken over $v, x \in \mathbf{R}$. We notice that for any Schwartz h_Λ , by (2.3), we have

$$\begin{aligned} c_{\Lambda_j}^{-1} |\tilde{h}_\Lambda|^2 &\leq \frac{1}{2} M_j^4 \{2\Lambda(Z_j)^2 + \Lambda([X_j, Y_2])^2 + 2|\Lambda([X_j, Y_2])\Lambda(Z_j)|\} |\tilde{h}_\Lambda|^2 \\ &\leq \frac{3}{2} M_j^4 (|\tilde{Z}_j \tilde{h}_\Lambda|^2 + |[\tilde{X}_j, \tilde{Y}_2] \tilde{h}_\Lambda|^2). \end{aligned}$$

Applying this inequality to \tilde{h}_Λ equal subsequently to $\tilde{X}_j \tilde{g}_\Lambda$, $\tilde{V}_j \tilde{g}_\Lambda$, and $[\tilde{X}_j, \tilde{V}_j] \tilde{Y}_1 \tilde{g}_\Lambda$, we get from (2.4)

$$\begin{aligned} |\tilde{f}_{\Lambda,q}(s_\Lambda, v, x)|^2 &\leq \frac{3}{2} M_j^4 \sum_{k=1}^2 (\sup |\tilde{U}_{jk} \tilde{X}_j \tilde{g}_\Lambda|^2 + 2 \sup |\tilde{U}_{jk} \tilde{V}_j \tilde{g}_\Lambda|^2 + 2 \sup |\tilde{U}_{jk} [\tilde{X}_j, \tilde{V}_j] \tilde{Y}_1 \tilde{g}_\Lambda|^2 M_j^2) \end{aligned}$$

with $U_{j1} = Z_j$ and $U_{j2} = [X_j, Y_2]$. Next, instead of $\int \dots d\xi$ in (1.5) we use $\iint \dots dv dx$ and then we break this double integral into two integrals, one over the unit disk $x^2 + v^2 \leq 1$ and the other one over its complement, $x^2 + v^2 \geq 1$. (Here we are using (2.3).) And finally, instead of (1.6) we use the \mathbf{R}^2 -Sobolev inequality

$$\sup |f(v, x)|^2 \leq C \left(\|f\|^2 + \left\| \frac{\partial^2 f}{\partial \xi^2} \right\|^2 + \left\| \frac{\partial^2 f}{\partial \eta^2} \right\|^2 \right)$$

which we apply to each of the summands in (2.4). (We note that $|c_{\Lambda_j}|^{-1}$ grows at most polynomially in Λ .) For the estimates on $\tilde{U} \tilde{f}_{\Lambda,q}$, $U \in \mathcal{U}(\mathfrak{N})$, we proceed as in (1.8) with the same modifications as described above. All this procedure works fine, except that for a general \mathfrak{N} it is hard to tell whether we can actually pick X_j and V_j so that $[X_j, V_j]$ needed in (2.1) is central. It surely is central if \mathfrak{N} is of step two, and we have the following proposition.

PROPOSITION 1. *Let N be a simply connected step 2 nilpotent Lie group, and Γ a cocompact discrete subgroup of N . If $X, Y \in \mathfrak{N}$ commute, and both are non-Liouville vectors, then $D = X + iY$ is globally regular on $\Gamma \setminus N$.*

Another natural class of groups for which $[X_j, V_j] = 0$ is that of nilpotent semidirect products of Abelian groups (metabelian groups). For let $\mathfrak{N} = \mathfrak{A} \ltimes \mathfrak{B}$ with $\mathfrak{A} \simeq \mathbf{R}^p$, $\mathfrak{B} \simeq \mathbf{R}^q$. Then, if the above constructed X_j and V_j are such that $X_j = \xi_1 + \eta_1$ and $V_j = \xi_2 + \eta_2$ with some $\xi_i \in \mathfrak{A}$, $\eta_i \in \mathfrak{B}$, $i = 1, 2$, we can simply replace them by $X'_j = \xi_1$ and $V'_j = \xi_2$. We summarize these facts as the following

THEOREM 2_C. *Let N be a metabelian nilpotent Lie group with a cocompact discrete subgroup Γ . Let $D = Y_1 + iY_2$ with $Y_1, Y_2 \in \mathfrak{N}_{k-1}$ —the next to the last step of the lower central series of \mathfrak{N} , the Lie algebra of N . Suppose both Y_1 and Y_2 are non-Liouville vectors (relative to Γ). Then D is globally regular on $\Gamma \setminus N$. If N is of step less than 3 assume moreover that Y_1 and Y_2 commute.*

Of course X_j and V_j may commute even if \mathfrak{N} is not metabelian.

EXAMPLE 1. Let \mathfrak{N} be the nilpotent Lie algebra of strictly upper triangular $n \times n$ real matrices, $n \geq 4$. Then \mathfrak{N} is of step $n - 1$ and the next to the last stage of \mathfrak{N} is spanned by the two vectors $E_{1,n-1}$ and $E_{2,n}$ (E_{kl} denotes the matrix which has 1 on the crossing of k th row with l th column and zeros otherwise) and the one-dimensional center by E_{nn} . $D = E_{1,n-1} + iE_{2,n}$ is globally regular on any nilmanifold $\Gamma \setminus \exp \mathfrak{N}$ such that $E_{1,n-1}, E_{2,n}$ are non-Liouville with respect to the Γ rational basis. The (commuting) external vectors X_1 and V_1 can be chosen to be $-E_{n-1,n}$ and $E_{1,2}$ respectively.

This covers (infinite-dimensional π corresponding to) Λ in general position: $\Lambda(E_{1n}) \neq 0$. However there are also exceptional orbits with $\dim \pi = \infty$. For these, $\Lambda(E_{1n}) = 0$ and passage to the quotient puts $\overline{E}_{1,n-1}$ and $\overline{E}_{2,n}$ in the center of $\overline{\mathfrak{N}}$, and so the required Sobolev estimates for such Λ are easier, and do not require a choice of V_j .

We will now quote two examples of 3-step nilpotent metabelian groups which will also be useful to illustrate our next theorems.

EXAMPLE 2. $\mathfrak{N} = \mathfrak{A} \ltimes \mathfrak{B}$ where $\mathfrak{A} = \mathbf{R}X_1 \oplus \mathbf{R}X_2$, $\mathfrak{B} = \mathbf{R}$ -span of $\{U_1, U_2, U_{12}, Y_1, Y_2, Z\}$ with all nontrivial bracket products generated by $[X_j, U_j] = Y_j$, $[X_j, Y_j] = Z$, and $[X_j, U_{12}] = Y_{3-j}$, $j = 1, 2$ [17]. Here $\mathfrak{N}_1 = [\mathfrak{N}, \mathfrak{N}] = \mathbf{R}Y_1 \oplus \mathbf{R}Y_2 \oplus \mathbf{R}Z$ is the next to the last stage of the lower central series and $N = \exp \mathfrak{N}$.

EXAMPLE 3. \mathfrak{N} = subalgebra of \mathfrak{N} of Example 2 that does not contain U_{12} , and $N = \exp \mathfrak{N}$.

In each of these two examples $D = Y_1 + iY_2$ is globally regular, the commuting external vectors being X_1 and X_2 , if Y_1 and Y_2 are non-Liouville with respect to some Γ -rational basis.

REMARK. We notice that the model we just built to prove Theorem 2_C, shows that for $D = X_j + iV_j$, j -fixed, $[X_j, V_j] = 0$, we can produce estimates similar to the ones we obtained for $D = Y_1 + iY_2$. To see this we apply the (Euclidean) Fourier transform to (2.2) obtaining

$$(2.5) \quad \begin{aligned} \hat{\pi}_\Lambda(X_j) &= i\xi, & \hat{\pi}_\Lambda(Y_1) &= -\Lambda(Z_j) \frac{\partial}{\partial \xi}, & \hat{\pi}_\Lambda(V_j) &= i\eta, \\ \hat{\pi}_\Lambda(Y_2) &= -\Lambda(Z_j) \frac{\partial}{\partial \eta} - \Lambda([X_j, Y_2]) \frac{\partial}{\partial \xi}, & \text{and } \hat{\pi}_\Lambda(D) &= i\xi - \eta. \end{aligned}$$

Now we can repeat the argument of Theorem 2_C using $\partial/\partial\xi = -\Lambda(Z_j)^{-1}\hat{Y}_1$ and $\partial/\partial\eta = -\Lambda(Z_j)^{-1}(Y_2 - Y_1[X_j, Y_2]\Lambda(Z_j)^{-1})^\wedge$. In particular we have sufficient L^2 -estimates on f_Λ for the global regularity of $D = X_1 + iV_1$ in Example 3, and in Examples 4 and 5 to follow, provided $\Lambda(Z) \neq 0$ there.

Now we would like to determine the algebraic conditions on (commuting) vector fields $X, V \in \mathfrak{N}$ that would guarantee the existence of a Schrödinger model for π_Λ satisfying (2.1), which would in turn, as described above, imply global regularity of $D = X + iV$. To make X the first external vector in our model, as in the proof of Theorem 1, we pick $Y_j \in \mathfrak{N}_{k-1}$, and a non-Liouville $Z_j \in \mathfrak{N}_k$ such that $[X, Y_j] = Z_j$ assuming therefore that X has the non-Liouville supplementation property. Next, to use V as a second external vector we need two things: (1°) \bar{V} to be in the centralizer of \bar{Y}_j in $\bar{\mathfrak{N}}$, and (2°) \bar{V} to bracket a vector, say Y'_j , in the next to the last step of the lower central series of $\mathfrak{Z}(\bar{Y}_1)$, or of $\bar{\mathfrak{N}}$ in particular, onto \bar{Z}_j . The following condition—a property of a pair of vectors X and V , stronger than the requirement that both X and V have the non-Liouville supplementation property—is sufficient for 1° and 2° above to hold (cf. also Example 4 at the end of this section).

(2.6) *Property (P_k)*. There exists a set Z_1, \dots, Z_m of non-Liouville vectors in \mathfrak{N}_k such that every rational subspace V of \mathfrak{N}_k of codimension 1 can be supplemented to \mathfrak{N}_k by some Z_j , $1 \leq j \leq m$, and also a set of vectors $Y_1, \dots, Y_m, Y'_1, \dots, Y'_m$ in \mathfrak{N}_{k-1} such that for each $1 \leq j \leq m$, $[X, Y_j] = Z_j$, $[V, Y'_j] = Z_j$ and either $[V, Y_j] = 0$ or $[X, Y'_j] = 0$, for all $j = 1, \dots, m$. (By Remark 1 of §0, Property (P_k) is invariant under rational automorphisms of \mathfrak{N} .) There is also, in this context, a stronger version of the rational supplementation property for the pair of vectors X and V (which implies (P_k)):

$$(P'_k) \quad \begin{array}{ll} [X, \mathfrak{N}_{k-1}] = \mathfrak{N}_k & \text{and} \quad [V, \ker(\text{ad}_X | \mathfrak{N}_{k-1})] = \mathfrak{N}_k \\ [V, \mathfrak{N}_{k-1}] = \mathfrak{N}_k & \text{and} \quad [X, \ker(\text{ad}_V | \mathfrak{N}_{k-1})] = \mathfrak{N}_k. \end{array}$$

Thus we have the following complex version of Theorem 1.

THEOREM 1_C. *If X and $V \in \mathfrak{N}$, an n -step nilpotent Lie algebra, commute and have Property (P_k) or (P'_k), $k = 1, \dots, n - 1$, then $D = X + iV$ is globally regular on every compact nilmanifold $\Gamma \backslash N$, $N = \exp \mathfrak{N}$.*

We notice that the pair X_1, V_1 of Example 1 satisfies (P'_{n-1}). Also, the pair X_1, X_2 of Examples 2 and 3 satisfies (P'₂). In Example 2 (P₁) holds but (P'₁) does not. In Example 3 neither (P₁) nor (P'₁) is satisfied, nevertheless $D = X_1 + iX_2$ there is globally regular if X_1 and X_2 are algebraic. This is because \bar{X}_1 and \bar{X}_2 both are algebraic vectors in the step 2 Lie algebra $\bar{\mathfrak{N}} = \mathfrak{N}/\mathbf{R}Z$ and we are in the situation of Theorem 2_C. In fact, we can state the following

COROLLARY. *Let N be a 3-step nilpotent Lie group possessing a cocompact discrete subgroup Γ . If $X_1, X_2 \in \mathfrak{N}$ satisfy (P₂) (or (P'₂)), $[X_1, X_2] = 0$, and both X_1 and X_2 are non-Liouville vectors (with respect to $\log \Gamma$), then $D = X_1 + iX_2$ is globally regular on $\Gamma \backslash N$.*

We end this section with the example showing that the assumption that both X and V have the supplementation property is not sufficient for global regularity of $D = X + iV$.

EXAMPLE 4. Let \mathfrak{N} be the 2-step nilpotent Lie algebra spanned by X_1, X_2, Y, Z with the commutation relations generated by $[X_j, Y] = Z, j = 1, 2$. Let $N = \exp \mathfrak{N}$ and let Γ , a discrete cocompact subgroup, be such that $\log \Gamma$ contains X_1, X_2, Y, Z . Let $D = X_1 + i(X_2 + \beta Z)$, where β is a Liouville number. Then, clearly, both X_1 and $X_2 + \beta Z$, have supplementation. However, the algebra \mathfrak{N} is in fact a direct product of a Heisenberg algebra spanned by X_1, Y, Z , and of \mathbf{R}^1 spanned by $X_2 - X_1$. Therefore the center of \mathfrak{N} is spanned by Z and $Z_2 = X_2 - X_1$, and the infinite-dimensional representations of N corresponding to $\Lambda \in \mathfrak{N}^*$ integral on the center act on $L^2(\mathbf{R})$ with $\pi_\Lambda(X_1) = \partial/\partial x, \pi_\Lambda(Y) = i\Lambda(Z)x, \pi_\Lambda(Z) = i\Lambda(Z), \pi_\Lambda(Z_2) = i\Lambda(Z_2)$. Consequently,

$$\pi_\Lambda(D) = (1 + i) \frac{\partial}{\partial x} - [\Lambda(Z_2) + \beta\Lambda(Z)],$$

which, after applying the Fourier transform, becomes

$$\hat{\pi}_\Lambda(D) = (i - 1)\xi - [\Lambda(Z_2) + \beta\Lambda(Z)].$$

As in Example 1 of §1 we can show that D is not globally regular on $\Gamma \setminus N$.

3. 3-step nilmanifolds, $[X, Y] = 0$. In this section we will prove the following “complex combination” of Theorems 1 and 2. As we needed Property (P_k) in Theorem 1_C and \mathfrak{N} to be metabelian in Theorem 2_C, here we also need some additional commutation properties of \mathfrak{N} .

THEOREM 3. *Let N be a 3-step nilpotent Lie group with Lie algebra \mathfrak{N} and a cocompact discrete subgroup Γ . If $X, Y \in \mathfrak{N}$ commute, X has the non-Liouville supplementation property, $Y \in [\mathfrak{N}, \mathfrak{N}]$ is a non-Liouville vector and moreover, $[\ker(\text{ad}_X^2), Y] = [\mathfrak{N}, Y]$ then $D = X + iY$ is globally regular on the compact nilmanifold $\Gamma \setminus N$.*

Since the proof of Theorem 3 for some $\Lambda \in (\Gamma \setminus N)^\wedge$ requires the estimates on step 2 quotient groups of N , we will state and prove its stronger step 2 version first.

THEOREM 3'. *Let N be a step 2 nilpotent Lie group with Lie algebra \mathfrak{N} and a cocompact discrete subgroup Γ . If $X, Y \in \mathfrak{N}$ commute, X has the non-Liouville supplementation property, and Y is a non-Liouville vector, then $D = X + iY$ is globally regular on $\Gamma \setminus N$.*

PROOF OF THEOREM 3'. Let $\Lambda \in (\Gamma \setminus N)^\wedge$ and let Z_1, \dots, Z_n be a rational basis of \mathfrak{Z} . We reduce \mathfrak{N} to an algebra $\overline{\mathfrak{N}}$ with 1-dimensional center $\overline{\mathfrak{Z}}$ spanned by \overline{Z}_j , say. We may assume $\overline{Y} \notin \overline{\mathfrak{Z}}$, otherwise we would have applied the estimates from the proof of Theorem 1' which are unaffected by the passage from \mathfrak{N} to $\overline{\mathfrak{N}}$. (It is in using Theorem 1' that we need Y non-Liouville.) There exists $\overline{X}_j \in \overline{\mathfrak{N}}$ such that $[\overline{X}_j, \overline{Y}] = \overline{Z}_j$. We choose X_j to be the first external vector in our construction of the Schrödinger model for π_Λ with a special maximal subordinate to Λ subalgebra \mathfrak{M} . Clearly $\overline{X} \in \mathfrak{Z}(\overline{Y})$. Two cases are now possible.

Case I. $\overline{X} \in \mathfrak{Z}(\mathfrak{Z}(\overline{Y})^-)$. Then $\pi_\Lambda(X_j) = \partial/\partial x, \pi_\Lambda(Y) = i\Lambda(Z_j)x, \pi_\Lambda(X) = i\Lambda([X, X_j])x$, and $\pi_\Lambda(D) = \{i\Lambda([X, X_j]) - \Lambda(Z_j)\}x, \Lambda(Z_j) \neq 0$. We use the same estimates as we used in (2.1).

Case II. $\overline{X} \notin \mathfrak{Z}(\mathfrak{Z}(\overline{Y})^-)$. We choose X to be the second external vector. There exists $\overline{Y}_j \in \mathfrak{Z}(\overline{Y})^-$ such that $[\overline{X}, \overline{Y}_j] = \overline{Z}_j$. In this model

$$\begin{aligned} \pi_\Lambda(X_j) &= \frac{\partial}{\partial v}, & \pi_\Lambda(Y) &= i\Lambda(Z_j)v, & \pi_\Lambda(Y_j) &= i\Lambda([X_j, Y_j])v + i\Lambda(Z_j)x, \\ \pi_\Lambda(X) &= \frac{\partial}{\partial x} + i\Lambda([X, X_j])v, & \text{and} & & \pi_\Lambda(D) &= \frac{\partial}{\partial x} + i\Lambda([X, X_j])v - \Lambda(Z_j)v. \end{aligned}$$

Taking the Euclidean partial Fourier transform $\widehat{}$ with respect to the x variable, we get

$$\begin{aligned} \hat{\pi}_\Lambda(X_j) &= \frac{\partial}{\partial v}, & \hat{\pi}_\Lambda(Y) &= i\Lambda(Z_j)v, & \hat{\pi}_\Lambda(X) &= i\xi + i\Lambda([X, X_j])v, \\ \hat{\pi}_\Lambda(Y_j) &= i\Lambda([X_j, Y_j])v - \Lambda(Z_j)\frac{\partial}{\partial \xi}, & \text{and} & & \\ \hat{\pi}_\Lambda(D) &= i(\xi + \Lambda([X, X_j])v) - \Lambda(Z_j)v \equiv P_\Lambda(\xi, v). \end{aligned}$$

As in (2.3), $|P_\Lambda(\xi, v)|^2 \geq c_{\Lambda_j}(\xi^2 + v^2)$ for all real ξ and v , where the lowest eigenvalue c_{Λ_j} of the quadratic form $|P_\Lambda(\xi, v)|^2$ satisfies the inequality

$$c_{\Lambda_j} \geq \Lambda(Z_j)^2 \{ \Lambda([X, X_j])^2 + \Lambda(Z_j)^2 + 1 \}^{-1}.$$

Now we apply the argument of §2 stated after (2.3) with $\partial/\partial v = \pi_\Lambda(X_j)$ and $\partial/\partial \xi = \hat{\pi}_\Lambda\{-\Lambda(Z_j)^{-1}(Y_j + i\Lambda(Z_j)^{-1}[X_j, Y_j]Y)\}$.

PROOF OF THEOREM 3. Let Z_1, \dots, Z_m be the set of non-Liouville vectors in \mathfrak{N}_2 described in Definition (0.2) and let $\Lambda \in (\Gamma \backslash N)^\wedge$ be such that $\Lambda(Z_j) \neq 0$ for some $1 \leq j \leq m$. (Otherwise we would have applied Theorem 3'.) We may assume that \overline{Y} is not in the center of $\overline{\mathfrak{N}}$. (Otherwise we would use the estimates from the proof of Theorem 1', which are unaffected by the passage from \mathfrak{N} to $\overline{\mathfrak{N}}$.) Hence there exists \overline{X}_j in $\overline{\mathfrak{N}} \sim [\overline{\mathfrak{N}}, \overline{\mathfrak{N}}]$ that brackets \overline{Y} onto \overline{Z}_j . We choose X_j (any preimage of \overline{X}_j in \mathfrak{N}) in $\ker(\text{ad}_{X_j}^2)$ to be the first external vector in our construction of a special \mathfrak{M}_Λ . Now $\overline{X} \in \mathfrak{Z}(\overline{Y})$. By the non-Liouville supplementation property of X , there exists $Y_j \in [\mathfrak{N}, \mathfrak{N}]$ such that $[X, Y_j] = Z_j$. We notice that $\overline{Y}_j \in \mathfrak{Z}(\overline{Y})^-$ since Y and Y_j , both being in $[\mathfrak{N}, \mathfrak{N}]$ of a step 3 algebra \mathfrak{N} , commute. Also, $[\overline{Y}_j, \mathfrak{Z}(\overline{Y})^-] = \mathbf{R}\overline{Z}_j$, the center of $\mathfrak{Z}(\overline{Y})^-$. Thus X can be chosen to be the second external vector. Let $N = \exp \mathfrak{M}_\Lambda \cdots \exp \mathbf{R}S_k \cdots \exp \mathbf{R}X \exp \mathbf{R}X_j$ be the parametrization of N obtained by this construction. Λ has already been used in passing to the quotient algebra $\overline{\mathfrak{N}}$. However, by applying $\text{Ad}^* \exp t\overline{X}_j$, we can make $\Lambda(Y) = 0$, and since $\overline{X} \in \mathfrak{Z}(\overline{Y})$, we will not disturb this arrangement if we use $\text{Ad}^* \exp t'\overline{X}$ to make $\Lambda(Y_2) = 0$ too. Also, Λ remains rational on the center. Thus we have

$$\pi_\Lambda(X_j) = \frac{\partial}{\partial v}, \quad \pi_\Lambda(Y) = i\Lambda(Z_j)v, \quad \pi_\Lambda(Y_j) = i\Lambda([X_j, Y_j])v + i\Lambda(Z_j)x,$$

and

$$\pi_\Lambda(X) = \frac{\partial}{\partial x} + i \left\{ \frac{1}{2}v^2\Lambda([X_j, X, X_j]) + v \left(\sum_{k=1}^l s_k\Lambda([X, X_j]S_k) + \Lambda([X, X_j]) \right) \right\}.$$

(Here we use the fact that \mathfrak{M}_Λ can be taken both to be special and to contain $[\mathfrak{N}, \mathfrak{N}]$ simultaneously.) Taking the Euclidean partial Fourier transform with respect to the

x -variable we obtain

$$\hat{\pi}_\Lambda(X_j) = \frac{\partial}{\partial v}, \quad \hat{\pi}_\Lambda(Y) = i\Lambda(Z_j)v, \quad \hat{\pi}_\Lambda(Y_j) = i\Lambda([X_j, Y_j])v - \Lambda(Z_j)\frac{\partial}{\partial \xi},$$

and

$$\hat{\pi}_\Lambda(X) = i \left\{ \xi + \frac{1}{2}v^2\Lambda([[X_j, X]X_j]) + v \left(\sum_{k=1}^l s_k\Lambda([[X, X_j], S_k]) + \Lambda([X, X_j]) \right) \right\}.$$

Hence $\partial/\partial v = \hat{\pi}_\Lambda(X_j)$, $\partial/\partial \xi = \hat{\pi}_\Lambda\{-\Lambda(Z_j)^{-1}(i\Lambda(Z_j)^{-1}[X_j, Y_j]Y + Y_j)\}$ and $\hat{\pi}_\Lambda(D)$ acts on $L^2(\mathbf{R}^{l+2})$ as a multiplication by a polynomial $P_{\Lambda_j}(s_\Lambda, \xi, v)$.

Letting

$$A = A(v) = \frac{1}{2}v\Lambda([[X_j, X], X_j]) + \sum_{k=1}^l s_k\Lambda([[X, X_j], S_k]) + \Lambda([X, X_j]),$$

we get

$$|P_{\Lambda_j}(s_\Lambda, \xi, v)|^2 = \Lambda(Z_j)^2v^2 + (Av + \xi)^2 = Q(A(v), v, \xi).$$

Introducing a dummy variable v^1 , and letting $c_{\Lambda,j}(v^1)$ be the minimum eigenvalue of the quadratic form $Q(A(v^1), v, \xi)$ in (v, ξ) , we have $Q(A(v^1), v, \xi) \geq c_{\Lambda,j}(v^1)(v^2 + \xi^2)$. Now we let $v^1 = v$, and we use (2.3) with $a = \Lambda(Z_j^2) + A^2$, $b = A$, and $c = 1$ to calculate $c_{\Lambda,j}$. The radical in the resulting denominator in (2.3) is less than $\Lambda(Z_j)^2 + A^2 + 1$, so we have $c_{\Lambda,j} \geq \Lambda(Z_j)^2(\Lambda(Z_j)^2 + A^2 + 1)^{-1}$.

We get the $L^2(\mathbf{R}^{l+2})$ estimates on \tilde{f}_Λ in essentially the same manner as in (2.3) of §2, the only difference being that in (2.4), before taking the sup, instead of M_j of (1.1') we use (1.1) and we write the multiplication by A as an action of $\hat{\pi}_\Lambda(U_j) + \Lambda(Z_j)^{-1}\hat{\pi}_\Lambda(V_j)$ for some $U_j, V_j \in \mathcal{U}(\mathfrak{N})$. Since $\hat{\pi}_\Lambda(-iX) = A \cdot v + \xi$, and $\hat{\pi}_\Lambda(X_j) = \partial/\partial v$, we have

$$\hat{\pi}_\Lambda([X_j, -iX]) = v\frac{\partial}{\partial v}A + A.$$

But $v(\partial/\partial v)A = (v/2)\Lambda([[X_1, X], X_j])$ so

$$A = \hat{\pi}_\Lambda([X_j, -iX]) + \frac{1}{2}\Lambda(Z_j)^{-1}[[X_j, X]X_j]Y = \hat{\pi}_\Lambda(U_j) + \Lambda(Z_j)^{-1}\hat{\pi}_\Lambda(V_j),$$

because $\hat{\pi}_\Lambda([[X_j, X], X_j]) = i\Lambda([[X_j, X], X_j])$, and $\pi_\Lambda(Y) = i\Lambda(Z_j)v$.

REMARK 1. The assumption $[\ker(\text{ad}_X^2), Y] = [\mathfrak{N}, Y]$ forces $[[X, X_j], X]$ to be 0 in the Schrödinger model we construct. Without this assumption we would have

$$\begin{aligned} \pi_\Lambda(X) = \frac{\partial}{\partial x} + i \left\{ \frac{v^2}{2} ([[X_j, X], X_j]) + xv\Lambda([[X, X_j], X]) \right. \\ \left. + v \left[\sum_{k=1}^l s_k\Lambda([[X, X_j], S_k]) + \Lambda([X, X_j]) \right] \right\}. \end{aligned}$$

We were unable to get satisfactory estimates when the “ xv ”-term is present; nor were we able to provide a simpler than the described above parametrization of a 3-step \mathfrak{N} .

REMARK 2. If in Example 2 of §5 we consider the operator $D = X_1 + iY_2$ the hypothesis on $\ker \text{ad}_X^2$ in Theorem 3 is satisfied.

4. 3-step nilmanifolds, $[X, Y]$ central. In this section we will prove a version of Theorems 3 and 3' of the previous section for $[X, Y] \neq 0$, central. We remark that the proof of this case is quite different from that used in the case of $[X, Y] = 0$. However, the case $[X, Y] = 0$ must be proved first, since it is an essential preliminary to proving the case with $[X, Y] \neq 0$. The reason for this is that, even if $[X, Y] \neq 0$, there are $\Lambda \in (\Gamma \setminus N)^\wedge$ such that $\Lambda([X, Y]) = 0$, which forces us to fall back upon the commuting case which is already proved. Also, Part II of the proof of Theorem 4 in the present paper corrects an oversight in part (B) of the proof of Theorem (3.13) in [5]. The right-hand side of inequality (3.16) on p. 548 there should have contained terms of the form $|\tilde{U}_n \cdots \tilde{U}_1 \tilde{f}_{\Lambda, q}(s_\Lambda, 0)|$. Consequently, what we call Lemma 1 in this section is missing there.

THEOREM 4. *Let N be a step 3 or step 2 nilpotent Lie group with Lie algebra \mathfrak{N} and a cocompact discrete subgroup Γ . If $X, Y \in \mathfrak{N}$, if X has the non-Liouville supplementation property, and Y is in the next to the last step of the lower central series of \mathfrak{N} (which is the commutator $[\mathfrak{N}, \mathfrak{N}]$ if \mathfrak{N} is of step 3 and \mathfrak{N} itself if \mathfrak{N} is of step 2), then $D = X + iY$ is globally regular on the compact nilmanifold $\Gamma \setminus N$ provided both Y and $[X, Y]$ are non-Liouville vectors. By the assumption, $Z = [X, Y]$ is central. Moreover, if \mathfrak{N} is of step 3 with the center more than 1 dimensional we assume that $[\ker(\text{ad}_X^2), Y] = [\mathfrak{N}, Y]$.*

PROOF. Let $\Lambda \in (\Gamma \setminus N)^\wedge$ and assume $\lambda := \Lambda(Z) \neq 0$. Otherwise we would have applied the estimates of Theorem 3, or 3' to \tilde{f}_Λ on the quotient group N/W , where $W = \exp \mathfrak{W}$ and \mathfrak{W} is the smallest rational subspace of \mathfrak{N} containing Z . Since $\ker(\Lambda|_{\mathfrak{Z}})$ is a rational subspace containing Z , it follows that $\mathfrak{W} \subset \ker(\Lambda|_{\mathfrak{Z}})$. Since $\mathfrak{W} \subset \mathfrak{Z}$, \mathfrak{W} is also a (rational) ideal of \mathfrak{N} . Notice that in case of \mathfrak{Z} being 1 dimensional, N/W is of step 2 and we apply Theorem 3', while in case of $\dim \mathfrak{W} < \dim \mathfrak{Z}$, N/W may still be of step 3 thus forcing us to apply Theorem 3 with its additional assumption $[\ker(\text{ad}_X^2), Y] = [\mathfrak{N}, Y]$, or actually with $[\ker(\text{ad}_X^2), \bar{Y}] = [\bar{\mathfrak{N}}, \bar{Y}]$ where $\bar{\mathfrak{N}} = \mathfrak{N}/\mathfrak{W}$. We choose X to be the first external vector in our construction of \mathfrak{M}_Λ , and thus we have

$$(4.0) \quad \pi_\Lambda(X) = \frac{\partial}{\partial x}, \quad \pi_\Lambda(Y) = i\lambda x, \quad \pi_\Lambda(Z) = i\lambda,$$

and $\pi_\Lambda(D) = \partial/\partial x - \lambda x$. Hence

$$(4.1) \quad \tilde{f}_\Lambda(s_\Lambda, x) = \exp\left(\frac{\lambda x^2}{2}\right) \left(\int_0^x \tilde{g}_\Lambda(s_\Lambda, t) \exp\left(\frac{-\lambda t^2}{2}\right) dt + c(s_\Lambda)\right),$$

where $c(s_\Lambda)$ is a constant depending on s_Λ , is the general solution of $\pi_\Lambda(D)\tilde{f}_\Lambda = \tilde{g}_\Lambda$. We will break the proof now into two parts, I and II, depending on whether λ is positive or negative.

PART I ($\lambda > 0$). We are interested in the solutions \tilde{f}_Λ of the form (4.1) which are in the Schwartz space $S(\mathbf{R}^d)$. This, in particular, means that $\lim_{x \rightarrow \pm\infty} \tilde{f}_\Lambda(s_\Lambda, x) = 0$, which implies

$$c(s_\Lambda) = -\int_0^\infty \tilde{g}_\Lambda(s_\Lambda, t) \exp\left(\frac{-\lambda t^2}{2}\right) dt = -\int_0^{-\infty} \cdots$$

That is,

$$(4.2) \quad \begin{aligned} \tilde{f}_\Lambda(s_\Lambda, x) &= - \int_x^\infty \tilde{g}_\Lambda(s_\Lambda, t) \exp \left[\frac{\lambda(x^2 - t^2)}{2} \right] dt \\ &= \int_{-\infty}^x \dots \end{aligned}$$

As a rule, we will work with the first integral if $x \geq 0$ and with the second one if $x < 0$. To get the estimates on the $L^2(\mathbf{R}^d)$ norm of \tilde{f}_Λ , we will need the following estimate on $\tilde{g}_\Lambda(s_\Lambda, x)$, which we obtain as in [7, p. 545] except that instead of δ there, which came from (1.1'), we use (1.1). In the notation of (1.1), but replacing n by m and d by k , take the \sup_t and obtain, by Sobolev's inequality, with Sobolev constant 2,

$$(4.3) \quad \begin{aligned} |\tilde{g}_\Lambda(s_\Lambda, t)|^2 &\leq 2c^{4k} \sum_{l=0}^1 \|(\tilde{e}_1^2 + \dots + \tilde{e}_m^2)^{4k} \tilde{X}^l (\tilde{Z}^2 + \tilde{Y}^2)^k \tilde{g}_\Lambda(s_\Lambda, \cdot)\|^2 \\ &\quad \cdot \begin{cases} t^{-4k}, & \text{for } |t| \geq 1, \\ 1, & \text{for } |t| < 1, \end{cases} \end{aligned}$$

where $\|h(s_\Lambda, \cdot)\|^2 = \int_{-\infty}^\infty |h(s_\Lambda, t)|^2 dt$. Of course (4.3) has nothing to do with solvability of $\hat{\pi}(D)\hat{f}_\Lambda = \hat{g}_\Lambda$, and works for any $\hat{g}_\Lambda \in S(\mathbf{R}^d)$; thus it is also true for $\tilde{U}\tilde{g}_\Lambda$, for all $U \in \mathcal{U}(\mathfrak{N})$.

Now let $\mathfrak{N} \supset \mathfrak{N}_1 \supset \dots \supset \mathfrak{N}_k \supset \{0\}$ be the lower central series of \mathfrak{N} : so $[\mathfrak{N}, \mathfrak{N}_i] \equiv \mathfrak{N}_{i+1}$. If $U \in \mathfrak{N}$, define the rank of U to be $k - i$, where $i = \text{Max}\{j | U \in \mathfrak{N}_j\}$. Clearly $\text{rank}([U, V]) < \text{rank } U$ for any $U, V \in \mathfrak{N}$. Thus $U \in \mathfrak{Z}$ if $\text{rank } U = 0$. If the monomial $A = U_m \dots U_1 \in \mathcal{U}(\mathfrak{N}) =$ the universal enveloping algebra of \mathfrak{N} , define the rank of (the factorization of) A to be $\sum_1^m \text{rank } U_i$.

If $Df_{\Lambda, q} = g_{\Lambda, q}$, then

$$(4.4) \quad D(U_n \dots U_1 f_{\Lambda, q}) = U_n \dots U_1 g_{\Lambda, q} + \sum_{l=1}^n U_n \dots U_{l+1} [D, U_l] U_{l-1} \dots U_1 f_{\Lambda, q}$$

and

$$\text{rank}(U_n \dots U_{l+1} [D, U_l] U_{l-1} \dots U_1) < \text{rank}(U_n \dots U_1).$$

Observe that $\text{rank}(U_n \dots U_1) = 0 \Rightarrow U_n \dots U_1 \in \mathcal{U}(\mathfrak{Z})$.

Inductive hypothesis. If $\text{rank}(U_n \dots U_1) \leq N$, then, for each $k = 1, 2, 3, \dots$,

$$(I_N) \quad \begin{aligned} |\tilde{U}_n \dots \tilde{U}_1 \tilde{f}_{\Lambda, q}(s_\Lambda, x)| &\leq \sum_{l=1}^{N+1} \text{Min}(1, x^{-4k+l}) \\ &\quad \cdot \sum_p c_{pl} \left(\sum_{r=0}^1 \|\tilde{X}^r (\tilde{e}_1^2 + \dots + \tilde{e}_m^2)^{4k} (\tilde{Z}^2 + \tilde{Y}^2)^k \tilde{U}_{pl} \tilde{g}_\Lambda(s_\Lambda, \cdot)\|^2 \right)^{1/2}, \end{aligned}$$

where the sum \sum_p is finite, and $\text{rank } U_{pl} = l$. (In fact, each U_{pl} is a product of U_j 's, $[D, U_j]$'s and $[D[DU_j]]$'s, $1 \leq j \leq n$.) The constants c_{pl} depend on c of (1.1), on the Sobolev constant, on k , and on U_1, \dots, U_n , but do not depend on Λ .

REMARK. (I_N) implies the desired estimate on the $L^2(\mathbf{R}^d)$ norm of $\tilde{U}_n \dots \tilde{U}_1 \tilde{f}_\Lambda$ ($= L^2(\Gamma \setminus N)$ norm of $U_n \dots U_1 f_\Lambda$) if k is sufficiently large—we just square (I_N) and integrate it $ds_\Lambda dx$ over \mathbf{R}^d . We will show now that the estimates (I_N) hold.

Initial case. For $N = 0$ (I_N) reads

$$(I_0) \quad |\tilde{U}_n \cdots \tilde{U}_1 \tilde{f}_\Lambda(s_\Lambda, x)| \leq \text{Min}(1, |x|^{-4k+1}) \cdot \sum_p c_p \left(\sum_{r=0}^1 \|\tilde{X}^r (\tilde{e}_1^2 + \cdots + \tilde{e}_m^2)^{4k} (\tilde{Z}^2 + \tilde{Y}^2)^k \tilde{U}_p \tilde{g}_\Lambda(s_\Lambda, \cdot)\|^2 \right)^{1/2},$$

which is a square root of (4.3) combined with (4.2), plus the observation that $\exp[\lambda(x^2 - t^2)/2] \leq 1$ under the integral sign of (4.2) and that $U_n \cdots U_1$ commutes with D .

Inductive step. To see that $(I_n) \Rightarrow (I_{N+1})$, let $U = U_n \cdots U_1$ be of rank $N + 1$. By (4.3), $\tilde{f}_1 = \tilde{U} \tilde{f}_\Lambda$ is the unique Schwartz solution of

$$\pi_\Lambda(D) \tilde{f}_1 = \tilde{U} \tilde{g}_\Lambda + \sum_{q=1}^n \tilde{U}_n \cdots [\tilde{D}, \tilde{U}_q] \cdots \tilde{U}_1 \tilde{f}_\Lambda \equiv \tilde{h}_\Lambda$$

and therefore is given by (4.2) with \tilde{g}_Λ there replaced by the above \tilde{h}_Λ . That is,

$$(4.5) \quad \tilde{U} \tilde{f}_\Lambda(s_\Lambda, x) = \int_{-\infty}^x \left\{ \tilde{U} \tilde{g}_\Lambda(s_\Lambda, t) + \sum_{q=1}^n \tilde{U}_n \cdots [\tilde{D}, \tilde{U}_q] \cdots \tilde{U}_1 \tilde{f}_\Lambda(s_\Lambda, t) \right\} \cdot \exp[\lambda(x^2 - t^2)/2] dt.$$

The first term of the above sum is estimated by an expression of the same form as the one on the right-hand side of the inequality (I_0) . For the other terms we apply (I_N) to each $\tilde{U}_n \cdots [\tilde{D}, \tilde{U}_q] \cdots \tilde{U}_1 \tilde{f}_\Lambda$, $q = 1, \dots, n$ (which are of rank $\leq N$), and we group the terms. The integration with respect to t of $\text{Min}(1, |t|^{-2k+l})$ in (4.4) changes the constants c_{lp} and increases l by 1.

PART II ($\lambda < 0$). Here we can pick $c(s_\Lambda)$ of (4.1) to be 0, so that

$$(4.6) \quad \tilde{f}_\Lambda(s_\Lambda, x) = \int_0^x \tilde{g}_\Lambda(s_\Lambda, t) \exp \left[\frac{\lambda(x^2 - t^2)}{2} \right] dt.$$

Since we have picked a solution, we do not know a priori that the function \tilde{f}_Λ defined by (4.6) is Schwartz. Clearly it is in $C^\infty(\mathbf{R}^d)$. We know, by the hypothesis (cf. Definition (0.1)), that there is a function $c(s_\Lambda)$ for which the function \tilde{f}_Λ^c corresponding to $c(s_\Lambda)$ under (4.1) is Schwartz. And \tilde{f}_Λ defined by (4.6) differs from \tilde{f}_Λ^c by $\exp(\lambda x^2/2)c(s_\Lambda)$. Clearly, $c(s_\Lambda) = \tilde{f}_\Lambda^c(s_\Lambda, 0)$ is Schwartz on \mathbf{R}^{d-1} . Hence $\exp(\lambda x^2/2)c(s_\Lambda) \in S(\mathbf{R}^d)$, and \tilde{f}_Λ must be Schwartz being a difference of two Schwartz functions. Now (4.4) and (4.1) yield the following expression for $\tilde{U} \tilde{f}_\Lambda$, $U = U_n \cdots U_1 \in \mathcal{U}(\mathfrak{N})$, $\text{rank}(U) = N$:

$$(4.5') \quad \tilde{U} \tilde{f}_\Lambda(s_\Lambda, x) = \int_0^x \left\{ \tilde{U} \tilde{g}_\Lambda(s_\Lambda, t) + \sum_{q=1}^n \tilde{U}_n \cdots [\tilde{D}, \tilde{U}_q] \cdots \tilde{U}_1 \tilde{f}_\Lambda(s_\Lambda, t) \right\} \cdot \exp[\lambda(x^2 - t^2)/2] dt + \tilde{U} \tilde{f}_\Lambda(s_\Lambda, 0) \exp(\lambda x^2/2).$$

Taking the sup over t of each of the summands of the integrand in (4.4'), next applying the integral estimate

$$\left| \int_0^x \exp \left[\frac{\lambda(x^2 - t^2)}{2} \right] dt \right| \leq 1 - \lambda^{-1} \quad (\lambda < 0, x \in \mathbf{R}),$$

and then the Sobolev inequality (1.6) to $\sup_t |\tilde{U}\tilde{g}_\Lambda(s_\Lambda, t)|$, we obtain

$$\begin{aligned} |\tilde{U}\tilde{f}_\Lambda(s_\Lambda, x)| &\leq (1 - \lambda^{-1}) \sup_t \left(|\tilde{U}\tilde{g}_\Lambda(s_\Lambda, t)| + \sum_{q=1}^n |\tilde{U}_n \cdots [\tilde{D}, \tilde{U}_q] \cdots \tilde{U}_1 \tilde{f}_\Lambda(s_\Lambda, t)| \right) \\ &\quad + \tilde{U}\tilde{f}_\Lambda(s_\Lambda, 0) \exp\left(\frac{\lambda x^2}{2}\right) \\ &\leq (1 - \lambda^{-1}) \left(2 \sum_{r=0}^1 \|\tilde{X}^r \tilde{U}\tilde{g}_\Lambda(s_\Lambda, \cdot)\|^2 + \sum_{q=1}^n \sup_t |\tilde{U}_n \cdots [\tilde{D}, \tilde{U}_q] \cdots \tilde{U}_1 \tilde{f}_\Lambda(s_\Lambda, t)| \right) \\ &\quad + \tilde{U}\tilde{f}_\Lambda(s_\Lambda, 0) \exp(\lambda x^2/2). \end{aligned}$$

Next we return to (4.5') for each summand on the right involving $\tilde{f}_\Lambda(s_\Lambda, t)$. We iterate this procedure until there is no $\tilde{f}_\Lambda(s_\Lambda, t)$ on the right-hand side. Then we obtain

$$(4.7) \quad |\tilde{U}\tilde{f}_\Lambda(s_\Lambda, x)| \leq \sum_{l=0}^N (1 - \lambda^{-1})^{N-l} \sum_p c_{pl} \cdot \left\{ (1 - \lambda^{-1}) \left(\sum_{r=0}^1 \|\tilde{X}^r \tilde{U}_{pl} \tilde{g}_\Lambda(s_\Lambda, \cdot)\|^2 \right)^{1/2} + |\tilde{U}_{pl} \tilde{f}_\Lambda(s_\Lambda, 0)| \right\}$$

where the sum \sum_p is finite, $\text{rank } U_{pl} = l$ (each U_{pl} being a product of U_j 's, $[D, U_j]$'s and $[D[D, U_j]]$, $1 \leq j \leq n$), and the constants c_{pl} do not depend on Λ .

To get an L^2 -estimate on $\tilde{U}\tilde{f}_\Lambda$, we add two copies of (4.7), one with $\tilde{Y}^2\tilde{U}$ in place of \tilde{U} , and the other with $\tilde{Z}^2\tilde{U}$ replacing \tilde{U} . This gives the $\lambda^2(1 + x^2)$ factor on the left-hand side of the following inequality (4.8):

$$(4.8) \quad |(\lambda^2 + \lambda^2 x^2) \tilde{U}\tilde{f}_\Lambda(x, s_\Lambda)| \leq \|\tilde{Z}^2\tilde{U}\tilde{f}_\Lambda(\cdot, s_\Lambda)\|_\infty + \|\tilde{Y}^2\tilde{U}\tilde{f}_\Lambda(\cdot, s_\Lambda)\|_\infty$$

\leq a finite linear combination (with coefficients of the form $\text{const. } \lambda^{-l}$) of the terms of the form $\|\tilde{X}^r \tilde{U}_{lp} \tilde{g}_\Lambda(\cdot, s_\Lambda)\|$, $r = 0, 1$, and of the form $|\tilde{U}_{pl} \tilde{f}_\Lambda(s_\Lambda, 0)|$.

Next, we square (4.8), divide by $\lambda^4(1 + x^2)^2$, which allows us to integrate the right-hand side of the so transformed (4.7) $dx ds_\Lambda$ over \mathbf{R}^d to get

$$\begin{aligned} \|\tilde{U}\tilde{f}_\Lambda\|^2 &\leq \text{finite linear combination of } \|\tilde{X}^r \tilde{U}_{l,p} \tilde{g}_\Lambda\|^2 \cdot \lambda^{-2q} \\ &\quad \text{and of } \int_{\mathbf{R}^{d-1}} |\tilde{U}_{lp} \tilde{f}_\Lambda(s_\Lambda, 0)|^2 ds_\Lambda \cdot \lambda^{-2q}. \end{aligned}$$

By (1.1), λ^{-1} can be absorbed into U_{pl} . Since $\sum_\Lambda \|\tilde{X}^r \tilde{U}_{lq} \tilde{g}_\Lambda\|^2 < \infty$, for $g \in C^\infty(\Gamma \setminus N)$, all we need to show to complete the proof of Theorem 4 is that $\sum_\Lambda \|\tilde{U}\tilde{f}_\Lambda(s_\Lambda, 0)\|_{L^2(s_\Lambda)}^2 < \infty$ for every fixed $U \in \mathcal{U}(\mathfrak{M})$.

LEMMA 1. $\sum_\Lambda \int_{\mathbf{R}^{d-1}} |\tilde{U}\tilde{f}_\Lambda(s_\Lambda, 0)|^2 ds_\Lambda < \infty$, for all $U \in \mathcal{U}(\mathfrak{M})$.

We will reduce that proof of Lemma 1 to the following

LEMMA 2. $\sum_\Lambda \int_{\mathbf{R}^{d-1}} |\tilde{U}\tilde{g}_\Lambda(s_\Lambda, 0)|^2 ds_\Lambda < \infty$, for every $U \in \mathcal{U}(\mathfrak{M})$ and $g \in C^\infty(\Gamma \setminus N)$.

PROOF OF LEMMA 2.

$$\begin{aligned} \int |\tilde{U}\tilde{g}_\Lambda(s_\Lambda, 0)|^2 ds_\Lambda &\leq \int \sup_t |\tilde{U}\tilde{g}_\Lambda(s_\Lambda, t)|^2 ds_\Lambda \\ &\leq \int c \sum_{l=0}^1 \|\tilde{X}^l \tilde{U}\tilde{g}_\Lambda(s_\Lambda, \cdot)\|^2 ds_\Lambda = c \sum_{l=0}^1 \|\tilde{X}^l \tilde{U}\tilde{g}_\Lambda\|^2. \end{aligned}$$

Since $X^l U g \in C^\infty(\Gamma \setminus N)$, the sum over $\Lambda \in (\Gamma \setminus N)^\wedge$ of the above inequality is finite. (We notice that $\sup_t |\tilde{U}\tilde{g}_\Lambda(t, s_\Lambda)|$ is s_Λ -measurable.)

PROOF THAT LEMMA 2 \Rightarrow LEMMA 1. First we observe that any $U \in \mathfrak{N}$ can be written $\bar{U} = a\bar{X} + \bar{W}$ for some $a \in \mathbf{R}$ and $\bar{W} \in \mathfrak{Z}(\bar{Y})$. Let $U = U_m \cdots U_1 \in \mathcal{U}(\mathfrak{N})$ with $\bar{U}_j = a_j \bar{X} + \bar{W}_j$, $a_j \in \mathbf{R}$, $\bar{W}_j \in \mathfrak{Z}(\bar{Y})$, $j = 1, \dots, m$. Then

$$\begin{aligned} \bar{U} &= (a_m \bar{X} + \bar{W}_m) \cdots (a_1 \bar{X} + \bar{W}_1) \\ &= \text{a (finite) linear combination of mixed products} \\ &\quad \text{of } \bar{X}\text{'s and } \bar{W}_j\text{'s} \\ &= \text{a (finite) linear combination of (ordered) products} \\ &\quad \text{of the form } \bar{V}_p \cdots \bar{V}_1 \bar{X}^k \text{ with } k \geq 0, \\ &\quad \text{and some } \bar{V}_j \in \mathfrak{Z}(\bar{Y}) \text{ (} V_j \text{ not necessarily } = W_j\text{)} \\ &\quad \text{since } \mathfrak{Z}(Y) \text{ is an ideal in } \bar{\mathfrak{N}}. \end{aligned}$$

Therefore it is sufficient to prove Lemma 1 for $U \in \mathcal{U}(\mathfrak{N})$ of the form $U = V_p \cdots V_2 V_1 X^k$, $k \geq 0$, $\bar{V}_j \in \mathfrak{Z}(\bar{Y})$, and in fact we will show that

LEMMA 3. $\tilde{V}_p \cdots \tilde{V}_1 \tilde{X}^k \tilde{f}_\Lambda(s_\Lambda, 0) = \tilde{\nu} \tilde{g}_\Lambda(s_\Lambda, 0)$ for some $\tilde{\nu} \in \mathcal{U}(\mathfrak{N})$, \tilde{f}_Λ given by (4.6), and $\bar{V}_j \in \mathfrak{Z}(\bar{Y})$, $1 \leq j \leq p$.

Step 1. $\tilde{X}\tilde{Y}^p \tilde{f}_\Lambda =$ a linear combination of the terms of the form $\tilde{Z}^r \tilde{Y}^q \tilde{f}_\Lambda$, $q \leq p + 1$, and $\tilde{Y}^p \tilde{g}_\Lambda$.

PROOF OF STEP 1 (INDUCTION ON p). For $p = 0$, it follows from (4.0) and (4.6) that $\tilde{X}\tilde{f}_\Lambda = \tilde{g}_\Lambda - i\tilde{Y}\tilde{f}_\Lambda$. Next we notice that $XY^{p+1} = (YX + [X, Y])Y^p = Y(XY^p) + ZY^p$ and we apply the inductive hypothesis to XY^p .

Step 2.

$$(4.9) \quad \tilde{X}^k \tilde{f}_\Lambda = \tilde{U}\tilde{g}_\Lambda + \text{a finite linear combination of } \tilde{Y}^p \tilde{Z}^r \tilde{f}_\Lambda, \quad \text{for some } U \in \mathcal{U}(\mathfrak{N}).$$

PROOF OF STEP 2 (INDUCTION ON k). For $k = 1$ this is just the $p = 0$ case of the proof of Step 1. All we need to check now is that $\tilde{X}(\tilde{Y}^p \tilde{Z}^r \tilde{f}_\Lambda)$ is again of the form (4.9), but this is just Step 1 applied to $\tilde{Z}^r \tilde{f}_\Lambda$.

Step 3. $(\tilde{V}_p \cdots \tilde{V}_1 \tilde{f}_\Lambda)(s_\Lambda, 0) = 0$.

PROOF OF STEP 3. In our Schrödinger model for π_Λ , since \mathfrak{M}_Λ contains the commutator $[\mathfrak{N}, \mathfrak{N}]$, each \tilde{V}_j is of the form $\tilde{V}_j = \varepsilon_j (\partial/\partial v_j) + iP_j(s_\Lambda, x)$, where $\partial/\partial v_j$ is a directional derivative in the s_Λ -space, P_j is a real polynomial, and $\varepsilon_j = 1$ if $V_j \notin \mathfrak{M}_\Lambda$ and $\varepsilon_j = 0$ if $V_j \in \mathfrak{M}_\Lambda$. Hence

$$\tilde{V}_j \tilde{f}_\Lambda(s_\Lambda, x) = \int_0^x \varepsilon_j \left(\frac{\partial \tilde{g}_\Lambda}{\partial v_j} \right) (s_\Lambda, t) \cdot \exp \left[\frac{\lambda(x^2 - t^2)}{2} \right] dt + iP_j(s_\Lambda, x) \tilde{f}_\Lambda(s_\Lambda, x),$$

and clearly $\tilde{V}_p \cdots \tilde{V}_1 \tilde{f}_\Lambda(s_\Lambda, 0) = 0$, since \tilde{f}_Λ itself is given by (4.5).

PROOF OF LEMMA 3. Since all the \tilde{V}_j 's commute with \tilde{Y} , by Step 2, we have

$$\begin{aligned} \tilde{V}_p \cdots \tilde{V}_1 \tilde{X}^k \tilde{f}_\Lambda(s_\Lambda, 0) &= \tilde{V}_p \cdots \tilde{V}_1 (\tilde{U} \tilde{g}_\Lambda + \text{a linear combination of } \tilde{Y}^p \tilde{Z}^r \tilde{f}_\Lambda)(s_\Lambda, 0) \\ &= \tilde{V}_p \cdots \tilde{V}_1 \tilde{U} \tilde{g}_\Lambda(s_\Lambda, 0) \\ &\quad + \text{a linear combination of } (\tilde{Y}^p (\tilde{V}_p \cdots \tilde{V}_1 \tilde{Z}^r \tilde{f}_\Lambda))(s_\Lambda, 0) \\ &\quad \text{with } p \geq 1 \\ &\quad + \text{a linear combination of } \tilde{V}_p \cdots \tilde{V}_1 \tilde{Z}^r \tilde{f}_\Lambda(s_\Lambda, 0). \end{aligned}$$

In the above sum the last term is zero by Step 3, and the second term is zero since \tilde{Y} is just a multiplication by $i\lambda x$.

COROLLARY 1. *Let N be a simply connected 2-step nilpotent Lie group, Γ a cocompact, discrete subgroup of N . If $X, Y \in \mathfrak{N}$ both are algebraic (resp. non-Liouville) vectors, then $D = X + iY$ is globally regular on $\Gamma \setminus N$ (resp. provided $[X, Y]$ is a non-Liouville vector too).*

PROOF. Let $\Lambda \in (\Gamma \setminus N)^\wedge$. If $\Lambda([X, Y]) = 0$, this is Proposition 1 of §2 applied to $N/\exp \mathfrak{W}$, where \mathfrak{W} is the smallest rational subspace of \mathfrak{N} containing Z . If $\Lambda([X, Y]) \neq 0$, after reducing \mathfrak{N} to $\overline{\mathfrak{N}}$ with 1-dimensional center spanned by $\overline{Z} = [X, Y]^-$, either \overline{X} or \overline{Y} can be used as the first external vector and we obtain the estimates now as in the proof of Theorem 4. (If X and Y are algebraic, then $[X, Y]$ is algebraic too.)

COROLLARY 2. *Let N be a simply connected 2-step nilpotent Lie group, and Γ a cocompact, discrete subgroup of N . If $X, Y \in \mathfrak{N}$ satisfy (P_1) , then $D = X + iY$ is globally regular on $\Gamma \setminus N$ provided $[X, Y]$ is a non-Liouville vector.*

PROOF. If $\Lambda([X, Y]) = 0$ for $\Lambda \in (\Gamma \setminus N)^\wedge$, then this is Theorem 1C applied to $N/\exp \mathfrak{W}$, \mathfrak{W} being the smallest rational subspace of \mathfrak{N} containing $[X, Y]$, as in the beginning of the proof of Theorem 4. If $\Lambda([X, Y]) \neq 0$, we choose X to be the first external vector and the rest of the proof is the same as that of Theorem 4.

5. Examples and further developments. In [7] we gave an example, based on a suggestion by Corwin (cf. [4]), to show how global regularity could fail in a 3-step group for $D = X + iY$, where X and Y generate the 3-step chain group. If the Kirillov orbits had been flat, then that parameter, summation over which yields the counterexample, would have been eliminated. In our next example, we show however that flattening the Kirillov orbits, by the addition of 1 to the dimension of the group [20] does not correct this analytic failure. (This should be compared also with Example 2 in §1.) Thus the fact that all 2-step groups have flat orbits does not seem to be relevant to the problem of global regularity.

EXAMPLE 1. Let \mathfrak{N} be the Lie algebra spanned by the vectors V, W, X, Y , and Z with all nonzero bracket products generated by $[W, X] = Y, [W, Y] = Z, [V, X] = Z$. Thus \mathfrak{N} is of step 3 with flat Kirillov orbits and 1-dimensional center [20]. W and X generate a step 3 subalgebra of \mathfrak{N} . Let $N = \exp \mathfrak{N}$. There is a series of representations $\pi_\lambda \in \hat{N}, \lambda \in \mathbf{R} \sim \{0\}$, realized on $L^2(\mathbf{R}^2)$ in such a way that $\pi_\lambda(W) = \partial/\partial w$ and $\pi_\lambda(X) = 2\pi i\lambda(v + w^2/3)$ [8]. Let Γ be a cocompact discrete

subgroup of N such that $Z \in \log \Gamma$. Then $\pi_\lambda \in (\Gamma \setminus N)^\wedge$ for $\lambda = \pm 1, \pm 2, \dots$. Let ϕ and ψ be $C^\infty(\mathbf{R})$ nonnegative functions such that

$$\begin{aligned} \phi(w) &= \begin{cases} 1 & \text{for } 0 \leq w \leq 3, \\ 0 & \text{for } w \notin [-1, 4], \end{cases} \\ \psi(v) &= \begin{cases} 1 & \text{for } -n - 1 \leq v \leq -n, \\ 0 & \text{for } v \notin [-2 - n, -n + 1]. \end{cases} \end{aligned}$$

Fix $\varepsilon > 0$ and $n > 0$ and for each positive integer λ define

$$\tilde{g}_\lambda(w, v) = \phi(w)\psi(v)e^{2\pi\lambda\varepsilon v}$$

and let $g_\lambda, \lambda = 1, 2, 3, \dots$, be the corresponding functions in $C^\infty(\Gamma \setminus N)$, each in a different primary summand of $L^2(\Gamma \setminus N)$. Then $g = \sum_\lambda g_\lambda$ is smooth on $\Gamma \setminus N$, for

$$\begin{aligned} \sum_{\lambda=1}^\infty \|Ug_\lambda\|_{L^2(\Gamma \setminus N)} &= \sum_{\lambda=1}^\infty \|\tilde{U}\tilde{g}_\lambda\|_{L^2(\mathbf{R}^2)} \\ &\leq \sum_{\lambda=1}^\infty \|(\text{differential operator with polynomial} \\ &\qquad\qquad\qquad \text{in } w, v \text{ and } \lambda \text{ coefficients}) \phi(w)\psi(v)\|_2 e^{2\pi\lambda\varepsilon(-n+1)} \\ &\leq \sum_{\lambda=1}^\infty |(\text{polynomial in } \lambda)| e^{2\pi\lambda\varepsilon(-n+1)} < \infty \end{aligned}$$

for every fixed $U \in \mathcal{U}(\mathfrak{N})$ and $n > 1$.

Let $D = W + iX$. Then

$$\pi_\lambda(D) = \frac{\partial}{\partial w} - 2\pi\lambda \left(v + \frac{w^2}{3} \right)$$

and

$$\tilde{f}_\lambda(w, v) = - \int_w^\infty \tilde{g}_\lambda(t, v) \exp 2\pi\lambda \left[\frac{(w^3 - t^3)}{9} + v(w - t) \right] dt$$

is the only Schwartz solution of $\pi_\lambda(D)\tilde{f}_\lambda = \tilde{g}_\lambda$. \tilde{f}_λ is indeed Schwartz, since \tilde{g}_λ is compactly supported, and $\exp(2\pi\lambda w^3/9) \rightarrow 0$ as $w \rightarrow -\infty$. On the other hand, by a change of variables $t = w + u$, if $0 \leq w \leq 1$ and $-1 - n \leq v \leq -n$

$$\begin{aligned} |\tilde{f}_\lambda(w, v)| &= e^{2\pi\lambda\varepsilon v}\psi(v) \int_0^\infty \phi(w + u) \exp \left[\frac{-2\pi\lambda(u^3 + 3u^2w + 3uw^2)}{9} \right] e^{-2\pi\lambda v u} du \\ &\geq \sup \exp \left[2\pi\lambda \left(\varepsilon v - \frac{(u^3 + 3u^2w + 3uw^2)}{9} - v u \right) \right] \\ &\geq \exp[2\pi\lambda(-\varepsilon - n\varepsilon - 3 + n)] \end{aligned}$$

where the sup is taken over $0 \leq w \leq 1, 1 \leq u \leq 2$. For $n = 5$ and $\varepsilon = \frac{1}{6}$ we thus have $|\tilde{f}_\lambda(w, v)| \geq \exp 2\pi\lambda$ and $\|f_\lambda\|_{L^2(\Gamma \setminus N)} = \|\tilde{f}_\lambda\|_{L^2(\mathbf{R}^2)} \geq \exp 2\pi\lambda$, i.e. $\sum_{\lambda=1}^\infty f_\lambda$ does not converge in $L^2(\Gamma \setminus N)$. This completes the Example 2.

Parts (a) and (b) of the next Example 2 (which in fact is an “implant” of Example 2 of §1 on the group N of Example 2 of §2) indicate how the assumptions of Theorems 2_C and 3, respectively, can possibly be relaxed. Part (c) of Example 2 deals with Theorem 4.

EXAMPLE 2. Let N and Γ be as in the Example 2 of §2 and let $\Lambda \in (\Gamma \setminus N)^\wedge$ be such that $\Lambda(Z) \neq 0$. Then $\pi_\Lambda(X_j) = \partial/\partial x_j$, $\pi_\Lambda(Y_j) = i\Lambda(Z)x_j$, $\pi_\Lambda(U_j) = i\Lambda(U_j) + i\Lambda(Z)x_j^2/2$, $j = 1, 2$, and $\pi_\Lambda(U_{12}) = i\Lambda(U_2) + i\Lambda(Z)x_1x_2$ [17]. Consider now the following operators:

- (a) $D_1 = X_1 + i(U_2 + \beta_2 Y_2 + \gamma_2 Z)$,
- (b) $D_2 = (U_1 + \beta_1 Y_1 + \gamma_1 Z) + i(U_2 + \beta_2 Y_2 + \gamma_2 Z)$,
- (c) $D_3 = X_1 - iU_{12}$.

For Λ as above we thus have

- (a) $\pi_\Lambda(D_1) = \partial/\partial x_1 - \Lambda(Z)(x_2 - x_{2+})(x_2 - x_{2-})$,
- (b) $\pi_\Lambda(D_2) = i\Lambda(Z)[(x_1 - x_{1+})(x_1 - x_{1-}) + i(x_2 - x_{2+})(x_2 - x_{2-})]$ where $x_{j\pm} = -\beta_j \pm (\beta_j^2 - 2\gamma_j - 2\Lambda(U_j)/\Lambda(Z))^{1/2}$, $j = 1, 2$,
- (c) $\pi_\Lambda(D_3) = \partial/\partial x_1 + \Lambda(Z)x_1x_2 + \Lambda(U_{12})$.

The operator D_1 is such that X_1 has supplementation, and $U_2 + \beta_2 Y_2 + \gamma_2 Z \notin [\mathfrak{N}, \mathfrak{N}]$ and does not have supplementation. Using estimates similar to those of Example 2 of §2 one can show that D_1 is globally regular provided neither $\beta_2^2/2 - \gamma_2$ nor β_2 is a Liouville number. (The requirement on β_2 is needed for global regularity of \overline{D}_1 on $N/\exp \mathbf{R}Z$, which follows now from Theorem 3'.)

The operator D_2 is such that neither its real nor its imaginary part have supplementation, $U_j + \beta_j Y_j + \gamma_j Z \notin [\mathfrak{N}, \mathfrak{N}]$, $j = 1, 2$. Again, the estimates of Example 2 and of Proposition 1, both of §2, show that D_2 is globally regular, provided neither of the following four numbers, $\beta_j^2/2 - \gamma_j, \beta_j$, $j = 1, 2$, is Liouville.

Finally, the operator D_3 is such that X_1 has supplementation, $U_{12} \notin [\mathfrak{N}, \mathfrak{N}]$ is an algebraic vector, and $[X_1, U_{12}]$ is *not central* in \mathfrak{N} . Applying to $\pi_\Lambda(D)$ of (c) the Euclidean partial Fourier transform with respect to the x_2 variable we obtain

(c') $\hat{\pi}_\Lambda(D_3) = \partial/\partial x_1 + i\Lambda(Z)x_1\partial/\partial\eta + \Lambda(U_{12})$, which (for $\Lambda(U_{12}) = 0$) is the nonhypoelliptic and nonlocally solvable operator of Mizohata [14, Appendix] and Nirenberg and Treves [16]. We were unable to settle the question of global regularity of D_3 . (Clearly, \overline{D}_3 is globally regular on $N/\exp \mathbf{R}Z$ by Theorem 4.)

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