TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 301, Number 1, May 1987

CONVERGENCE OF SERIES OF SCALAR- AND VECTOR-VALUED RANDOM VARIABLES AND A SUBSEQUENCE PRINCIPLE IN L_2

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ABSTRACT. Let $(d_n)_{n=1}^\infty$ be a martingale difference sequence in $L_0(X)$, where X is a uniformly convex Banach space. We investigate a necessary condition for convergence of the series $\sum_{n=1}^\infty a_n d_n$. We also prove a related subsequence principle for the convergence of a series of square-integrable scalar random variables.

Introduction. Let $(d_n)_{n=1}^\infty$ be an orthonormal sequence of independent random variables and let $(a_n)_{n=1}^\infty$ be a sequence of real numbers. In [**14**] Marcinkiewicz and Zygmund proved that if $E|d_n| \geq \delta > 0$ for all $n \geq 1$ then $\sum_{n=1}^\infty a_n^2 < \infty$ whenever $\sum_{n=1}^\infty a_n d_n$ converges almost surely. This theorem has been extended to the case of martingale difference sequences by Chow [**4**]. In §1 the almost sure convergence of the series $\sum_{n=1}^\infty a_n d_n$ is considered when $(d_n)_{n=1}^\infty$ is a bounded sequence in L_0. Necessary and sufficient conditions are given on such a sequence of independent random variables to be able to conclude that $\sum_{n=1}^\infty a_n^2 < \infty$ whenever $\sum_{n=1}^\infty a_n d_n$ converges almost surely. The same question is treated in §2 for a vector-valued martingale difference sequence $(d_n)_{n=1}^\infty$ in $L_0(X)$ (here X is a Banach space). When $(d_n)_{n=1}^\infty$ is adapted to a regular sequence of σ-fields and X is a q-convex Banach space, necessary and sufficient conditions on $(d_n)_{n=1}^\infty$ are given to be able to conclude that $\sum_{n=1}^\infty |a_n|^q < \infty$ whenever $\sum_{n=1}^\infty a_n d_n$ has bounded partial sums almost surely (or with high probability).

In §3 the theorem of Chow mentioned above is used to deduce a subsequence principle for random variables in L_2 which is related to some theorems of Revesz. A consequence of this is that any orthonormal sequence $(\phi_n)_{n=1}^\infty$ which is bounded away from zero in probability will contain a subsequence $(\phi_{n_k})_{n=1}^\infty$ with the following property: $\sum_{k=1}^\infty a_k^2 < \infty$ whenever $\sum_{k=1}^\infty a_k \phi_{n_k}$ converges almost surely (or merely whenever $\sum_{k=1}^\infty a_k \phi_{n_k}$ has bounded partial sums with high probability). The section closes with an abstract version of a theorem of Zygmund on lacunary Fourier coefficients.

The last part gives some vectorial extensions of a theorem of Aldous and Fremlim [**1**] stating that $\sum_{n=1}^\infty a_n^2 < \infty$ whenever $\sum_{n=1}^\infty a_n d_n$ converges in L_1 and $(d_n)_{n=1}^\infty$ is a uniformly integrable normalized martingale difference sequence. Some subsequence principles are then obtained for martingale difference sequences in $L_1(X)$

Received by the editors July 5, 1985 and, in revised form, April 28, 1986. Presented to the Society November 1, 1985 at the special session on Banach spaces held in Columbia, Missouri.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 60B12; Secondary 60G42, 46B20.

Key words and phrases. Martingale, uniformly convex, subsequence.

when X is a q-convex Banach space. A rather more complete picture is given for sequences in $L_p(X)$ for $p > 1$.

1. Almost sure convergence of a series of independent random variables. We start with some notation. Let $(\Omega, \mathcal{F}, P)$ be a probability space. If $A \in \mathcal{F}$, then $I(A)$ denotes the indicator function of A. The term "random variable" is used to mean an element of $L_0(\Omega)$. We say that a set S of random variables is bounded in probability if S is a bounded subset of $L_0(\Omega)$, i.e., if for each $\varepsilon > 0$ there exists M such that $P(|f| > M) < \varepsilon$ for all $f \in S$. We write $\mathbf{E}f$ for the expectation of f when $f \in L_1(\Omega)$ and $\operatorname{var}(f)$ for the variance of f when $f \in L_2(\Omega)$.

THEOREM 1.1. *Let $(d_n)_{n=1}^\infty$ be a sequence of independent random variables which is bounded in probability. Then the following are equivalent:*
(i) *$(d_n)_{n=1}^\infty$ contains no subsequence converging in probability;*
(ii) *$\sum_{n=1}^\infty a_n^2 < \infty$ whenever $\sum_{n=1}^\infty a_n d_n$ converges almost surely.*

PROOF. We first assume (i) and deduce (ii). Since $(d_n)_{n=1}^\infty$ is bounded in probability and contains no subsequence converging in probability it follows that there exists $\varepsilon > 0$ such that for all real numbers a and for all sufficiently large n we have $P(|d_n - a| > \varepsilon) > \varepsilon$. Suppose that $(a_n)_{n=1}^\infty$ is a real sequence such that $\sum_{n=1}^\infty a_n d_n$ converges almost surely. Then there exists $M > 0$ such that

$$P\left(\sup_{n\geq 1}\left|\sum_{k=1}^n a_k d_k\right| > M\right) < \frac{\varepsilon}{2}, \quad \text{whence } P\left(\sup_{n\geq 1}|a_n d_n| \leq 2M\right) > 1 - \frac{\varepsilon}{2}.$$

By Kolmogorov's three series theorem

$$\sum_{n=1}^\infty a_n^2 \operatorname{var}(d_n I(|a_n d_n| \leq 2M)) < \infty.$$

But

$$P(\{|a_n d_n| \leq 2M\} \cap \{|d_n - E(d_n I(|a_n d_n| \leq 2M))| \geq \varepsilon\}) > \frac{\varepsilon}{2}$$

for all sufficiently large n, and so $\operatorname{var}(d_n I(|a_n d_n| \leq 2M)) > \varepsilon^3/2$ for all sufficiently large n. Thus $\sum_{n=1}^\infty a_n^2 < \infty$, which proves (ii).

Now suppose that (i) fails. Then there exists a subsequence $(d_{n_k})_{k=1}^\infty$ and a real number b such that $P(|d_{n_k} - b| > 2^{-k}) < 2^{-k}$. Let $\sum_{k=1}^\infty a_k$ be any conditionally convergent series of real numbers. By the Borel-Cantelli lemma $\sum_{k=1}^\infty a_k(d_{n_k} - b)$ converges almost surely, and so $\sum_{k=1}^\infty a_k d_{n_k}$ converges almost surely. □

REMARK. Let $(d_n)_{n=1}^\infty$ be a uniformly integrable sequence of independent random variables in $L_1(\Omega)$ with $E|d_n| = 1$ and $Ed_n = 0$. Then $(d_n)_{n=1}^\infty$ must satisfy (i), and so we deduce the theorem of Chow and Teicher [**5**, p. 117] that $\sum_{n=1}^\infty a_n^2 < \infty$ whenever $\sum_{n=1}^\infty a_n d_n$ converges almost surely.

COROLLARY 1.2. *Let $(d_n)_{n=1}^\infty$ be a sequence of independent random variables which is bounded in probability. Then the following are equivalent:*
(i) *$\sum_{n=1}^\infty a_n^2 < \infty$ whenever $\sum_{n=1}^\infty a_n d_n$ converges almost surely;*
(ii) *$\sum_{n=1}^\infty a_n^2 < \infty$ whenever $\sum_{n=1}^\infty a_n d_{\pi(n)}$ converges almost surely for some permutation π of $\mathbf{N}$.*

PROOF. Clearly (ii) implies (i). Suppose that (i) holds; then by Theorem 1 $(d_n)_{n=1}^\infty$ contains no subsequence which converges in L_0. If π is a permutation of $\mathbf{N}$, then $(d_{\pi(n)})_{n=1}^{\infty}$ also contains no subsequence which converges in L_0. So $\sum_{n=1}^{\infty} a_n^2 < \infty$ whenever $\sum_{n=1}^{\infty} a_n d_{\pi(n)}$ converges almost surely. □

In the next corollary let $(X, \| \ \|)$ denote a quasi-Banach function space of random variables in $L_0(\Omega)$; that is, $(X, \| \ \|)$ has the following properties:

(i) $g \in X$ and $\|g\| = \|f\|$ whenever $f \in X$ and g and f have the same distribution;

(ii) the inclusion mapping of X into $L_0(\Omega)$ is continuous (the quasi-norm is assumed to satisfy $\|x + y\| \leq C(\|x\| + \|y\|)$ for all $x, y \in X$ and some constant $C \geq 1$).

A sequence $(x_n)_{n=1}^{\infty}$ in X is said to satisfy a lower q-estimate, where $0 < q < \infty$, if

$$\left\| \sum_{n=1}^{\infty} a_n x_n \right\| \geq C \left(\sum_{n=1}^{\infty} |a_n|^q \right)^{1/q}$$

for some $C > 0$ and for all real sequences $(a_n)_{n=1}^{\infty}$.

COROLLARY 1.3. *Let $(d_n)_{n=1}^{\infty}$ be a sequence of independent random variables in X which is bounded in probability and contains no subsequence converging in probability. Then $(d_n)_{n=1}^{\infty}$ satisfies a lower 2-estimate.*

PROOF. Let $(d_n^{(1)})_{n=1}^{\infty}$ and $(d_n^{(2)})_{n=1}^{\infty}$ be independent copies of $(d_n)_{n=1}^{\infty}$. The symmetry of $(d_n^{(1)} - d_n^{(2)})_{n=1}^{\infty}$ and the first property of X imply that

$$\left\| \sum_{k=1}^{m} a_k (d_k^{(1)} - d_k^{(2)}) \right\| \leq 2C \left\| \sum_{1}^{n} a_k (d_k^{(1)} - d_k^{(2)}) \right\|$$

for all $1 \leq m \leq n$ and reals $a_1, \ldots, a_n$. It follows that $(d_n^{(1)} - d_n^{(2)})_{n=1}^{\infty}$ is a Schauder basis of its closed linear span $[d_n^{(1)} - d_n^{(2)}]_{n=1}^{\infty}$ (see e.g. [**12**]). Now suppose that the series $\sum_{n=1}^{\infty} a_n (d_n^{(1)} - d_n^{(2)})$ converges in X. Then by the second property of X the series converges in L_0 and so converges almost surely because the terms are independent. Since $(d_n^{(1)} - d_n^{(2)})_{n=1}^{\infty}$ is bounded away from zero in probability it follows from Theorem 1 that $\sum_{n=1}^{\infty} a_n^2 < \infty$. Now define

$$T \colon [d_n^{(1)} - d_n^{(2)}]_{n=1}^{\infty} \to l_2 \quad \text{by } T \left(\sum_{n=1}^{\infty} a_n (d_n^{(1)} - d_n^{(2)}) \right) = (a_n)_{n=1}^{\infty}.$$

Then T is bounded by the Banach-Steinhaus theorem, and so $(d_n^{(1)} - d_n^{(2)})_{n=1}^{\infty}$ satisfies a lower 2-estimate. But

$$\left\| \sum_{k=1}^{\infty} a_k d_k \right\| \geq \frac{1}{2C} \left\| \sum_{k=1}^{\infty} a_k (d_k^{(1)} - d_k^{(2)}) \right\|,$$

whence $(d_n)_{n=1}^{\infty}$ satisfies a lower 2-estimate. □

REMARK. Consideration of a sequence of constant random variables shows that the hypothesis that $(d_n)_{n=1}^{\infty}$ contains no subsequence which converges in probability cannot be eliminated. If $(d_n)_{n=1}^{\infty}$ is bounded in X, then $(d_n)_{n=1}^{\infty}$ is bounded in probability by the second property of X. Finally, the hypotheses are met by a nondegenerate independent identically distributed sequence.

2. Almost sure convergence of a vector-valued martingale with respect to a regular sequence of σ-fields. Let X be a Banach space. Then $L_0(X)$ denotes the collection of all equivalence classes of measurable functions $f\colon \Omega \to X$ having essentially separable range. For $0 < p \leq \infty$, $L_p(X)$ is the collection of those functions f such that

$$\|f\|_p = \left(\int |f|^p \, dP\right)^{1/p} < \infty \quad \text{if } 0 < p < \infty$$

and

$$\|f\|_\infty = \operatorname{ess\,sup} \|f(\omega)\| < \infty \quad \text{if } p = \infty.$$

Let $(\mathcal{F}_n)_{n=0}^\infty$ be an increasing sequence of σ-fields contained in $\mathcal{F}$ and let $(d_n)_{n=1}^\infty$ be a sequence in $L_1(X)$. Say that $(d_n)_{n=1}^\infty$ is a martingale difference sequence (with respect to $(\mathcal{F}_n)_{n=0}^\infty$) if d_n is measurable with respect to $\mathcal{F}_n$ and $E(d_n \| \mathcal{F}_{n-1}) = 0$ for all $n > 1$. An increasing sequence of atomic σ-fields (i.e., σ-fields generated by a countable set of disjoint atoms) $(\mathcal{F}_n)_{n=0}^\infty$ is said to be regular (see e.g., [**21**, p. 83]) if there exists a constant α such that $P(E_{n+1})/P(E_n) \geq \alpha$ for all $n \geq 0$ and for all atoms $E_n \in \mathcal{F}_n$, $E_{n+1} \in \mathcal{F}_{n+1}$ such that $P(E_n) > 0$, $P(E_{n+1}) > 0$ and $E_{n+1} \subset E_n$ (this is called the Vitali-Chow condition in [**16**]).

Note that when $(\mathcal{F}_n)_{n=0}^\infty$ is regular and f is merely measurable with respect to $\mathcal{F}_n$ then $E(f \| \mathcal{F}_{n-1})$ still makes sense. Further, a real martingale difference sequence with respect to a regular filtration $(\mathcal{F}_n)_{n=0}^\infty$ is regular in the sense of Marcinkiewicz and Zygmund (regular MZ); that is, there exists $\delta > 0$ such that $\delta E^{1/2}(d_n^2 \| \mathcal{F}_{n-1}) \leq E(|d_n| \, \| \mathcal{F}_{n-1})$ [**21**, p. 80]. A regular MZ martingale difference sequence is said to be normed if $E(d_n^2 \| \mathcal{F}_{n-1}) = 1$ almost surely. The convergence of martingale transforms of normed regular MZ martingale difference sequences is considered in [**4**] and [**9**].

PROPOSITION 2.1. *Let $(d_n)_{n=1}^\infty$ be a martingale difference sequence in $L_0(X)$ with respect to a regular sequence of σ-fields $(\mathcal{F}_n)_{n=0}^\infty$. Suppose further that $(d_n)_{n=1}^\infty$ is bounded away from zero in probability. Then there exists $\eta > 0$ with the following property: whenever $(a_n)_{n=1}^\infty$ is a real sequence such that*

$$P\left(\sup_{n\geq 1} \left\| \sum_{k=1}^n a_k d_k \right\| = \infty\right) < \eta$$

then there exists a martingale difference sequence $(\tilde{d}_n)_{n=1}^\infty$ which is bounded away from zero in probability such that $(\sum_{k=1}^n a_k \tilde{d}_k)_{n=1}^\infty$ is uniformly bounded in $L_\infty(X)$.

PROOF. Choose $\varepsilon > 0$ such that $(d_n I(A))_{n=1}^\infty$ is bounded away from zero in probability whenever $P(A) > 1 - \varepsilon$. Suppose that $(a_n)_{n=1}^\infty$ is a real sequence such that

$$P\left(\sup_{n\geq 1} \left\| \sum_{k=1}^n a_k d_k \right\| = \infty\right) < \eta.$$

There exists $M > 0$ such that

$$P\left(\sup_{n\geq 1} \left\| \sum_{k=1}^n a_k d_k \right\| > M\right) < \eta.$$

For each $n \geq 1$, define e_n thus: for $\omega \in A$, where A is an atom of $\mathcal{F}_{n-1}$, let $e_n(\omega) = \sup_{\omega \in A} \|a_n d_n(\omega)\|$. Then $(e_n)_{n=1}^\infty$ is a predictable sequence. It follows from the regularity of $(\mathcal{F}_n)_{n=0}^\infty$ that

$$P\left(\sup_{n\geq 1} e_n(\omega) > 2M\right) \leq \frac{1}{\alpha} P\left(\sup_{n\geq 1} \|a_n d_n\| > 2M\right) < \frac{\eta}{\alpha}.$$

Define the stopping time

$$\tau(\omega) = \inf\left\{n \colon e_n(\omega) > 2M \text{ or } \left\|\sum_{k=1}^n a_k d_k\right\| > M\right\}.$$

Then

$$\begin{aligned} P(\tau(\omega) < \infty) &\leq P\left(\sup_{n\geq 1} e_n(\omega) > 2M\right) + P\left(\sup_{n\geq 1}\left\|\sum_{k=1}^n a_k d_k\right\| > M\right) \\ &< \frac{\eta}{\alpha} + \eta < \varepsilon \end{aligned}$$

provided η is sufficiently small. Let $\tilde{d}_n = d_n I(\tau \leq n)$. Then $\|\sum_{k=1}^n a_k \tilde{d}_k\| \leq 3M$ and $(\tilde{d}_k)_{k=1}^\infty$ is bounded away from zero in probability. □

We now need to recall some facts from the theory of Banach spaces. The modulus of convexity $\delta_X(\varepsilon)$ of a Banach space X is defined for all $0 < \varepsilon \leq 2$ by

$$\delta_X(\varepsilon) = \inf\{1 - \|(x+y)/2\| \colon \|x\| = \|y\| = 1,\ \|x - y\| = \varepsilon\}.$$

X is said to be uniformly convex if $\delta_X(\varepsilon) > 0$ for all $0 < \varepsilon \leq 2$. Suppose that $2 \leq q < \infty$; X is said to be q-convex if X admits an equivalent norm whose modulus of convexity δ satisfies $\delta(\varepsilon) \geq C\varepsilon^q$ for some $C > 0$. In particular, the function space $L_p(S, \Sigma, \mu)$, where (S, Σ, μ) is a measure space, is $\max(2, p)$-convex for each $1 < p < \infty$. More generally, every superreflexive Banach space (see [**10**] for some characterizations of superreflexivity) is q-convex for some $2 \leq q < \infty$ [**17**].

Finally, recall that a sequence $(x_n)_{n=1}^\infty$ is said to be a monotone basic sequence (e.g., [**12**]) if

$$\left\|\sum_{k=1}^n a_k x_k\right\| \leq \left\|\sum_{k=1}^m a_k x_k\right\| \quad \text{for all } 1 \leq n \leq m < \infty$$

and all scalars $a_1, \ldots, a_m$.

THEOREM 2.2. *Let X be a q-convex Banach space and let $(d_n)_{n=1}^\infty$ be a martingale difference sequence in $L_0(X)$ with respect to a regular sequence of σ-fields $(\mathcal{F}_n)_{n=0}^\infty$. Then the following are equivalent:*

(i) *$(d_n)_{n=1}^\infty$ is bounded away from zero in probability;*

(ii) *there exists $\eta > 0$ such that $\sum_{n=1}^\infty |a_n|^q < \infty$ whenever*

$$P\left(\sup_{n\geq 1}\left\|\sum_{k=1}^n a_k d_k\right\| = \infty\right) < \eta.$$

PROOF. Let $(f_n)_{n=1}^\infty$ be any sequence in $L_0(X)$ and $(a_n)_{n=1}^\infty$ any sequence of scalars. It is easily seen that if $(f_n)_{n=1}^\infty$ is not bounded away from zero in probability then there is a subsequence $(f_{n_k})_{k=1}^\infty$ such that $\sum_{k=1}^\infty a_k f_{n_k}$ converges almost surely.

Thus (ii) implies (i). Suppose that (i) holds; then there exists $\eta > 0$ satisfying the conclusion of Proposition 2.1. Let $(a_n)_{n=1}^{\infty}$ be a real sequence such that

$$P\left(\sup_{n\geq 1}\left\|\sum_{k=1}^{n} a_k d_k\right\| = \infty\right) < \eta.$$

There exists a martingale difference sequence $(\tilde{d}_n)_{n=1}^{\infty}$ bounded away from zero in probability such that $(\sum_{k=1}^{n} a_k \tilde{d}_k)_{n=1}^{\infty}$ is uniformly bounded in $L_\infty(X)$. In particular, $(\tilde{d}_n)_{n=1}^{\infty}$ is a monotone basic sequence in $L_2(X)$ with

$$\inf_{n\geq 1}\|\tilde{d}_n\|_2 > 0 \quad \text{and} \quad \sup_{n\geq 1}\left\|\sum_{k=1}^{n} a_k \tilde{d}_k\right\|_2 < \infty.$$

$L_2(X)$ is itself q-convex (see [**8**]) and a monotone basic sequence in a q-convex space satisfies a lower q-estimate [**17**], and so $\sum_{n=1}^{\infty} |a_n|^q < \infty$. □

COROLLARY 2.3. *Let X be a q-convex Banach space and let $(d_n)_{n=1}^{\infty}$ be a dyadic martingale difference sequence in $L_0(X)$ which is bounded away from zero in probability. Then $\sum_{n=1}^{\infty} |a_n|^q < \infty$ whenever $\sum_{n=1}^{\infty} a_n d_n$ converges almost surely.*

3. Subsequence principles for square-integrable random variables. The following result is an immediate consequence of a summability theorem of Chow [**4**, Theorem 3].

THEOREM A. *Let $(d_n)_{n=1}^{\infty}$ be a martingale difference sequence which is normalized in $L_2(\Omega)$ and satisfies $E|d_n| \geq c$ for some $c > 0$ and for all $n \geq 1$. Then there exists $\varepsilon > 0$ such that*

$$\sup_{n\geq 1} P\left\{\left|\sum_{k=1}^{n} a_k d_k\right| > K\right\} \geq \varepsilon \quad \textit{for all } K > 0$$

whenever $\sum_{n=1}^{\infty} a_n^2 = \infty$. In particular,

$$P\left\{\sup_{n\geq 1}\left|\sum_{k=1}^{n} a_k d_k\right| = \infty\right\} \geq \varepsilon \quad \textit{whenever} \quad \sum_{n=1}^{\infty} a_n^2 = \infty.$$

We use Theorem A to deduce the following subsequence principle for almost sure convergence of square-integrable random variables.

THEOREM 3.1. *Let $(f_n)_{n=1}^{\infty}$ be a normalized sequence in $L_2(\Omega)$ having no subsequence convergent in $L_1(\Omega)$. Then there exists $f \in L_2(\Omega)$, $\varepsilon > 0$, and a subsequence $(f_{n_k})_{n=1}^{\infty}$ with the following properties:*

(i) *$\sum_{k=1}^{\infty} a_k(f_{n_k} - f)$ converges almost surely and in $L_2(\Omega)$ whenever $\sum_{k=1}^{\infty} a_k^2 < \infty$;*

(ii) *$P(\sup_{n\geq 1} |\sum_{k=1}^{n} a_k(f_{n_k} - f)| = \infty) > \varepsilon$ whenever $\sum_{n=1}^{\infty} a_n^2 = \infty$.*

PROOF. Bounded subsets of $L_2(\Omega)$ are weakly sequentially compact, and so there exist $f \in L_2(\Omega)$ and a subsequence $(f_{n_k})_{k=1}^{\infty}$ such that $(f_{n_k} - f)_{k=1}^{\infty}$ is weakly null. Since $(f_n)_{n=1}^{\infty}$ has no subsequence convergent in $L_1(\Omega)$ we may assume that $\|f_{n_k} - f\|_1 > 2c$ for some $c > 0$ and for all $k \geq 1$. By a well-known argument (e.g., [**3**, p. 243]) we may also assume by passing to a further subsequence that there exists a martingale difference sequence (g_k) of simple functions such that $\sum_{k=1}^\infty \|f_{n_k} - f - g_k\|_2 < c$. By Theorem A there exists $\tilde{\varepsilon} > 0$ such that

$$\sum_{k=1}^\infty a_k^2 < \infty \quad \text{whenever } P\left(\sup_{n\geq 1}\left|\sum_{k-1}^n a_k g_k\right| = \infty\right) < \tilde{\varepsilon}.$$

By Hölder's inequality $(f_{n_k} - f)_{k=1}^\infty$ is uniformly integrable, and so there exists $\varepsilon > 0$ such that $\int |f_{n_k} - f| I(A)\, dP < c$ for all $k \geq 1$ whenever $P(A) < \varepsilon$; moreover, we may assume that $\varepsilon \leq \tilde{\varepsilon}$. Suppose now that $(a_k)_{k=1}^\infty$ is a real sequence and that

$$P\left(\sup_{m\geq 1}\left|\sum_{k=1}^m a_k(f_{n_k} - f)\right| = \infty\right) < \varepsilon.$$

Then there exists $M > 0$ such that $P(A) > 1 - \varepsilon$, where

$$A = \left\{\sup_{m\geq 1}\left|\sum_{k=1}^m a_k(f_{n_k} - f)\right| \leq M\right\}.$$

So

$$2MP(A) \geq \int |a_k(f_{n_k} - f)| I(A)\, dP \geq c|a_k|.$$

Hence $\sup_{k\geq 1} |a_k| < \infty$, and it follows that $\sum_{k=1}^\infty a_k(f_{n_k} - f - g_k)$ converges absolutely almost surely. So

$$P\left(\sup_{n\geq 1}\left|\sum_{k=1}^n a_k g_k\right| = \infty\right) < \varepsilon \leq \tilde{\varepsilon},$$

whence $\sum_{n=1}^\infty a_n^2 < \infty$. This completes the proof of (ii).

Property (i) is a well-known theorem of Revesz (see [**18**]) and follows easily from the martingale convergence theorem (see [**3**]). Indeed, $\sum_{n=1}^\infty a_n g_n$ converges almost surely and in $L_2(\Omega)$ whenever $\sum_{n=1}^\infty a_n^2 < \infty$, and so the same is true of $\sum_{k=1}^\infty a_k(f_{n_k} - f)$. □

REMARK. The hypothesis that $(f_n)_{n=1}^\infty$ contains no subsequence convergent in L_1 is used only in the proof of property (ii). Revesz proved in [**19**] that something like property (ii) could be made to work for the case in which $(f_n)_{n=1}^\infty$ is a uniformly bounded sequence in $L_\infty(\Omega)$.

Combining Theorem 3.1 with the proof of "(i) implies (ii)" in Theorem 2.2 gives the following result.

THEOREM 3.2. *Let $(f_n)_{n=1}^\infty$ be weakly null in $L_2(\Omega)$. Then the following are equivalent:*

(i) *$(f_n)_{n=1}^\infty$ contains a subsequence which is bounded away from zero in probability;*

(ii) *there exists $\varepsilon > 0$ and a subsequence $(f_{n_k})_{k=1}^\infty$ with the following properties:*

(a) *$\sum_{k=1}^\infty a_k f_{n_k}$ converges almost surely and in $L_2(\Omega)$ whenever $\sum_{k=1}^\infty a_k^2 < \infty$;*

(b) *$P\{\sup_{m\geq 1} |\sum_{k=1}^m a_k f_{n_k}| = \infty\} > \varepsilon$ whenever $\sum_{k=1}^\infty a_k^2 = \infty$.*

REMARK. The last result applies, in particular, to an arbitrary orthonormal system in $L_2(\Omega)$. In this setting (i) corresponds to the fact that every orthonormal system $(\phi_n)_{n=1}^\infty$ contains a subsystem $(\phi_{n_k})_{k=1}^\infty$ that is a system of convergence (meaning $\sum_{k=1}^\infty a_k \phi_{n_k}$ converges whenever $\sum_{k=1}^\infty a_k^2 < \infty$ (see [**2**, p. 156])).

Whereas (ii) resembles the fact that a lacunary trigonometric series

$$\sum_{k=1}^{\infty}(a_{n_k}\cos n_k t + b_{n_k}\sin n_k t)$$

diverges almost everywhere when $\sum_{k=1}^{\infty}(a_{n_k}^2 + b_{n_k}^2) = \infty$ [**22**, p. 203]. (An increasing sequence of positive integers $(n_k)_{k=1}^{\infty}$ is lacunary if $n_{k+1}/n_k > t$ for some $t > 1$ and for all k.)

To conclude this part we prove an abstract version of a related theorem of Zygmund on lacunary Fourier coefficients [**23**, p. 132]. Let $(\phi_n)_{n=1}^{\infty}$ be a uniformly bounded orthonormal system in $L_2(0,1)$. For $f \in L_1(0,1)$, let $\hat{f}(n) = \int f\phi_n\,dt$ for all $n \geq 1$.

THEOREM 3.3. *Let $(\phi_n)_{n=1}^{\infty}$ be a uniformly bounded orthonormal system. Every sequence of positive integers $(n_k)_{k=1}^{\infty}$ contains a subsequence $(n'_k)_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty}\hat{f}(n'_k)^2 < \infty$ whenever $f \in L_p(0,1)$ for some $p > 1$. Moreover, there exists $f \in L_1(0,1)$ such that $\sum_{k=1}^{\infty}\hat{f}(n_k)^2 = \infty$.*

PROOF. By [**11**, Corollary 6] and a diagonal argument the subsequence $(n'_k)_{k=1}^{\infty}$ may be chosen so that $(\phi_{n'_k})_{k=1}^{\infty}$ is equivalent to the unit vector basis of l_2 in L_p for all $p > 2$; that is, there exists $C_p > 0$ such that for all $m \geq 1$ and for all scalars $a_1, \ldots, a_m$, we have

$$\frac{1}{C_p}\left(\sum_{k=1}^{m} a_k^2\right)^{1/2} \leq \left\|\sum_{k=1}^{m} a_k\phi_{n'_k}\right\|_p \leq C_p\left(\sum_{k=1}^{m} a_k^2\right)^{1/2}.$$

By the results of [**11**] the same is true for all $p > 0$. Let $P(f) = \sum_{k=1}^{\infty}\hat{f}(n'_k)\phi_{n'_k}$. Then P is an orthogonal projection on L_2 and so is bounded. Since the L_2 and L_p norms are equivalent on the closed linear span of $(\phi_{n'_k})_{k=1}^{\infty}$ it follows that P is bounded on $L_p(0,1)$ for all $p \geq 2$. For $1 < p < 2$, $P\colon L_p(0,1) \to L_p(0,1)$ is bounded because it is the adjoint of $P\colon L_q(0,1) \to L_q(0,1)$, where $1/p + 1/q = 1$, which is bounded. This proves the first part of the proposition.

To show the last part, suppose on the contrary that $\sum_{k=1}^{\infty}\hat{f}(n_k)^2 < \infty$ for all $f \in L_1$, and so, in particular, that $\sum_{k=1}^{\infty}\hat{f}(n'_k)^2 < \infty$. But then by the Banach-Steinhaus theorem P is a bounded projection on $L_1(0,1)$ whose range is $[\varphi_{n'_k}]_{k=1}^{\infty}$, which is impossible because $L_1(0,1)$ contains no complemented subspace isomorphic to a Hilbert space. □

REMARK. A Rademacher-like property of subsequences of random variables in L_p is also proved in [**15**, Lemma 2.1].

4. Convergence in $L_1(X)$. The following is a vectorial generalization of [**1**, §4]. Since the proof is essentially the same it has been omitted.

PROPOSITION 4.1. *Let $(d_n)_{n=1}^{\infty}$ be a uniformly integrable martingale difference sequence normalized in $L_1(X)$. Suppose that $(a_n)_{n=1}^{\infty}$ is a real sequence such that $\sum_{n=1}^{\infty} a_n d_n$ converges in $L_1(X)$. Then there exists a martingale difference sequence $(\tilde{d}_n)_{n=1}^{\infty}$, bounded away from zero in $L_1(X)$, such that $(\sum_{k=1}^{n} a_k\tilde{d}_k)_{n=1}^{\infty}$ is uniformly bounded in $L_\infty(X)$.*

THEOREM 4.2. *Suppose that X is a q-convex Banach space and that $(d_n)_{n=1}^{\infty}$ is a uniformly integrable martingale difference sequence normalized in $L_1(X)$. Then $(d_n)_{n=1}^{\infty}$ satisfies a lower q-estimate.*

PROOF. This follows from Proposition 4.1 together with the proof of Theorem 2.2. □

THEOREM 4.3. *Suppose that X is a q-convex Banach space and that $(d_n)_{n=1}^{\infty}$ is a martingale difference sequence normalized in $L_1(X)$. There exists a subsequence $(d_{n_k})_{k=1}^{\infty}$ which satisfies a lower q-estimate.*

PROOF. If $(d_n)_{n=1}^{\infty}$ is uniformly integrable then $(d_n)_{n=1}^{\infty}$ satisfies a lower q-estimate. Otherwise, by the results of [**20**], one can extract a subsequence $(d_{n_k})_{k=1}^{\infty}$ equivalent to the unit vector basis of the sequence space l_1. Then $(d_{n_k})_{k=1}^{\infty}$ satisfies a lower 1-estimate, and a fortiori a lower q-estimate. □

It is possible that Theorem 4.3 remains valid for arbitrary sequences in $L_1(X)$ which are not relatively compact, and this is the case in $L_p(X)$ for $1 < p \leq q$. For if X is q-convex then $L_p(X)$ is q-convex for $1 < p \leq q$, and so it follows from the next proposition that any normalized sequence $(f_n)_{n=1}^{\infty}$ in $L_p(X)$ which is not relatively compact contains a subsequence satisfying a lower q-estimate.

PROPOSITION 4.4. *Suppose that X is a q-convex Banach space and that $(x_n)_{n=1}^{\infty}$ is a normalized sequence in X which is not relatively compact. Then $(x_n)_{n=1}^{\infty}$ contains a subsequence satisfying a lower q-estimate.*

PROOF. By passing to a subsequence we may assume that $(x_n)_{n=1}^{\infty}$ contains no norm convergent subsequence. Since X is reflexive its bounded subsets are relatively weakly sequentially compact, and so we may further assume that there exists x in X such that $(x_n - x)_{n=1}^{\infty}$ is weakly null (and bounded away from zero). By [**6**, Proposition 2.4] $(x_n - x)_{n=1}^{\infty}$ contains a subsequence $(x_{n_k} - x)_{k=1}^{\infty}$ satisfying a lower q-estimate, and by [**1**, §2] there exists $m \geq 1$ such that $(x_{n_k})_{k=m}^{\infty}$ satisfies a lower q-estimate. □

ACKNOWLEDGMENT. I am grateful to the referee for many helpful suggestions and for supplying the far superior proof of Proposition 2.1.

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