CONVERGENCE OF SERIES
OF SCALAR- AND VECTOR-VALUED RANDOM VARIABLES
AND A SUBSEQUENCE PRINCIPLE IN $L_2$

S. J. DILWORTH

ABSTRACT. Let $(d_n)_{n=1}^\infty$ be a martingale difference sequence in $L_0(X)$, where $X$ is a uniformly convex Banach space. We investigate a necessary condition for convergence of the series $\sum_{n=1}^\infty a_n d_n$. We also prove a related subsequence principle for the convergence of a series of square-integrable scalar random variables.

Introduction. Let $(d_n)_{n=1}^\infty$ be an orthonormal sequence of independent random variables and let $(a_n)_{n=1}^\infty$ be a sequence of real numbers. In [14] Marcinkiewicz and Zygmund proved that if $E|d_n| \geq \delta > 0$ for all $n \geq 1$ then $\sum_{n=1}^\infty a_n^2 < \infty$ whenever $\sum_{n=1}^\infty a_n d_n$ converges almost surely. This theorem has been extended to the case of martingale difference sequences by Chow [4]. In §1 the almost sure convergence of the series $\sum_{n=1}^\infty a_n d_n$ is considered when $(d_n)_{n=1}^\infty$ is a bounded sequence in $L_0$. Necessary and sufficient conditions are given on such a sequence of independent random variables to be able to conclude that $\sum_{n=1}^\infty a_n^2 < \infty$ whenever $\sum_{n=1}^\infty a_n d_n$ converges almost surely. The same question is treated in §2 for a vector-valued martingale difference sequence $(d_n)_{n=1}^\infty$ in $L_0(X)$ (here $X$ is a Banach space). When $(d_n)_{n=1}^\infty$ is adapted to a regular sequence of $\sigma$-fields and $X$ is a $q$-convex Banach space, necessary and sufficient conditions on $(d_n)_{n=1}^\infty$ are given to be able to conclude that $\sum_{n=1}^\infty |a_n|^q < \infty$ whenever $\sum_{n=1}^\infty a_n d_n$ has bounded partial sums almost surely (or with high probability).

In §3 the theorem of Chow mentioned above is used to deduce a subsequence principle for random variables in $L_2$ which is related to some theorems of Revesz. A consequence of this is that any orthonormal sequence $(\phi_n)_{n=1}^\infty$ which is bounded away from zero in probability will contain a subsequence $(\phi_{n_k})_{k=1}^\infty$ with the following property: $\sum_{k=1}^\infty a_k^2 < \infty$ whenever $\sum_{k=1}^\infty a_k \phi_{n_k}$ converges almost surely (or merely whenever $\sum_{k=1}^\infty a_k \phi_{n_k}$ has bounded partial sums with high probability). The section closes with an abstract version of a theorem of Zygmund on lacunary Fourier coefficients.

The last part gives some vectorial extensions of a theorem of Aldous and Fremlin [1] stating that $\sum_{n=1}^\infty a_n^2 < \infty$ whenever $\sum_{n=1}^\infty a_n d_n$ converges in $L_1$ and $(d_n)_{n=1}^\infty$ is a uniformly integrable normalized martingale difference sequence. Some subsequence principles are then obtained for martingale difference sequences in $L_1(X)$.
when $X$ is a $q$-convex Banach space. A rather more complete picture is given for sequences in $L_p(X)$ for $p > 1$.

1. **Almost sure convergence of a series of independent random variables.** We start with some notation. Let $(\Omega, \mathcal{F}, P)$ be a probability space. If $A \in \mathcal{F}$, then $I(A)$ denotes the indicator function of $A$. The term “random variable” is used to mean an element of $L_0(\Omega)$. We say that a set $S$ of random variables is bounded in probability if $S$ is a bounded subset of $L_0(\Omega)$, i.e., if for each $\varepsilon > 0$ there exists $M$ such that $P(|f| > M) < \varepsilon$ for all $f \in S$. We write $Ef$ for the expectation of $f$ when $f \in L_1(\Omega)$ and $\text{var}(f)$ for the variance of $f$ when $f \in L_2(\Omega)$.

**Theorem 1.1.** Let $(d_n)_{n=1}^\infty$ be a sequence of independent random variables which is bounded in probability. Then the following are equivalent:

(i) $(d_n)_{n=1}^\infty$ contains no subsequence converging in probability;

(ii) $\sum_{n=1}^\infty a_n^2 < \infty$ whenever $\sum_{n=1}^\infty a_n d_n$ converges almost surely.

**Proof.** We first assume (i) and deduce (ii). Since $(d_n)_{n=1}^\infty$ is bounded in probability and contains no subsequence converging in probability it follows that there exists $\varepsilon > 0$ such that for all real numbers $a$ and for all sufficiently large $n$ we have $P(|d_n - a| > \varepsilon) > \varepsilon$. Suppose that $(a_n)_{n=1}^\infty$ is a real sequence such that $\sum_{n=1}^\infty a_n d_n$ converges almost surely. Then there exists $M > 0$ such that

$$P\left(\sup_{n \geq 1} \left| \sum_{k=1}^n a_k d_k \right| > M\right) < \frac{\varepsilon}{2},$$

whence

$$P\left(\sup_{n \geq 1} |a_n d_n| \leq 2M\right) > 1 - \frac{\varepsilon}{2}.$$ 

By Kolmogorov’s three series theorem

$$\sum_{n=1}^\infty a_n^2 \text{var}(d_n I(|a_n d_n| \leq 2M)) < \infty.$$

But

$$P(\{|a_n d_n| \leq 2M\} \cap \{|d_n - E(d_n I(|a_n d_n| \leq 2M)|) \geq \varepsilon\}) > \frac{\varepsilon}{2}$$

for all sufficiently large $n$, and so

$$\text{var}(d_n I(|a_n d_n| \leq 2M)) > \varepsilon^2/2$$

for all sufficiently large $n$. Thus $\sum_{n=1}^\infty a_n^2 < \infty$, which proves (ii).

Now suppose that (i) fails. Then there exists a subsequence $(d_{n_k})_{k=1}^\infty$ and a real number $b$ such that $P(|d_{n_k} - b| > 2^{-k}) < 2^{-k}$. Let $\sum_{k=1}^\infty a_k$ be any conditionally convergent series of real numbers. By the Borel-Cantelli lemma $\sum_{k=1}^\infty a_k (d_{n_k} - b)$ converges almost surely, and so $\sum_{k=1}^\infty a_k d_{n_k}$ converges almost surely. $\square$

**Remark.** Let $(d_n)_{n=1}^\infty$ be a uniformly integrable sequence of independent random variables in $L_1(\Omega)$ with $E|d_n| = 1$ and $Ed_n = 0$. Then $(d_n)_{n=1}^\infty$ must satisfy (i), and so we deduce the theorem of Chow and Teicher [5, p. 117] that $\sum_{n=1}^\infty a_n^2 < \infty$ whenever $\sum_{n=1}^\infty a_n d_n$ converges almost surely.

**Corollary 1.2.** Let $(d_n)_{n=1}^\infty$ be a sequence of independent random variables which is bounded in probability. Then the following are equivalent:

(i) $\sum_{n=1}^\infty a_n^2 < \infty$ whenever $\sum_{n=1}^\infty a_n d_n$ converges almost surely;

(ii) $\sum_{n=1}^\infty a_n^2 < \infty$ whenever $\sum_{n=1}^\infty a_n d_{\pi(n)}$ converges almost surely for some permutation $\pi$ of $N$.

**Proof.** Clearly (ii) implies (i). Suppose that (i) holds; then by Theorem 1 $(d_n)_{n=1}^\infty$ contains no subsequence which converges in $L_0$. If $\pi$ is a permutation
of N, then \((d_{\pi(n)})_{n=1}^{\infty}\) also contains no subsequence which converges in \(L_0\). So 
\[ \sum_{n=1}^{\infty} a_n^2 < \infty \] whenever \(\sum_{n=1}^{\infty} a_n d_{\pi(n)}\) converges almost surely. □

In the next corollary let \((X, \| \cdot \|)\) denote a quasi-Banach function space of random variables in \(L_0(\Omega)\); that is, \((X, \| \cdot \|)\) has the following properties:

(i) \(g \in X\) and \(\|g\| = \|f\|\) whenever \(f \in X\) and \(g\) and \(f\) have the same distribution;

(ii) the inclusion mapping of \(X\) into \(L_0(\Omega)\) is continuous (the quasi-norm is assumed to satisfy \(\|x + y\| \leq C(\|x\| + \|y\|)\) for all \(x, y \in X\) and some constant \(C \geq 1\)).

A sequence \((x_n)_{n=1}^{\infty}\) in \(X\) is said to satisfy a lower \(q\)-estimate, where \(0 < q < \infty\), if
\[
\left\| \sum_{n=1}^{\infty} a_n x_n \right\| \geq C \left( \sum_{n=1}^{\infty} |a_n|^q \right)^{1/q}
\]
for some \(C > 0\) and for all real sequences \((a_n)_{n=1}^{\infty}\).

**COROLLARY 1.3.** Let \((d_n)_{n=1}^{\infty}\) be a sequence of independent random variables in \(X\) which is bounded in probability and contains no subsequence converging in probability. Then \((d_n)_{n=1}^{\infty}\) satisfies a lower 2-estimate.

**PROOF.** Let \((d_n^{(1)})_{n=1}^{\infty}\) and \((d_n^{(2)})_{n=1}^{\infty}\) be independent copies of \((d_n)_{n=1}^{\infty}\). The symmetry of \((d_n^{(1)} - d_n^{(2)})_{n=1}^{\infty}\) and the first property of \(X\) imply that
\[
\left\| \sum_{k=1}^{m} a_k (d_k^{(1)} - d_k^{(2)}) \right\| \leq 2C \left\| \sum_{k=1}^{n} a_k (d_k^{(1)} - d_k^{(2)}) \right\|
\]
for all \(1 \leq m \leq n\) and reals \(a_1, \ldots, a_n\). It follows that \((d_n^{(1)} - d_n^{(2)})_{n=1}^{\infty}\) is a Schauder basis of its closed linear span \((d_n^{(1)} - d_n^{(2)})_{n=1}^{\infty}\) (see e.g. [12]). Now suppose that the series \(\sum_{n=1}^{\infty} a_n (d_n^{(1)} - d_n^{(2)})\) converges in \(X\). Then by the second property of \(X\) the series converges in \(L_0\) and so converges almost surely because the terms are independent. Since \((d_n^{(1)} - d_n^{(2)})_{n=1}^{\infty}\) is bounded away from zero in probability it follows from Theorem 1 that \(\sum_{n=1}^{\infty} a_n^2 < \infty\). Now define
\[
T: [d_n^{(1)} - d_n^{(2)})_{n=1}^{\infty} \to l_2 \text{ by } T \left( \sum_{n=1}^{\infty} a_n (d_n^{(1)} - d_n^{(2)}) \right) = (a_n)_{n=1}^{\infty}.
\]
Then \(T\) is bounded by the Banach-Steinhaus theorem, and so \((d_n^{(1)} - d_n^{(2)})_{n=1}^{\infty}\) satisfies a lower 2-estimate. But
\[
\left\| \sum_{k=1}^{\infty} a_k d_k \right\| \geq \frac{1}{2C} \left\| \sum_{k=1}^{\infty} a_k (d_k^{(1)} - d_k^{(2)}) \right\|
\]
whence \((d_n)_{n=1}^{\infty}\) satisfies a lower 2-estimate. □

**REMARK.** Consideration of a sequence of constant random variables shows that the hypothesis that \((d_n)_{n=1}^{\infty}\) contains no subsequence which converges in probability cannot be eliminated. If \((d_n)_{n=1}^{\infty}\) is bounded in \(X\), then \((d_n)_{n=1}^{\infty}\) is bounded in probability by the second property of \(X\). Finally, the hypotheses are met by a nondegenerate independent identically distributed sequence.
2. Almost sure convergence of a vector-valued martingale with respect to a regular sequence of \( \sigma \)-fields. Let \( X \) be a Banach space. Then \( L_0(X) \) denotes the collection of all equivalence classes of measurable functions \( f: \Omega \rightarrow X \) having essentially separable range. For \( 0 < p \leq \infty \), \( L_p(X) \) is the collection of those functions \( f \) such that

\[
\|f\|_p = \left( \int |f|^p \, dP \right)^{1/p} < \infty \quad \text{if } 0 < p < \infty
\]

and

\[
\|f\|_\infty = \text{ess sup} \|f(\omega)\| < \infty \quad \text{if } p = \infty.
\]

Let \( (\mathcal{F}_n)_{n=0}^{\infty} \) be an increasing sequence of \( \sigma \)-fields contained in \( \mathcal{F} \) and let \( (d_n)_{n=1}^{\infty} \) be a sequence in \( L_1(X) \). Say that \( (d_n)_{n=1}^{\infty} \) is a martingale difference sequence (with respect to \( (\mathcal{F}_n)_{n=0}^{\infty} \)) if \( d_n \) is measurable with respect to \( \mathcal{F}_n \) and \( E(d_n|\mathcal{F}_{n-1}) = 0 \) for all \( n > 1 \). An increasing sequence of atomic \( \sigma \)-fields (i.e., \( \sigma \)-fields generated by a countable set of disjoint atoms) \( (\mathcal{F}_n)_{n=0}^{\infty} \) is said to be regular (see e.g., [21, p. 83]) if there exists a constant \( \alpha \) such that \( P(E_{n+1})/P(E_n) \geq \alpha \) for all \( n \geq 0 \) and for all atoms \( E_n \in \mathcal{F}_n, E_{n+1} \in \mathcal{F}_{n+1} \) such that \( P(E_n) > 0, P(E_{n+1}) > 0 \) and \( E_{n+1} \subset E_n \) (this is called the Vitali-Chow condition in [16]).

Note that when \( (\mathcal{F}_n)_{n=0}^{\infty} \) is regular and \( f \) is merely measurable with respect to \( \mathcal{F}_n \) then \( E(f|\mathcal{F}_{n-1}) \) still makes sense. Further, a real martingale difference sequence with respect to a regular filtration \( (\mathcal{F}_n)_{n=0}^{\infty} \) is regular in the sense of Marcinkiewicz and Zygmund (regular MZ); that is, there exists \( \delta > 0 \) such that \( \delta E^{1/2}(d_n^2|\mathcal{F}_{n-1}) \leq E(|d_n|\mathcal{F}_{n-1}) \) [21, p. 80]. A regular MZ martingale difference sequence is said to be normed if \( E(d_n^2|\mathcal{F}_{n-1}) = 1 \) almost surely. The convergence of martingale transforms of normed regular MZ martingale difference sequences is considered in [4] and [9].

**Proposition 2.1.** Let \( (d_n)_{n=1}^{\infty} \) be a martingale difference sequence in \( L_0(X) \) with respect to a regular sequence of \( \sigma \)-fields \( (\mathcal{F}_n)_{n=0}^{\infty} \). Suppose further that \( (d_n)_{n=1}^{\infty} \) is bounded away from zero in probability. Then there exists \( \eta > 0 \) with the following property: whenever \( (a_n)_{n=1}^{\infty} \) is a real sequence such that

\[
P \left( \sup_{n \geq 1} \left\| \sum_{k=1}^{n} a_k d_k \right\| = \infty \right) < \eta
\]

then there exists a martingale difference sequence \( (\tilde{d}_n)_{n=1}^{\infty} \) which is bounded away from zero in probability such that \( (\sum_{k=1}^{n} a_k \tilde{d}_k)_{n=1}^{\infty} \) is uniformly bounded in \( L_\infty(X) \).

**Proof.** Choose \( \varepsilon > 0 \) such that \( (d_n I(A))_{n=1}^{\infty} \) is bounded away from zero in probability whenever \( P(A) > 1 - \varepsilon \). Suppose that \( (a_n)_{n=1}^{\infty} \) is a real sequence such that

\[
P \left( \sup_{n \geq 1} \left\| \sum_{k=1}^{n} a_k d_k \right\| = \infty \right) < \eta.
\]

There exists \( M > 0 \) such that

\[
P \left( \sup_{n \geq 1} \left\| \sum_{k=1}^{n} a_k d_k \right\| > M \right) < \eta.
\]
For each $n \geq 1$, define $e_n$ thus: for $\omega \in A$, where $A$ is an atom of $\mathcal{F}_{n-1}$, let $e_n(\omega) = \sup_{\omega \in A} \| a_n d_n(\omega) \|$. Then $(e_n)_{n=1}^\infty$ is a predictable sequence. It follows from the regularity of $(\mathcal{F}_n)_{n=0}^\infty$ that

$$P \left( \sup_{n \geq 1} e_n(\omega) > 2M \right) \leq \frac{1}{\alpha} P \left( \sup_{n \geq 1} \| a_n d_n \| > 2M \right) < \frac{\eta}{\alpha}. $$

Define the stopping time

$$\tau(\omega) = \inf \left\{ n: e_n(\omega) > 2M \text{ or } \sum_{k=1}^n a_k d_k \right\}. $$

Then

$$P(\tau(\omega) < \infty) \leq P \left( \sup_{n \geq 1} e_n(\omega) > 2M \right) + P \left( \sum_{k=1}^n a_k d_k > M \right) < \frac{\eta}{\alpha} + \eta < \varepsilon$$

provided $\eta$ is sufficiently small. Let $\tilde{d}_n = d_n I(\tau \leq n)$. Then $\sum_{k=1}^n a_k \tilde{d}_k \leq 3M$ and $(\tilde{d}_k)_{k=1}^\infty$ is bounded away from zero in probability. \(\square\)

We now need to recall some facts from the theory of Banach spaces. The modulus of convexity $\delta_X(\varepsilon)$ of a Banach space $X$ is defined for all $0 < \varepsilon \leq 2$ by

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \| (x+y)/2 \|: \| x \| = \| y \| = 1, \| x - y \| = \varepsilon \right\}. $$

$X$ is said to be uniformly convex if $\delta_X(\varepsilon) > 0$ for all $0 < \varepsilon \leq 2$. Suppose that $2 \leq q < \infty$; $X$ is said to be $q$-convex if $X$ admits an equivalent norm whose modulus of convexity $\delta$ satisfies $\delta(\varepsilon) \geq C\varepsilon^q$ for some $C > 0$. In particular, the function space $L_p(S, \Sigma, \mu)$, where $(S, \Sigma, \mu)$ is a measure space, is max$(2, p)$-convex for each $1 < p < \infty$. More generally, every superreflexive Banach space (see [10] for some characterizations of superreflexivity) is $q$-convex for some $2 \leq q < \infty$ [17].

Finally, recall that a sequence $(x_n)_{n=1}^\infty$ is said to be a monotone basic sequence (e.g., [12]) if

$$\left\| \sum_{k=1}^n a_k x_k \right\| \leq \left\| \sum_{k=1}^m a_k x_k \right\| \quad \text{for all } 1 \leq n \leq m < \infty$$

and all scalars $a_1, \ldots, a_m$.

**Theorem 2.2.** Let $X$ be a $q$-convex Banach space and let $(d_n)_{n=1}^\infty$ be a martingale difference sequence in $L_0(X)$ with respect to a regular sequence of $\sigma$-fields $(\mathcal{F}_n)_{n=0}^\infty$. Then the following are equivalent:

(i) $(d_n)_{n=1}^\infty$ is bounded away from zero in probability,

(ii) there exists $\eta > 0$ such that $\sum_{n=1}^\infty |a_n|^q < \infty$ whenever

$$P \left( \sup_{n \geq 1} \left\| \sum_{k=1}^n a_k d_k \right\| = \infty \right) < \eta. $$

**Proof.** Let $(f_n)_{n=1}^\infty$ be any sequence in $L_0(X)$ and $(a_n)_{n=1}^\infty$ any sequence of scalars. It is easily seen that if $(f_n)_{n=1}^\infty$ is not bounded away from zero in probability then there is a subsequence $(f_{n_k})_{k=1}^\infty$ such that $\sum_{k=1}^\infty a_k f_{n_k}$ converges almost surely.
Thus (ii) implies (i). Suppose that (i) holds; then there exists \( \eta > 0 \) satisfying the conclusion of Proposition 2.1. Let \( (a_n)_{n=1}^\infty \) be a real sequence such that

\[
P \left( \sup_{n \geq 1} \left| \sum_{k=1}^{n} a_k d_k \right| = \infty \right) < \eta.
\]

There exists a martingale difference sequence \( (\tilde{d}_n)_{n=1}^\infty \) bounded away from zero in probability such that \( \left( \sum_{k=1}^{n} a_k \tilde{d}_k \right)_{n=1}^\infty \) is uniformly bounded in \( L_\infty(X) \). In particular, \( (\tilde{d}_n)_{n=1}^\infty \) is a monotone basic sequence in \( L_2(X) \) with

\[
\inf_{n \geq 1} \| \tilde{d}_n \|_2 > 0 \quad \text{and} \quad \sup_{n \geq 1} \left\| \sum_{k=1}^{n} a_k \tilde{d}_k \right\|_2 < \infty.
\]

\( L_2(X) \) is itself \( q \)-convex (see [8]) and a monotone basic sequence in a \( q \)-convex space satisfies a lower \( q \)-estimate [17], and so \( \sum_{n=1}^{\infty} |a_n|^q < \infty \). \( \square \)

**Corollary 2.3.** Let \( X \) be a \( q \)-convex Banach space and let \( (d_n)_{n=1}^\infty \) be a dyadic martingale difference sequence in \( L_0(X) \) which is bounded away from zero in probability. Then \( \sum_{n=1}^{\infty} |a_n|^q < \infty \) whenever \( \sum_{n=1}^{\infty} a_n d_n \) converges almost surely.

**3. Subsequence principles for square-integrable random variables.** The following result is an immediate consequence of a summability theorem of Chow [4, Theorem 3].

**Theorem A.** Let \( (d_n)_{n=1}^\infty \) be a martingale difference sequence which is normalized in \( L_2(\Omega) \) and satisfies \( E|d_n| \geq c \) for some \( c > 0 \) and for all \( n \geq 1 \). Then there exists \( \varepsilon > 0 \) such that

\[
\sup_{n \geq 1} P \left\{ \left| \sum_{k=1}^{n} a_k d_k \right| > K \right\} \geq \varepsilon \quad \text{for all} \quad K > 0
\]

whenever \( \sum_{n=1}^{\infty} a_n^2 = \infty \). In particular,

\[
P \left\{ \sup_{n \geq 1} \left| \sum_{k=1}^{n} a_k d_k \right| = \infty \right\} \geq \varepsilon \quad \text{whenever} \quad \sum_{n=1}^{\infty} a_n^2 = \infty.
\]

We use Theorem A to deduce the following subsequence principle for almost sure convergence of square-integrable random variables.

**Theorem 3.1.** Let \( (f_n)_{n=1}^{\infty} \) be a normalized sequence in \( L_2(\Omega) \) having no subsequence convergent in \( L_1(\Omega) \). Then there exists \( f \in L_2(\Omega) \), \( \varepsilon > 0 \), and a subsequence \( (f_{n_k})_{k=1}^{\infty} \) with the following properties:

(i) \( \sum_{k=1}^{\infty} a_k (f_{n_k} - f) \) converges almost surely and in \( L_2(\Omega) \) whenever \( \sum_{k=1}^{\infty} a_k^2 < \infty \);

(ii) \( P(\sup_{n \geq 1} |\sum_{k=1}^{n} a_k (f_{n_k} - f)| = \infty) > \varepsilon \) whenever \( \sum_{n=1}^{\infty} a_n^2 = \infty \).

**Proof.** Bounded subsets of \( L_2(\Omega) \) are weakly sequentially compact, and so there exist \( f \in L_2(\Omega) \) and a subsequence \( (f_{n_k})_{k=1}^{\infty} \) such that \( (f_{n_k} - f)_{k=1}^{\infty} \) is weakly null. Since \( (f_n)_{n=1}^{\infty} \) has no subsequence convergent in \( L_1(\Omega) \) we may assume that \( \|f_{n_k} - f\|_1 > 2c \) for some \( c > 0 \) and for all \( k \geq 1 \). By a well-known argument (e.g., [3, p. 243]) we may also assume by passing to a further subsequence that
there exists a martingale difference sequence \((g_k)\) of simple functions such that 
\[
\sum_{k=1}^{\infty} \|f_{n_k} - f - g_k\|_2 < c.
\]
By Theorem A there exists \(\tilde{\varepsilon} > 0\) such that 
\[
\sum_{k=1}^{\infty} a_k^2 < \infty \quad \text{whenever} \quad P \left( \sup_{n \geq 1} \left| \sum_{k=1}^{n} a_k g_k \right| = \infty \right) < \tilde{\varepsilon}.
\]

By Hölder’s inequality \((f_{n_k} - f)_{k=1}^{\infty}\) is uniformly integrable, and so there exists \(\varepsilon > 0\) such that 
\[
\int |f_{n_k} - f| I(A) \, dP < c \quad \text{for all} \quad k \geq 1 \quad \text{whenever} \quad P(A) < \varepsilon; \quad \text{moreover,}
\]
we may assume that \(\varepsilon \leq \tilde{\varepsilon}\). Suppose now that \((a_k)_{k=1}^{\infty}\) is a real sequence and that 
\[
P \left( \sup_{m \geq 1} \left| \sum_{k=1}^{m} a_k (f_{n_k} - f) \right| = \infty \right) < \varepsilon.
\]

Then there exists \(M > 0\) such that \(P(A) > 1 - \varepsilon\), where 
\[
A = \left\{ \sup_{m \geq 1} \left| \sum_{k=1}^{m} a_k (f_{n_k} - f) \right| \leq M \right\}.
\]

So 
\[
2MP(A) \geq \int |a_k (f_{n_k} - f)| I(A) \, dP \geq c|a_k|.
\]

Hence \(\sup_{k \geq 1} |a_k| < \infty\), and it follows that \(\sum_{k=1}^{\infty} a_k (f_{n_k} - f - g_k)\) converges absolutely almost surely. So 
\[
P \left( \sup_{n \geq 1} \left| \sum_{k=1}^{n} a_k g_k \right| = \infty \right) < \varepsilon \leq \tilde{\varepsilon},
\]
whence \(\sum_{n=1}^{\infty} a_n^2 < \infty\). This completes the proof of (ii).

Property (i) is a well-known theorem of Revesz (see [18]) and follows easily from the martingale convergence theorem (see [3]). Indeed, \(\sum_{n=1}^{\infty} a_n g_n\) converges almost surely and in \(L_2(\Omega)\) whenever \(\sum_{n=1}^{\infty} a_n^2 < \infty\), and so the same is true of \(\sum_{k=1}^{\infty} a_k (f_{n_k} - f)\).

**Remark.** The hypothesis that \((f_n)_{n=1}^{\infty}\) contains no subsequence convergent in \(L_1\) is used only in the proof of property (ii). Revesz proved in [19] that something like property (ii) could be made to work for the case in which \((f_n)_{n=1}^{\infty}\) is a uniformly bounded sequence in \(L_\infty(\Omega)\).

Combining Theorem 3.1 with the proof of “(i) implies (ii)” in Theorem 2.2 gives the following result.

**Theorem 3.2.** Let \((f_n)_{n=1}^{\infty}\) be weakly null in \(L_2(\Omega)\). Then the following are equivalent:

(i) \((f_n)_{n=1}^{\infty}\) contains a subsequence which is bounded away from zero in probability;

(ii) there exists \(\varepsilon > 0\) and a subsequence \((f_{n_k})_{k=1}^{\infty}\) with the following properties:

(a) \(\sum_{k=1}^{\infty} a_k f_{n_k}\) converges almost surely and in \(L_2(\Omega)\) whenever \(\sum_{k=1}^{\infty} a_k^2 < \infty\);

(b) \(P\{ \sup_{m \geq 1} | \sum_{k=1}^{m} a_k f_{n_k} | = \infty \} > \varepsilon \) whenever \(\sum_{k=1}^{\infty} a_k^2 = \infty\).

**Remark.** The last result applies, in particular, to an arbitrary orthonormal system in \(L_2(\Omega)\). In this setting (ii) corresponds to the fact that every orthonormal system \((\phi_n)_{n=1}^{\infty}\) contains a subsystem \((\phi_{n_k})_{k=1}^{\infty}\) that is a system of convergence (meaning \(\sum_{k=1}^{\infty} a_k \phi_{n_k}\) converges whenever \(\sum_{k=1}^{\infty} a_k^2 < \infty\) (see [2, p. 156])).
Whereas (ii) resembles the fact that a lacunary trigonometric series
\[ \sum_{k=1}^{\infty} (a_{n_k} \cos n_k t + b_{n_k} \sin n_k t) \]
diverges almost everywhere when \( \sum_{k=1}^{\infty} (a_{n_k}^2 + b_{n_k}^2) = \infty \) [22, p. 203]. (An increasing sequence of positive integers \((n_k)_{k=1}^{\infty}\) is lacunary if \( n_{k+1}/n_k > t \) for some \( t > 1 \) and for all \( k \).)

To conclude this part we prove an abstract version of a related theorem of Zygmund on lacunary Fourier coefficients [23, p. 132]. Let \((\phi_n)_{n=1}^{\infty}\) be a uniformly bounded orthonormal system in \( L_2(0,1) \). For \( f \in L_1(0,1) \), let \( \hat{f}(n) = \int f \phi_n dt \) for all \( n \geq 1 \).

**Theorem 3.3.** Let \((\phi_n)_{n=1}^{\infty}\) be a uniformly bounded orthonormal system. Every sequence of positive integers \((n_k)_{k=1}^{\infty}\) contains a subsequence \((n_{k_m})_{m=1}^{\infty}\) such that \( \sum_{m=1}^{\infty} \hat{f}(n_{k_m})^2 < \infty \) whenever \( f \in L_p(0,1) \) for some \( p > 1 \). Moreover, there exists \( f \in L_1(0,1) \) such that \( \sum_{k=1}^{\infty} \hat{f}(n_k)^2 = \infty \).

**Proof.** By [11, Corollary 6] and a diagonal argument the subsequence \((n_{k_m})_{m=1}^{\infty}\) may be chosen so that \((\phi_{n_{k_m}})_{m=1}^{\infty}\) is equivalent to the unit vector basis of \( l_2 \) in \( L_p \) for all \( p > 2 \); that is, there exists \( C_p > 0 \) such that for all \( m \geq 1 \) and for all scalars \( a_1, \ldots, a_m \), we have
\[
\frac{1}{C_p} \left( \sum_{k=1}^{m} a_k^2 \right)^{1/2} \leq \left\| \sum_{k=1}^{m} a_k \phi_{n_{k_m}} \right\|_p \leq C_p \left( \sum_{k=1}^{m} a_k^2 \right)^{1/2}.
\]

By the results of [11] the same is true for all \( p > 0 \). Let \( P(f) = \sum_{k=1}^{\infty} \hat{f}(n_{k_m}) \phi_{n_{k_m}} \). Then \( P \) is an orthogonal projection on \( L_2 \) and so is bounded. Since the \( L_2 \) and \( L_p \) norms are equivalent on the closed linear span of \((\phi_{n_{k_m}})_{m=1}^{\infty}\) it follows that \( P \) is bounded on \( L_p(0,1) \) for all \( p > 2 \). For \( 1 < p < 2 \), \( P: L_p(0,1) \rightarrow L_p(0,1) \) is bounded because it is the adjoint of \( P: L_q(0,1) \rightarrow L_q(0,1) \), where \( 1/p + 1/q = 1 \), which is bounded. This proves the first part of the proposition.

To show the last part, suppose on the contrary that \( \sum_{k=1}^{\infty} \hat{f}(n_k)^2 < \infty \) for all \( f \in L_1 \), and so, in particular, that \( \sum_{k=1}^{\infty} \hat{f}(n_k')^2 < \infty \). But then by the Banach-Steinhaus theorem \( P \) is a bounded projection on \( L_1(0,1) \) whose range is \( \{ \varphi_{n_k'} \}_{k=1}^{\infty} \), which is impossible because \( L_1(0,1) \) contains no complemented subspace isomorphic to a Hilbert space. \( \square \)

**Remark.** A Rademacher-like property of subsequences of random variables in \( L_p \) is also proved in [15, Lemma 2.1].

4. **Convergence in \( L_1(X) \).** The following is a vectorial generalization of [1, §4]. Since the proof is essentially the same it has been omitted.

**Proposition 4.1.** Let \((d_n)_{n=1}^{\infty}\) be a uniformly integrable martingale difference sequence normalized in \( L_1(X) \). Suppose that \((a_n)_{n=1}^{\infty}\) is a real sequence such that \( \sum_{n=1}^{\infty} a_n d_n \) converges in \( L_1(X) \). Then there exists a martingale difference sequence \((d_n)_{n=1}^{\infty}\), bounded away from zero in \( L_1(X) \), such that \( (\sum_{k=1}^{n} a_k d_k)_{n=1}^{\infty} \) is uniformly bounded in \( L_\infty(X) \).
THEOREM 4.2. Suppose that $X$ is a $q$-convex Banach space and that $(d_n)_{n=1}^{\infty}$ is a uniformly integrable martingale difference sequence normalized in $L_1(X)$. Then $(d_n)_{n=1}^{\infty}$ satisfies a lower $q$-estimate.

PROOF. This follows from Proposition 4.1 together with the proof of Theorem 2.2. □

THEOREM 4.3. Suppose that $X$ is a $q$-convex Banach space and that $(d_n)_{n=1}^{\infty}$ is a martingale difference sequence normalized in $L_1(X)$. There exists a subsequence $(d_{n_k})_{k=1}^{\infty}$ which satisfies a lower $q$-estimate.

PROOF. If $(d_n)_{n=1}^{\infty}$ is uniformly integrable then $(d_n)_{n=1}^{\infty}$ satisfies a lower $q$-estimate. Otherwise, by the results of [20], one can extract a subsequence $(d_{n_k})_{k=1}^{\infty}$ equivalent to the unit vector basis of the sequence space $l_1$. Then $(d_{n_k})_{k=1}^{\infty}$ satisfies a lower 1-estimate, and a fortiori a lower $q$-estimate. □

It is possible that Theorem 4.3 remains valid for arbitrary sequences in $L_1(X)$ which are not relatively compact, and this is the case in $L_p(X)$ for $1 < p \leq q$. For if $X$ is $q$-convex then $L_p(X)$ is $q$-convex for $1 < p \leq q$, and so it follows from the next proposition that any normalized sequence $(f_n)_{n=1}^{\infty}$ in $L_p(X)$ which is not relatively compact contains a subsequence satisfying a lower $q$-estimate.

PROPOSITION 4.4. Suppose that $X$ is a $q$-convex Banach space and that $(x_n)_{n=1}^{\infty}$ is a normalized sequence in $X$ which is not relatively compact. Then $(x_n)_{n=1}^{\infty}$ contains a subsequence satisfying a lower $q$-estimate.

PROOF. By passing to a subsequence we may assume that $(x_n)_{n=1}^{\infty}$ contains no norm convergent subsequence. Since $X$ is reflexive its bounded subsets are relatively weakly sequentially compact, and so we may further assume that there exists $x$ in $X$ such that $(x_n - x)_{n=1}^{\infty}$ is weakly null (and bounded away from zero). By [6, Proposition 2.4] $(x_n - x)_{n=1}^{\infty}$ contains a subsequence $(x_{n_k} - x)_{k=1}^{\infty}$ satisfying a lower $q$-estimate, and by [1, §2] there exists $m \geq 1$ such that $(x_{n_k})_{k=m}^{\infty}$ satisfies a lower $q$-estimate. □

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS AT AUSTIN, AUSTIN, TEXAS 78712