TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 301, Number 2, June 1987

INFINITELY MANY TRAVELING WAVE SOLUTIONS OF A GRADIENT SYSTEM

DAVID TERMAN

ABSTRACT. We consider a system of equations of the form $u_t = u_{xx} + \nabla F(u)$. A traveling wave solution of this system is one of the form $u(x,t) = U(z)$, $z = x + \theta t$. Sufficient conditions on $F(u)$ are given to guarantee the existence of infinitely many traveling wave solutions.

1. Introduction.

A. *Statement of the problem.* Consider the system of reaction-diffusion equations

$$u_{1t} = u_{1xx} + f_1(u_1, u_2), \quad u_{2t} = u_{2xx} + f_2(u_1, u_2) \tag{1A.1}$$

where u_1 and u_2 are functions of $(x,t) \in \mathbf{R} \times \mathbf{R}^+$. We assume that f_1 and f_2 are derived from some potential. That is, there exists a function $F \in C^2(\mathbf{R}^2)$ such that

$$f_i(u_1, u_2) = \frac{\partial F}{\partial u_i}(u_1, u_2), \qquad i = 1, 2, \tag{1A.2}$$

for each $(u_1, u_2) \in \mathbf{R}^2$. By a traveling wave solution of (1A.1) we mean a nonconstant, bounded solution of the form

$$\big(u_1(x,t), u_2(x,t)\big) = \big(U_1(z), U_2(z)\big), \qquad z = x + \theta t. \tag{1A.3}$$

A traveling wave solution corresponds to a solution which appears to be traveling with constant shape and velocity. Our goal is to prove that for a certain class of potentials, there exists infinitely many traveling wave solutions of (1A.1).

We shall assume that the graph of $F(U)$ is as shown in Figure 1. Precise assumptions of F will be given shortly. For now we assume that F has at least three local maxima. These are at $(U_1, U_2) = A, B$ and C with $F(A) < F(B) < F(C)$. We will be interested in traveling wave solutions which satisfy

$$\lim_{z \to -\infty} \big(U_1(z), U_2(z)\big) = A \quad \text{and} \quad \lim_{z \to -\infty} \big(U_1(z), U_2(z)\big) = B. \tag{1A.4}$$

Motivation for studying this problem will be given shortly.

Received by the editors January 16, 1986 and, in revised form, May 15, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 35B35, 35K55; Secondary 35B40.
Supported in part by the National Science Foundation under grant #DMS 8401719.

0002-9947/87 $1.00 + $.25 per page

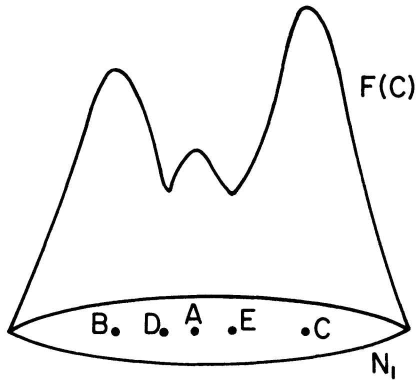

FIGURE 1

If $(U_1(z), U_2(z))$ is a traveling wave solution of (1A.1) and $V_i(z) = U_i'(z)$, $i = 1, 2$, then (U_1, U_2, V_1, V_2) satisfies the first order system of ordinary differential equations

$$\begin{aligned} U_1' &= V_1, \quad V_1' = \theta V_1 - F_{u_1}(U_1, U_2), \\ U_2' &= V_2, \quad V_2' = \theta V_2 - F_{u_2}(U, U_2) \end{aligned} \tag{1A.5}$$

together with the boundary conditions

$$\begin{aligned} &\text{(a)} \quad \lim_{z \to -\infty} (U_1(z), U_2(z), V_1(z), V_2(z)) = (A, \mathcal{O}), \\ &\text{(b)} \quad \lim_{z \to +\infty} (U_1(z), U_2(z), V_1(z), V_2(z)) = (B, \mathcal{O}). \end{aligned} \tag{1A.6}$$

Here we have set $\mathcal{O} = (0, 0)$. For convenience we let $U = (U_1, U_2)$ and $V = (V_1, V_2)$.

Our goal is to prove that for certain assumptions on F there exist infinitely many θ for which there exists a solution of (1A.5), (1A.6).

B. *Assumptions on F*. We assume that the graph of $F(U)$ is as shown in Figure 1. The precise assumptions on F are as follows:

(F1) $F \in C^2(\mathbf{R}^2)$.

(F2) F has at least three nondegenerate local maxima. These are at $A = (A_1, A_2)$, $B = (B_1, B_2)$, and $C = (C_1, C_2)$. F has at least two saddles at $D = (D_1, D_2)$ and $E = (E_1, E_2)$.

(F3) $F(A) < F(B) < F(C)$ and $B_1 < D_1 < A_1 < E_1 < C_1$. Moreover there exists an α_0 such that if α is any critical point of F with $\alpha \notin \{A, B, C\}$, then $F(\alpha) < F(A) - \alpha_0$.

(F4) There exists W such that if $K < W$, then $\{U : F(U) \geqslant K\}$ is convex.

(F5) If $U_1 = D_1$ or E_1, then $(\partial F/\partial U_1)(U_1, U_2) = 0$ for all $U_2 \in \mathbf{R}$.

(F6) Let

$$\begin{aligned} N_1 &= \{U : F(U) \geqslant W\}, \\ X_1 &= \{U \in N_1 : U_1 < D_1\}, \\ X_2 &= \{U \in N_1 : D_1 < U_1 < E_1\}, \\ X_3 &= \{U \in N_1 : E_1 < U_1\}. \end{aligned} \tag{1B.1}$$

Suppose that $(U(z), V(z))$ is a bounded solution of (1A.5) with $\theta = 0$ which satisfies the following for $i = 1, 2$, or 3,

(a) $U(z) \in X_i$ for all $z \in \mathbf{R}$,

(b) $F(U(z)) > F(A) - \alpha_0$ for some $z \in \mathbf{R}$,

where α_0 was defined in (F3). Then $U(z)$ is identically equal to one of the critical points A, B, or C, and $V(z) = \mathcal{O}$ for all $z \in \mathbf{R}$.

Remarks concerning these assumptions are certainly in order. These remarks are given in §1E. First, we state our main result and then try to motivate the results and assumptions on F by briefly discussing the scalar equation $u_t = u_{xx} + g(u)$.

C. *The main result.*

THEOREM 1. *Assume that* f_1 *and* f_2 *satisfy* (1A.2) *and* $F(U_1, U_2)$ *satisfies* (F1) *to* (F6). *Then there exists infinitely many traveling wave solutions of* (1A.1) *which satisfy* (1A.4).

REMARK. By infinitely many traveling waves we mean that there exist infinitely many values of θ for which there exists a solution of (1A.5), (1A.6).

D. *The scalar equation.* This paper is a generalization of previous work on the scalar equation

$$u_t = u_{xx} + g(u). \tag{1D.1}$$

In [3], a rather complete description of which waves exist is given for general nonlinearities $g(u)$. In [3], we used graph theory to prove the existence of traveling waves. In this paper we demonstrate what takes the place of the directed graphs and arrays of integers, which were used in [3], for higher dimensional gradient systems.

E. *Remarks on the assumptions on F.*

REMARK 1. (F4) will be used to prove that the set of bounded solutions of (1A.5) is compact.

REMARK 2. (F6) guarantees that the set of bounded solutions of (1A.5) is not too bizarre. One may think of (1A.5) with $\theta = 0$ as describing the motion of a ball rolling along the landscape defined by the graph of F. There may exist bounded solutions of (1A.5) with $\theta = 0$, because the ball may roll back and forth between the mountain peaks given by $F(A)$, $F(B)$, and $F(C)$. Assumption (F6) implies that these are the only bounded solutions, besides the critical points, which lie above $F(A) - \alpha_0$. This condition will be satisfied if the level sets of $F(U)$ are sufficiently nice (some starlike condition, for example). It is not clear whether or not this condition is necessary to guarantee the existence of infinitely many traveling wave solutions.

REMARK 3. (F5) is perhaps the most unreasonable assumption. It can be weakened slightly as follows. Let

$$l_D = \{U \in N_1 : U_1 = D_1\}, \quad l_E = \{U \in N_1 : U_1 = E_1\}. \tag{1E.1}$$

Then (F5) implies that if $U \in l_D \cap N_1$ or $U \in l_E \cap N_1$, then $\nabla F(U)$ is tangent to l_D or l_E, respectively. Our result remains true if this property holds for some lines l_D and l_E through D and E, not necessarily the ones given in (1E.1). We choose l_D and l_E as in (1E.1) only for convenience. We do feel that our method of proof should carry over to a more general assumption than (F5). In [**4**, Appendix 5] we describe how one should be able to weaken (F5).

F. *Description of the proof*. The proof of Theorem 1 is quite geometrical. The purpose of this subsection is to introduce the basic geometrical features of the proof. Note that each solution of (1A.5) corresponds to a trajectory in four-dimensional phase space. The boundary conditions (1A.6) imply that we are looking for a value of θ for which there exists a trajectory in phase space which approaches the equilibrium $(A, \mathcal{O})$ as $z \to -\infty$ and the equilibrium $(B, \mathcal{O})$ as $z \to +\infty$. Hence, we are looking for a trajectory which lies in both W_A, the unstable manifold at $(A, \mathcal{O})$, and W_B^s, the stable manifold at $(B, \mathcal{O})$.

The first step in the proof of Theorem 1 is to obtain apriori bounds on the bounded solutions of (1A.5). This is done in §2. We prove that there exists a T such that no solutions of (1A.5), (1A.6) exist for $\theta > T$. We also construct a four-dimensional box, N, which contains all the bounded solutions of (1A.5).

We then construct a subset $\mathcal{E}$ of the boundary of N with the property that each nontrivial trajectory in W_A, for $\theta < T$, can only leave N through $\mathcal{E}$. The most interesting feature of $\mathcal{E}$ is that it has four topological holes.

Now choose θ_0 so that no solutions of (1A.5), (1A.6) exist for $\theta = \theta_0$. We prove that each nontrivial trajectory in W_A must leave N. Because the dimension of W_A is two, the places where W_A leaves N define a curve, $\Lambda(\theta_0)$, in $\mathcal{E}$. We define an algebraic object, $\Gamma(\theta_0)$, which describes how $\Lambda(\theta_0)$ winds around the four holes in $\mathcal{E}$. $\Gamma(\theta_0)$ will be an element of F_4, the free group on four elements. The definition of $\Gamma(\theta_0)$ is given in §4B. In §4B we set things up a bit more generally than described here in anticipation of future papers in which we characterize the solutions of (1A.5), (1A.6) by how many times they wind around in phase space. This notion of winding number will play an important role in this paper, as we describe shortly. The winding number is defined precisely in §3B.

The algebraic invariant, $\Gamma(\theta_0)$, will have the following important property:

PROPOSITION A (SEE PROPOSITION 4B.2). *Suppose that* $\theta_0 < \theta_1$ *are chosen so that there exist no solutions of* (1A.5), (1A.6) *for* $\theta \in [\theta_0, \theta_1]$. *Then* $\Gamma(\theta_0) = \Gamma(\theta_1)$.

The next step in the proof of Theorem 1 is to assign to each element $\Gamma \in F_4$ a positive integer $|\Gamma|$. We prove

PROPOSITION B (SEE PROPOSITION 5A.1). *Let M be a positive integer. There exists* θ_M *such that if* $\theta_0 < \theta_M$ *and no solutions of* (1A.5), (1A.6) *exist for* $\theta = \theta_0$, *then* $|\Gamma(\theta_0)| > M$.

Theorem 1 then follows from Proposition A and Proposition B.

The key steps in the proof of Proposition B are Proposition 3C.1 and Proposition 4C.1. In Proposition 4C.1 we give a relationship between $|\Gamma(\theta_0)|$ and the winding number of trajectories in W_A for $\theta = \theta_0$. In Proposition 3C.1 we prove that if θ_0 is very small then there must exist a trajectory in W_A, for $\theta = \theta_0$, with large winding number.

2. The isolating neighborhood.

A. *Basic definitions and a preliminary result.* Recall the set N_1 defined in (1B.1). Let

$$N_2 = \{(U,V): U \in N_1,\ \|V\| \leqslant \bar{V}\} \tag{2A.1}$$

where $\bar{V}$ is a large number to be determined. Let

$$P_D = \{(U,V): U_1 = D_1 \text{ and } V_1 = 0\},$$
$$P_E = \{(U,V): U_1 = E_1 \text{ and } V_1 = 0\},$$
$$N = N_2 \backslash (P_D \cup P_E).$$

REMARK. N is topologically a four-dimensional box with two holes, P_D and P_E, removed. This is topologically equivalent to a two-dimensional disc with two points removed.

We first prove

LEMMA 2A.1. *For all θ, P_D and P_E are invariant with respect to the flow defined by* (1A.5).

PROOF. We wish to show that if $(U(z_0), V(z_0)) \in P_D$ or P_E for some z_0, then $(U(z), V(z)) \in P_D$ or P_E for all z. However, from (1A.5) and (F5) we conclude that on P_D or P_E,

$$U_1' = V_1 = 0$$

and

$$V_1' = \theta V_1 - F_{u_1}(u_1, u_2) = 0.$$

These two equalities prove the lemma. An immediate consequence of the lemma is

COROLLARY 2A.2. *For each θ, the unstable manifold at $(A, \mathcal{O})$ does not intersect P_D or P_E.*

We wish to prove that all bounded solutions of (1A.4) lie in N_2 if $\bar{V}$ is sufficiently large. Together with the corollary, this will imply that solutions of (1A.5), (1A.6) must lie in N. We begin with

LEMMA 2A.3. *The projection onto U-space of every bounded solution of* (1A.5) *lies in N_1. Moreover, there cannot exist a solution of* (1A.5) *whose projection onto U-space is internally tangent to ∂N_1, the boundary of N_1.*

PROOF. The proof follows that given in Conley [**1**]. Choose $K \leqslant W$ and suppose that $(U(z), V(z))$ is a solution of (1A.5) which satisfies for some z_0, $F(U(z_0)) = K$, and

$$\left.\frac{\partial F}{\partial z}\right|_{z_0} = \langle \nabla F(U(z_0)), V(z_0)\rangle = 0.$$

Then

$$\left.\frac{d^2F}{dz^2}\right|_{z_0} = d^2F(U) + \theta\langle\nabla F(U(z_0)), V(z_0)\rangle - \langle\nabla F(U(z_0)), \nabla F(U(z_0))\rangle < 0 \tag{2A.2}$$

since the assumption that the level set $\{F(V) \geqslant K\}$ is convex implies that

$$d^2F(\xi) < 0 \quad \text{if } \xi \neq 0 \text{ and } \langle\nabla F, \xi\rangle = 0.$$

This implies that there cannot exist any internal tangencies on the level set $\{F(U) = K\}$ for all $K \leqslant W$.

On any solution which leaves the set where $F(U) \geqslant W$ there is a point where $F < W$ and either $dF/dz < 0$ or $dF/dz > 0$. Suppose that $F(U(z_0)) < W$ and $dF/dz|_{z_0} < 0$. Then (2A.2) implies that $F(U(z))$ is strictly decreasing for $z \geqslant z_0$. Therefore, if the solution is bounded, it would have to go to a rest point in the set where $F < W$. Since there are no rest points, the solution must be unbounded in forward time. A similar argument shows that if $F(U(z_0)) < W$ and $dF/dz|_{z_0} > 0$, then the solution is unbounded in backward time.

REMARK. The proof of this last result shows that if $U(z)$ leaves N_1 in forward or backward time, then it can never return to N_1.

B. *The energy H*. Consider the function $H(U, V) = \frac{1}{2}\langle V, V\rangle^2 + F(U)$ where $\langle V, V\rangle$ is the usual inner product in $\mathbf{R}^2$. If $(U(z), V(z))$ is a solution of (1A.5) we sometimes write $H(z) = H(U(z), V(z))$. An important fact is that on solutions of (1A.5),

$$H'(z) = \theta\langle V(z), V(z)\rangle^2. \tag{2B.1}$$

Therefore, if $\theta \neq 0$, then

$$H'(z) > 0 \tag{2B.2}$$

and $H(z)$ is increasing on solutions of (1A.5). An immediate consequence of this is

PROPOSITION 2B.1. *The only bounded solutions of* (1A.5) *with* $\theta > 0$ *are critical points or trajectories which connect critical points.*

Note that if $(U(z), V(z))$ is a solution of (1A.5), (1A.6), then

$$\lim_{z \to -\infty} H(z) = F(A) \quad \text{and} \quad \lim_{z \to +\infty} H(z) = F(B). \tag{2B.3}$$

C. *A bound on* θ. In this section we prove

LEMMA 2C.1. *There exists T such that no solutions of* (1A.5), (1A.6) *exist with* $\theta \geqslant T$.

PROOF. For $\lambda > 0$, let

$$S_\lambda = \{(U, V) : |V_1| \geqslant \lambda|U_1 - A_1| \text{ and } |V_2| \geqslant \lambda|U_2 - A_2|\}.$$

We prove that given λ there exists T_λ such that if $\theta \geqslant T_\lambda$, W_{A^θ} is the unstable manifold of (1A.5) at $(A, \mathcal{O})$, and $(U(z), V(z)) \in W_{A^\theta}$, then $(U(z), V(z)) \in S_\lambda$. Because every solution of (1A.5), (1A.6) must lie in W_{A^θ}, and $(B, \mathcal{O}) \notin S_\lambda$, for any $\lambda > 0$, this will imply the desired result.

We first prove that there exists T_λ such that if $\theta \geqslant T_\lambda$, then S_λ is positively invariant for the flow (1A.5). To prove this we show that on the boundary of S_λ, the vector field given by the right side of (1A.5) points into S_λ.

There are many cases to consider. Suppose, for example, that $V_1 = \lambda(U_1 - A_1)$, $U_1 > A_1$, and $|V_2| \geqslant \lambda|U_2 - A_2|$. Let $n = (\lambda, -1)$ be a vector outwardly normal to $\{(U_1, V_1): V_1 = \lambda(U_1 - A_1)\}$. Then

$$\begin{aligned} n \cdot (U_1', V_1') &= \lambda V_1 - \theta V_1 + F_{u_1}(U) \\ &= (\lambda - \theta)(U_1 - A_1) + F_{u_1}(U) < 0 \end{aligned}$$

for θ sufficiently large. A similar proof works in the other cases.

To complete the proof of the lemma we show that T_λ can be chosen so that if $\theta \geqslant T_\lambda$ and $(U(z), V(z)) \in W_{A^\theta}$, then there exists z_0 such that $(U(z), V(z)) \in S_\lambda$ for $z < z_0$. This is proved by linearizing (1A.5) at $(A, \mathcal{O})$ and showing that the positive eigenvectors point into S_λ.

If we set $A' = (A, \mathcal{O})$ and

$$\left.\frac{\partial^2 F}{\partial u_1^2}\right|_{A'} = a, \qquad \left.\frac{\partial^2 F}{\partial u_1 \partial u_2}\right|_{A'} = b, \qquad \left.\frac{\partial^2 F}{\partial u_2^2}\right|_{A'} = c,$$

then the linear system at A' is

$$\text{(2C.1)} \qquad \begin{bmatrix} U_1 \\ U_2 \\ V_1 \\ V_2 \end{bmatrix}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a & -b & \theta & 0 \\ -b & -c & 0 & \theta \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ V_1 \\ V_2 \end{bmatrix}.$$

To compute the eigenvalues and eigenvectors of this system let

$$M = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

be the Hessian matrix of F at A. Since $F(U)$ obtains a local maximum at A it follows that M has negative eigenvalues, which we denote by $-\lambda_1$ and $-\lambda_2$. The eigenvalues of (2C.1) are then σ_1^+, σ_1^-, σ_2^+, σ_2^- where for $i = 1, 2$, σ_i^+ and σ_i^- are roots of the polynomial $\sigma^2 - \sigma\theta - \lambda_i = 0$. Since this polynomial has one positive and one negative root we may assume that

$$\text{(2C.2)} \qquad \sigma_1^- < 0 < \sigma_1^+ \quad \text{and} \quad \sigma_2^- < 0 < \sigma_2^+.$$

Let W_1 and W_2 be eigenvectors of M corresponding to $-\lambda_1$ and $-\lambda_2$, respectively. Then eigenvectors corresponding to $\sigma_1^\pm$ and $\sigma_2^\pm$ are then

$$\text{(2C.3)} \qquad p_i^\pm = (w_i, \sigma_i^\pm w_i), \qquad i = 1, 2.$$

Because $\sigma_i^+ \to \infty$, $i = 1, 2$, as $\theta \to \infty$ it follows that p_1^+ and p_2^+ point into S_λ if θ is sufficiently large. Because W_{A^θ} is tangent to the linear space spanned by p_1^+ and p_2^+ this implies the desired result.

REMARK. Our proof proved more than Lemma 2C.1. We also proved

COROLLARY 2C.2. *Fix* $\delta > 0$ *and let* $A_\delta = \{U: \|U - A\| = \delta\}$. *Given* $\lambda > 0$ *there exists* M *such that if* $\theta > M$, $(U(z), V(z)) \in W_{A^\theta}$, $U(z_0) \in A_\delta$, *and* $\|U(z) - A\| < \delta$ *for* $z < z_0$, *then* $\|V(z_0)\| > \lambda$.

PROOF. Let $\lambda_1 = 2\lambda/\delta$ and $M = T_{\lambda_1}$, as in the previous lemma. If $\theta > M$, $(U(z), V(z)) \in W_{A^\theta}$, and $\|U(z) - A\| = \delta$. Then either $|U_1 - A_1| > \delta/2$ or $|U_2 - A_2| > \delta/2$. From the previous lemma, either $\|V_1\| \geqslant \lambda_1\delta/2 = \lambda$ or $\|V_2\| \geqslant \lambda_1\delta/2 = \lambda$.

D. *A bound on* V. Recall that in the definition of N we needed a constant $\bar{V}$ which has not been defined yet. $\bar{V}$ will serve as an a priori bound on $\|V\|$ of solutions of (1A.5), (1A.6). That such a bound exists follows from

PROPOSITION 2D.1. *Assume that* $\theta < T$. $\bar{V}$ *can be chosen so that if* $U(z_0) \in N_1$ *and* $\|V(z_0)\| \geqslant \bar{V}$, *then* $U(z)$ *leaves* N_1 *in backward time.*

PROOF. Suppose that $\|V(z_0)\| > \bar{V}$ where $\bar{V}$ is to be determined. Then either

$$V_1(z_0) > \tfrac{1}{2}\bar{V}, \qquad V_1(z_0) < -\tfrac{1}{2}\bar{V}, \qquad V_2(z_0) > \tfrac{1}{2}\bar{V}, \quad \text{or} \quad V_2(z_0) < -\tfrac{1}{2}\bar{V}. \tag{2D.1}$$

We suppose that $V_1(z_0) > \frac{1}{2}\bar{V}$, and for convenience $z_0 = 0$. Choose M_1 so that $\|\nabla F(V)\| < M$ in N_1. Then, from (1A.5),

$$V_1' = \theta V_1 - F_{u_1}(U) \leqslant TV_1 + M_1$$

as long as $V_1 \geqslant 0$. Therefore, if $V_1 \geqslant 0$,

$$\left[e^{-Tz}V_1\right]' \leqslant e^{-Tz}M_1.$$

Integrate this equation for $-z_1 \leqslant z \leqslant 0$ to obtain

$$V_1(-z_1) \geqslant e^{-Tz_1}V_1(0) - \frac{1}{T}M_1(1 - e^{-Tz_1}) \geqslant \frac{1}{2}e^{-T}\bar{V} - \frac{1}{T}M_1$$

as long as $0 \leqslant z_1 \leqslant 1$ and $V_1(z) \geqslant 0$ for $-1 \leqslant z \leqslant 0$. This last statement is true if $\frac{1}{2}e^{-T}\bar{V} - (1/T)M_1 > 0$ or $\bar{V} > (2/T)M_1e^{+T}$, which we assume to be true. Therefore

$$U_1'(z) = V_1(z) \geqslant \frac{1}{2}e^{-T}\bar{V} - \frac{1}{T}M_1$$

for $-1 \leqslant z \leqslant 0$. This implies that

$$U_1(-1) \leqslant U_1(0) - \left(\frac{1}{2}e^{-T}\bar{V} - \frac{1}{T}M_1\right). \tag{2D.2}$$

Let M_2 = diameter of N_1 and choose $\bar{V}$ so that

$$\frac{1}{2}e^{-T}\bar{V} - \frac{1}{T}M_1 > M_2. \tag{2D.3}$$

Then (2D.2) and (2D.3) imply that $U_1(-1) \notin N_1$. Similar arguments hold for the other cases in (2D.1).

An immediate consequence of this result, Corollary 2A.2, and Lemma 2A.3 is

COROLLARY 2D.2. *If* $(U(z), V(z))$ *is a solution of* (1A.5), (1A.6), *then* $\theta < T$ *and* $(U(z), V(z)) \in N$ *for all* z.

E. *The critical point* C. From (2B.2) and (2B.3) it follows that $H(z) < F(B) < F(C)$ on all solutions of (1A.5), (1A.6). This implies that there exists δ such that if

$(U(z), V(z))$ is a solution of (1A.5), (1A.6) then $U(z) \notin \{U: \|U - C\| < \delta\} = C_\delta$ for each z. Hence, the values of $F(U)$ in C_δ do not matter if we are only interested in solutions of (1A.5), (1A.6). In particular, $F(U)$ may be chosen to be arbitrarily large in C_δ. We change $F(U)$ in C_δ so that if $(U(z), V(z)) \in W_{A^\theta}$ with $\theta < T$, then $U(z) \neq C$ for all z. This is possible for the following reason. Suppose that $(U(z), V(z)) \in W_{A^\theta}$ and $U(z_0) = C$ for some z_0. If $F(C)$ is very large, then we must have $\|V(z_1)\|$ very large for some $z_1 < z_0$. However, as Proposition 2D.1 shows, if $\|V(z_1)\|$ is too large, then $U(z)$ will leave N_1 in backward time. The remark in §2A implies that $U(z)$ can never return to N_1, in backward time, after leaving N_1. This contradicts the assumption that $(U(z), V(z)) \in W_{A^\theta}$. It is very tedious to make this all precise so we do not give the details.

3. The local unstable manifold at $(A, \mathcal{O})$ and the winding number.

A. *A parameterization of W_{A^θ}.* As before, let W_{A^θ} be the unstable manifold at $(A, \mathcal{O})$ for a particular θ. As we saw in §2C, $\dim W_{A^\theta} = 2$ for all θ. The eigenvalues of the linearized equations at $(A, \mathcal{O})$ are given in (2C.2), and their corresponding eigenvectors given in (2C.3). We conclude from the Stable Manifold Theorem (see **[2]**),

THEOREM 3A.1. *Near $A' = (A, \mathcal{O})$, W_{A^θ} is a C^2 injectively immersed, two-dimensional manifold. Moreover, the tangent space to W_{A^θ} at A' is the linear subspace spanned by p_1^+ and p_2^+.*

An important consequence of this theorem is

PROPOSITION 3A.2. *There exists $\delta > 0$ such that if $A_\delta = \{U: \|A - U\| = \delta\}$ then for each θ, we have that*

(a) *For each $U_0 \in A_\delta$ there exists a unique $V_0 \in \mathbf{R}^2$ such that $(U_0, V_0) \in W_{A^\theta}$.*

(b) *If $(U(z), V(z))$ is any nontrivial trajectory in W_{A^θ} then there exists a unique z_0 such that $U(z_0) \in A_\delta$.*

This proposition implies that for each θ we may parametrize the nontrivial trajectories in W_{A^θ} by the points on A_δ. Let us parametrize the points on A_δ by the angle φ. Let

(3A.1) $$D_1 = \{(\varphi, \theta) \colon 0 \leqslant \varphi < 2\pi, 0 < \theta \leqslant T\}.$$

Then to each $(\varphi, \theta) \in D_1$ there corresponds a unique trajectory in W_{A^θ}. If $(\theta, \varphi) = d$, then we denote this trajectory by $\gamma(d)(z) = (U(d)(z), V(d)(z))$. Here, z is the independent variable along the trajectory.

B. *The winding number.* We wish to define the number of times trajectories in W_{A^θ} wind around P_D and P_E. To do this let

$$Q_D = \{(U, V) \colon U_1 = D_1,\ V_1 < 0, \text{ and } U \in N\},$$
$$Q_E = \{(U, V) \colon U_1 = E_1,\ V_1 > 0, \text{ and } U \in N\}.$$

If $(U(z), V(z)) \in W_{A^\theta}$, let

$$h(U, V) = \operatorname{card}\{z : (U(z), V(z)) \in Q_D \cup Q_E\}.$$

By card X we mean the cardinality of the set X. If $d \in \mathcal{D}_1$, let

(3B.1) $$h(d) = h(\gamma(d)(z)).$$

REMARK. $h(d)$ counts the number of times $\gamma(d)(z)$ intersects $Q_D \cup Q_E$ which is equal to the number of times $\gamma(d)(z)$ winds around P_D and P_E. This notion of winding number may seem complicated because it involves trajectories in four-dimensional space. However, one can compute $h(U, V)$ by considering $U(z)$ in the two-dimensional state space. Recall that $h(U, V)$ equals the number of times (U, V) intersects Q_D and Q_E. Now $(U(z_0), V(z_0)) \in Q_D$ if and only if $U(z_0) \in l_D$ and at $z = z_0$, $U(z)$ crosses l_D from right to left. Similarly, $(U(z_0), V(z_0)) \in Q_E$ if and only if $U(z_0) \in l_E$ and at $z = z_0$, $U(z)$ crosses l_E from left to right.

C. *θ near* 0. Crucial to the proof of the theorem is the following result.

PROPOSITION 3C.1. *Given M there exists θ_M such that if $0 < \theta < \theta_M$, $0 \leqslant \varphi < 2\pi$, $d = (\varphi, \theta)$, and $U(d)(z_0) = B$ for some z_0, then $h(d) > M$.*

The proof of this result is quite technical so we save the proof for Appendix A.

4. An algebraic invariant.

A. *A preliminary result.*

LEMMA 4A.1. *Fix $\theta \in [0, T]$ and $q \in \partial X_2$, where X_2 was defined in* (1B.1). *Then there exists $\varphi = \varphi(\theta, q)$ such that $U(\varphi(\theta, q), \theta)(z_0) = q$ for some z_0 and $U(\varphi(\theta, q), \theta)(z) \in X_2$ for $z < z_0$. Moreover, $\varphi(\theta, q)$ can be chosen to depend continuously on θ and q.*

The proof of this result is quite technical so we save the proof for Appendix B.

Let q_1 and q be any points on ∂X_2 as shown in Figure 2. That is, q_1 is on the top side of N_1 and q_2 is on the bottom side. From Lemma 4A.1 there exists continuous functions $\varphi_1(\theta)$ and $\varphi_2(\theta)$ such that for $i = 1, 2$, $U(\varphi_i(\theta), \theta)(z)$ leaves X_2 through q_i. We assume, without loss of generality, that $\varphi_1(\theta) < \varphi_2(\theta)$ for all θ. Let

$$\begin{aligned} D &= \{(\varphi, \theta) \in D_1 : \varphi_1(\theta) \leqslant \varphi \leqslant \varphi_2(\theta)\}, \\ X &= \{d \in D : \gamma(d)(z) \to (B, \mathcal{O}) \text{ as } z \to \infty\}, \\ Y &= D \setminus X. \end{aligned}$$

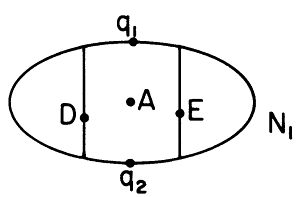

FIGURE 2

Note that X corresponds to solutions of (1A.5) and (1A.6). We wish to prove that X is an infinite set.

B. Γ *and* Γ^*. Suppose that $y \in Y$, which was defined in the preceding section. We claim that $\gamma(y)(z)$ must leave N. From Propositin 2B.1 and the fact that $H(z)$ is increasing on solutions, it follows that if $\gamma(y)(z)$ does not leave N, then either $\lim_{z \to \infty} \gamma(y)(z) = (B, \mathcal{O})$ or $\lim_{z \to \infty} \gamma(d)(z) = (C, \mathcal{O})$. By the definition of Y, we have that $\lim_{z\to\infty}\gamma(y)(z) \neq (B,\mathcal{O})$. Moreover, the remarks in §2E imply that $\lim_{z\to\infty}\gamma(y)(z) \neq (C,\mathcal{O})$. Hence, $\gamma(y)(z)$ leaves N. Let

$$\mathscr{E} = \{(U,V) \in \partial N : \|V\| < \bar{V}\} \setminus (P_D \cup P_E).$$

From Proposition 2D.1 we conclude that if $y \in Y$, then $\gamma(y)(z)$ must leave N through $\mathscr{E}$. Hence, we have a mapping $\Lambda : Y \to \mathscr{E}$ defined by $\Lambda(y) =$ the place where $\gamma(y)(z)$ leaves N. From Lemma 2A.3 it follows that Λ is continuous.

Let I be the unit interval and $\mathscr{G}$ the set of functions $g : I \to Y$ such that

(a) g is continuous,

(b) $g(0) \in \{(\varphi,\theta) : \varphi = \varphi_1(\theta)\}$,

(c) $g(1) \in \{(\varphi,\theta) : \varphi = \varphi_2(\theta)\}$.

If $g \in \mathscr{G}$, then we have a continuous map $\Lambda \cdot g : I \to \mathscr{E}$. Note that $\mathscr{E}$ is topologically an annulus with four holes removed.

We now define two algebraic objects, $\Gamma^*(g)$ and $\Gamma(g)$, which indicate how the curve $(\Lambda \cdot g)(I)$ winds around the four holes. They will be elements of F_4, the set of words on the four elements α, β, γ, and δ.

We begin with some notation. For convenience we assume that N_1 is the square

$$N_1 = \{(U_1,V_2) : |U_1| \leqslant W, |U_2| \leqslant W\}.$$

Let

$$\begin{aligned}
E_1 &= \{(U,V) \in \mathscr{E} : U_1 > E_1\},\\
E_2 &= \{(U,V) \in \mathscr{E} : D_1 < U_1 < E_1,\ U_2 = W\},\\
E_3 &= \{(U,V) \in \mathscr{E} : U_1 < D_1\},\\
E_4 &= \{(U,V) \in \mathscr{E} : D_1 < U_1 < E_1,\ U_2 = -W\},\\
l_1 = l_\alpha^+ &= \{(U,V) : U_1 = E_1,\ U_2 = W,\ 0 < V_1 \leqslant \bar{V},\ V_2 = 0\},\\
l_2 = l_\alpha^- &= \{(U,V) : U_1 = E_1,\ U_2 = W,\ -\bar{V} \leqslant V_1 < 0,\ V_2 = 0\},\\
l_3 = l_\beta^+ &= \{(U,V) : U_1 = D_1,\ U_2 = W,\ 0 < V_1 \leqslant \bar{V},\ V_2 = 0\},\\
l_4 = l_\beta^- &= \{(U,V) : U_1 = D_1,\ U_2 = W,\ -\bar{V} \leqslant V_1 < 0,\ V_2 = 0\},\\
l_5 = l_\gamma^+ &= \{(U,V) : U_1 = D_1,\ U_2 = -W,\ 0 < V_1 \leqslant \bar{V},\ V_2 = 0\},\\
l_6 = l_\gamma^- &= \{(U,V) : U_1 = D_1,\ U_2 = -W,\ -\bar{V} \leqslant V_1 < 0,\ V_2 = 0\},\\
l_7 = l_\delta^+ &= \{(U,V) : U_1 = E_1,\ U_2 = -W,\ 0 < V_1 \leqslant \bar{V},\ V_2 = 0\},\\
l_8 = l_\delta^- &= \{(U,V) : U_1 = E_1,\ U_2 = -W,\ -\bar{V} \leqslant V_1 < 0,\ V_2 = 0\}.
\end{aligned}$$

Assume that $g \in \mathscr{G}$. Choose $\eta_k \in [0,1]$, $k = 1,2,\ldots,K$, such that

(a) $\eta_1 = 0$, $\eta_K = 1$,

(b) $\eta_k < \eta_{k+1}$ for all k,

(c) $(\Lambda \cdot g)(\eta_k) \in \bigcup_{i=1}^4 E_i$ for all k,

(d) $(\Lambda \cdot g)(\eta_k, \eta_{k+1})$ intersects at most one of the line segments l_i, $i = 1,\ldots,8$, for all k. We refer to $\eta^* = \{\eta_1,\ldots,\eta_K\}$ as a g-partition. It is not hard to prove that a g-partition does exist.

TABLE 1

$\Phi(\eta_k) \in$	$\Phi(\eta_{k+1}) \in$	$\phi(\eta)$ crosses for $\eta \in (\eta_k, \eta_{k+1})$	$\lambda(\eta_{k+1})$	$e(\eta_{k+1})$
E_2	E_1	l_α^+	α	1
E_1	E_2	l_α^+	α	-1
E_2	E_3	l_β^-	β	1
E_3	E_2	l_β^-	β	-1
E_3	E_4	l_γ^-	γ	-1
E_4	E_3	l_γ^-	γ	1
E_4	E_5	l_δ^+	δ	1
E_5	E_4	l_δ^+	δ	-1

We now define $\Gamma^*(g, \eta^*)$. First we define $\lambda_k = \lambda(\eta_k) \in \{\alpha, \beta, \gamma, \delta\}$ and $e_k = e(\eta_k) \in \{-1, 0, 1\}$. These quantities are determined by Table 1. In the table, we let, for $s \in [0,1]$, $\Phi(s) \equiv (\Lambda \cdot g)(s)$. For each case not shown in the chart we let $e_k = e(\eta_k) = 0$. For this case we do not define $\lambda_k = \lambda(\eta_k)$ because, as we shall see, since $e_k = 0$ the choice of λ_k does not matter. Then define

$$\Gamma^*(g, \eta^*) = \prod_{i=1}^{K} \lambda_i^{e_i} \overset{\text{def}}{=} \lambda_1^{e_1}\lambda_2^{e_2} \cdots \lambda_K^{e_K}.$$

An example is shown in Figure 3. In the figure, $\xi_k = \Phi(\eta_k)$. For this example

$$\Gamma^*(g, \eta^*) = \alpha\alpha^{-1}\alpha\beta\gamma^{-1}\delta^{-1}.$$

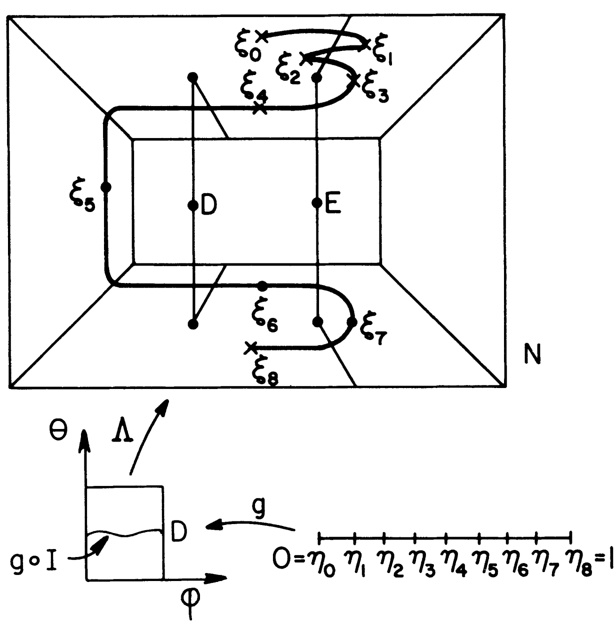

FIGURE 3

Note that there may be cancellations in $\Gamma^*(g,\eta^*)$. By $\Gamma(g)$ we mean the element of F_4 obtained by making all cancellations in $\Gamma^*(g,\eta^*)$. We shall see later that $\Gamma(g)$ does not depend on η^*. In the above example $\Gamma(g)=\alpha\beta\gamma^{-1}\delta^{-1}$.

By $\Gamma^*(g)$ we mean the subset of F_4 consisting of all elements which yield, after all cancellations, $\Gamma(g)$. Note that, for each η^*, $\Gamma^*(g,\eta^*)\in\Gamma^*(g)$ and $\Gamma(g)\in\Gamma^*(g)$. For the above example, $\Gamma^*(g)$ includes the elements $\alpha\beta\delta\delta^{-1}\gamma^{-1}\delta^{-1}$ and $\alpha\beta\beta^{-1}\gamma\delta\delta^{-1}\gamma^{-1}\beta\gamma^{-1}\delta^{-1}$.

The following two propositions will be important for the rest of the paper. Their proofs are tedious but straightforward. We do not give the details.

PROPOSITION 4B. *$\Gamma(g)$ does not depend on the choice of η^*.*

Before stating the next proposition we need the following definition.

DEFINITION. Suppose that $g_1, g_2\in\mathcal{G}$. We say that g_1 *is homotopic to* g_2 *relative to* Y, and write $g_1\sim g_2$, if there exists a continuous map $\Phi: I\times I\to Y$ such that

(a) $\Phi(s,0)=g_1(s)$ for $s\in I$,

(b) $\Phi(s,1)=g_2(s)$ for $s\in I$,

(c) $\Phi(\cdot,t)\in\mathcal{G}$ for each $t\in I$.

PROPOSITION 4B.2. *If $g_1\sim g_2$, then $\Gamma(g_1)=\Gamma(g_2)$.*

C. *The winding number revisited.* Recall that for each $d=(\varphi,\theta)\in D$ there corresponds a trajectory $\gamma(d)(z)\in W_{A^\theta}$. Moreover there is a winding number, $h(d)$, which counts the number of times $\gamma(d)(z)$ winds around P_D and P_E. Suppose that $g(s)\in\mathcal{G}$ and $s_0\in I$. In this section we derive a formula for $h_1(s_0)\equiv h(g(s_0))$ in terms of $\Gamma^*(g,\eta^*)$ for some g-partition η^*. First we need some notation.

Suppose that $\Gamma\in F_4$ is given by $\Gamma=\lambda_1^{e_1}\cdots\lambda_J^{e_J}$ where each $\lambda_i\in\{\alpha,\beta,\gamma,\delta\}$ and $e_i\in\{1,-1\}$. Let $\omega(\Gamma)=\sum_{i=1}^J e_i$. Let $g\in\mathcal{G}$ and η^* be a g-partition. Define the map $\Lambda_0(g,\eta^*): I\to F_4$ as follows. Suppose that $\eta^*=\{\eta_1,\ldots,\eta_K\}$ and $\eta_K\leqslant s<\eta_{k+1}$. Then define

$$\Lambda_0(g,\eta^*)(s)=\prod_{i=1}^{k}\lambda_i^{e_i}=\lambda_1^{e_1}\cdots\lambda_k^{e_k}.$$

The λ_i and e_i are defined as in the previous section. Finally, define $\Lambda_1(g,\eta^*): I\to Z^+$, where Z^+ is the set of nonnegative integers, by

$$\Lambda_1(g,\eta^*)(s)=\left[\omega\cdot\Lambda_0(g,\eta^*)\right](s).$$

We can now state the main result of this section.

PROPOSITION 4C.1. *Assume that $g\in\mathcal{G}$ and $\eta^*=(\eta_1,\ldots,\eta_K)$ is a g-partition. If $\eta_j\leqslant s_0<\eta_{j+1}$, then either*

$$h_1(s_0)=\Lambda_1(g,\eta^*)(\eta_j)\quad\text{or}\quad h_1(s_0)=\Lambda_1(g,\eta^*)(\eta_{j+1}).$$

PROOF. The proof is by induction on j. First assume that $j=1$. That is

$$0=\eta_1\leqslant s_0<\eta_2. \tag{4C.1}$$

By assumption, $\eta_1=0$ and $(\Lambda\cdot g)(\eta_1)\in E_2$ where E_2 was defined in the preceding section. There are a number of cases to consider. Suppose, for example, that

$$(\Lambda\cdot g)(s_0)\in E_2\quad\text{and}\quad(\Lambda\cdot g)(\eta_2)\in E_2. \tag{4C.2}$$

Because $(\Lambda \cdot g)(\eta_2) \in E_2$ it follows, from Table 1, that $e_1 = 0$ and, therefore,

$$\Lambda_1(g, \eta^*)(\eta_1) = \Lambda_1(g, \eta^*)(\eta_2) = 0.$$

Hence, we need to prove that $h_1(s_0) = 0$. Suppose that $h_1(s_0) > 0$. Then $\gamma(g(s_0)(z)$ must intersect $Q_D \cup Q_E$ at least once. Suppose that $\gamma(g(s_0))(z)$ intersects Q_D. The other case is similar. Recall that

$$Q_D = \{(U, V): U_1 = D_1, V_1 < 0, \text{ and } U \in N_1\}.$$

Because $(\Lambda \cdot g)(s_0) \in E_2$ it follows that $\gamma(g(s_0))(z)$ must also intersect

$$Q_D^+ = \{(U, V): U_1 = D_1, V_1 > 0, \text{ and } U \in N_1\}.$$

Let

$$s_1 = \inf\{s : \gamma(g(s))(z) \text{ intersects } Q_D\},$$
$$s_2 = \inf\{s : \gamma(g(s))(z) \text{ intersects } Q_D^+\}.$$

Clearly, $0 < s_1 < s_2 < s_0 < \xi_2$. Moreover, $(\Lambda \cdot g)(s_1) \in l_\beta^-$ and $(\Lambda \cdot g)(s_2) \in l_\beta^+$. This, however, contradicts the assumption that η^* is a g-partition.

There are other cases to consider besides (4C.2). We only consider one more. The rest are similar. Suppose that $(\Lambda \cdot g)(s_0) \in E_1$ and $(\Lambda \cdot g)(\eta_2) \in E_1$. Then, using Table 1, $\Lambda_1(g, \eta^*)(\eta_1) = 0$ and $\Lambda_1(g, \eta^*)(\eta_2) = 1$. We claim that $h_1(s_0) = 1$. Because $(\Lambda \cdot g)(\eta_1) \in E_2$ and $(\Lambda \cdot g)(s_0) \in E_1$ it is clear that $h_1(s_0) \neq 0$. Suppose that $h_1(s_0) > 1$. Then $\gamma(g(s_0))(z)$ must intersect $Q_D \cup Q_E$ at least twice. Because $\gamma(g(0))(z)$ does not intersect $Q_D \cup Q_E$ at all, this implies that there exist s_1, s_2 with $\eta_1 < s_1 < s_2 < s_0 < \eta_2$ such that $(\Lambda \cdot g)(s_1) \in l_i$ and $(\Lambda \cdot g)(s_2) \in l_j$ for $i \neq j$. This, however, contradicts the assumption that η^* is a g-partition.

To complete the proof of the proposition we must prove the induction step. That is, we assume that the proposition is true if $\eta_j \leqslant s_0 \leqslant \eta_{j+1}$ for $j < k$, and then prove the result if $j = k$. The proof of this is very similar to the proof just given so we do not include the details.

5. Completion of the proof of Theorem 1.

A. *Preliminaries.* Suppose that $g \in \mathscr{G}$, which was defined in the previous section, and

$$\Gamma(g) = \lambda_1^{e_1}\lambda_2^{e_2} \cdots \lambda_K^{e_K}$$

where, for each i, $\lambda_i \in \{\alpha, \beta, \gamma, \delta\}$ and $e_i \in \{-1, 1\}$. Define

$$|\Gamma(g)| = \sup_{1 \leqslant J \leqslant K} \sum_{i=1}^{J} e_i.$$

In the next subsection we prove

PROPOSITION 5A.1. *Let M be any positive integer and let θ_M be as in Proposition 3C.1. Suppose that $g \in \mathscr{G}$ is given by $g(s) = (\varphi(s), \theta(s))$ and $\theta(s) < \theta_M$ for each s. Then $|\Gamma(g)| > M$.*

In this section we show that Proposition 5A.1 implies that Theorem 1 is true.

Suppose that Theorem 1 is not true; that is, there exists only a finite number of θ's, say $\{\theta_1, \ldots, \theta_N\}$, for which there exists a solution of (1A.5), (1A.6). Let

$$\theta_0 = \tfrac{1}{2} \inf_{1 \leqslant j \leqslant N} \theta_j.$$

Then $\theta_0 > 0$. Choose $g_0 \in \mathcal{G}$ such that $g_0(s) = (\varphi(s), \theta(s))$ and $\theta(s) = \theta_0$ for all s. From Proposition 5A.1 there exists $\theta^* < \theta_0$ such that if $g_1(s) \in \mathcal{G}$ is such that $g_1(s) = (\varphi_1(s), \theta_1(s))$ and $\theta_1(s) = \theta^*$ for all s, then $|\Gamma(g_1)| > |\Gamma(g_0)| + 1$. But $g_0(s)$ and $g_1(s)$ are clearly homotopic relative to Y. From Proposition 4G.2 it follows that $\Gamma(g_0) = \Gamma(g_1)$. Hence,

$$|\Gamma(g_0)| = |\Gamma(g_1)| > |\Gamma(g_0)| + 1.$$

This is clearly impossible, thus proving the theorem.

B. *Proof of Proposition* 5A.1. Let $\gamma(g(s))(z) = (U(g(s))(z), V(g(s))(z))$. Now $U(g(0))(z)$ leaves X_2 through its top side without ever crossing l_D or l_E, and $U(g(1))(z)$ leaves X_2 through its bottom side without ever crossing l_D or l_E. Moreover, by the remarks in §2E, $U(g(s))(z) \neq C$ for all s and z. Since the curves $U(g(s))(\cdot)$ vary continuously with s this implies that there exist s_0 and z_0 such that $U(g(s_0))(z_0) = B$. Because $\theta(s_0) < \theta_M$, Proposition 3C.1 implies that $h_1(s_0) = h(g(s_0)) > M$. By Proposition 4C.1, if η^* is a g_0-partition and $\eta_j \leqslant s_0 < \eta_j$, then either

$$h_1(s_0) = \Lambda_1(g, \eta^*)(\eta_j) \quad \text{or} \quad h_1(s_0) = \Lambda_1(g, \eta^*)(\eta_{j+1}).$$

From the definitions this implies that $|\Gamma^*(g, \eta^*)| > M$. We must show that this implies that $|\Gamma(g)| > M$.

Note that $\Gamma(g)$ is obtained from $\Gamma^*(g, \eta^*)$ by a finite number of cancellations. We show that after each cancellation the index is still greater than M. More precisely, suppose that $\Gamma^*(g, \eta^*)$ is of the form

$$\Gamma^*(g, \eta^*) = \Gamma_A^* \lambda_k^{e_k} \lambda_{k+1}^{e_{k+1}} \Gamma_B^* \tag{5B.1}$$

where Γ_A^*, $\Gamma_B^* \in F_4$, $\lambda_k = \lambda_{k+1} \in \{\alpha, \beta, \gamma, \delta\}$, and $e_k = -e_{k+1} \in \{-1, 1\}$. Let $\Gamma' = \Gamma_A^* \Gamma_B^*$. We prove that $|\Gamma'| > M$. Since $\Gamma(g)$ is obtained from $\Gamma^*(g, \eta^*)$ by a finite number of such cancellations, this will prove the result.

Note that if $e_k = -1$, then $|\Gamma'| = |\Gamma^*(g, \eta^*)| > M$. Hence, we may assume that $e_k = +1$.

There are four cases to consider. Either $\lambda_k = \alpha, \beta, \gamma$, or δ. First suppose that $\lambda_k = \alpha$. We then consider two subcases. These are

$$\begin{aligned} &\text{(a) } h(g(\eta_0)) > M \text{ for some } \eta_0 < \eta_{k-1}. \\ &\text{(b) } h(g(\eta)) \leqslant M \text{ for all } \eta < \eta_{k-1}. \end{aligned} \tag{5B.2}$$

Suppose (5B.2a). Choose $j < k - 1$ such that $\eta_j \leqslant \eta_0 < \eta_{j+1}$. Then, from Proposition 4C.1, either

$$h(g(\eta_0)) = \sum_{i=1}^{j} e_i > M, \quad \text{or} \quad h(g(\eta_0)) = \sum_{i=1}^{j+1} e_i > M.$$

In either case, it follows that $|\Gamma'| > M$. This is what we wished to prove.

Now suppose that (5B.2b) holds. We first show that (5B.2b) implies that $U(g(\eta))(z) \neq B$ for all z and $\eta < \eta_{k+1}$. Suppose, for the sake of contradiction, that $U(g(\eta_0))(z_0) = B$ for some z_0 and $\eta_0 < \eta_{k+1}$. Let

$$t(\eta_0) = \sup\{z < z_0 : U(g(\eta_0))(z) \in l_D\}.$$

Because $\lim_{z \to -\infty} U(g(\eta_0)(z) = A$ and $U(g(\eta_0))(z_0) = B$, it is clear that $t(\eta_0)$ is well defined. For η close to η_0 there exists a continuous function $t(\eta)$ such that $U(g(\eta))(t(\eta)) \in l_D$. Let J be the maximum subset of $[0,1]$ such that $t(\eta)$ is a well defined, continuous function. Let $\zeta = \inf\{\eta : \eta \in J\}$. Because $U(g(\eta_0))(z_0) = B$ we conclude from Proposition 3C.1 that $h(g(\eta_0)) > M$. It follows that $h(g(\eta)) > M$ for all $\eta \in J$. In particular, $h(g(\zeta)) > M$. If $\zeta < \eta_{k-1}$, then we have a contradiction of (5B.2b). Therefore, assume that $\eta_{k-1} \leqslant \zeta < \eta_{k+1}$. Clearly, $U(g(\zeta))(z)$ leaves N_1 through $l_D \cap \partial N_1$. Hence, $(\Lambda \cdot g)(\zeta) \in l_\beta^- \cup l_\gamma^- = l_4 \cup l_6$, where Λ, l_β^-, l_γ^-, l_4 and l_6 were defined in §4B. However, from the definition of a g-partition, $(\Lambda \cdot g)(\eta)$ can cross at most one of the lines $l_1 - l_8$ for $\eta \in (\eta_k, \eta_{k+1})$. Because $\lambda_k = \alpha$ and $\lambda_{k+1} = \alpha$ we have, from Table 1, that $(\Lambda \cdot g)(\eta)$ crosses $l_\alpha^+ = l_1$ for some $\eta \in [\eta_{k-1}, \eta_k)$ and another $\eta \in [\eta_k, \eta_{k+1})$. This gives the desired contradiction. We have now shown that if (5B.2b) holds, then $U(g(\eta))(z) \neq B$ for all z and $\eta < \eta_{k+1}$.

It is clear that $U(g(\eta_0))(z_0) = B$ for some η_0 and some z_0. If (5B.2b) holds then $\eta_0 > \eta_{k+1}$. Suppose that $\eta_j \leqslant \eta_0 < \eta_{j+1}$ for some $j \geqslant k+1$. Because $U(g(\eta_0)) = B$ it follows that $h(g(\eta_0)) > M$. From Proposition 4C.1 we conclude that either $\Lambda_1(g, \eta^*)(\eta_j) > M$ or $\Lambda_1(g, \eta^*)(\eta_{j+1}) > M$. Both of these inequalities imply that $|\Gamma'| > M$.

It remains to consider the cases $\lambda_k = \beta$, γ, and δ. The proofs in each of these cases is similar to the one just given so we do not include the details.

Appendix A. Proof of Proposition 3C.1.

Idea of the proof. Suppose that $(U(z), V(z)) \in W_{A^\theta}$ and $U(z_0) = B$ for some z_0. Then while $(U(z), V(z)) \in N$, $H(z)$ must increase from $F(A)$ to $F(B)$. Recall that $H'(z) = \theta \|V\|^2 \leqslant \theta \bar{V}^2$ as long as $(U, V) \in N$. Then $H(z)$ increases very slowly if θ is very small. Since $H(z)$ must increase from $F(A)$ to $F(B)$ this implies that $(U(z), V(z))$ must spend a long time in N. We shall use hypothesis (F6) to conclude that $U(z)$ must move back and forth between the mountain peaks defined by $F(A)$, $F(B)$, and $F(C)$ a large number of times. Together with the remarks in §3B this implies the desired result.

Proof of Proposition 3C.1. Let $\gamma(d)(z) = (U(z), V(z))$. We shall prove that there exists θ_0 such that if $\theta < \theta_0$ and $U(z_1) \in X_i$, $i = 1, 2$, or 3, then there exists $z_2 > z_1$ such that $U(z_2) \notin X_i$ and $H(z_2) - H(z_1) \leqslant F(B)/4M$. This proves the proposition because it implies that $U(z)$ must change regions (X_1, X_2, or X_3) at least $4M$ times if $U(z_0) = F(B)$ for some z_0. Together with the remark in §3B this implies the desired result.

We begin with a rather technical result. For $r > 0$ let $A_r = \{U: \|U - A\| \leqslant r\}$.

Lemma A.1. *There exists θ_1, h_0, r such that if $\theta < \theta_1$, $(U(z_1), V(z_1)) \in \partial A_r$, $\langle U(z_1), V(z_1) \rangle > 0$, and $H(z_1) < h_0$, then there exists $z_2 > z_1$ such that $U(z_2) \notin X_2$ and $U(z) \notin A_r$ for $z \in (z_1, z_2)$.*

Proof of Lemma A.1. Suppose that the lemma is not true. Then for each $n > 0$ there exists θ_n, h_n, r_n which all approach zero as $n \to \infty$, solutions $(U_n(z), V_n(z))$ of (1A.5) with $\theta = \theta_n$ and z_n such that $(U_n(z_n), V_n(z_n)) \in \partial A r_n$, $\langle U_n(z_n), V_n(z_n) \rangle > 0$,

$H(z_n) = h_n$, and each $U_n(z)$ returns to Ar_n before leaving X_2. By compactness, this implies that some subsequence of $\{U_n(z), V_n(z)\}$ converges to a solution $(U(z), V(z))$ of (1A.5) with $\theta = 0$ such that $H(z) = 0$ along $(U(z), V(z))$ and $\lim_{z \to \pm\infty} U(z) = A$. Hence, $(U(z), V(z))$ is a bounded solution of (1A.5) such that $U(z)$ lies entirely in X_2. If we show that $U(z) \neq A$ for some z, then this will contradict (F6).

Let δ be as in Proposition 3A.2. The assumption that $\langle U_n(z), V_n(z)\rangle > 0$ for each n and the saddle point property at $(A, \mathcal{O})$ imply that for each n there exists z_n' such that $\|U_n(z_n') - A\| = \delta$. Hence, there exists z' such that $\|U(z') - A\| = \delta$. In particular, $U(z') \neq A$ and the proof of the lemma is complete. In what follows we assume that $h_0 < F(B)$.

We now return to the proof of Proposition 3C.1.

First suppose that $i = 1$ or 3. Assumption (F6) implies that there exists T_1 such that if $\theta = 0$, $0 < H(z_1) < F(B)/2$ and $U(z_1) \in X_i$, then $U(z) \notin X_i$ for some $z \in (z_1, z_1 + T_1)$. By continuous dependence of solutions on a parameter it follows that there exists θ_0 such that if $\theta < \theta_0$, $0 < H(z_1) < F(B)/2$ and $U(z_1) \in X_i$, then $U(z) \notin X_i$ for some $z \in (z_1, z_1 + 2T_1)$. Let

$$z_2 = \inf\{ z > z_1 : \ U(z) \notin X_i \}.$$

Then

$$H(z_2) - H(z_1) = \int_{z_1}^{z_2} H'(z)\, dz = \int_{z_1}^{z_2} \theta \langle V(z), V(z)\rangle^2 dz$$
$$\leqslant \theta_0 \bar{V}^2 2T_1 \leqslant h_0/12M < F(B)/4M,$$

if

$$\theta_0 \leqslant \frac{h_0}{24M} \bar{V}^2 T_1 \tag{A.1}$$

which we assume to be true.

It remains to consider the case when $U(z_1) \in X_2$ for some z_1. Note that $U(z)$ starts out in X_2 (because $\lim_{z \to -\infty} U(z) = A$). If $\theta = 0$, then (F6) implies that there exists ξ_0 such that $U(\xi_0) \notin X_2$. Moreover, $H(\xi_0) = 0$. Hence, θ_0 can be chosen such that if $0 < \theta < \theta_0$, then there exists ξ_1 such that $U(\xi_1) \notin X_2$ and $H(\xi_1) < h_0/12M$.

Let

$$\eta_1 = \inf\{ z : U(z) \notin A_r \} \quad \text{and} \quad \eta_2 = \inf\{ z : U(z) \notin X_2 \}.$$

Let

$$\gamma = \min\{ \|\gamma_1 - \gamma_2\| : \ \gamma_1 \in A, \ \gamma_2 \notin X \}.$$

Since $\|V(z)\| \leqslant \bar{V}$ it follows that $\eta_2 - \eta_1 \geqslant \gamma/\bar{V}$. Let

$$-\lambda = \sup\{ F(U(z)) : U(z) \in \partial Ar \} < 0$$

because we are assuming that $F(A) = 0$. Then, for $z \in (\eta_1, \eta_2)$,

$$0 < H(z) = \tfrac{1}{2}F(U) + \|V(z)\|^2 \leqslant -\tfrac{1}{2}\lambda + \|V(z)\|^2,$$

or

$$\|V(z)\|^2 > +\tfrac{1}{2}\lambda.$$

It follows that

$$H(\eta_2) \geqslant H(\eta_2) - H(\eta_1) = \int_{\eta_1}^{\eta_2} H'(\eta)\,d\eta$$
$$\geqslant \int_{\eta_1}^{\eta_2} \theta\langle V(\eta), V(\eta)\rangle\,d\eta$$
$$\geqslant \int_{\eta_1}^{\eta_2} \frac{\theta\lambda}{2}\,d\eta \geqslant \frac{\theta\lambda\gamma}{2\overline{V}}.$$

Therefore,

$$H(z) \geqslant \frac{1}{2}\frac{\theta\lambda\gamma}{\overline{V}} \quad \text{if } z > \eta_2. \tag{A.2}$$

Now suppose that $z_1 > \eta_2$ and $U(z_1) \in X_2$. There are two cases to consider. These are, (1) $U(z_2) \notin X_2$ for some $z_2 > z_1$ and $U(z) \notin A_r$ for $z \in (z_1, z_2)$, and (2) $U(z_2) \in A_r$ for some $z_2 > z_1$ and $U(z) \in X_2$ for $z \in (z_1, z_2)$.

As before, (F6) and the continuous dependence of solutions on a parameter imply that θ_0 and T_1 can be chosen so that if $\theta < \theta_0$, $z_1 > \eta_2$, $U(z_1) \in X_2$, and $U(z) \notin A_r$ for $z \in (z_1, z_1 + T_1)$, then there exists z_2 such that $z_1 < z_2 < z_1 + T_1$ and $U(z_2) \notin X_2$. As before, $H(z_2) - H(z_1) < h_0/12M$ if (A.1) is satisfied.

It remains to consider the case $U(z_1) \in X_2$, $z_1 > \eta_2$, and there exists $z_2 > z_1$ such that $U(z_2) \in A_r$ and $U(z) \in X_2$ for $z \in (z_1, z_2)$. We assume that

$$z_2 = \inf\{z > z_1 : U(z) \in A_r\}.$$

As before, θ_0 and T_1 can be chosen so that if $\theta < \theta_0$, then $z_2 - z_1 < T_1$. Moreover, $H(z_2) - H(z_1) \leqslant h_0/12M$ if θ_0 is sufficiently small. Because $z_2 > \eta_2$, we have, from (A.2), that $H(z_2) \geqslant \lambda\theta\gamma/2\overline{V}$. Hence, as long as $U(z) \in A_r$, $\|V(z)\|^2 \geqslant F(U)/2 + \|V\|^2 \geqslant H(z) \geqslant \theta\lambda\gamma/2\overline{V} \equiv \theta K$. This implies that $U(z)$ must pass through Ar. If we let $z_3 = \inf\{z > z_2 : U(z) \notin A_r\}$, then

$$z_3 - z_2 \leqslant \frac{\text{diameter of } Ar}{\inf\|V(z)\|} \leqslant \frac{2r}{\theta^{1/2}K^{1/2}}.$$

Moreover,

$$H(z_3) - H(z_2) = \int_{z_2}^{z_3} H'(z)\,dz \leqslant \theta\overline{V}^2(z_3 - z_2)$$
$$\leqslant \theta\overline{V}^2\frac{2r}{\theta^{1/2}K^{1/2}} \leqslant \theta_0^{1/2}\frac{2\overline{V}^2 r}{K^{1/2}} \leqslant \frac{h_0}{12M}$$

if θ_0 is sufficiently small.

From Lemma A.1 there exists $z_4 > z_3$ such that $U(z_4) \notin X_2$ and $U(z) \notin A_r$ for $z \in (z_3, z_4)$. We assume that $z_4 = \inf\{z > z_3 : U(z) \notin X_2\}$. As before, θ_0 and T_1 can be chosen so that if $\theta < \theta_0$, then $z_4 - z_3 < T_1$. Hence, if (A1.1) holds, then

$$H(z_4) - H(z_3) \leqslant h_0/12M.$$

We can now estimate the change of $H(z)$ as $U(z)$ crosses X_2 in the case that $U(z)$ crosses A_r. Let z_1, z_2, z_3 and z_4 be as above. Then

$$H(z_4) - H(z_1) = [H(z_4) - H(z_3)] + [H(z_3) - H(z_2)] + [H(z_2) - H(z_1)]$$
$$\leqslant h_0/4M \leqslant F(B)/4M,$$

which is what we wished to prove.

Appendix B. Proof of Lemma 4A.1. Let I be equal to the set of $\theta \in [0, T]$ for which the first part of the lemma is true. That is, if $\theta \in I$ and $q \in \partial X_2$, then there exists $\varphi(\theta, q)$ such that $U(\varphi(\theta, q), \theta)(z_0) = q$ for some z_0 and $U(\varphi(\theta, q), \theta)(z) \in X_2$ for $z < z_0$. We prove that I is nonempty, open, and closed.

We first prove that I is open. Assume that $\theta_0 \in I$. Note that for each $(\varphi, \theta) \in D_1$, if $\gamma(\varphi, \theta)(z) = (U(z), V(z))$, then $U(z)$ must leave X_2. This is because of assumption (F6). Moreover, if we set $Q(\varphi, \theta)$ equal to the place where $U(z)$ leaves X_2. then $Q(\varphi, \theta)$ is continuous. This is because $U(z)$ cannot be tangent to ∂X_2.

Fix $q_0 \in \partial X_2$, and let q_1 and q_2 be any other distinct points on ∂X_2. Choose φ_1 and φ_2 such that $Q(\varphi_1, \theta_0) = q_1$ and $Q(\varphi_2, \theta_0) = q_2$. Then φ_1 and φ_2 split $\Sigma = \{(\varphi, \theta)\colon\ \theta = \theta_0\}$ into two distinct subsets, if we think of Σ as a circle identifying $\varphi = 0$ with $\varphi = 2\pi$. We label these two subsets by Σ_1 and Σ_2. Moreover, q_1 and q_2 split ∂X_2 into two distinct subsets which we label as Y_1 and Y_2. We assume that $q_0 \in Y_1$. Then either $Q(\Sigma_1, \theta_0)$ covers Y_1 and $Q(\Sigma_2, \theta_0)$ covers Y_2, or $Q(\Sigma_1, \theta_0)$ covers Y_2 and $Q(\Sigma_2, \theta_0)$ covers Y_1. This follows from the assumption that $\theta_0 \in I$. Suppose that $Q(\Sigma_1, \theta_0)$ covers Y_1. Let q_3 and q_4 be any other points in Y_1 such that the portion of ∂X_2 which lies in Y_1 and between q_3 and q_4 contains q_0. Call this subset of ∂X_2, Y_3. Since $Q(\varphi, \theta)$ is continuous, there exists λ such that if $|\theta_0 - \theta| < \lambda$, then $Q(\Sigma_1, \theta)$ covers Y_3. Since $q_0 \in Y_3$, if $|\theta - \theta_0| < \lambda$ then $Q(\varphi, \theta) = q_0$ for some φ. Since ∂X_2 is compact this implies that λ can be chosen so that if $|\theta - \theta_0| < \lambda$, then $Q(\Sigma, \theta)$ covers ∂X_2. Hence, I is open.

We now prove that I is closed. Fix $q_0 \in \partial X_2$ and $\theta_0 \in [0, T]$. Assume there exists a sequence $\{\theta_k\}$ such that for each k, $\theta_k \in I$ and $\theta_k \to \theta_0$ as $k \to \infty$. Choose $\{\varphi_k\}$ so that $Q(\varphi_k, \theta_k) = q_0$. By compactness, some subsequence of $\{\varphi_k\}$ converges, to say φ_0. Clearly, $Q(\varphi_0, \theta_0) = q_0$. Since q_0 was arbitrary, $\theta_0 \in I$ and I is closed.

It remains to prove that I is nonempty. We prove that $\theta \in I$ if θ is sufficiently large. We will prove that there exists T_1 such that if $\theta > T_1$, $\varphi \in [0, 2\pi)$, $d = (\varphi, \theta)$ and $\gamma(d)(z) = (U(z), V(z))$, then $\|U(z) - A\|$ is increasing as long as $U(z) \in X_2$. In particular, $U(z) \neq A$ for each z. This certainly implies the desired result.

We wish to prove that there exists T_1 such that if $\theta > T_1$ and $(U(z), V(z)) \in W_{A^\theta}$, then $\|U(z) - A\|$ is increasing as long as $U(z) \in X_2$. Certainly there exists δ such that if $\|U(z) - A\| < \delta$ for $z < z_0$, and $\theta > 0$, then $\|U(z) - A\|$ is increasing for $z < z_0$. Let

$$M_1 = \sup\{\|\nabla F(U)\|\colon U \in X\},$$

$$M_2 = \sup\{|\langle U - A, \nabla F(U)\rangle|\colon U \in X_2\},$$

$$\lambda_1 = \max\{2M_1, 2M_2^{1/2}\}.$$

By Lemma 2C.2 there exists T_1 such that if $\theta > T_1$, $\|U(z_0) - A\| = \delta$ and $\|U(z) - A\| < \delta$ for $z < z_0$, then $\|V(z_0)\| > \lambda$. We assume that $T_1 > 1$. Then for each $z > z_0$, $\theta > T_1$,

$$\text{(5B.1)}\qquad \frac{d}{dz}\langle V, V\rangle = 2\theta\langle V, V\rangle - 2\langle V, \nabla F(U)\rangle \geqslant 2\theta\|V\|^2 - 2\|V\|M_1$$
$$\geqslant 2\|V\|(\theta\|V\| - M_1) \geqslant 2\|V\|(\lambda_1 - M_1) > 0$$

as long as $\|V\| > \lambda_1$ and $U(z) \in X_2$. However, at $z = z_0$, $\|V\| > \lambda_1$ and (5B.1) implies that $\|V\|$ is then increasing. Hence, $\|V(z)\| \geqslant \lambda$ for $z \geqslant z_0$, $U(z) \in X_2$. Another computation shows that for $z \geqslant z_0$, $\theta > T_1$, $U(z) \in X_2$,

$$\begin{aligned}\langle U - A, V\rangle' &= \langle V, V\rangle + \theta\langle U - A, V\rangle - \langle U - A, \nabla F(U)\rangle \\ &> \lambda_1^2 + T_1\langle U - A, V\rangle - M_2 > 3M_2 + T_1\langle U - A, V\rangle.\end{aligned}$$

Since at z_0,

$$\langle U - A, V\rangle = \frac{1}{2}\frac{d}{dz}\|U - A\|^2 > 0$$

we conclude that $\langle U - A, V\rangle$ is increasing. Therefore, $\langle U - A, V\rangle > 0$ for $z > z_0$ as long as $U(z) \in X_2$. Finally, if $z > z_0$ and $U(z) \in X_2$, then

$$\frac{d}{dz}\|U(z) - A\|^2 = 2\langle U - A, V\rangle > 0.$$

To complete the proof of Lemma 4A.1 we observe that our proof that I is open also implies that $\varphi(\theta, q)$ can be chosen to depend continuously on θ and q.

REFERENCES

1. C. Conley, *Isolated invariant sets and the Morse index*, CBMS Regional Conf. Ser. in Math., no. 38, Amer. Math. Soc., Providence, R. I., 1978.

2. P. Hartman, *Ordinary differential equations*, 2nd ed, Birkhauser, Boston, Mass., 1982.

3. D. Terman, *Directed graphs and traveling waves*, Trans. Amer. Math. Soc. **289** (1985), 809–847.

4. ______, *Infinitely many radial solutions of an elliptic system* (submitted).

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48824

Current address: Department of Mathematics, Ohio State University, Columbus, Ohio 43210