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STRONG MULTIPLICITY THEOREMS FOR $GL(n)$

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ABSTRACT. Let $\pi = \otimes\pi_v$ be a cuspidal automorphic representation of $GL(n, F_A)$, where F_A denotes the adeles of a number field F. Let E be a Galois extension of F and let $\{g\}$ denote a conjugacy class of the Galois group. The author considers those cuspidal automorphic representations which have local components π_v whenever the Frobenius of the prime v is $\{g\}$, showing that such representations are often easily described and finite in number. This generalizes a result of Moreno [Bull. Amer. Math. Soc. **11** (1984), pp. 180–182].

1. Notation. We first introduce the notation to be used throughout this paper. Let F_A denote the ring of adeles of a number field F. We denote by v a typical prime of F, usually finite, and by F_v the completion of F at v. When v is finite, we write Nv for the order of its residue field.

An automorphic (cuspidal) representation of $GL(n, F_A)$, denoted by π, is an irreducible component of the right regular representation of $GL(n, F_A)$ on the space of automorphic (cuspidal) forms. For the definition of an automorphic form see [**1, 19**]. We write $\tilde{\pi}$ for the contragredient representation to π.

The representation π factors into a product of local representations $\pi \simeq \otimes\pi_v$. Those local representations whose restriction to the maximal compact subgroup of $GL(n, F_v)$ contains the trivial representation are said to be unramified. Attached to π is a global L-function $L(s, \pi)$ which is a product of local L-functions $L_v(s, \pi_v)$, which for finite, unramified primes are of the form

$$L_v(s, \pi_v) = \prod_{i=1}^{n}(1 - \mu_{v,i}Nv^{-s})^{-1}.$$

The complex numbers $\mu_{v,i}$ are called eigenvalues of π_v. For finite, unramified primes, the central character of π at v is just $\prod_{i=1}^{n}\mu_{v,i}$ [**7**].

We remark that if χ is a unitary character of the ideles of F, then

$$\pi \times \chi = \pi \times (\chi \circ \det)$$

is cuspidal if π is.

We use the expressions $f(x) = O(g(x))$ and $f(x) \ll g(x)$ synonymously to mean there exists a constant C such that $|f(x)| \leq Cg(x)$ for x a sufficiently large real number. We let $o(g(x))$ represent a function $f(x)$ such that $\lim_{x\to\infty} f(x)/g(x) = 0$.

2. Statement of main results. Let E be a finite normal extension of the number field F with Galois group G. To each prime v of F which does not ramify

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in E, one can associate a conjugacy class of G, an element of which is known as a Frobenius automorphism associated to v. We denote this class by σv.

Consider the following question: Fix a conjugacy class $\{g\}$ of G. Suppose π and π' are two cuspidal automorphic representations of $GL(n, F_A)$ for which $\pi_v \simeq \pi'_v$ for primes v such that $\sigma v = \{g\}$. If we think of π as given, how many possibilities for π' exist?

We expect the general answer to be that $\pi' \simeq \pi \times \chi$ for χ an abelian character of G unless the liftings (base change) of π and π' to E are both noncuspidal. If $\pi' \simeq \pi \times \chi$, we say that π and π' are twist equivalent.

We could only prove the following results, for reasons sketched in the next section and fully explained in the rest of the paper.

THEOREM A. *Let π, π' be cuspidal automorphic representations of $GL(n, F_A)$. Suppose the eigenvalues of π and π' satisfy a weak Ramanujan-Petersson conjecture:*

$$\mu_{v,i} = o\left(Nv^{1/4}\right) \quad \text{and} \quad \mu'_{v,i} = o\left(Nv^{1/4}\right).$$

Let E be a finite abelian extension of F with Galois group G.

There exists a constant $B(\pi, \pi')$ such that if $\pi_v \simeq \pi'_v$ for all primes whose Frobenius is g with $Nv \leq B(\pi, \pi')$, then (at least) one of the following is true:

(i) *$\pi' \simeq \pi \times \chi$ for some character χ of G with $\chi(g) = 1$.*

(ii) *$\pi \simeq \pi \times \chi$ for some character χ of G with $\chi(g) \neq 1$ and $\pi' \simeq \pi' \times \chi'$ for some character χ' of G with $\chi'(g) \neq 1$.*

In either case, the central characters of π and π' differ by a character of the quotient group $G/\langle g \rangle$.

THEOREM B. *Let π, π' be cuspidal automorphic representations of $GL(3, F_A)$ or of $GL(2, F_A)$.*

Let E be a finite abelian extension of F with Galois group G.

The conclusions of Theorem A *hold.*

Now let E be a solvable extension of F with Galois group G. Let F_g be the largest subfield of E fixed by g which is obtained from a tower of abelian extensions starting with F. Let G_g be the subgroup of G fixing F_g.

THEOREM C. *Let π, π' be cuspidal automorphic representations of $GL(2, F_A)$.*

Let E be a solvable extension of F with Galois group G.

Suppose G_g has abelian commutator subgroup.

There exists a constant $B(\pi, \pi')$ such that, if $\pi_v \simeq \pi'_v$ for all primes whose Frobenius σv is $\{g\}$ when $Nv \leq B(\pi, \pi')$, then (at least) one of the following holds.

(i) *$\pi' \simeq \pi \times \chi$ for χ an abelian character of G with $\chi(g) = 1$.*

(ii) *The lifts π_E and π'_E of π and π' to automorphic representations of E are noncuspidal and $\pi_E \simeq \pi'_E$.*

(iii) *π_E and π'_E are both noncuspidal and $\sum_i \mu_{v,i} = \sum_i \mu'_{v,i} = 0$ for all v with $\sigma v = \{g^r\}$ where (r, order g)=1.*

In all cases, the central characters of π and π' differ by a character of G which is 1 on the group generated by g.

COROLLARY C1. *If G is solvable and $g = 1$, then Theorem* C *holds.*

COROLLARY C2. *If G has a normal abelian subgroup A such that G/A is nilpotent, then Theorem C holds for any g.*

The constant $B(\pi, \pi')$ will be effective in terms of the local representations for primes of F which are infinite or ramified for π, π', or in E.

We view these theorems as generalized or weighted versions of Dirichlet's theorem on primes in progression or of the Chebotarev density theorem.

3. Preliminaries. Outline of proof. Our main tool in proving Theorems A, B, and C is the Rankin-Selberg convolution L-function associated to two cuspidal automorphic representations π and π'. Written $L(s, \pi \times \tilde{\pi}')$, it is a product of local L-functions $L_v(s, \pi_v \times \tilde{\pi}'_v)$ which for finite primes unramified for π and π' are of the form

$$L_v(s, \pi_v \times \tilde{\pi}'_v) = \prod_{i,j=1}^{n} (1 - \mu_{v,i}\bar{\mu}'_{v,j} N v^{-s})^{-1}.$$

Let S be a finite set of primes of F which includes the infinite primes, the primes which ramify for π or π', and the primes which ramify in E. We form the product of local L-functions over the "good" primes

$$L_S(s, \pi \times \tilde{\pi}') = \prod_{v \notin S} L_v(s, \pi_v \times \tilde{\pi}'_v).$$

Among its properties, we shall make use of the following:

THEOREM 3.1 (JACQUET, SHALIKA [**13**]). (i) *$L_S(s, \pi \times \tilde{\pi}')$ converges absolutely for* $\operatorname{Re}(s) > 1$.

(ii) *Let $X = \{1 + it \colon \pi' \simeq \pi \times |\quad|_{F_A}^{it}$. The $L_S(s, \pi \times \tilde{\pi}')$ extends to a continuous function on the complement of X in* $\operatorname{Re}(s) \geq 1$. *If $s_0 \in X$, then*

$$\lim_{s \to s_0} (s - s_0) L_S(s, \pi \times \tilde{\pi}')$$

exists and is a finite nonzero number.

(iii) *$L_S(s, \pi \times \tilde{\pi}')$ has meromorphic continuation to* $\operatorname{Re}(s) > 0$.

THEOREM 3.2 (SHAHIDI [**22**, P. 353]). *$L_S(s, \pi \times \pi')$ is nonzero on* $\operatorname{Re}(s) = 1$.

We note that Theorem 3.1 is sufficient to show that π is determined by its L-function [**13**, I, §4].

In order to make the constants $B(\pi, \pi')$ effective in terms of the infinite and ramified primes, we shall need further properties of $L_S(s, \pi \times \tilde{\pi}')$ which require the following assumptions.

HYPOTHESIS 3.3. (i) *For ramified primes, $L_v(s, \pi_v \times \tilde{\pi}'_v)$ is Eulerian of degree at most n^2.*

(ii) *For infinite primes, $L_v(s, \pi_v \times \tilde{\pi}'_v)$ is a product of exponential and gamma functions.*

(iii) *$L_S(s, \pi \times \tilde{\pi}')$ has analytic continuation to the complex plane with the exception of a potential simple pole on the line* $\operatorname{Re} s = 1$.

(iv) $L(s, \pi \times \tilde{\pi}') = \prod_v L_v(s, \pi_v \times \tilde{\pi}'_v)$ *satisfies a functional equation of the form*

$$L(s, \pi \times \tilde{\pi}') = \varepsilon(s, \pi \times \tilde{\pi}') L(1 - s, \tilde{\pi} \times \pi'),$$

where $\varepsilon(s, \pi \times \tilde{\pi}')$ *is an exponential function which depends on the ramified primes.*

Hypothesis 3.3 is a theorem for $GL(2)$ [**9**]. In the general case, the meromorphic continuation and functional equation of $L(s, \pi \times \tilde{\pi}')$ have been announced in [**12**, p. 368], while the meromorphic continuation of $L_S(s, \pi \times \tilde{\pi}')$ has been asserted in [**13**, I, p. 554]. For a global viewpoint in the case of $GL(3,Q)$ see [**4**].

This paper has its origins in a paper of Moreno [**17, 18**].

The proof of Theorems A, B, and C basically mimics the proof of Dirichlet's theorem on primes in progression or, more generally, the Chebotarev density theorem. Classically, one considers the logarithmic derivative of the Artin L-functions formed by the characters of the Galois group G, specifically keeping track of poles at $s = 1$. One uses the orthogonality relations for characters to show that each conjugacy class of G contributes to the pole at $s = 1$ of the Dedekind zeta function of the field F, thus proving there are infinitely many primes in each class.

Although we follow these lines, substituting the Rankin-Selberg convolution L-function for the Dedekind zeta function, two new problems arise. First, classically, it is easy to show that prime powers do not contribute to the pole. In our case, the eigenvalues $\mu_{v,i}$ and $\mu'_{v,i}$, hence the prime power coefficients of the convolution L-functions, may in theory be fairly large. The Ramanujan-Petersson conjecture asserts they are not. For general $GL(n)$ we assume the weak Ramanujan-Petersson conjecture of Theorem A, a hypothesis which is not required for $GL(2)$ and $GL(3)$.

The second problem arises when we attempt to twist by the nonabelian characters of G. In general, the analytic behavior at $s = 1$ of these twisted Rankin-Selberg convolutions is unknown. A comprehensive theory of lifting or base change would remedy this situation. Currently, base change results are available only for special extensions of F and only for $GL(2)$ [**16, 11**].

4. A relation between coefficients and poles. In this section we formulate an asymptotic equation relating the coefficients of the Rankin-Selberg convolution to its behavior at $s = 1$. Making the assumptions of Hypotheses 3.3 (a theorem for $GL(2)$), a result of Moreno [**17, 18**] makes the constants effective in terms of the infinite and ramified primes of the automorphic representations involved.

We shall be dealing with the logarithmic derivative of $L_S(s, \pi \times \tilde{\pi}')$ which may be written

$$-\frac{L'_S}{L_S}(s, \pi \times \tilde{\pi}') = \sum_{v \notin S, m \geq 1} b_{v,m} \log Nv Nv^{-ms},$$

where $b_{v,m} = \sum_{i,j} \mu_{v,i}^m \bar{\mu}'^m_{v,j}$.

Let

$$\delta(\pi, \pi') = \begin{cases} 1 & \text{if } \pi \simeq \pi', \\ 0 & \text{if } \pi \not\simeq \pi'. \end{cases}$$

Theorems 3.1 and 3.2 combined with a standard Tauberian theorem [**15**, Chapter 15] are enough to yield the following ineffective result.

PROPOSITION 4.1. *Let π and π' be two cuspidal automorphic representations of $GL(n, F_A)$. Then*

$$\sum_{v \notin S, Nv^m \leq x} b_{v,m} \log Nv \sim \delta(\pi, \pi')x.$$

Moreno's effective version is obtained by integrating the product of the logarithmic derivative and a kernel function over the same rectangle(s) used in the proof of the Prime Number Theorem. The kernel function

$$k(s,x) = \left(\frac{x^{2s-2} - x^{s-1}}{s-1}\right)^2$$

is chosen for three reasons:

(i) within the critical strip, $0 \leq \mathrm{Re}(s) \leq 1$, it is "large" at $s = 1$,

(ii) it allows absolute integrability on vertical lines,

(iii) its inverse Mellin transform for $u > 0$,

$$\hat{k}(u,x) = \begin{cases} u^{-1}\log(u/x^2) & \text{if } x^2 \leq u \leq x^3, \\ u^{-1}\log(x^4/u) & \text{if } x^3 \leq u \leq x^4, \\ 0 & \text{otherwise} \end{cases}$$

is nonzero only for $u \leq x^4$.

We note that the $\log^2 x$ appearing in the following theorem is just $k(1,x)$.

THEOREM 4.2 (MORENO **[17, 18]**). *Let π and π' be cuspidal automorphic representations of $GL(n, F_A)$ satisfying Hypothesis* 3.3. *Then*

$$\sum_{v \notin S, m \geq 1} b_{v,m} \log Nv\, \hat{k}(Nv^m, x) = \delta(\pi, \pi') \log^2 x + O(1),$$

where the constant depends only on the infinite and ramified local factors of π and π'.

We point out that the left-hand sum is only over prime powers with $Nv^m \leq x^4$, hence is finite.

Returning briefly to the ineffective case, if we let

$$\hat{k}(Nv^m, x) = \begin{cases} (\log^2 x)/x^4, & x^2 \leq Nv^m \leq x^4, \\ 0, & \text{otherwise}, \end{cases}$$

then the conclusion of Proposition 4.1 may be reformulated as

$$\sum_{v \notin S,\ m \geq 1} b_{v,m} \log Nv\, \hat{k}(Nv^m, x) \sim \delta(\pi, \pi') \log^2 x.$$

We use this notational convenience throughout the paper. Despite this notation, we are merely keeping track of poles at $s = 1$ in the ineffective case.

5. Removal of prime powers. The asymptotic equation of Theorem 4.2 involves a sum over prime powers. We shall actually need the sum to be over primes and, therefore, would like to show that there is negligible contribution from powers greater than one. Although in the case of classical L-functions this is trivial, for the Rankin-Selberg convolutions we consider this is not always possible. A slightly weak form of the Ramanujan-Petersson conjecture asserts that, for any $\lambda > 0$, the eigenvalues $\mu_{v,i}$ should satisfy $\mu_{v,i} = O(Nv^{\lambda})$, where the implied constant depends on the automorphic representation π, but not on the prime v.

The current state of affairs indicated below, though a far cry from the Ramanujan-Petersson conjecture, suffices for $GL(2)$ and $GL(3)$. For general $GL(n)$ we shall be forced to assume the truth of the above form of the conjecture for some $\lambda < 1/4$.

PROPOSITION 5.1. *Let π be a cuspidal automorphic representation of $GL(n, F_A)$.*

(i) *$\mu_{v,i} = o(Nv^{1/2})$ with the implied constant depending on π and ineffective.*

(ii) *Assume further Hypothesis* 3.3. *Then $|\mu_{v,i}| \leq Nv^{1/2-\varepsilon}$ with ε effective in terms of the infinite components of π.*

PROOF. Part (i) is an immediate consequence of Proposition 4.1(ii), pointed out by Serre, follows from [**3**, Theorem 4.1] which estimates coefficients for Dirichlet series which satisfy a functional equation with gamma factors. □

We remark that, for $GL(2)$, the symmetric square lift of Shimura [**23**] and Gelbart and Jacquet [**5**] allows one to improve the exponent 1/2 of Proposition 5.1 to 1/4. In what follows, this result would improve constants were they ever actually computed, but is otherwise unnecessary.

Lemma 5.2 and Proposition 5.4 show when prime powers may be removed from the basic asymptotic relation and cover both the effective and the ineffective cases.

LEMMA 5.2. *Suppose the eigenvalues $\mu_{v,i}$ and $\mu'_{v,i}$ of cuspidal automorphic representations π and π' are $o(Nv^{1/4})$. Then*

$$\sum_{v \notin S} b_{v,1} \log Nv\, \hat{k}(Nv, x) \sim \delta(\pi, \pi') \log^2 x.$$

PROOF. We prove only the effective case, the proof in the ineffective case being similar.

A trivial estimate for the kernel function introduced in §4 yields

$$\hat{k}(u, x) \leq u^{-1} \log x \leq (2u)^{-1} \log u$$

when $\hat{k}(u, x) \neq 0$. Thus,

$$\left| \sum_{v \notin S,\ m>1} b_{v,m} \log Nv\ \hat{k}(Nv^m, x) \right| \leq \sum_{\substack{v \notin S.\ m>1 \\ x^2 \leq Nv^m \leq x^4}} |b_{v,m}| \log Nv\ Nv^{-m} \log Nv^m.$$

Replacing $|b_{v,m}| = \left|\sum_{i,j} \mu_{v,i}^m \bar{\mu}_{v,j}^{\prime m}\right|$ by $o(Nv^{m/2})$ and $\log Nv^m$ by $\log x^4$, we obtain

$$\left| \sum_{v \notin S,\ m>1} b_{v,m} \log Nv\ \hat{k}(Nv^m, x) \right| \ll \sum_{\substack{m>1 \\ x^2 \leq Nv^m \leq x^4}} \log Nv\ o(1) Nv^{-m/2} \log x.$$

Looking first at the terms with $m > 2$, we have

$$\begin{aligned} \sum_{\substack{m>2 \\ x^2 \leq Nv^m \leq x^4}} \log Nv\ o(1) Nv^{-m/2} \log x &\ll \left\{ \sum_{m>2} -\frac{\varsigma'}{\varsigma}\left(\frac{m}{2}\right) \right\} \cdot \log x \\ &\ll \log x, \end{aligned}$$

where ς denotes the Riemann zeta function.

For $m = 2$, we have

$$\sum_{x^2 \le Nv^2 \le x^4} \log Nv \; o(1) Nv^{-1} \log x = \left\{ \sum_{x \le Nv \le x^2} \log Nv \; o(1) Nv^{-1} \right\} \log x.$$

By a step which leads to the Prime Number Theorem [**2**, p. 5], this last expression is

$$o(\log x^2) \log x \ll o(\log^2 x).$$

We have shown that the contribution of prime powers is $o(\log^2 x)$ which means we can drop them from the asymptotic relation of §4, proving the lemma. □

The above assumption about eigenvalues is unnecessary for $GL(2)$ and $GL(3)$. There, at most one eigenvalue of the either two or three local eigenvalues can be large. If one actually is large, then $b_{v,m}$ may be approximated by $b_{v,1}^m$. We shall use the following fact whose elementary proof we leave to the reader.

LEMMA 5.3. *Let $z \neq 0$ be a complex number, k a positive integer, and $\delta = 0$ or 1. Define*

$$f_k(z) = z^k + \delta + \bar{z}^{-k}.$$

Then

$$|f_{k+1}(z)|^{2k/(k+1)} \le 16 |f_k(z)|^2.$$

We next show how prime powers may be removed for $GL(2)$ and $GL(3)$. Again our result includes both the effective and ineffective cases.

PROPOSITION 5.4. *Let π and π' be cuspidal automorphic representations of $GL(n, F_A)$, $n = 2$ or 3. Then*

$$\sum_{v \notin S} b_{v,1} \log Nv \; \hat{k}(Nv, x) \sim \delta(\pi, \pi') \log^2 x.$$

PROOF. Since

$$\begin{aligned} |b_{v,m}| &= \left| \left(\sum_i \mu_{v,i}^m \right) \left(\sum_j \bar{\mu}_{v,j}'^m \right) \right| \\ &\le \frac{1}{2} \left(\left| \sum_i \mu_{v,i}^m \right|^2 + \left| \sum_i \mu_{v,i}'^m \right|^2 \right), \end{aligned}$$

it will be sufficient to remove prime powers from the asymptotic relations of §4 for $\pi \simeq \pi'$.

For $GL(n)$, the fact that π is unitary implies $\{\mu_{v,i}^{-1}\} = \{\bar{\mu}_{v,i}\}$ for finite, unramified primes v [**13**]. For $GL(2)$ or $GL(3)$, if one eigenvalue, say $\mu_{v,1}$, satisfies $|\mu_{v,1}| > 1$, then another, say $\mu_{v,2}$, satisfies $\mu_{v,2} = 1/\bar{\mu}_{v,1}$. In the case of $GL(3)$, one would then have $|\mu_{v,3}| = 1$. Of course, the Ramanujan-Petersson conjecture asserts that such large eigenvalues do not exist.

Prime power coefficients for primes all of whose eigenvalues have absolute value one are easily removed as in the proof of Lemma 5.2

We complete the proof only in the effective case; the proof in the ineffective case is similar but easier.

Writing $\sum'$ for the sum over primes which have one eigenvalue with absolute value greater than one,

$$\sum_{v\notin S,\ m>1} b_{v,m}\log Nv\ \hat{k}(Nv^m,x) = \sum_{v\notin S,\ m>1}{}' b_{v,m}\log Nv\ \hat{k}(Nv^m,x) + o(\log^2 x).$$

For each prime involved in $\sum'$, we order the two or three eigenvalues by $|\mu_{v,1}| > 1 > |\mu_{v,2}|$ as above. For these primes, letting $\delta = 0$ for $GL(2)$ and $\delta = 1$ for $GL(3)$, we note that $b_{v,m} = |f_m(\mu_{v,1})|^2$ for $GL(2)$ and $b_{v,m} = |f_m(\mu_{v,1}/\mu_{v,3})|^2$ for $GL(3)$, where $f_m(z)$ is the function introduced in Lemma 5.3.

We also have

$$\begin{aligned}\hat{k}(u,x) &= (1+k^{-1})u^{-1/(k+1)}\hat{k}\left(u^{k/(k+1)}, x^{k/(k+1)}\right)\\ &\le 2u^{-1/(k+1)}\hat{k}\left(u^{k/(k+1)}, x^{k/(k+1)}\right).\end{aligned}$$

We use this kernel estimate and Lemma 5.3 with $k = 2m$ to estimate the restricted sum over odd powers

$$\begin{aligned}&\sum_{v\notin S,\ m>1}{}' b_{v,2m+1}\log Nv\ \hat{k}(Nv^{2m+1},x)\\ &\qquad \ll \sum_{v\notin S,\ m>1}{}' b_{v,2m+1}^{1/(2m+1)} Nv^{-1} b_{v,2m}\log Nv\ \hat{k}\left(Nv^{2m}, x^{2m/(2m+1)}\right).\end{aligned}$$

The coefficient estimate of Proposition 5.1 implies $b_{v,2m+1}^{1/(2m+1)}Nv^{-1} \ll 1$, yielding

$$\begin{aligned}&\sum_{v\notin S,\ m>1}{}' b_{v,m}\log Nv\ \hat{k}(Nv^m,x)\\ &\qquad \ll \sum_{v\notin S,\ m>1}{}' b_{v,2m}\log Nv\left[\hat{k}(Nv^{2m},x) + \hat{k}\left(Nv^{2m}, x^{2m/(2m+1)}\right)\right].\end{aligned}$$

We again use the above kernel estimate and Lemma 5.3, this time with $k = 1$, to estimate this new sum

$$\begin{aligned}&\sum_{v\notin S,\ m>1}{}' b_{v,m}\log Nv\ \hat{k}(Nv^m,x)\\ &\quad \ll \sum_{v\notin S,\ m>1}{}' b_{v,2m}^{1/2} Nv^{-m} b_{v,m}\log Nv\left[\hat{k}(Nv^m,x^{1/2}) + \hat{k}\left(Nv^m, x^{m/2(m+1)}\right)\right].\end{aligned}$$

We have $b_{v,2m}^{1/2}Nv^{-m} = o(1)$. Removing our restriction on primes yields

$$\begin{aligned}&\sum_{v\notin S,\ m>1}{}' b_{v,m}\log Nv\,\hat{k}(Nv^m,x)\\ &\qquad \ll o(1)\sum_{v\notin S,\ m\ge 1} b_{v,m}\log Nv\left[\hat{k}(Nv^m,x^{1/2}) + \hat{k}\left(Nv^m, x^{m/(2m+1)}\right)\right].\end{aligned}$$

Finally, since

$$\hat{k}\left(Nv^m, x^{m/(2m+1)}\right) \le \hat{k}(Nv^m, x^{1/3}) + \hat{k}(Nv^m, x^{1/2}),$$

the asymptotic relation of Theorem 4.2 implies

$$\sum_{v\notin S,\ m>1}{}' b_{v,m} \log Nv\, \hat{k}(Nv^m, x) \ll o(1)\left(\log^2 x^{1/2} + \log^2 x^{1/3} + \log^2 x^{1/2}\right)$$
$$\ll o(\log^2 x),$$

proving the proposition.

6. The abelian case. Having completed necessary preliminaries, we proceed to the main results of this paper. In this section we prove Theorems A and B which were stated back in §2. Theorem C and some additional results will be proved in the final, most important, section.

The constants will be effective for $GL(2)$ and effective in general if one assumes Hypothesis 3.3.

Let χ be a character of the abelian group $G = \mathrm{Gal}(E/F)$. Then $\pi \times \chi$ is also a cuspidal automorphic representation of $GL(n, F_A)$. For finite primes v which do not ramify for π, π', or E (the only primes we allow in our L-function L_S), the local eigenvalues are $\{\chi(v)\mu_{v,i}\}$ where $\chi(v) = \chi(\sigma v)$ is the value of χ at the Frobenius automorphism σv associated to v.

From now on, we will use $b_{v,m}$ for the coefficients of the logarithmic derivative of $L_S(s, \pi \times \tilde{\pi}')$ and $a_{v,m}$ for those of $L_S(s, \pi \times \tilde{\pi})$.

We apply Lemma 5.2 for Theorem A and Proposition 5.4 for Theorem B to $(\pi \times \chi) \times \tilde{\pi}$ and $(\pi \times \chi) \times \tilde{\pi}'$. Taking the difference yields

$$\sum_{v\notin S} \chi(v)(a_{v,1} - b_{v,1}) \log Nv\ \hat{k}(Nv, x) \sim [\delta(\pi \times \chi, \pi) - \delta(\pi \times \chi, \pi')] \log^2 x.$$

Multiplying the above by $\bar{\chi}(g)$ and summing over all characters of G, we obtain

$$\sum_{\sigma v = g} (a_{v,1} - b_{v,1}) \log Nv\ \hat{k}(Nv, x)$$
$$\sim |G|^{-1} \sum_{\chi} \bar{\chi}(g)[\delta(\pi \times \chi, \pi) - \delta(\pi \times \chi, \pi')] \log^2 x$$

from the orthogonality relations for characters.

The assumption that $\pi_v \simeq \pi'_v$ for $\sigma v = g$ and $Nv \leq x^4$ implies the left-hand side is zero. If x is sufficiently large, we must have

$$\sum_{\chi} \bar{\chi}(g)[\delta(\pi \times \chi, \pi) - \delta(\pi \times \chi, \pi')] = 0.$$

It is an easy exercise to show that

$$\sum_{\chi} \bar{\chi}(g)[\delta(\pi \times \chi, \pi) - \delta(\pi \times \chi, \pi')] = (1 - \bar{\psi}(g)) \sum_{\chi:\ \pi\times\chi\simeq\pi} \bar{\chi}(g),$$

where ψ is any character of G such that $\pi' \simeq \pi \times \psi$. By convention, $\psi(g) = 0$ if no such ψ exists. From this last equation, we see that either $\pi' \simeq \pi \times \psi$ with $\psi(g) = 1$ or $\pi \simeq \pi \times \chi$ with $\chi(g) \neq 1$. By interchanging the roles of π and π', we observe that if only the latter holds, then $\pi' \simeq \pi' \times \chi'$ with $\chi'(g) \neq 1$ as well.

The main parts of Theorems A and B are proved. To verify the concluding statement about the central characters, we simply observe that they agree for primes with $\sigma v = g$ since the local representations are equivalent.

7. The nonabelian case. When the Galois group G of E/F is not abelian, we would like to prove that, if $\pi_v \simeq \pi'_v$ for all v such that $\sigma v = \{g\}$ (the conjugacy class of g), then either π and π' are twist equivalent or the liftings of π and π' to E are both noncuspidal. The latter is the generalization of π and π' being twist equivalent to themselves as in part (ii) of Theorems A and B.

We encounter immediate problems trying to use twists and orthogonality relations to isolate a conjugacy class as we did in the abelian case. When χ is a nonabelian character, $\pi \times \chi$ is no longer an automorphic representation of $GL(n, F_A)$. The analytic behavior of $L_S(s, \pi \times \tilde{\pi}')$ at $s = 1$, hence the asymptotic behavior of

$$\sum_{v \notin S,\ m \leq 1} \chi(\sigma v^m) b_{v,m} \log Nv\ \hat{k}(Nv^m, x),$$

is no longer apparent.

Our attempt to circumvent this difficulty begins with a well-known theorem on representations of finite groups due to Artin.

THEOREM 7.1 (ARTIN [**21**, §9.2]). *Any character χ of a finite group G can be written as a linear combination with rational coefficients of characters induced from one-dimensional characters of subgroups.*

We may now write χ in the form $\sum_k t_k \operatorname{Ind}_{G_k}^G \psi_k$, where the t_k are rational and the ψ_k are one-dimensional characters of the subgroup G_k. Such an expression is not necessarily unique.

We have reduced the problem of asymptotic equations for twists to determining the asymptotic behavior of

$$\sum_{v \notin S,\ m \geq 1} (\operatorname{Ind}_{G_k}^G \psi_k)(\sigma v^m) b_{v,m} \log Nv\ \hat{k}(Nv^m, x).$$

We let F_k be the subfield of E fixed by G_k. Note that the dimension of $\operatorname{Ind}_{G_k}^G \psi_k$ is $|G|/|G_k| = [F_k : F]$.

We now examine the local L-functions $L_v(s, \operatorname{Ind}_{G_k}^G \psi_k \times \pi_v \times \tilde{\pi}'_v)$ at a prime $v \notin S$. Recall that a local L-function may be written as

$$L_v(s, \pi_v \times \tilde{\pi}'_v) = \prod_{i,j=1}^{n} (1 - \mu_{v,i}\bar{u}'_{v,j} Nv^{-s})^{-1} = \prod_{i,j=1}^{n} (1 - Nv^{-(s-s_{ij})})^{-1},$$

where s_{ij} is a complex number such that $Nv^{s_{ij}} = \mu_{v,i}\bar{\mu}'_{v,j}$. We observe that each of the n^2 factors of the Euler product has now been written as the translation, by $-s_{ij}$, of the local factor of the Dedekind zeta function of the number field F. Twisting by the character $\operatorname{Ind}_{G_k}^G \psi_k$ transforms each factor into the (translated) local factor of the Artin L-function $L(s - s_{ij}, \operatorname{Ind}_{G_k}^G \psi_k, E/F)$. Since each of these local factors is Eulerian of degree $|G|/|G_k|$, the total local factor,

$$L_v(s, \operatorname{Ind}_{G_k}^G \psi_k \times \pi_v \times \tilde{\pi}'_\pi) = \prod_{i,j=1}^{n} L_v(s - s_{ij}, \operatorname{Ind}_{G_k}^G \psi_k, E/F)$$

is Eulerian of degree $n^2|G|/|G_k|$. Because the above characters are induced, we may rewrite this as

$$L_v(s, \operatorname{Ind}_{G_k}^G \psi_k \times \pi_v \times \tilde{\pi}'_v) = \prod_{w|v} \prod_{i,j=1}^{n} L_w(s - s_{ij}, \psi_k, E/F_k),$$

where the first product on the right-hand side of the equation is over the primes w of F_k lying above v. For details see [**15**, Chapter 12].

The above shows that $L_S(s, \mathrm{Ind}_{G_k}^G \psi_k \times \pi \times \tilde{\pi}')$ may be thought of as an L-function for F_k which is Eulerian of degree n^2. Conjecturally, it always corresponds to the convolution L-function of two automorphic, not necessarily cuspidal, representations of $GL(n, (F_k)_A)$.

The automorphic representation corresponding to π is called the lifting, or base change, of π from F to F_k. Its L-function coincides with $L(s, \mathrm{Ind}_{G_k}^G 1 \times \pi)$. We write π_{F_k} for this conjectured representation. If G_k is normal in G, then $\mathrm{Ind}_{G_k}^G 1$ is just the character of the regular representation of G/G_k.

Base change has only been verified for certain extensions in the case of automorphic representations of $GL(2, F_A)$. We record what is known below.

THEOREM 7.2: CYCLIC BASE CHANGE (LANGLANDS [**16**]; SEE ALSO [**6**]). *Let π be an automorphic representation of $GL(2, F_A)$ and let E be a cyclic extension of F of prime degree. Then there exists an automorphic representation π_E of $GL(2, E_A)$ whose L-function coincides with that for $\pi \times \chi_{\mathrm{reg}}$, where χ_{reg} is the character of the regular representation of* $\mathrm{Gal}(E/F)$.

Suppose further that π is cuspidal. The lift π_E is then cuspidal unless E/F is quadratic and corresponds via its L-function to a representation of $GL(1, E_A)$.

The comment that noncuspidality generalizes $\pi \times \chi \simeq \pi$ is explained by the following:

THEOREM 7.3 (LABESSE, LANGLANDS [**14**, §6]). *Let π be a cuspidal automorphic representation of $GL(2, F_A)$ and let χ be a character of $GL(1, F_A)$. If $\pi \times \chi \simeq \pi$, then χ is of order 2 and π corresponds to a $GL(1)$ representation of the quadratic extension of F determined by χ.*

The preceding results are based on the trace formula. The converse theorem yields the special result:

THEOREM 7.4: CUBIC BASE CHANGE (JACQUET, PIATETSKI-SHAPIRO, SHALIKA [**11**]). *Let π' be an automorphic representation of $GL(2, F_A)$ and let E be a nonnormal cubic extension of F. Then there exists an automorphic representation π_E of $GL(2, E_A)$ whose L-function is the convolution of π and the automorphic representation of $GL(3, F_A)$ whose L-function is the Dedekind zeta function of E.*

Moreover, if π is cuspidal, then π_E is cuspidal unless π corresponds to a $GL(1)$ representation of the quadratic extension of F contained in the normal closure of E.

We are forced to restrict consideration to $GL(2)$. Since we consider all characters χ of G simultaneously, we add a subscript χ to t_k, G_k, ψ_k and F_k. If all the fields $F_{\chi k}$ arising from Theorem 7.1 are in towers of cyclic or towers of cyclic and cubic extensions, then we can obtain an asymptotic estimate for

$$\sum_{v \notin S,\ m \geq 1} \chi(\sigma v^m) b_{v,m} \log Nv\ \hat{k}(Nv^m, x).$$

The orthogonality relations for characters can then be used to isolate any conjugacy class of G.

Because the cubic lift is a fairly special case, we focus attention on discovering when cyclic towers exist. Before doing so, however, we mention that the cubic lift may be used to isolate any conjugacy class of a supersolvable group of order 2^i3^jm, where the subgroup of order m is either abelian or of order the product of two primes.

If G is not solvable, the intersection of all $G_{\chi k}$ must contain a normal, nonabelian group which is its own commutator. Since all the characters $\psi_{\chi k}$ must be 1 on this group, we cannot separate elements in the same coset of this group. Thus, we are forced to assume G is solvable. Even this is too weak an assumption.

We observe that $\{g\}$ can be isolated using cyclic base change if and only if the subspace of the space of class functions on G spanned by characters (not necessarily irreducible) induced from one-dimensional characters of subgroups in cyclic towers contains the characteristic function of $\{g\}$.

Let F_g be the largest subfield of E fixed by g which may be reached by a tower of cyclic extensions from F. Let G_g be the subgroup of G fixing F_g.

Writing G'_g for the commutator subgroup of G_g, the following proposition gives the group theoretic conditions under which the class of g may be isolated.

PROPOSITION 7.5. *The following are equivalent:*
(i) *$\{g\}$ may be isolated,*
(ii) $\{g\} \supset gG'_g$.

PROOF. (ii)$\Rightarrow$(i) We can use the abelian characters of G_g to obtain the characteristic function of gG'_g in the space of class functions of G_g. Inducing to G yields a multiple of the characteristic function of $\{g\}$ in G.

(i)$\Rightarrow$(ii) Conversely, any character induced up a cyclic tower must be constant on gG'_g, since any group in a cyclic tower either contains G_g or contains no member of the coset gG'_g. □

Before proving Theorem C, we provide examples of groups for which cyclic base change is not sufficient to isolate all conjugacy classes. The first such group is $SL(2, F_3)$, a group of order 24 whose commutator subgroup is the quaternions. The first example which is supersolvable is the group of order 54 whose commutator subgroup is the nonabelian group of order 27, all of whose elements are of order 3 [**8**, p. 57]. Use of cubic base change would allow one to isolate all conjugacy classes of this group; however, one could still not isolate all classes of the analogous group of order 250.

The remainder of this paper is devoted to the proof of Theorem C. We first show that under the hypotheses of either Theorem C or its corollaries, the conjugacy class of g may be isolated in the asymptotic relation of Proposition 5.4. As in the proof of Theorems A and B in §6, we classify the ways in which the various poles may cancel. Finally, we examine the implications of the different types of poles which may occur.

We begin by showing that if G'_g is abelian, the hypothesis in Theorem C, then the conjugacy class of g in G_g is gG'_g.

LEMMA 7.6. *Let N be a finite group such that N' is abelian and the quotient group N/N' is generated by gN'. Then $\{g\} = gN'$.*

PROOF. Let H be the subset of N' such that for h in H, gh is conjugate to g. Because N' is abelian, H is a subgroup of N'.

A product of terms of the form, $(g^in_1)(g^jn_2)(g^in_1)^{-1}(g^jn_2)^{-1}$, with n_1 and n_2 in N', is in H since g commutes with an element of N' up to an element of H. However, since products of the above from generate N', we have $H = N'$. □

Because any subgroup of a nilpotent group is contained in a cyclic tower, the assumption of Corollary 2 that G has a normal abelian subgroup A with G/A nilpotent allows one to isolate any conjugacy class of G. In fact, all representations of such a group are monomial representations induced up cyclic towers as the following adaptation of a result of Huppert [**20**, p. 63] shows.

LEMMA 7.7. *Suppose a finite group G has a normal abelian subgroup A such that G/A is nilpotent. Then all representations of G are induced from one-dimensional representations of subgroups whose fixed fields may be obtained from a tower of cyclic extensions.*

PROOF. We first observe that A may be taken to properly contain the center [**20**, p. 63].

Proceeding by induction on the order of G, we may assume the representation χ in question is faithful. Now, either χ restricted to A is isotypic or χ is induced from a representation θ of a subgroup N containing A which may be nonabelian [**21**].

In the first case, since A is abelian, χ restricted to A must be the sum of isomorphic one-dimensional representations. However, the image of A will then be in the center of the image of G, contradicting the inductive assumption that χ is faithful.

In the second case, since G/A is nilpotent, N is contained in a cyclic tower to G. Finally, applying the inductive hypothesis to N and θ proves the lemma. □

In Theorem C and its corollaries, we have assumed the group G_g has abelian commutator, which implies the conjugacy class of g in G_g is gG'_g. Thus, if we lift to the fixed field F_g of G_g, we can then isolate $\{g\}$ as in the abelian case.

A potential difficulty is that one of the automorphic representations π_{F_g} and π'_{F_g} may not be cuspidal. In this case, Theorem 7.2 tells us that $L(s,\pi_{F_g})$ and/or $L(s,\pi'_{F_g})$ will be the product of two $GL(1,(F_g)_A)$ L-functions. If just one, say π'_{F_g}, is not cuspidal, then the Rankin-Selberg convolutions $L(s,\psi\times\pi_{F_g}\times\tilde{\pi}'_{F_g})$ will be the product of two L-functions of twists of π_{F_g}. It will therefore be entire [**10**]. If both are not cuspidal, it will be the product of four $GL(1)$ L-functions.

The above allows us to generalize $\delta(\pi,\pi')$ to arbitrary, not just cuspidal, automorphic representations of $GL(2)$. Let $\delta(\pi,\pi')$ be the order of the pole of $L_S(s,\pi\times\tilde{\pi}')$ at $s=1$.

We consider base field F_g and the cyclic extension of F_g in E which is fixed by G'_g. If ψ denotes a character of the cyclic Galois group, which may also be viewed as an abelian character of G_g, then the asymptotic relation of Proposition 5.4 holds for $\pi_{F_g}\times\psi$ and π'_{F_g}. Primes of F_g of relative degree greater than one over F may be removed from the asymptotic relation by an argument similar to the proof of Proposition 5.4.

The orthogonality relations for abelian characters now yield

$$\sum_{\sigma v=\{g\}} (a_{v,1} - b_{v,1}) \log Nv \ \hat{k}(Nv, x)$$
$$\sim |G|^{-1}|\{g\}| \sum_{\psi} \bar{\psi}(g)[\delta(\pi_{F_g} \times \psi, \pi_{F_g}) - \delta(\pi_{F_g} \times \psi, \pi'_{F_g})] \log^2 x.$$

If $\pi_v \simeq \pi'_v$ for sufficiently many primes with $\sigma v = \{g\}$, as is assumed in Theorem C, we must have

$$\sum_{\psi} \bar{\psi}(g)[\delta(\pi_{F_g} \times \psi, \pi_{F_g}) - \delta(\pi_{F_g} \times \psi, \pi'_{F_g})] = 0.$$

If ψ is the trivial character, then $\delta(\pi_{F_g} \times \psi, \pi_{F_g}) > 0$. Thus we conclude that either $\delta(\pi_{F_g} \times \psi, \pi_{F_g}) > 0$ for a character ψ with $\psi(g) \neq 1$ or $\delta(\pi_{F_g} \times \psi, \pi'_{F_g}) > 0$ for some ψ.

If only the former holds, we see that

$$\sum_{\sigma v=\{g\}} a_{v,1} \log Nv \ \hat{k}(Nv, x) = o(\log^2 x).$$

If π_{F_g} is cuspidal, then $\pi_{F_g} \times \psi \simeq \pi_{F_g}$, implying that $\operatorname{Tr} \pi_v = \mu_{v,1} + \mu_{v,2} = 0$ for $\sigma v = \{g\}$. In addition, Theorem 7.3 implies π_E is noncuspidal.

If π_{F_g} is noncuspidal, then $L(s, \psi \times \pi_{F_g} \times \tilde{\pi}_{F_g})$ is the product of four $GL(1, (F_g)_A)$ representations. Since at least one of the four has a pole at $s = 1$, at least one of $\psi(\sigma v)\mu_{v,i}\bar{\mu}_{v,j}$ $(i, j = 1, 2)$ is equal to one. If $\sigma v = \{g\}$, we must have $\mu_{v,2} = \psi(g)\mu_{v,1}$ or $\mu_{v,1} = \psi(g)\mu_{v,2}$ since $\psi(g) \neq 1$. Because $a_{v,1} = |1 + \psi(g)|^2$ in either case,

$$\sum_{\sigma v=\{g\}} a_{v,1} \log Nv \ \hat{k}(Nv, x) \sim |G|^{-1}|\{g\}| \ |1 + \psi(g)|^2 \log^2 x.$$

Therefore $1 + \psi(g) = 0$, hence $\operatorname{Tr} \pi_v = 0$ in this case as well. Of course, π_E is noncuspidal in this case. Interchanging the roles of π and π', we see the same holds for $\operatorname{Tr} \pi'_v$ and π'_E. We have proved that if only $\delta(\pi_{F_g} \times \psi, \pi_{F_g}) > 0$ with $\psi(g) \neq 1$ holds, then conclusion (iii) of Theorem C holds.

Suppose now that $\delta(\pi_{F_g} \times \psi, \pi'_{F_g}) > 0$. If π_{F_g}, hence also π'_{F_g}, is cuspidal, then it follows immediately that $\pi_E \simeq \pi'_E$. If π_{F_g}, hence also π'_{F_g}, is noncuspidal, then $L(s, \psi \times \pi_{F_g} \times \tilde{\pi}'_{F_g})$ is the product of four $GL(1, (F_g)_A)$ L-functions, at least one of which has a pole at $s = 1$. This implies that

$$\{\psi(\sigma v)\mu_{v,1}, \ \psi(\sigma v)\mu_{v,2}\} = \{\mu'_{v,1}, \mu'_{v,2}\}$$

whenever $\sigma v \in G_g$ since these primes split completely in F_g. Considering the case when σv is the identity, we see $\pi_E \simeq \pi'_E$.

Since the statement about central characters is trivial ("base change" for $GL(1)$), to complete the proof of Theorem C, we need only show that if $\pi_E \simeq \pi'_E$ with both cuspidal, the $\pi' \simeq \pi \times \chi$ with $\chi(g) = 1$. We prove this in the following proposition.

PROPOSITION 7.8. *Let π and π' be cuspidal automorphic representations of $GL(2, F_A)$. Let E be a solvable extension of F. If $\pi_E \simeq \pi'_E$ and the lifts are cuspidal, the $\pi' \simeq \pi \times \chi$ for χ an abelian character of* $\operatorname{Gal}(E/F)$.

PROOF. We proceed by induction on the order of the group, the proposition following immediately from base change theory if G is abelian.

By the inductive hypothesis, any proper normal subgroup of G must be abelian. If the quotient group G/G' requires three of more generators, any two must commute with each other as well as with G' , forcing G to be abelian. We may then assume G/G' is generated by either one or two elements. Using similar reasoning, it is easy to see that G/G' must be of prime power order, say p^r.

Therefore, any maximal normal subgroup of G is abelian of index p. Lemma 7.7 now implies all nonabelian characters are monomial and induced from such a normal subgroup, hence of degree p. Write $\chi = \mathrm{Ind}_{G_\chi}^G \psi_\chi$. Note that $G_\chi = G$ and $\psi_\chi = \chi$ if χ is abelian.

Isolating the identity of G now yields the asymptotic relation

$$\sum_{v \notin S,\ \sigma v = 1} (a_{v,1} - b_{v,1}) \log Nv\ \hat{k}(Nv, x) \sim |G|^{-1} c \log^2 x,$$

where

$$c = \sum_\chi \bar{\chi}(1)[\delta(\pi_{F_\chi} \times \psi_\chi, \pi_{F_\chi}) - \delta(\pi_{F_\chi} \times \psi_\chi, \pi'_{F_\chi})].$$

Because we have assumed π_E is cuspidal, $\delta(\pi_{F_\chi} \times \psi_\chi, \pi_{F_\chi}) = 0$ unless χ is the trivial character of G. Hence,

$$c = 1 - \sum_\chi \bar{\chi}(1)\delta(\pi_{F_\chi} \times \psi_\chi, \pi'_{F_\chi}).$$

Since $\pi_E \simeq \pi'_E$, $a_{v,1} = b_{v,1}$ for $\sigma v = 1$. Thus, the left-hand side of the asymptotic relation is zero, hence $c = 0$. Looking at the above expression for c, the fact that $\bar{\chi}(1) = p$ for all nonabelian characters implies $\pi' \simeq \pi \times \chi$ for some abelian character χ, proving the proposition. □

We note that if $\chi(g) \neq 1$, then $\mathrm{Tr}\,\pi_v = \mathrm{Tr}\,\pi'_v = 0$ for $\sigma v = \{g\}$ and the lifts to E will be noncuspidal. This completes the proof of Theorem C as well as its corollaries.

We conclude the paper with a chart illustrating the necessity of cuspidality in Proposition 7.8 as well as examples of conclusions (ii) and (iii) of Theorem C. Suppose the Galois group G of E/F is the dihedral group of order 10 with $x^5 = y^2 = 1$ and $yxy^{-1} = x^4$. We list its character table below.

Conjugacy Class	χ_0	χ_1	θ	ψ
1	1	1	2	2
x, x^4	1	1	$(\sqrt{5}-1)/2$	$(-\sqrt{5}-1)/2$
x^2, x^3	1	1	$(-\sqrt{5}-1)/2$	$(\sqrt{5}-1)/2$
yx^i	1	-1	0	0

By the converse theorem for $GL(2)$ [**10**, §11], both θ and ψ will correspond to cuspidal automorphic representations of $GL(2, F_A)$.

Note first that if $\sigma v = 1$, then $\theta(v) = \psi(v) = 2$ and yet the representations fail to be twist equivalent. The L-function of their lifts to E will be the square of the Dedekind zeta function $\varsigma_E(s)$. The lifts to E are therefore equivalent and noncuspidal.

To illustrate Theorem C(iii), take a second extension E' of F with Galois group G and the same quadratic subfield as E. Then θ and θ' will coincide on the conjugacy class of y since those primes are exactly the primes which are inert in the quadratic subfield. Yet the primes for which $\theta(v) = 2$ and those for which $\theta'(v) = 2$ are the primes which split completely in E and E', hence are very different sets [**15**, Chapter 8].

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