PROOF OF A CONJECTURE OF KOSTANT

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ABSTRACT. Let \( \mathfrak{g}_0 = \mathfrak{t}_0 + \mathfrak{p}_0 \) be a Cartan decomposition of a semisimple real Lie algebra and \( \mathfrak{g} = \mathfrak{t} + \mathfrak{p} \) its complexification. Denote by \( G \) the adjoint group of \( \mathfrak{g} \) and by \( G_0, K, K_0 \) the connected subgroups of \( G \) with respective Lie algebras \( \mathfrak{g}_0, \mathfrak{t}, \mathfrak{t}_0 \). A conjecture of Kostant asserts that there is a bijection between the \( G_0 \)-conjugacy classes of nilpotent elements in \( \mathfrak{g}_0 \) and the \( K \)-orbits of nilpotent elements in \( \mathfrak{p} \) which is given explicitly by the so-called Cayley transformation. This conjecture is proved in the paper.

1. Introduction. Let \( \mathfrak{g}_0 = \mathfrak{t}_0 + \mathfrak{p}_0 \) be a Cartan decomposition of a real semisimple Lie algebra and let \( \mathfrak{g} = \mathfrak{t} + \mathfrak{p} \) be its complexification. Denote by \( G \) the adjoint group of \( \mathfrak{g} \) and by \( G_0, K, K_0 \) the connected Lie subgroup of \( G \) with \( \mathfrak{t} \) resp., \( \mathfrak{g}_0, \mathfrak{t}_0 \) as its Lie algebra. We consider the adjoint action of \( G \) and \( G_0 \) and their restrictions to the subgroups \( K \) and \( K_0 \), respectively.

According to D. King [7] it was conjectured by B. Kostant that there is a bijection between the \( G_0 \)-conjugacy classes of nilpotent elements in \( \mathfrak{g}_0 \) and the \( K \)-conjugacy classes of nilpotent elements in \( \mathfrak{p} \) given explicitly by the so-called Cayley transformation. Of course it suffices to consider the case when \( \mathfrak{g}_0 \) is simple If \( \mathfrak{g}_0 \) is of classical type then the conjecture has been verified recently by D. King [7] using case by case considerations.

In this paper we give a proof of Kostant’s conjecture (in full generality) by a completely different method. Our proof is based on Vinberg’s work on the classification of nilpotent elements in graded Lie algebras.

The tables of nilpotent \( K \)-orbits in \( \mathfrak{p} \) for exceptional simple Lie algebras \( \mathfrak{g} \) will be submitted for publication elsewhere. These tables then can be considered as a classification of nilpotent \( G_0 \)-conjugacy classes in \( \mathfrak{g}_0 \). When \( \mathfrak{g}_0 \) is of Cartan type EV this was accomplished by Antonyan [1], but he does not indicate which nilpotent \( K \)-orbits in \( \mathfrak{p} \) belong to the same \( G \)-orbit.

I would like to thank D. King for sending me his preprint [7]. It was this preprint that prompted me to look for a direct proof of Kostant’s conjecture.

2. Notations and definitions. \( \mathfrak{g}_0 \) will be a finite-dimensional real Lie algebra and \( \mathfrak{g}_0 = \mathfrak{t}_0 + \mathfrak{p}_0 \) its Cartan decomposition. Its complexification will be written as \( \mathfrak{g} = \mathfrak{t} + \mathfrak{p} \).

\( G \) denotes the adjoint group of \( \mathfrak{g} \) and for each subalgebra of \( \mathfrak{g} \), denoted by a German letter (possibly with a subscript), the corresponding connected Lie subgroup

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of $G$ will be denoted by the corresponding uppercase italic letter (and the same subscript).

If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\alpha$ a root of $(\mathfrak{g}, \mathfrak{h})$ then $\mathfrak{g}^{\alpha}$ will denote the corresponding root space of $\mathfrak{g}$.

Let $\mathfrak{s} = \bigoplus \mathfrak{s}_k$ be a $\mathbb{Z}$-graded complex semisimple Lie algebra. Then there is a unique element $H \in \mathfrak{s}$ such that $\mathfrak{s}_k = \{X \in \mathfrak{s} : [H, X] = kX\}$ for all $k \in \mathbb{Z}$. Clearly $H \in \mathfrak{s}_0$ and we call $H$ the defining element of this $\mathbb{Z}$-graded algebra. Since $H$ determines the gradation of $\mathfrak{s}$ we shall refer to this $\mathbb{Z}$-graded Lie algebra as $(\mathfrak{s}, H)$.

If $\mathfrak{s} = \bigoplus \mathfrak{s}_k$ is $\mathbb{Z}$-graded then by using the canonical surjection $\mathbb{Z} \rightarrow \mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ we obtain a $\mathbb{Z}_2$-grading of $\mathfrak{s}$ which we call the associated $\mathbb{Z}_2$-grading.

The algebra $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$ is a $\mathbb{Z}_2$-graded Lie algebra. A $\mathbb{Z}$-graded subalgebra of this $\mathbb{Z}_2$-graded algebra is a $\mathbb{Z}$-graded subalgebra $\mathfrak{s} = \bigoplus \mathfrak{s}_k$ of $\mathfrak{g}$ such that $\mathfrak{s}_k \subseteq \mathfrak{t}$ for $k$ even and $\mathfrak{s}_k \subseteq \mathfrak{p}$ for $k$ odd.

By $\theta$ we denote the automorphism of $\mathfrak{g}_0$ which is 1 on $\mathfrak{t}_0$ and $-1$ on $\mathfrak{p}_0$. We also denote by $\theta$ its extension to a complex automorphism of $\mathfrak{g}$.

By $\sigma$ we denote the conjugation of $\mathfrak{g}$ with respect to its real form $\mathfrak{g}_0$. If $\mathfrak{s}$ is a $\sigma$-stable subalgebra of $\mathfrak{g}$ then by $\mathfrak{s}^{\sigma}$ we denote the subalgebra of $\mathfrak{s}$ consisting of elements of $\mathfrak{s}$ fixed by $\sigma$.

A $\mathbb{Z}$-graded semisimple Lie algebra $\mathfrak{g} = \bigoplus \mathfrak{s}_k$ is called locally flat if $\dim \mathfrak{s}_0 = \dim \mathfrak{s}_1$. In that case the group $S_0$ has precisely one open orbit in $\mathfrak{s}_1$ under the adjoint action and we shall refer to any element of that orbit as a generic element of $\mathfrak{s}_1$. For each generic element $X \in \mathfrak{s}_1$ the centralizer of $X$ in $S_0$ is finite. If this centralizer is trivial then we say that this $\mathbb{Z}$-graded algebra is flat. These definitions are due to Vinberg [11].

A subalgebra of $\mathfrak{g}$ is called regular if it is normalized by some Cartan subalgebra of $\mathfrak{g}$. A nonzero nilpotent element $X \in \mathfrak{g}$ and its $G$-conjugacy class $G \cdot X$ are said to be semiregular (in $\mathfrak{g}$) if $G \cdot X$ does not meet any proper regular semisimple subalgebra of $\mathfrak{g}$. Given any nonzero nilpotent element $X \in \mathfrak{g}$ there exists a regular semisimple subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that $G \cdot X \cap \mathfrak{s}$ is nonempty and every element of this intersection is semiregular in $\mathfrak{s}$. Dynkin’s classification of nilpotent $G$-conjugacy classes of $\mathfrak{g}$ is based on the classification of semiregular nilpotent classes. The semiregular nilpotent $G$-conjugacy classes are also discussed by Elkington [6].

Let $X \neq 0$ be a nilpotent element of $\mathfrak{g}$. By a theorem of Morozov there exist $H, Y \in \mathfrak{g}$ such that

$$[X, Y] = -H, \quad [H, X] = 2X, \quad [H, Y] = -2Y.$$  

Following Bourbaki [3] we shall call such triple $(X, H, Y)$ an $\mathfrak{s}l_2$-triple. (Usually one replaces the equality $[X, Y] = -H$ by $[X, Y] = H$ in the above definition but we make this departure in order to conform with the terminology of [3].)

A real Cayley triple is an $\mathfrak{s}l_2$-triple $(E, H, F)$ in $\mathfrak{g}_0$ such that $\theta(E) = F$. This implies that $\theta(F) = E$, $\theta(H) = -H$ and consequently $H \in \mathfrak{p}_0$, $E + F \in \mathfrak{t}_0$, and $E - F \in \mathfrak{p}_0$.

A complex Cayley triple is an $\mathfrak{s}l_2$-triple $(E, H, F)$ in $\mathfrak{g}$ such that $E, F \in \mathfrak{p}$ and $\sigma(E) = -F$. It follows that $\sigma(F) = -E$, $\sigma(H) = -H$, $H \in \mathfrak{t}_0$, $E + F \in \mathfrak{t}_0$, and $E - F \in \mathfrak{p}_0$. 
Clearly $K_0$ acts by adjoint action on both real and complex Cayley triples. The Cayley transform $c$ is a map from real to complex Cayley triples defined by

$$c(E, H, F) = \left( \frac{1}{2}(H + iF - iE), i(E + F), \frac{1}{2}(-H + iF - iE) \right).$$

It is easy to check that this map is bijective and $K_0$-equivariant and that its inverse is given by

$$c^{-1}(E, H, F) = \left( \frac{1}{2}(E + F - H), E - F, -\frac{1}{2}(E + F + H) \right).$$

Hence $c$ induces a bijection $\tilde{c}$ from the set of $K_0$-conjugacy classes of real Cayley triples to the set of $K_0$-conjugacy classes of complex Cayley triples.

An $\mathfrak{s}_2$-triple $(E, H, F)$ in $\mathfrak{g}$ is called normal if $E, F \in \mathfrak{p}$ and $H \in \mathfrak{k}$. They have been studied extensively by Kostant and Rallis [8].

3. Some known results. Define a map $\phi$ from the set of $K_0$-conjugacy classes of real Cayley triples to the set of nonzero nilpotent $G_0$-orbits in $\mathfrak{g}_0$ by assigning to the class containing the real Cayley triple $(E, H, F)$ the orbit $G_0 \cdot E$. It is shown by King [7, Lemma 1.1] that $\phi$ is surjective.

Each $K_0$-conjugacy class of complex Cayley triples is contained in a unique $K$-conjugacy class of normal $\mathfrak{s}_2$-triples. Hence the inclusion relation defines a map $\psi_0$ from the set of $K_0$-conjugacy classes of complex Cayley triples to the set of $K$-conjugacy classes of normal $\mathfrak{s}_2$-triples. King shows that $\psi \circ c \circ \phi^{-1}$ is a well-defined map from the set of nonzero nilpotent $G_0$-orbits in $\mathfrak{g}_0$ to the set of $K$-orbits of normal $\mathfrak{s}_2$-triples (the proof is in the paragraph following Remark 1.1). He also shows that this map is injective. His proof of this fact is based on a theorem of Kostant and Rao the proof of which was published by D. Barbasch [2, Proposition 3.1]. These proofs will not be reproduced here. In the Addendum we show that $\phi$ is also injective.

Let $\psi_1$ be the map from the set of $K$-conjugacy classes of normal $\mathfrak{s}_2$-triples to the set of nonzero nilpotent $K$-orbits in $\mathfrak{p}$ which assigns to the class containing the normal $\mathfrak{s}_2$-triple $(E, H, F)$ the orbit $K \cdot E$. Kostant and Rallis [8, Proposition 4] have shown that $\psi_1$ is injective.

Now we can state Kostant's conjecture: The map $\psi_1 \circ \psi_0 \circ c \circ \phi^{-1}$ from nonzero nilpotent $G_0$-orbits in $\mathfrak{g}_0$ to nonzero nilpotent $K$-orbits in $\mathfrak{p}$ is bijective. Some partial results in connection with this conjecture have been obtained by L. Preiss-Rothschild [9].

From the results stated above we know that this map is injective. This is the content of Proposition 1.2 in [7].

In order to complete the proof of the conjecture it remains to prove that $\psi_0$ is also surjective, i.e., that every $K$-conjugacy class of normal $\mathfrak{s}_2$-triples in $\mathfrak{g}$ contains a complex Cayley triple. Equivalently, it suffices to show that the map $\psi_1 = \psi_1 \circ \psi_0$ is surjective. That will be accomplished in §5.

The following lemma will be needed for our proof. The validity of this lemma follows from the description of nilpotent $G$-orbits in $\mathfrak{g}$, which was accomplished by Dynkin [5] (see also [6]) and the description of flat Lie algebras in [11 or 12].

**Lemma 1.** Let $(E, H, F)$ be an $\mathfrak{s}_2$-triple in $\mathfrak{g}$ with $E$ a semiregular nilpotent in $\mathfrak{g}$. Then $\text{ad}(H/2)$ has integer eigenvalues, the $\mathbb{Z}$-graded Lie algebra $(\mathfrak{g}, H/2) = \bigoplus_{k \in \mathbb{Z}} s_k$ is flat, and $E$ is a generic element of $\mathfrak{s}_1$. 

4. Basic Lemma. For the proof of the basic lemma we need the following technical lemma.

**Lemma 2.** Assume that \( \text{rank } \mathfrak{k} = \text{rank } \mathfrak{g} \), fix a Cartan subalgebra \( \mathfrak{h}_0 \) of \( \mathfrak{k}_0 \) and let \( \mathfrak{h} = \mathfrak{h}_0 + i\mathfrak{h}_0 \). Let \( R \) be the root system of \( (\mathfrak{g}, \mathfrak{h}) \) and \( \mathfrak{R}^{(0)} = \{ \alpha \in R : \mathfrak{g}^\alpha \subset \mathfrak{k} \} \), \( \mathfrak{R}^{(1)} = \{ \alpha \in R : \mathfrak{g}^\alpha \subset \mathfrak{p} \} \).

Then there exists a Chevalley system \( (X_\alpha), \alpha \in R \), of \( (\mathfrak{g}, \mathfrak{h}) \) such that

\[
\sigma(X_\alpha) = (-1)^k X_{-\alpha}, \quad \alpha \in \mathfrak{R}^{(k)}.
\]

(For the definition of Chevalley systems see [3, Chapitre VIII, §3, no. 4, p. 84].)

**Proof.** Let \( (Y_\alpha), \alpha \in R \), be any Chevalley system of \( (\mathfrak{g}, \mathfrak{h}) \).

The \( \mathbb{R} \)-span of \( \mathfrak{h}_0 \) and the vectors \( Y_\alpha + Y_{-\alpha}, i(Y_\alpha - Y_{-\alpha}), \alpha \in \mathfrak{R}^{(0)} \); and \( i(Y_\alpha + Y_{-\alpha}), Y_\alpha - Y_{-\alpha}, \alpha \in \mathfrak{R}^{(1)} \); are a real form \( \mathfrak{g}^\# \) of \( \mathfrak{g} \) isomorphic to \( \mathfrak{g}_0 \). Choose an isomorphism \( \tau : \mathfrak{g}^\# \rightarrow \mathfrak{g}_0 \) such that \( \tau(\mathfrak{h}_0) = \mathfrak{h}_0 \) and extend \( \tau \) to an automorphism of \( \mathfrak{g} \). Set \( X_\alpha = \tau(Y_\alpha) \). Then \( (X_\alpha), \alpha \in R \), is a Chevalley system of \( (\mathfrak{g}, \mathfrak{h}) \) having the required properties.

Now we can prove our basic lemma.

**Lemma 3.** Let \( (\mathfrak{g}, \mathfrak{H}/2) = \bigoplus \mathfrak{s}_k, k \in \mathbb{Z} \), be a simple flat complex Lie algebra and assume that the associated \( \mathbb{Z}_2 \)-grading on \( \mathfrak{g} \) coincides with \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \). Then there exists \( X \in \mathfrak{s}_1 \) such that

\[
[X, \sigma(X)] = H.
\]

**Proof.** Fix a Cartan subalgebra \( \mathfrak{h}_0 \) of \( \mathfrak{k}_0 \) such that \( \mathfrak{h}_0 \subset \mathfrak{s}_0 \) and set \( \mathfrak{h} = \mathfrak{h}_0 + i\mathfrak{h}_0 \). Let \( R \) be the root system of \( (\mathfrak{g}, \mathfrak{h}) \) and for \( k \in \mathbb{Z} \) let

\[
R_k = \{ \alpha \in R : \mathfrak{g}^\alpha \subset \mathfrak{s}_k \}. \]

Also define

\[
\mathfrak{R}^{(0)} = \bigcup_{k \in \mathbb{Z}} R_{2k}, \quad \mathfrak{R}^{(1)} = \bigcup_{k \in \mathbb{Z}} R_{2k+1}.
\]

By Lemma 2 there exists a Chevalley system \( (X_\alpha), \alpha \in R \), such that

\[
\sigma(X_\alpha) = (-1)^k X_{-\alpha}, \quad \alpha \in \mathfrak{R}^{(k)}.
\]

For \( \alpha \in R_1 \) let \( Y_\alpha = -X_{-\alpha} \). Let \( H_\alpha \) be the unique element of \( [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] \) such that \( \alpha(H_\alpha) = 2 \). Recall that \( [X_\alpha, X_{-\alpha}] = -H_\alpha \), \( \alpha \in R \), and so \( [X_\alpha, Y_\alpha] = H_\alpha \) for \( \alpha \in R_1 \).

We shall seek a solution of equation (1) in the form

\[
X = \sum_{\alpha \in R_1} \lambda_\alpha X_\alpha
\]

with all \( \lambda_\alpha \) real. Then

\[
\sigma(X) = \sum_{\alpha \in R_1} \lambda_\alpha Y_\alpha
\]

and equation (1) can be written as

\[
\sum_{\alpha, \beta \in R_1} \lambda_\alpha \lambda_\beta [X_\alpha, Y_\beta] = H.
\]
Assume first that our flat Lie algebra \((g, H/2)\) is principal, i.e., that \(R_1\) is a base, say \(B\), of \(R\). In that case \(\alpha - \beta \notin R\) for \(\alpha, \beta \in B\) and so (2) becomes

\[
\sum_{\alpha \in B} \lambda_{\alpha}^2 H_{\alpha} = H.
\]

Since \(\alpha(H) = 2\) for all \(\alpha \in B\), this equation is equivalent to the system

\[
\sum_{\beta \in B} \alpha(H_{\beta}) \lambda_{\beta}^2 = 2, \quad \alpha \in B.
\]

By a theorem of Vinberg [10, Theorem 3] the unique solution \(\mu_{\beta}, \beta \in R\), of the system of linear equations

\[
\sum_{\beta \in B} \alpha(H_{\beta}) \mu_{\beta} = 2, \quad \alpha \in B,
\]

is positive in the sense that \(\mu_{\beta} > 0\) for each \(\beta \in B\). It follows that the system (3) has a real solution.

Next assume that \((g, H/2)\) is the simple flat Lie algebra \(D_{n+m+1}(a_m)\), \(n > m \geq 1\), see [5 or 6]. In this case we shall use the notations for roots, the Chevalley system, etc., given in Bourbaki [3, Chapitre VIII, §13, no. 4, pp. 206–212]. Then \(H/2\) is the diagonal matrix of order \(2n + 2m + 2\) whose diagonal entries are the integers \(n, n-1, \ldots, -n\) and \(m, m-1, \ldots, -m\) arranged in nonincreasing order. The set \(R_1\) consists of the roots

\[
\begin{align*}
\varepsilon_i - \varepsilon_{i+1}, & \quad 1 \leq i \leq n - m; \quad \varepsilon_{n-m} - \varepsilon_{n-m+2}; \\
\varepsilon_{n-m+2k-1} - \varepsilon_{n-m+2k}, & \quad \varepsilon_{n-m+2k} - \varepsilon_{n-m+2k+1}, \quad 1 \leq k \leq m; \\
\varepsilon_{n-m+2k-1} - \varepsilon_{n-m+2k+2}, & \quad \varepsilon_{n-m+2k} - \varepsilon_{n-m+2k+2}, \quad 1 \leq k \leq m - 1;
\end{align*}
\]

and

\[
\varepsilon_{n+m-1} + \varepsilon_{n+m+1}; \quad \varepsilon_{n+m} + \varepsilon_{n+m+1}.
\]

In this case some of the \(\lambda_{\alpha}\) can be taken to be zero. An explicit solution of equation (1) is provided by

\[
X = \lambda_1 X_{\varepsilon_1} - \varepsilon_2 + \lambda_2 X_{\varepsilon_2} - \varepsilon_3 + \cdots + \lambda_{n-m} X_{\varepsilon_{n-m}} - \varepsilon_{n-m+1} \\
+ \mu_1 X_{\varepsilon_{n-m+1}} - \varepsilon_{n-m+3} + \nu_1 X_{\varepsilon_{n-m+2}} - \varepsilon_{n-m+4} \\
+ \mu_2 X_{\varepsilon_{n-m+3}} - \varepsilon_{n-m+5} + \nu_2 X_{\varepsilon_{n-m+4}} - \varepsilon_{n-m+6} \\
+ \cdots \\
+ \mu_{m-1} X_{\varepsilon_{n+m-3}} - \varepsilon_{n+m-1} + \nu_{m-1} X_{\varepsilon_{n+m-2}} - \varepsilon_{n+m} \\
+ \rho_1 X_{\varepsilon_{n+m-1}} - \varepsilon_{n+m+1} + \sigma_1 X_{\varepsilon_{n+m-1}} - \varepsilon_{n+m+1} \\
+ \rho_2 X_{\varepsilon_{n+m}} - \varepsilon_{n+m+1} + \sigma_2 X_{\varepsilon_{n+m}} + \varepsilon_{n+m+1}
\]

where

\[
\begin{align*}
\lambda_k^2 &= k(2n - k + 1), \quad 1 \leq k \leq n - m; \\
\mu_k^2 &= (n - m)(n + m + 1) + k(2m - k + 1), \quad 1 \leq k \leq m - 1; \\
\nu_k^2 &= k(2m - k + 1), \quad 1 \leq k \leq m - 1; \\
\rho_1 + i\rho_2 &= \pm \sqrt{z}, \quad \sigma_1 + i\sigma_2 = \pm \sqrt{w} \text{ with } z \text{ and } w \text{ complex numbers satisfying } \\
z + w = (n - m)(n + m + 1), \quad |z| = |w| = m^2 - 3m + 4 + \frac{1}{2}(n - m)(n + m + 1).
\end{align*}
\]
We omit the routine details of the verification of this claim.

There remain five exceptional simple flat Lie algebras to be dealt with, namely, \(E_8(a_1), E_8(a_2), E_7(a_1), E_7(a_2),\) and \(E_6(a_1)\). In order to exhibit an explicit solution of (1) it will be convenient to use the Chevalley system of \(E_8\) constructed in our paper [4]. For the convenience of the reader we shall review some basic facts about this Chevalley system.

The Lie algebra \(E_8\) is realized as \(\mathbb{Z}\)-graded algebra \(g = \bigoplus_{k \in \mathbb{Z}} s_k\) where \(s_0 = V \otimes V^*, s_1 = \wedge^3 V, s_{-1} = \wedge^3 V^*, s_2 = \wedge^2 V, s_{-2} = \wedge^2 V^*, s_3 = V, s_4 = V^*\) and \(s_k = 0\) otherwise. Here \(V\) denotes a complex vector space of dimension 8 with a fixed basis \(e_k, 1 \leq k \leq 8\), and \(V^*\) its dual space with the dual basis \(e_k^*, 1 \leq k \leq 8\).

For the definition of the Lie bracket see [4]. We mention only that \(s_0 \cong \mathfrak{gl}(V)\), that the action of \(s_0\) on each \(s_k\) is the standard one, and that

\[
[a \wedge b \wedge c, f \wedge g \wedge h] = -\begin{vmatrix}
f(a) & f(b) & f(c) & f \\
g(a) & g(b) & g(c) & g \\
h(a) & h(b) & h(c) & h \\
a & b & c & 1/3
\end{vmatrix}
\]

where \(a, b, c \in V, f, g, h \in V^*\) and when evaluating this determinant the product of, say, \(a\) and \(f\) should be written as \(a \otimes f\).

Using the formula (4) one finds that

\[
[e_{ij}, e^{jk}] = e_i^r
\]

if \(i \neq r, i < j < k,\) and \(r < j\).

Writing \(e_i^j = e_i \otimes e^j\), the subspace \(h\) spanned by the elements \(e_i^j, 1 \leq i \leq 8,\) is a Cartan subalgebra of \(s_0\) and of \(g\). The Chevalley system of \((g, h)\) is given by the elements:

\[
e_i^j, -e_j^i \quad (1 \leq i < j \leq 8);
\]
\[
e_i, -e^i \quad (1 \leq i \leq 8);
\]
\[
e_{i,j,k}, -e_i^{j,k} \quad (1 \leq i < j < k \leq 8);
\]
\[
e_{i,j}, e_j^i \quad (1 \leq i < j \leq 8);
\]

where \(e_{i,j,k} = e_i \wedge e_j \wedge e_k, e_i^{j,k} = e_i \wedge e^j \wedge e^k,\) etc.

In [4] \(\lambda_i, 1 \leq i \leq 8,\) is a basis of \(h^*\) dual to the basis \(e_i^j, 1 \leq i \leq 8,\) of \(h\). A base \(B\) of the root system \(R\) of \((g, h)\) consists of the roots \(\lambda_i - \lambda_{i+1}, 1 \leq i \leq 7,\) and the root \(\lambda_6 + \lambda_7 + \lambda_8:\)

\[
\lambda_1 - \lambda_2 \quad \lambda_2 - \lambda_3 \quad \lambda_3 - \lambda_4 \quad \lambda_4 - \lambda_5 \quad \lambda_5 - \lambda_6 \quad \lambda_6 - \lambda_7 \quad \lambda_7 - \lambda_8
\]

\[
\lambda_6 + \lambda_7 + \lambda_8
\]

For \(\alpha \in B\) the elements \(H_\alpha \in h\) are given by

\[
h_i := H_{\lambda_i - \lambda_{i+1}} = e_i - e_i^{i+1}, \quad 1 \leq i \leq 7,
\]
\[
h_8 := H_{\lambda_6 + \lambda_7 + \lambda_8} = -\frac{1}{3} + e_6^6 + e_7^7 + e_8^8,
\]

whence \(-\frac{1}{3}\) means \(-\frac{1}{3} \sum_{i=1}^{8} e_i^i\).

Case \(E_8(a_1)\). In this case we have

\[
H = 46h_1 + 90h_2 + 132h_3 + 172h_4 + 210h_5 + 142h_6 + 72h_7 + 106h_8,
\]
and $R_1$ consists of the roots $\lambda_i - \lambda_{i+1}, i \neq 5, 6 + \lambda_7 + \lambda_8, \lambda_4 - \lambda_6, \lambda_5 - \lambda_7$, and $\lambda_5 + \lambda_7 + \lambda_8$.

An explicit solution of (1) is given by

$$X = \sqrt{46}e_1^2 + \sqrt{90}e_2^3 + \sqrt{132}e_3^4 + \sigma_1 e_4^5 + \sqrt{72}e_5^8 + \sqrt{106}e_6^7 + \rho_2 e_8^6 + \sigma_2 e_5^7,$$

where $\rho_1 + i\rho_2 = \pm \sqrt{z}$, $\sigma_1 - i\sigma_2 = \pm \sqrt{w}$ and $z$ and $w$ are complex numbers such that $|z| = 172$, $|w| = 142$, $z + w = -106$.

Case $E_6(a_2)$. We have

$$H = 38h_1 + 74h_2 + 108h_3 + 142h_4 + 174h_5 + 118h_6 + 60h_7 + 88h_8,$$

and $R_1$ consists of the roots $\lambda_i - \lambda_{i+1}, i \neq 3, 5; \lambda_6 + \lambda_7 + \lambda_8, \lambda_2 - \lambda_4, \lambda_3 - \lambda_5, \lambda_4 - \lambda_6, \lambda_5 - \lambda_7$, and $\lambda_5 + \lambda_7 + \lambda_8$. A solution of (1) is provided by

$$X = \sqrt{38}e_1^2 + \sqrt{74}e_2^3 + \sqrt{34}e_3^5 + \sigma_1 e_4^7 + \sqrt{60}e_5^8 + \sigma_1 e_6^7 + \sqrt{108}e_8^6 + \rho_2 e_5^7 + \sigma_2 e_5^7,$$

where $\rho_1 + i\rho_2 = \pm \sqrt{z}$, $\sigma_1 - i\sigma_2 = \pm \sqrt{w}$ and $z$ and $w$ are complex numbers such that $|z| = 118$, $|w| = 88$, $z + w = 74$.

Formula (5) is useful when one checks that $X$ is indeed a solution.

Case $E_7(a_1)$. We have

$$H = 21h_2 + 40h_3 + 57h_4 + 72h_5 + 50h_6 + 26h_7 + 37h_8,$$

and $R_1$ is the same as in case $E_6(a_1)$ except that the root $\lambda_1 - \lambda_2$ should be omitted. A solution $X$ of (1) is given by

$$X = \sqrt{21}e_2^3 + \sqrt{40}e_4^3 + \sigma_1 e_6^4 + \rho_1 e_7^6 + \sqrt{26}e_7^8 + \sqrt{37}e_8^6 + \sigma_2 e_6^7 + \rho_2 e_5^7,$$

where $\rho_1 + i\rho_2 = \pm \sqrt{z}$, $\sigma_1 - i\sigma_2 = \pm \sqrt{w}$ and $z$ and $w$ are complex numbers satisfying $|z| = 50$, $|w| = 57$, $z + w = -37$.

Case $E_7(a_2)$. Now

$$H = 17h_2 + 32h_3 + 47h_4 + 60h_5 + 42h_6 + 22h_7 + 31h_8$$

and $R_1$ is the same as in the case $E_6(a_2)$ except that the root $\lambda_1 - \lambda_2$ should be omitted. A solution $X$ of (1) is given by

$$X = \sqrt{17}e_2^3 + \sqrt{15}e_4^3 + \rho_1 e_6^4 + \sqrt{22}e_7^8 + \sqrt{32}e_8^6 + \rho_2 e_5^7 + \sigma_2 e_5^7,$$

where $\rho_1 + i\rho_2 = \pm \sqrt{z}$, $\sigma_1 + i\sigma_2 = \pm \sqrt{w}$ and $z$ and $w$ are complex numbers satisfying $|z| = 42$, $|w| = 31$, $z + w = 17$.

Case $E_6(a_1)$. In this case

$$H = 12h_3 + 22h_4 + 30h_5 + 22h_6 + 12h_7 + 16h_8,$$

and $R_1$ is the same as in case $E_6(a_1)$ except that the roots $\lambda_1 - \lambda_2$ and $\lambda_2 - \lambda_3$ should be omitted. A solution $X$ of (1) is given by

$$X = \sqrt{12}e_3^3 + \rho_1 e_4^2 + \sigma_1 e_6^7 + \sqrt{12}e_7^8 + 4e_6^7 + \rho_2 e_4^6 + \sigma_2 e_5^7,$$

where $\rho_1 + i\rho_2 = \pm \sqrt{z}$, $\sigma_1 - i\sigma_2 = \pm \sqrt{w}$, and $z$ and $w$ are complex numbers satisfying $|z| = |w| = 22$, $z + w = -16$. This completes the proof of the lemma.
5. **Proof that \( \psi \) is surjective.** Let \( E \) be a nonzero nilpotent element in \( p \).
We have to show that there exists a complex Cayley triple \((X, H, Y)\) such that \( X \in K \cdot E \). The proof is by induction on the dimension of \( g \).

We can embed \( E \) in a normal \( \mathfrak{sl}_2 \)-triple \((E, H, F)\). Since \( H \) is a real semisimple element it is \( K \)-conjugate to an element of \( i\mathfrak{k}_0 \). Hence by replacing this triple by a suitable \( K \)-conjugate we may assume that \( H \in i\mathfrak{k}_0 \).

Let \( s = \bigoplus s_k, \ k \in \mathbb{Z}, \) be the \( \mathbb{Z} \)-graded subalgebra of the \( \mathbb{Z}_2 \)-graded algebra \( g = \mathfrak{k} + p \) defined as follows:

\[
s_k = \{ X \in \mathfrak{k} : [H, X] = 2kX \}
\]

for \( k \) even and

\[
s_k = \{ X \in p : [H, X] = 2kX \}
\]

for \( k \) odd. Clearly \( E \in s_1 \) and by a result of Vinberg [12, Lemma 2] \( s \) is reductive. If \( s \neq g \) then the induction hypothesis can be applied to the associated \( \mathbb{Z}_2 \)-graded algebra \( s = s \cap \mathfrak{k} \oplus s \cap p \) and the element \( E \).

Hence we may assume that \( s = g \). Since the centralizer of \( H \) in \( s \) is \( s_0 \), \( s = g \), and \( s_0 \subset \mathfrak{k} \), it follows that \( \text{rank } \mathfrak{k} = \text{rank } g \). Let us fix a Cartan subalgebra \( \mathfrak{h}_0 \) of \( \mathfrak{k}_0 \) such that \( iH \in \mathfrak{h}_0 \). Set \( \mathfrak{h} = \mathfrak{h}_0 + i\mathfrak{k}_0 \).

By a theorem of Vinberg and Elaśvili [13, p. 223] there exist \( X \in K \cdot E \) and a regular semisimple subalgebra \( t \) of \( g \) normalized by \( \mathfrak{h} \) such that \( X \in t \) and \( X \) is a semiregular nilpotent element of \( t \).

Since \( \mathfrak{h} \) normalizes \( t \), it follows that the \( \mathbb{Z}_2 \)-grading \( g = \mathfrak{k} + p \) induces a \( \mathbb{Z}_2 \)-grading on \( t \), i.e., that \( t = t \cap \mathfrak{k} \oplus t \cap p \). If \( t \neq g \) then the inductive hypothesis can be applied to \( t \) and \( X \). Hence we may assume that \( t = g \), i.e., that \( E \) is a semiregular nilpotent element of \( g \).

By Lemma 1 the \( \mathbb{Z} \)-graded algebra \( g = s = \bigoplus s_k \) is flat and \( E \) is a generic element of \( s_1 \). Since \( H \in i\mathfrak{k}_0 \subset i\mathfrak{k}_0 \), we have \( \sigma(H) = -H \) and consequently \( \sigma(s_k) = s_{-k} \) for all \( k \).

Assume that there is an \( X \in s_1 \) such that \( [X, \sigma(X)] = H \). Then \((X, H, -\sigma(X))\) is a complex Cayley triple and by a result of Kostant and Rallis [8, Lemma 4] the normal \( \mathfrak{sl}_2 \)-triples \((E, H, F)\) and \((X, H, -\sigma(X))\) are \( K \)-conjugate.

Hence it suffices to prove the existence of an element \( X \in s_1 \) such that \( [X, \sigma(X)] = H \). Since every flat Lie algebra is a direct product of simple flat Lie algebras we have

\[
(g, H/2) = (g^{(1)}, H_1/2) \times \cdots \times (g^{(m)}, H_m/2)
\]

where each \( (g^{(k)}, H_k/2) \) is a simple flat Lie algebra and \( H = H_1 + \cdots + H_m \). This shows that without any loss of generality we may now assume that \( g \) is simple.

In Lemma 3 we have shown that in the case of simple flat Lie algebras the equation \( [X, \sigma(X)] = H \) indeed has a solution for \( X \). This completes the proof of the conjecture.

**Addendum (February 1987).** The maps \( \phi \) and \( \psi_0 \) defined in §3 are in fact bijective. In view of the results mentioned there and our main theorem, the claim follows from the following proposition.

**Proposition.** The map \( \phi \) is injective.

**Proof.** Let \((E, H, F)\) and \((E', H', F')\) be two real Cayley triples with \( E' \in G_0 \cdot E \). By [3, Chapter VIII, §11, Lemma 4] these triples are \( G_0 \)-conjugate. By
using [9, Proposition 1.1] it follows that $E' - F' \in K_0 \cdot (E - F)$. Hence we may assume that $E' - F' = E - F = Z$, say. Let $G_0^Z$ (resp., $K_0^Z$) be the centralizer of $Z$ in $G_0$ (resp., $K_0$). Fix a maximal compact subgroup $M$ of $G_0^Z$ containing $K_0^Z$. If $x \in M$ write $x = y \exp(X)$ with $y \in K_0$ and $X \in p_0$. By using an argument of L. Preiss-Rothschild [9, Proof of Proposition 1.1] it follows from $\exp(X) \cdot Z = y^{-1} \cdot Z$ that $y^{-1} \cdot Z = Z$. Hence $y \in K_0^Z$, $\exp(X) \in M$ and since $M$ is compact we must have $X = 0$. Thus $M = K_0^Z$.

By [8, p. 779] $g_0^Z = \mathfrak{t}_0^Z \oplus \mathfrak{p}_0^Z$ is a Cartan decomposition of $g_0^Z$ and consequently $G_0^Z = K_0^Z \cdot \exp(p_0^Z)$.

If $a \in G_0$ is an element which maps the triple $(E, H, F)$ to $(E', H', F')$ then $a \in G_0^Z$ and $a \cdot (E + F) = E' + F'$. Write $a = b \exp(Y)$ with $b \in K_0^Z$ and $Y \in p_0^Z$. Then by applying the above mentioned argument to $\exp(Y) \cdot (E + F) = b^{-1} \cdot (E' + F')$ we infer that $b^{-1} \cdot (E' + F') = E + F$. Thus $b \in K_0$ sends $(E, H, F)$ to $(E', F', H')$.

ADDED IN PROOF. After this paper was written D. King informed me that Jiro Sekiguchi had also proved Kostant’s conjecture (by a different method) in a preprint entitled Remarks on real nilpotent orbits of a symmetric pair.

REFERENCES


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