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PROOF OF A CONJECTURE OF KOSTANT

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ABSTRACT. Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be a Cartan decomposition of a semisimple real Lie algebra and $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ its complexification. Denote by G the adjoint group of $\mathfrak{g}$ and by G_0, K, K_0 the connected subgroups of G with respective Lie algebras $\mathfrak{g}_0, \mathfrak{k}, \mathfrak{k}_0$. A conjecture of Kostant asserts that there is a bijection between the G_0-conjugacy classes of nilpotent elements in $\mathfrak{g}_0$ and the K-orbits of nilpotent elements in $\mathfrak{p}$ which is given explicitly by the so-called Cayley transformation. This conjecture is proved in the paper.

1. Introduction. Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be a Cartan decomposition of a real semisimple Lie algebra and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be its complexification. Denote by G the adjoint group of $\mathfrak{g}$ and by K resp., G_0, K_0 the connected Lie subgroup of G with $\mathfrak{k}$ resp., $\mathfrak{g}_0, \mathfrak{k}_0$ as its Lie algebra. We consider the adjoint action of G and G_0 and their restrictions to the subgroups K and K_0, respectively.

According to D. King [**7**] it was conjectured by B. Kostant that there is a bijection between the G_0-conjugacy classes of nilpotent elements in $\mathfrak{g}_0$ and the K-conjugacy classes of nilpotent elements in $\mathfrak{p}$ given explicitly by the so-called Cayley transformation. Of course it suffices to consider the case when $\mathfrak{g}_0$ is simple If $\mathfrak{g}_0$ is of classical type then the conjecture has been verified recently by D. King [**7**] using case by case considerations.

In this paper we give a proof of Kostant's conjecture (in full generality) by a completely different method. Our proof is based on Vinberg's work on the classification of nilpotent elements in graded Lie algebras.

The tables of nilpotent K-orbits in $\mathfrak{p}$ for exceptional simple Lie algebras $\mathfrak{g}$ will be submitted for publication elsewhere. These tables then can be considered as a classification of nilpotent G_0-conjugacy classes in $\mathfrak{g}_0$. When $\mathfrak{g}_0$ is of Cartan type EV this was accomplished by Antonyan [**1**], but he does not indicate which nilpotent K-orbits in $\mathfrak{p}$ belong to the same G-orbit.

I would like to thank D. King for sending me his preprint [**7**]. It was this preprint that prompted me to look for a direct proof of Kostant's conjecture.

2. Notations and definitions. $\mathfrak{g}_0$ will be a finite-dimensional real Lie algebra and $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ its Cartan decomposition. Its complexification will be written as $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$.

G denotes the adjoint group of $\mathfrak{g}$ and for each subalgebra of $\mathfrak{g}$, denoted by a german letter (possibly with a subscript), the corresponding connected Lie subgroup

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of G will be denoted by the corresponding uppercase italic letter (and the same subscript).

If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and α a root of $(\mathfrak{g}, \mathfrak{h})$ then $\mathfrak{g}^\alpha$ will denote the corresponding root space of $\mathfrak{g}$.

Let $\mathfrak{s} = \bigoplus \mathfrak{s}_k$ be a Z-graded complex semisimple Lie algebra. Then there is a unique element $H \in \mathfrak{s}$ such that $\mathfrak{s}_k = \{X \in \mathfrak{s}\colon [H, X] = kX\}$ for all $k \in \mathbf{Z}$. Clearly $H \in \mathfrak{s}_0$ and we call H the defining element of this $\mathbf{Z}$-graded algebra. Since H determines the gradation of $\mathfrak{s}$ we shall refer to this $\mathbf{Z}$-graded Lie algebra as $(\mathfrak{s}, H)$.

If $\mathfrak{s} = \bigoplus \mathfrak{s}_k$ is $\mathbf{Z}$-graded then by using the canonical surjection $\mathbf{Z} \to \mathbf{Z}_2 := \mathbf{Z}/2\mathbf{Z}$ we obtain a $\mathbf{Z}_2$-grading of $\mathfrak{s}$ which we call the *associated* $\mathbf{Z}_2$*-grading.*

The algebra $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a $\mathbf{Z}_2$-graded Lie algebra. A $\mathbf{Z}$-graded subalgebra of this $\mathbf{Z}_2$-graded algebra is a $\mathbf{Z}$-graded subalgebra $\mathfrak{s} = \bigoplus \mathfrak{s}_k$ of $\mathfrak{g}$ such that $\mathfrak{s}_k \subset \mathfrak{k}$ for k even and $\mathfrak{s}_k \subset \mathfrak{p}$ for k odd.

By θ we denote the automorphism of $\mathfrak{g}_0$ which is 1 on $\mathfrak{k}_0$ and -1 on $\mathfrak{p}_0$. We also denote by θ its extension to a complex automorphism of $\mathfrak{g}$.

By σ we denote the conjugation of $\mathfrak{g}$ with respect to its real form $\mathfrak{g}_0$. If $\mathfrak{s}$ is a σ-stable subalgebra of $\mathfrak{g}$ then by $\mathfrak{s}^\sigma$ we denote the subalgebra of $\mathfrak{s}$ consisting of elements of $\mathfrak{s}$ fixed by σ.

A $\mathbf{Z}$-graded semisimple Lie algebra $\mathfrak{g} = \bigoplus \mathfrak{s}_k$ is called *locally flat* if $\dim \mathfrak{s}_0 = \dim \mathfrak{s}_1$. In that case the group S_0 has precisely one open orbit in $\mathfrak{s}_1$ under the adjoint action and we shall refer to any element of that orbit as a *generic element* of $\mathfrak{s}_1$. For each generic element $X \in \mathfrak{s}_1$ the centralizer of X in S_0 is finite. If this centralizer is trivial then we say that this $\mathbf{Z}$-graded algebra is *flat.* These definitions are due to Vinberg [**11**].

A subalgebra of $\mathfrak{g}$ is called *regular* if it is normalized by some Cartan subalgebra of $\mathfrak{g}$. A nonzero nilpotent element $X \in \mathfrak{g}$ and its G-conjugacy class $G \cdot X$ are said to be *semiregular* (in $\mathfrak{g}$) if $G \cdot X$ does not meet any proper regular semisimple subalgebra of $\mathfrak{g}$. Given any nonzero nilpotent element $X \in \mathfrak{g}$ there exists a regular semisimple subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that $G \cdot X \cap \mathfrak{s}$ is nonempty and every element of this intersection is semiregular in $\mathfrak{s}$. Dynkin's classification of nilpotent G-conjugacy classes of $\mathfrak{g}$ is based on the classification of semiregular nilpotent classes. The semiregular nilpotent G-conjugacy classes are also discussed by Elkington [**6**].

Let $X \neq 0$ be a nilpotent element of $\mathfrak{g}$. By a theorem of Morozov there exist $H, Y \in \mathfrak{g}$ such that

$$[X, Y] = -H, \quad [H, X] = 2X, \quad [H, Y] = -2Y.$$

Following Bourbaki [**3**] we shall call such triple (X, H, Y) an $\mathfrak{sl}_2$-triple. (Usually one replaces the equality $[X, Y] = -H$ by $[X, Y] = H$ in the above definition but we make this departure in order to conform with the terminology of [**3**].)

A *real Cayley triple* is an $\mathfrak{sl}_2$-triple (E, H, F) in $\mathfrak{g}_0$ such that $\theta(E) = F$. This implies that $\theta(F) = E$, $\theta(H) = -H$ and consequently $H \in \mathfrak{p}_0$, $E + F \in \mathfrak{k}_0$, and $E - F \in \mathfrak{p}_0$.

A *complex Cayley triple* is an $\mathfrak{sl}_2$-triple (E, H, F) in $\mathfrak{g}$ such that $E, F \in \mathfrak{p}$ and $\sigma(E) = -F$. It follows that $\sigma(F) = -E$, $\sigma(H) = -H$, $H \in i\mathfrak{k}_0$, $E + F \in i\mathfrak{p}_0$, and $E - F \in \mathfrak{p}_0$.

Clearly K_0 acts by adjoint action on both real and complex Cayley triples. *The Cayley transfrom c is a map from real to complex Cayley triples defined by*

$$c(E,H,F) = \left(\tfrac{1}{2}(H+iF-iE), i(E+F), \tfrac{1}{2}(-H+iF-iE)\right).$$

It is easy to check that this map is bijective and K_0-equivariant and that its inverse is given by

$$c^{-1}(E,H,F) = \left(\tfrac{i}{2}(E+F-H), E-F, -\tfrac{i}{2}(E+F+H)\right).$$

Hence c induces a bijection $\bar{c}$ from the set of K_0-conjugacy classes of real Cayley triples to the set of K_0-conjugacy classes of complex Cayley triples.

An $\mathfrak{sl}_2$-triple (E,H,F) in $\mathfrak{g}$ is called *normal* if $E, F \in \mathfrak{p}$ and $H \in \mathfrak{k}$. They have been studied extensively by Kostant and Rallis [**8**].

3. Some known results. Define a map ϕ from the set of K_0-conjugacy classes of real Cayley triples to the set of nonzero nilpotent G_0-orbits in $\mathfrak{g}_0$ by assigning to the class containing the real Cayley triple (E,H,F) the orbit $G_0 \cdot E$. It is shown by King [**7**, Lemma 1.1] that ϕ is surjective.

Each K_0-conjugacy class of complex Cayley triples is contained in a unique K-conjugacy class of normal $\mathfrak{sl}_2$-triples. Hence the inclusion relation defines a map ψ_0 from the set of K_0-conjugacy classes of complex Cayley triples to the set of K-conjugacy classes of normal $\mathfrak{sl}_2$-triples. King shows that $\psi \circ \bar{c} \circ \phi^{-1}$ is a well-defined map from the set of nonzero nilpotent G_0-orbits in $\mathfrak{g}_0$ to the set of K-orbits of normal $\mathfrak{sl}_2$-triples (the proof is in the paragraph following Remark 1.1). He also shows that this map is injective. His proof of this fact is based on a theorem of Kostant and Rao the proof of which was published by D. Barbasch [**2**, Proposition 3.1]. These proofs will not be reproduced here. In the Addendum we show that ϕ is also injective.

Let ψ_1 be the map from the set of K-conjugacy classes of normal $\mathfrak{sl}_2$-triples to the set of nonzero nilpotent K-orbits in $\mathfrak{p}$ which assigns to the class containing the normal $\mathfrak{sl}_2$-triple (E,H,F) the orbit $K \cdot E$. Kostant and Rallis [**8**, Proposition 4] have shown that ψ_1 is bijective.

Now we can state Kostant's conjecture: The map $\psi_1 \circ \psi_0 \circ \bar{c} \circ \phi^{-1}$ from nonzero nilpotent G_0-orbits in $\mathfrak{g}_0$ to nonzero nilpotent K-orbits in $\mathfrak{p}$ is bijective. Some partial results in connection with this conjecture have been obtained by L. Preiss-Rothschild [**9**].

From the results stated above we know that this map is injective. This is the content of Proposition 1.2 in [**7**].

In order to complete the proof of the conjecture it remains to prove that ψ_0 is also surjective, i.e., that every K-conjugacy class of normal $\mathfrak{sl}_2$-triples in $\mathfrak{g}$ contains a complex Cayley triple. Equivalently, it suffices to show that the map $\psi := \psi_1 \circ \psi_0$ is surjective. That will be accomplished in §5.

The following lemma will be needed for our proof. The validity of this lemma follows from the description of nilpotent G-orbits in $\mathfrak{g}$, which was accomplished by Dynkin [**5**] (see also [**6**]) and the description of flat Lie algebras in [**11** or **12**].

LEMMA 1. *Let (E,H,F) be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ with E a semiregular nilpotent in $\mathfrak{g}$. Then* $\mathrm{ad}(H/2)$ *has integer eigenvalues, the* $\mathbf{Z}$*-graded Lie algebra* $(\mathfrak{g}, H/2) = \bigoplus_{k\in\mathbf{Z}} \mathfrak{s}_k$ *is flat, and E is a generic element of* $\mathfrak{s}_1$.

4. Basic lemma. For the proof of the basic lemma we need the following technical lemma.

LEMMA 2. *Assume that* $\operatorname{rank}\mathfrak{k} = \operatorname{rank}\mathfrak{g}$, *fix a Cartan subalgebra* $\mathfrak{h}_0$ *of* $\mathfrak{k}_0$ *and let* $\mathfrak{h} = \mathfrak{h}_0 + i\mathfrak{h}_0$. *Let* R *be the root system of* $(\mathfrak{g}, \mathfrak{h})$ *and*

$$R^{(0)} = \{\alpha \in R \colon \mathfrak{g}^\alpha \subset \mathfrak{k}\}, \quad R^{(1)} = \{\alpha \in R \colon \mathfrak{g}^\alpha \subset \mathfrak{p}\}.$$

Then there exists a Chevalley system (X_α), $\alpha \in R$, *of* $(\mathfrak{g}, \mathfrak{h})$ *such that*

$$\sigma(X_\alpha) = (-1)^k X_{-\alpha}, \qquad \alpha \in R^{(k)}.$$

(*For the definition of Chevalley systems see* [**3**, *Chapitre* VIII, §3, *no.* 4, *p.* 84].)

PROOF. Let (Y_α), $\alpha \in R$, be any Chevalley system of $(\mathfrak{g}, \mathfrak{h})$.

The $\mathbf{R}$-span of $\mathfrak{h}_0$ and the vectors $Y_\alpha + Y_{-\alpha}$, $i(Y_\alpha - Y_{-\alpha})$, $\alpha \in R^{(0)}$; and $i(Y_\alpha + Y_{-\alpha})$, $Y_\alpha - Y_{-\alpha}$, $\alpha \in R^{(1)}$; are a real form $\mathfrak{g}^{\#}$ of $\mathfrak{g}$ isomorphic to $\mathfrak{g}_0$. Choose an isomorphism $\tau \colon \mathfrak{g}^{\#} \to \mathfrak{g}_0$ such that $\tau(\mathfrak{h}_0) = \mathfrak{h}_0$ and extend τ to an automorphism of $\mathfrak{g}$. Set $X_\alpha = \tau(Y_\alpha)$. Then (X_α), $\alpha \in R$, is a Chevalley system of $(\mathfrak{g}, \mathfrak{h})$ having the required properties.

Now we can prove our basic lemma.

LEMMA 3. *Let* $(\mathfrak{g}, H/2) = \bigoplus \mathfrak{s}_k$, $k \in \mathbf{Z}$, *be a simple flat complex Lie algebra and assume that the associated* $\mathbf{Z}_2$*-grading on* $\mathfrak{g}$ *coincides with* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. *Then there exists* $X \in \mathfrak{s}_1$ *such that*

$$[X, \sigma(X)] = H. \tag{1}$$

PROOF. Fix a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{k}_0$ such that $\mathfrak{h}_0 \subset \mathfrak{s}_0$ and set $\mathfrak{h} = \mathfrak{h}_0 + i\mathfrak{h}_0$. Let $\mathbf{R}$ be the root system of $(\mathfrak{g}, \mathfrak{h})$ and for $k \in \mathbf{Z}$ let

$$R_k = \{\alpha \in R \colon \mathfrak{g}^\alpha \subset \mathfrak{s}_k\}.$$

Also define

$$R^{(0)} = \bigcup_{k \in \mathbf{Z}} R_{2k}, \quad R^{(1)} = \bigcup_{k \in \mathbf{Z}} R_{2k+1}.$$

By Lemma 2 there exists a Chevalley system (X_α), $\alpha \in R$, such that

$$\sigma(X_\alpha) = (-1)^k X_{-\alpha}, \qquad \alpha \in R^{(k)}.$$

For $\alpha \in R_1$ let $Y_\alpha = -X_{-\alpha}$. Let H_α be the unique element of $[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ such that $\alpha(H_\alpha) = 2$. Recall that $[X_\alpha, X_{-\alpha}] = -H_\alpha$, $\alpha \in R$, and so $[X_\alpha, Y_\alpha] = H_\alpha$ for $\alpha \in R_1$.

We shall seek a solution of equation (1) in the form

$$X = \sum_{\alpha \in R_1} \lambda_\alpha X_\alpha$$

with all λ_α real. Then

$$\sigma(X) = \sum_{\alpha \in R_1} \lambda_\alpha Y_\alpha$$

and equation (1) can be written as

$$\sum_{\alpha, \beta \in R_1} \lambda_\alpha \lambda_\beta [X_\alpha, Y_\beta] = H. \tag{2}$$

Assume first that our flat Lie algebra $(\mathfrak{g}, H/2)$ is principal, i.e., that R_1 is a base, say B, of R. In that case $\alpha - \beta \notin R$ for $\alpha, \beta \in B$ and so (2) becomes

$$\sum_{\alpha \in B} \lambda_\alpha^2 H_\alpha = H.$$

Since $\alpha(H) = 2$ for all $\alpha \in B$, this equation is equivalent to the system

$$\sum_{\beta \in B} \alpha(H_\beta)\lambda_\beta^2 = 2, \qquad \alpha \in B. \tag{3}$$

By a theorem of Vinberg [**10**, Theorem 3] the unique solution μ_β, $\beta \in R$, of the system of linear equations

$$\sum_{\beta \in B} \alpha(H_\beta)\mu_\beta = 2, \qquad \alpha \in B,$$

is positive in the sense that $\mu_\beta > 0$ for each $\beta \in B$. It follows that the system (3) has a real solution.

Next assume that $(\mathfrak{g}, H/2)$ is the simple flat Lie algebra $D_{n+m+1}(a_m)$, $n > m \geq 1$, see [**5** or **6**]. In this case we shall use the notations for roots, the Chevalley system, etc., given in Bourbaki [**3**, Chapitre VIII, §13, no. 4, pp. 206–212]. Then $H/2$ is the diagonal matrix of order $2n + 2m + 2$ whose diagonal entries are the integers $n, n-1, \ldots, -n$ and $m, m-1, \ldots, -m$ arranged in nonincreasing order. The set R_1 consists of the roots

$$\begin{array}{lll}
\varepsilon_i - \varepsilon_{i+1}, \quad 1 \leq i \leq n-m; & \varepsilon_{n-m} - \varepsilon_{n-m+2}; & \\
\varepsilon_{n-m+2k-1} - \varepsilon_{n-m+2k+1}, & \varepsilon_{n-m+2k} - \varepsilon_{n-m+2k+1}, & 1 \leq k \leq m; \\
\varepsilon_{n-m+2k-1} - \varepsilon_{n-m+2k+2}, & \varepsilon_{n-m+2k} - \varepsilon_{n-m+2k+2}, & 1 \leq k \leq m-1;
\end{array}$$

and

$$\varepsilon_{n+m-1} + \varepsilon_{n+m+1}, \quad \varepsilon_{n+m} + \varepsilon_{n+m+1}.$$

In this case some of the λ_α can be taken to be zero. An explicit solution of equation (1) is provided by

$$\begin{aligned}
X = {} & \lambda_1 X_{\varepsilon_1 - \varepsilon_2} + \lambda_2 X_{\varepsilon_2 - \varepsilon_3} + \cdots + \lambda_{n-m} X_{\varepsilon_{n-m} - \varepsilon_{n-m+1}} \\
& + \mu_1 X_{\varepsilon_{n-m+1} - \varepsilon_{n-m+3}} + \nu_1 X_{\varepsilon_{n-m+2} - \varepsilon_{n-m+4}} \\
& + \mu_2 X_{\varepsilon_{n-m+3} - \varepsilon_{n-m+5}} + \nu_2 X_{\varepsilon_{n-m+4} - \varepsilon_{n-m+6}} \\
& + \cdots \\
& + \mu_{m-1} X_{\varepsilon_{n+m-3} - \varepsilon_{n+m-1}} + \nu_{m-1} X_{\varepsilon_{n+m-2} - \varepsilon_{n+m}} \\
& + \rho_1 X_{\varepsilon_{n+m-1} - \varepsilon_{n+m+1}} + \sigma_1 X_{\varepsilon_{n+m-1} - \varepsilon_{n+m+1}} \\
& + \rho_2 X_{\varepsilon_{n+m} - \varepsilon_{n+m+1}} + \sigma_2 X_{\varepsilon_{n+m} + \varepsilon_{n+m+1}}
\end{aligned}$$

where

$$\begin{aligned}
&\lambda_k^2 = k(2n - k + 1), \qquad 1 \leq k \leq n - m; \\
&\mu_k^2 = (n-m)(n+m+1) + k(2m - k + 1), \qquad 1 \leq k \leq m - 1; \\
&\nu_k^2 = k(2m - k + 1), \qquad 1 \leq k \leq m - 1;
\end{aligned}$$

and $\rho_1 + i\rho_2 = \pm\sqrt{z}$, $\sigma_1 + i\sigma_2 = \pm\sqrt{w}$ with z and w complex numbers satisfying $z + w = (n-m)(n+m+1)$, $|z| = |w| = m^2 - 3m + 4 + \frac{1}{2}(n-m)(n+m+1)$.

We omit the routine details of the verification of this claim.

There remain five exceptional simple flat Lie algebras to be dealt with, namely, $E_8(a_1), E_8(a_2), E_7(a_1), E_7(a_2)$, and $E_6(a_1)$. In order to exhibit an explicit solution of (1) it will be convenient to use the Chevalley system of E_8 constructed in our paper [**4**]. For the convenience of the reader we shall review some basic facts about this Chevalley system.

The Lie algebra E_8 is realized as $\mathbf{Z}$-graded algebra $\mathfrak{g} = \bigoplus_{k\in\mathbf{Z}} \mathfrak{s}_k$ where $\mathfrak{s}_0 = V\otimes V^*$, $\mathfrak{s}_1 = \bigwedge^3 V$, $\mathfrak{s}_{-1} = \bigwedge^3 V^*$, $\mathfrak{s}_2 = \bigwedge^2 V$, $\mathfrak{s}_{-2} = \bigwedge^2 V^*$, $\mathfrak{s}_3 = V$, $\mathfrak{s}_4 = V^*$ and $\mathfrak{s}_k = 0$ otherwise. Here V denotes a complex vector space of dimension 8 with a fixed basis e_k, $1 \le k \le 8$, and V^* its dual space with the dual basis e^k, $1 \le k \le 8$.

For the definition of the Lie bracket see [**4**]. We mention only that $\mathfrak{s}_0 \cong \mathfrak{gl}(V)$, that the action of $\mathfrak{s}_0$ on each $\mathfrak{s}_k$ is the standard one, and that

$$[a\wedge b\wedge c, f\wedge g\wedge h] = -\begin{vmatrix} f(a) & f(b) & f(c) & f \\ g(a) & g(b) & g(c) & g \\ h(a) & h(b) & h(c) & h \\ a & b & c & 1/3 \end{vmatrix} \tag{4}$$

where $a,b,c \in V$, $f,g,h\in V^*$ and when evaluating this determinant the product of, say, a and f should be written as $a\otimes f$.

Using the formula (4) one finds that

$$[e_{ijk}, e^{rjk}] = e_i^r \tag{5}$$

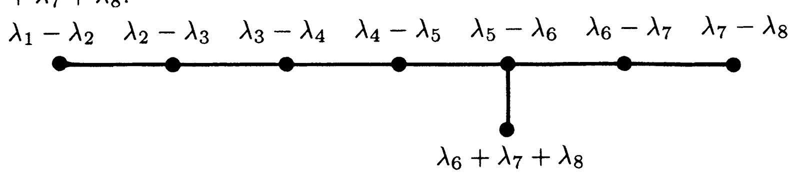
if $i \neq r$, $i < j < k$, and $r < j$.

Writing $e_i^j = e_i \otimes e^j$, the subspace $\mathfrak{h}$ spanned by the elements e_i^i, $1 \le i \le 8$, is a Cartan subalgebra of $\mathfrak{s}_0$ and of $\mathfrak{g}$. The Chevalley system of $(\mathfrak{g},\mathfrak{h})$ is given by the elements:

$$\begin{array}{ll} e_i^j, -e_j^i & (1 \le i < j \le 8); \\ e_i, -e^i & (1 \le i \le 8); \\ e_{ijk}, -e^{ijk} & (1 \le i < j < k \le 8); \\ e_{ij}, e^{ij} & (1 \le i < j \le 8); \end{array}$$

where $e_{ijk} = e_i\wedge e_j\wedge e_k$, $e^{ijk} = e^i \wedge e^j \wedge e^k$, etc.

In [**4**] λ_i, $1\le i \le 8$, is a basis of $\mathfrak{h}^*$ dual to the basis $e_i^i, 1 \le i \le 8$, of $\mathfrak{h}$. A base B of the root system R of $(\mathfrak{g},\mathfrak{h})$ consists of the roots $\lambda_i - \lambda_{i+1}$, $1 \le i \le 7$, and the root $\lambda_6+\lambda_7+\lambda_8$:



For $\alpha \in B$ the elements $H_\alpha \in \mathfrak{h}$ are given by

$$\begin{aligned} h_i &:= H_{\lambda_i-\lambda_{i+1}} = e_i^i - e_{i+1}^{i+1}, \qquad 1 \le i \le 7, \\ h_8 &:= H_{\lambda_6+\lambda_7+\lambda_8} = -\tfrac{1}{3} + e_6^6 + e_7^7 + e_8^8, \end{aligned}$$

whence $-\frac{1}{3}$ means $-\frac{1}{3}\cdot \sum_{i=1}^8 e_i^i$.

Case $E_8(a_1)$. In this case we have

$$H = 46h_1 + 90h_2 + 132h_3 + 172h_4 + 210h_5 + 142h_6 + 72h_7 + 106h_8,$$

and R_1 consists of the roots $\lambda_i - \lambda_{i+1}$, $i \neq 5$, $\lambda_6 + \lambda_7 + \lambda_8$, $\lambda_4 - \lambda_6$, $\lambda_5 - \lambda_7$, and $\lambda_5 + \lambda_7 + \lambda_8$.

An explicit solution of (1) is given by

$$X = \sqrt{46}e_1^2 + \sqrt{90}e_2^3 + \sqrt{132}e_3^4 + \rho_1 e_4^5 + \sigma_1 e_6^7 + \sqrt{72}e_7^8 + \sqrt{106}e_{678} + \rho_2 e_4^6 + \sigma_2 e_5^7$$

where $\rho_1 + i\rho_2 = \pm\sqrt{z}$, $\sigma_1 - i\sigma_2 = \pm\sqrt{w}$ and z and w are complex numbers such that $|z| = 172$, $|w| = 142$, $z + w = -106$.

Case $E_8(a_2)$. We have

$$H = 38h_1 + 74h_2 + 108h_3 + 142h_4 + 174h_5 + 118h_6 + 60h_7 + 88h_8,$$

R_1 consists of the roots $\lambda_i - \lambda_{i+1}$, $i \neq 3, 5$; $\lambda_6 + \lambda_7 + \lambda_8$, $\lambda_2 - \lambda_4$, $\lambda_3 - \lambda_5$, $\lambda_4 - \lambda_6$, $\lambda_5 - \lambda_7$, and $\lambda_5 + \lambda_7 + \lambda_8$. A solution of (1) is provided by

$$X = \sqrt{38}e_1^2 + \sqrt{74}e_2^3 + \sqrt{34}e_4^5 + \rho_1 e_6^7 + \sqrt{60}e_7^8 + \sigma_1 e_{678} + \sqrt{108}e_3^6 + \rho_2 e_5^7 + \sigma_2 e_{578}$$

where $\rho_1 + i\rho_2 = \pm\sqrt{z}$, $\sigma_1 + i\sigma_2 = \pm\sqrt{w}$ and z and w are complex numbers such that $|z| = 118$, $|w| = 88$, $z + w = 74$.

Formula (5) is useful when one checks that X is indeed a solution.

Case $E_7(a_1)$. We have

$$H = 21h_2 + 40h_3 + 57h_4 + 72h_5 + 50h_6 + 26h_7 + 37h_8,$$

and R_1 is the same as in case $E_8(a_1)$ except that the root $\lambda_1 - \lambda_2$ should be omitted. A solution X of (1) is given by

$$X = \sqrt{21}e_2^3 + \sqrt{40}e_3^4 + \sigma_1 e_4^5 + \rho_1 e_6^7 + \sqrt{26}e_7^8 + \sqrt{37}e_{678} + \sigma_2 e_4^6 + \rho_2 e_5^7,$$

where $\rho_1 + i\rho_2 = \pm\sqrt{z}$, $\sigma_1 - i\sigma_2 = \pm\sqrt{w}$ and z and w are complex numbers satisfying $|z| = 50$, $|w| = 57$, $z + w = -37$.

Case $E_7(a_2)$. Now

$$H = 17h_2 + 32h_3 + 47h_4 + 60h_5 + 42h_6 + 22h_7 + 31h_8$$

and R_1 is the same as in the case $E_8(a_2)$ except that the root $\lambda_1 - \lambda_2$ should be omitted. A solution X of (1) is given by

$$X = \sqrt{17}e_2^3 + \sqrt{15}e_4^5 + \rho_1 e_6^7 + \sqrt{22}e_7^8 + \sigma_1 e_{678} + \sqrt{32}e_3^6 + \rho_2 e_5^7 + \sigma_2 e_{578},$$

where $\rho_1 + i\rho_2 = \pm\sqrt{z}$, $\sigma_1 + i\sigma_2 = \pm\sqrt{w}$ and z and w are complex numbers satisfying $|z| = 42$, $|w| = 31$, $z + w = 17$.

Case $E_6(a_1)$. In this case

$$H = 12h_3 + 22h_4 + 30h_5 + 22h_6 + 12h_7 + 16h_8,$$

and R_1 is the same as in case $E_8(a_1)$ except that the roots $\lambda_1 - \lambda_2$ and $\lambda_2 - \lambda_3$ should be omitted. A solution X of (1) is given by

$$X = \sqrt{12}e_3^4 + \rho_1 e_4^5 + \sigma_1 e_6^7 + \sqrt{12}e_7^8 + 4e_{678} + \rho_2 e_4^6 + \sigma_2 e_5^7,$$

where $\rho_1 + i\rho_2 = \pm\sqrt{z}$, $\sigma_1 - i\sigma_2 = \pm\sqrt{w}$, and z and w are complex numbers satisfying $|z| = |w| = 22$, $z + w = -16$. This completes the proof of the lemma.

5. Proof that ψ is surjective. Let E be a nonzero nilpotent element in $\mathfrak{p}$. We have to show that there exists a complex Cayley triple (X, H, Y) such that $X \in K \cdot E$. The proof is by induction on the dimension of $\mathfrak{g}$.

We can embed E in a normal $\mathfrak{sl}_2$-triple (E, H, F). Since H is a real semisimple element it is K-conjugate to an element of $i\mathfrak{k}_0$. Hence by replacing this triple by a suitable K-conjugate we may assume that $H \in i\mathfrak{k}_0$.

Let $\mathfrak{s} = \bigoplus \mathfrak{s}_k$, $k \in \mathbf{Z}$, be the $\mathbf{Z}$-graded subalgebra of the $\mathbf{Z}_2$-graded algebra $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ defined as follows:

$$\mathfrak{s}_k = \{X \in \mathfrak{k}\colon [H, X] = 2kX\}$$

for k even and

$$\mathfrak{s}_k = \{X \in \mathfrak{p}\colon [H, X] = 2kX\}$$

for k odd. Clearly $E \in \mathfrak{s}_1$ and by a result of Vinberg [**12**, Lemma 2] $\mathfrak{s}$ is reductive. If $\mathfrak{s} \neq \mathfrak{g}$ then the induction hypothesis can be applied to the associated $\mathbf{Z}_2$-graded algebra $\mathfrak{s} = \mathfrak{s} \cap \mathfrak{k} \oplus \mathfrak{s} \cap \mathfrak{p}$ and the element E.

Hence we may assume that $\mathfrak{s} = \mathfrak{g}$. Since the centralizer of H in $\mathfrak{s}$ is $\mathfrak{s}_0$, $\mathfrak{s} = \mathfrak{g}$, and $\mathfrak{s}_0 \subset \mathfrak{k}$, it follows that $\operatorname{rank} \mathfrak{k} = \operatorname{rank} \mathfrak{g}$. Let us fix a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{k}_0$ such that $iH \in \mathfrak{h}_0$. Set $\mathfrak{h} = \mathfrak{h}_0 + i\mathfrak{h}_0$.

By a theorem of Vinberg and Elašvili [**13**, p. 223] there exist $X \in K \cdot E$ and a regular semisimple subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ normalized by $\mathfrak{h}$ such that $X \in \mathfrak{t}$ and X is a semiregular nilpotent element of $\mathfrak{t}$.

Since $\mathfrak{h}$ normalizes $\mathfrak{t}$, it follows that the $\mathbf{Z}_2$-grading $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ induces a $\mathbf{Z}_2$-grading on $\mathfrak{t}$, i.e., that $\mathfrak{t} = \mathfrak{t} \cap \mathfrak{k} \oplus \mathfrak{t} \cap \mathfrak{p}$. If $\mathfrak{t} \neq \mathfrak{g}$ then the inductive hypothesis can be applied to $\mathfrak{t}$ and X. Hence we may assume that $\mathfrak{t} = \mathfrak{g}$, i.e., that E is a semiregular nilpotent element of $\mathfrak{g}$.

By Lemma 1 the $\mathbf{Z}$-graded algebra $\mathfrak{g} = \mathfrak{s} = \bigoplus \mathfrak{s}_k$ is flat and E is a generic element of $\mathfrak{s}_1$. Since $H \in i\mathfrak{h}_0 \subset i\mathfrak{k}_0$, we have $\sigma(H) = -H$ and consequently $\sigma(\mathfrak{s}_k) = \mathfrak{s}_{-k}$ for all k.

Assume that there is an $X \in \mathfrak{s}_1$ such that $[X, \sigma(X)] = H$. Then $(X, H, -\sigma(X))$ is a complex Cayley triple and by a result of Kostant and Rallis [**8**, Lemma 4] the normal $\mathfrak{sl}_2$-triples (E, H, F) and $(X, H, -\sigma(X))$ are K-conjugate.

Hence it suffices to prove the existence of an element $X \in \mathfrak{s}_1$ such that $[X, \sigma(X)] = H$. Since every flat Lie algebra is a direct product of simple flat Lie algebras we have

$$(\mathfrak{g}, H/2) = (\mathfrak{g}^{(1)}, H_1/2) \times \cdots \times (\mathfrak{g}^{(m)}, H_m/2)$$

where each $(\mathfrak{g}^{(k)}, H_k/2)$ is a simple flat Lie algebra and $H = H_1 + \cdots + H_m$. This shows that without any loss of generality we may now assume that $\mathfrak{g}$ is simple.

In Lemma 3 we have shown that in the case of simple flat Lie algebras the equation $[X, \sigma(X)] = H$ indeed has a solution for X. This completes the proof of the conjecture.

Addendum (February 1987). The maps ϕ and ψ_0 defined in §3 are in fact bijective. In view of the results mentioned there and our main theorem, the claim follows from the following proposition.

PROPOSITION. *The map ϕ is injective.*

PROOF. Let (E, H, F) and (E', H', F') be two real Cayley triples with $E' \in G_0 \cdot E$. By [**3**, Chapter VIII, §11, Lemma 4] these triples are G_0-conjugate. By using [**9**, Proposition 1.1] it follows that $E' - F' \in K_0 \cdot (E - F)$. Hence we may assume that $E' - F' = E - F = Z$, say. Let G_0^Z (resp., K_0^Z) be the centralizer of Z in G_0 (resp., K_0). Fix a maximal compact subgroup M of G_0^Z containing K_0^Z. If $x \in M$ write $x = y\exp(X)$ with $y \in K_0$ and $X \in \mathfrak{p}_0$. By using an argument of L. Preiss-Rothschild [**9**, Proof of Proposition 1.1] it follows from $\exp(X) \cdot Z = y^{-1} \cdot Z$ that $y^{-1} \cdot Z = Z$. Hence $y \in K_0^Z$, $\exp(X) \in M$ and since M is compact we must have $X = 0$. Thus $M = K_0^Z$.

By [**8**, p. 779] $\mathfrak{g}_0^Z = \mathfrak{k}_0^Z \oplus \mathfrak{p}_0^Z$ is a Cartan decomposition of $\mathfrak{g}_0^Z$ and consequently $G_0^Z = K_0^Z \cdot \exp(\mathfrak{p}_0^Z)$.

If $a \in G_0$ is an element which maps the triple (E, H, F) to (E', H', F') then $a \in G_0^Z$ and $a \cdot (E + F) = E' + F'$. Write $a = b\exp(Y)$ with $b \in K_0^Z$ and $Y \in \mathfrak{p}_0^Z$. Then by applying the above mentioned argument to $\exp(Y) \cdot (E+F) = b^{-1} \cdot (E'+F')$ we infer that $b^{-1} \cdot (E' + F') = E + F$. Thus $b \in K_0$ sends (E, H, F) to (E', F', H').

ADDED IN PROOF. After this paper was written D. King informed me that Jiro Sekiguchi had also proved Kostant's conjecture (by a different method) in a preprint entitled *Remarks on real nilpotent orbits of a symmetric pair.*

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