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VMO, ESV, AND TOEPLITZ OPERATORS ON THE BERGMAN SPACE

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ABSTRACT. This paper studies the largest C^*-subalgebra Q of $L^\infty(\mathbf{D})$ such that the Toeplitz operators T_f on the Bergman space $L_a^2(\mathbf{D})$ with symbols f in Q have a symbol calculus modulo the compact operators. Q is characterized by a condition of vanishing mean oscillation near the boundary. I also give several other necessary and sufficient conditions for a bounded function to be in Q. After decomposing Q in a "nice" way, I study the Fredholm theory of Toeplitz operators with symbols in Q. The essential spectrum of $T_f (f \in Q)$ is shown to be connected and computable in terms of the Stone-Cěch compactification of $\mathbf{D}$. The results in this article partially answer a question posed in [**3**] and give several new necessary and sufficient conditions for a bounded analytic function on the open unit disc to be in the little Bloch space $\mathcal{B}_0$.

1. Introduction. Let $\mathbf{D}$ be the open unit disc in the complex plane C. Consider the Bergman space $L_a^2(\mathbf{D})$ of analytic functions in $L^2(\mathbf{D}, dA)$, where $dA = \frac{1}{\pi} r\, dr\, d\theta$ is the normalized area measure on $\mathbf{D}$. For any function f in $L^\infty(\mathbf{D}, dA)$, the Toeplitz operator $T_f\colon L_a^2(\mathbf{D}) \to L_a^2(\mathbf{D})$ and the Hankel operator $H_f\colon L_a^2(\mathbf{D}) \to L^2(\mathbf{D})$ are defined by

$$T_f g = P(fg), \quad H_f g = (I - P)(fg), \qquad g \in L_a^2(\mathbf{D}),$$

where $P\colon L^2(\mathbf{D}) \to L_a^2(\mathbf{D})$ is the orthogonal projection. It is well known that Toeplitz operators and Hankel operators are related by

$$T_{|f|^2} - T_{\bar f} T_f = H_f^* H_f.$$

In [**3**], Sheldon Axler raised the question of characterizing the functions $f \in L^\infty(\mathbf{D})$ such that H_f is compact. This is equivalent to characterizing functions $f \in L^\infty(\mathbf{D})$ such that the semi-self-commutator $T_{|f|^2} - T_{\bar f} T_f$ is compact. Axler answered a special case of this problem in [**1**]. He proved that for any analytic function f on $\mathbf{D}$, $H_{\bar f}$ is compact if and only if $f \in \mathcal{B}_0$, the "little Bloch" space.

Recall that for Toeplitz operators T_f and Hankel operators H_f ($f \in L^\infty(S^1)$) on the Hardy space H^2 of the unit circle S^1, it is well known [**15**] that H_f is compact if and only if $f \in C(S^1) + H^\infty$; H_f and $H_{\bar f}$ are compact if and only if $f \in \mathrm{VMO}(S^1)$ [**22, 23**]. For Toeplitz operators T_f and Hankel operators H_f ($f \in L^\infty(C^n)$) on the Bergman space $L_a^2(C^n, d\mu)$ of C^n with the Gaussian measure $d\mu$, L. A. Coburn and C. A. Berger in [**7**] proved that H_f is compact if and only if H_f and $H_{\bar f}$ are compact if and only if $f = f_1 + f_2$ with $f_1 \in \mathrm{ESV}(C^n)$ and $T_{|f_2|}$ compact.

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In this paper, we introduce a new space $\mathrm{VMO}_\partial(\mathbf{D})$ of integrable functions on $\mathbf{D}$ and use it to characterize the functions $f \in L^\infty(\mathbf{D})$ such that H_f and $H_{\bar f}$ are compact. $\mathrm{VMO}_\partial(\mathbf{D})$, roughly speaking, is the space of integrable functions on $\mathbf{D}$ with vanishing mean oscillation near the boundary of $\mathbf{D}$. The usual area VMO [**10**] fails to work in this situation because a Toeplitz operator T_f on $L^2_a(\mathbf{D})$, up to a compact perturbation, only depends on the behavior of f near the boundary of $\mathbf{D}$ [**2**, **6**, **9**, **16**]. The (mean) oscillation of f inside $\mathbf{D}$ does not affect T_f in the Calkin algebra.

In §§3 and 4, we study the space $\mathrm{VMO}_\partial(\mathbf{D})$ and one of its important subspaces, $\mathrm{ESV}(\mathbf{D})$. Several equivalent conditions for a function to be in $\mathrm{VMO}_\partial(\mathbf{D})$ or $\mathrm{ESV}(\mathbf{D})$ are proved. We also prove that for $f \in H^\infty(\mathbf{D})$, $f \in \mathrm{VMO}_\partial(\mathbf{D})$ if and only if $f \in \mathrm{ESV}(\mathbf{D})$ if and only if $f \in \mathcal{B}_0$. The so-called Berezin symbol [**7**, **5**] serves as a basic tool to study $\mathrm{VMO}_\partial(\mathbf{D})$ and Toeplitz operators. Some basic properties of Berezin symbol are first established in §2. §5 is devoted to the proof of the main theorem: For $f \in L^\infty(\mathbf{D})$, H_f and $H_{\bar f}$ are compact if and only if f is in $\mathrm{VMO}_\partial(\mathbf{D})$. Notice that this theorem also solves the "symbol calculus" problem of finding the largest C^*-subalgebra Q of $L^\infty(\mathbf{D})$ such that the map $\xi\colon Q \to \mathcal{B}(L^2_a(\mathbf{D}))/\mathcal{K}$ defined by $\xi(f) = T_f + \mathcal{K}$ is a C^*-algebra homomorphism, where $\mathcal{K}$ is the compact ideal of the full algebra $\mathcal{B}(L^2_a(\mathbf{D}))$ of bounded linear operators on $L^2_a(\mathbf{D})$. $\mathcal{B}(L^2_a(\mathbf{D}))/\mathcal{K}$ is the Calkin algebra. The theorem simply says that $Q = L^\infty(\mathbf{D}) \cap \mathrm{VMO}_\partial(\mathbf{D})$. In §6 we discuss the Fredholm theory of Toeplitz operators with symbols in Q. The conformal invariance of VMO_∂ is discussed in §7. §8 concludes the paper with some open problems and possible generalizations.

2. The Berezin symbol of Toeplitz operators. Recall that $L^2_a(\mathbf{D})$ has reproducing kernel

$$K(z, \bar w) = 1/(1 - z\bar w)^2.$$

For any $w \in \mathbf{D}$, we can define a unit vector k_w in $L^2_a(\mathbf{D})$ by

$$k_w(z) = \frac{K(z,\bar w)}{\sqrt{K(w,\bar w)}} = \frac{1-|w|^2}{(1-z\bar w)^2}, \qquad z \in \mathbf{D}.$$

The k_w's are called the normalized reproducing kernels.

Now given any bounded linear operator S on $L^2_a(\mathbf{D})$, we define a bounded continuous function $\tilde S$ on D [**5**] by

$$\tilde S(z) = \langle S k_z, k_z\rangle, \qquad z \in \mathbf{D}.$$

$\tilde S$ is called the Berezin symbol of S. For any function $f \in L^\infty(\mathbf{D}, dA)$, we define $\tilde f = \tilde T_f$, so that

$$\tilde f(z) = \langle T_f k_z, k_z\rangle = \langle f k_z, k_z\rangle = \int_{\mathbf{D}} f(w)|k_z(w)|^2\, dA(w).$$

We also call $\tilde f$ the Berezin symbol of f. Notice that

$$|\tilde f(z)| = |\langle T_f k_z, k_z\rangle| \le \|T_f k_z\|\, \|k_z\| \le \|T_f\| \le \|f\|_\infty,$$

so $\|\tilde f\|_\infty \le \|f\|_\infty$. The map $f \mapsto \tilde f\colon L^\infty(\mathbf{D}, dA) \to L^\infty(\mathbf{D}, dA)$ is linear, contractive (hence continuous), and order-preserving. It is easy to see that $\tilde{\bar f} = \bar{\tilde f}$ for any

$f \in L^\infty(\mathbf{D})$. Actually, we have $\tilde{S}^* = \bar{\tilde{S}}$ for any bounded linear operator S on $L_a^2(\mathbf{D})$.

Let $H^\infty(\mathbf{D})$ denote the Banach algebra of bounded holomorphic functions on $\mathbf{D}$. We have

PROPOSITION 1. *For any $f \in H^\infty(\mathbf{D})$ and $z \in \mathbf{D}$, $T_{\bar{f}} k_z = \bar{f}(z) k_z$.*

PROOF. First recall the reproducing property of $K(z, \bar{w})$:

$$f(z) = \int_{\mathbf{D}} K(z, \bar{w}) f(w)\, dA(w)$$

for any $f \in L_a^2(\mathbf{D})$ and $z \in \mathbf{D}$. The Toeplitz operator T_f is an integral operator:

$$(T_f g)(z) = \int_{\mathbf{D}} K(z, \bar{w}) f(w) g(w)\, dA(w)$$

for any $f \in L^\infty(\mathbf{D})$ and $g \in L_a^2(\mathbf{D})$. Now if $f \in H^\infty(\mathbf{D})$, we have

$$\begin{aligned}
(T_{\bar{f}} K(\cdot, \bar{z}))(w) &= \int_{\mathbf{D}} K(w, \bar{u}) K(u, \bar{z}) \bar{f}(u) dA(u) \\
&= \overline{\int_{\mathbf{D}} \overline{K(w, \bar{u})}\, \overline{K(u, \bar{z})} f(u) dA(u)} \\
&= \overline{\int_{\mathbf{D}} K(u, \bar{w}) K(z, \bar{u}) f(u) dA(u)} \\
&= \overline{K(z, \bar{w}) f(z)} = \bar{f}(z) K(w, \bar{z}),
\end{aligned}$$

so

$$T_{\bar{f}} K(\cdot, \bar{z}) = \bar{f}(z) K(\cdot, \bar{z}).$$

Dividing both sides by $\sqrt{K(z, \bar{z})}$, we get $T_{\bar{f}} k_z = \bar{f}(z) k_z$.

PROPOSITION 2. *For any $f \in H^\infty(\mathbf{D}) + \overline{H^\infty(\mathbf{D})}$, $\tilde{f} = f$.*

PROOF. Since the map $f \mapsto \tilde{f}$ is linear and conjugation-preserving, it suffices to prove the result for $f \in \overline{H^\infty(\mathbf{D})}$. But in this case, we have $T_f k_z = f(z) k_z$ by Proposition 1. Thus

$$\tilde{f}(z) = \langle T_f k_z, k_z \rangle = \langle f(z) k_z, k_z \rangle = f(z) \langle k_z, k_z \rangle = f(z).$$

PROPOSITION 3. *For any $f \in L^\infty(\mathbf{D})$, the following are equivalent:*
(1) $\lim_{|z| \to 1^-} (\widetilde{fg}(z) - \tilde{f}(z) \tilde{g}(z)) = 0$ *for all* $g \in L^\infty(\mathbf{D})$;
(2) $\lim_{|z| \to 1^-} (\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2) = 0$.

PROOF. First it is easy to establish the following two identities:

$$\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2 = \frac{1}{2} \int_{\mathbf{D}} \int_{\mathbf{D}} |f(w) - f(u)|^2 |k_z(w)|^2 |k_z(u)|^2 dA(w) dA(u);$$

$$\begin{aligned}
&\widetilde{fg}(z) - \tilde{f}(z) \tilde{g}(z) \\
&\quad = \frac{1}{2} \int_{\mathbf{D}} \int_{\mathbf{D}} (f(u) - f(w))(g(u) - g(w)) |k_z(u)|^2 |k_z(w)|^2 dA(w) dA(u).
\end{aligned}$$

Then the Cauchy-Schwarz inequality gives

$$|\widetilde{fg}(z) - \tilde{f}(z) \tilde{g}(z)|^2 \le (\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2)(\widetilde{|g|^2}(z) - |\tilde{g}(z)|^2).$$

Now the desired result follows easily from this inequality.

COROLLARY.

$$\tilde{Q} = \left\{ f \in L^\infty(\mathbf{D}) \middle| \lim_{|z|\to 1^-} (\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2) = 0 \right\}$$

is a C^*-subalgebra of $L^\infty(\mathbf{D})$.

PROOF. Let $f_1, f_2 \in \tilde{Q}$. Then for any $g \in L^\infty(\mathbf{D})$, we have

$$\widetilde{f_i g}(z) - \tilde{f}_i(z)\tilde{g}(z) \to 0 \quad \text{as } |z| \to 1^-, \ i = 1, 2.$$

So

$$\begin{aligned} &\widetilde{(f_1 + f_2)g}(z) - \widetilde{(f_1 + f_2)}(z)\tilde{g}(z) \\ &\qquad = \widetilde{f_1 g}(z) - \tilde{f}_1(z)\tilde{g}(z) + \widetilde{f_2 g}(z) - \tilde{f}_2(z)\tilde{g}(z) \to 0 \end{aligned}$$

as $|z| \to 1^-$, and

$$\begin{aligned} &\widetilde{(f_1 f_2)g}(z) - \widetilde{f_1 f_2}(z)\tilde{g}(z) \\ &\qquad = \widetilde{f_1(f_2 g)}(z) - \tilde{f}_1(z)\widetilde{f_2 g}(z) + \tilde{f}_1(z)(\widetilde{f_2 g}(z) - \tilde{f}_2(z)\tilde{g}(z)) \\ &\qquad\quad - (\widetilde{f_1 f_2}(z) - \tilde{f}_1(z)\tilde{f}_2(z))\tilde{g}(z) \to 0 \quad \text{as } |z| \to 1^-, \end{aligned}$$

thus f_1+f_2 and $f_1 f_2$ are in $\tilde{Q}$ by the previous proposition. $\tilde{Q}$ is obviously selfadjoint and closed under scalar multiplication. $\tilde{Q}$ is norm-closed since $f \mapsto \tilde{f}$ is continuous. Therefore, $\tilde{Q}$ is a C^*-subalgebra of $L^\infty(\mathbf{D})$.

Before going on, we have some remarks:

(1) $k_z \to 0$ weakly in $L^2_a(\mathbf{D})$ as $|z| \to 1^-$, so if S is a compact operator on $L^2_a(\mathbf{D})$, then

$$\tilde{S}(z) = \langle S k_z, k_z \rangle \to 0 \quad \text{as } |z| \to 1^-.$$

(2) If f is a polynomial in z, then it is well known that $T_{|f|^2} - T_f T_{\bar{f}}$ is compact [**9**], so

$$\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2 = \widetilde{|f|^2}(z) - |f(z)|^2 = \langle (T_{|f|^2} - T_f T_{\bar{f}})k_z, k_z \rangle \to 0 \quad \text{as } |z| \to 1^-.$$

Propositions 1 and 2 are used here. Thus $\tilde{Q}$ contains all the polynomials. By the Stone-Weierstrass theorem, $\tilde{Q}$ contains $C(\overline{\mathbf{D}})$, the algebra of all continuous complex-valued functions on the closed disc $\overline{\mathbf{D}}$.

(3) Propositions 1–3 extend to general domains in C^n without change of proofs. However, for an arbitrary domain Ω in C^n, one does not have $k_z \to 0$ weakly as z goes to the boundary of Ω.

Let $\tilde{B} = \{ f \in L^\infty(\mathbf{D}) | \tilde{f}(z) \to 0 \text{ as } |z| \to 1^- \}$.

PROPOSITION 4. *$\tilde{Q} \cap \tilde{B}$ is a closed selfadjoint ideal of $\tilde{Q}$, and the following conditions are all equivalent:*

(1) $f \in \tilde{Q} \cap \tilde{B}$;

(2) $|f| \in \tilde{B}$;

(3) $|f|^2 \in \tilde{B}$.

PROOF. If $f \in \tilde{Q} \cap \tilde{B}$ and $g \in \tilde{Q}$, then $\widetilde{fg}(z) = (\widetilde{fg}(z) - \tilde{f}(z)\tilde{g}(z)) + \tilde{f}(z)\tilde{g}(z) \to 0$ as $|z| \to 1$, so $fg \in \tilde{Q} \cap \tilde{B}$. Thus $\tilde{Q} \cap \tilde{B}$ is an ideal in $\tilde{Q}$. The selfadjointness and closedness (in the sup-norm topology) of $\tilde{Q} \cap \tilde{B}$ are trivial.

Next we prove that (1)–(3) are all equivalent.

(2)$\Leftrightarrow$ (3) follows from the following inequalities:

$$\begin{aligned}\widetilde{|f|^2}(z) &= \int_{\mathbf{D}} |f(w)|^2 |k_z(w)|^2 dA(w)\\ &\leq \|f\|_\infty \int_{\mathbf{D}} |f(w)||k_z(w)|^2 dA(w) = \|f\|_\infty \widetilde{|f|}(z);\\ \widetilde{|f|}(z) &= \int_{\mathbf{D}} |f(w)||k_z(w)|^2 dA(w)\\ &\leq \sqrt{\int_{\mathbf{D}} |f(w)|^2 |k_z(w)|^2 dA(w)} = \sqrt{\widetilde{|f|^2}(z)}.\end{aligned}$$

(3) $\Rightarrow$ (1). Suppose $|f|^2 \in \tilde{B}$, i.e. $\widetilde{|f|^2}(z) \to 0$ as $|z| \to 1^-$. Then

$$0 \leq \widetilde{|f|^2}(z) - |\tilde{f}(z)|^2 \leq \widetilde{|f|^2}(z) \to 0 \qquad (|z| \to 1^-),$$

so $f \in \tilde{Q}$. But $|\tilde{f}(z)| \leq \widetilde{|f|}(z) \to 0$ $(|z| \to 1^-)$, so we have $f \in \tilde{Q} \cap \tilde{B}$.

(1) $\Rightarrow$ (3). If $f \in \tilde{Q} \cap \tilde{B}$, then $\tilde{f}(z) \to 0$ and $\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2 \to 0$ as $|z| \to 1^-$, so $\widetilde{|f|^2}(z) \to 0$ $(|z| \to 1^-)$, i.e. $|f|^2 \in \tilde{B}$.

In [**7**], Coburn and Berger pointed out that for Toeplitz operators on $L_a^2(C^n, d\mu)$, where $d\mu$ is the so-called Gaussian measure on C^n, the Berezin symbol $\tilde{f}$ is just the solution of the heat equation on $C^n = \mathbf{R}^{2n}$ at time $t = \frac{1}{2}$ with initial values f. We expect that the same thing happens on the unit disc, but no such equation has been found yet.

3. $\mathrm{VMO}_\partial(\mathbf{D})$. For any $z \in \mathbf{D}$, let

$$S_z = \{w \in \mathbf{D} \,|\, |w| \geq |z|, |\arg w - \arg z| \leq 1 - |z|\}.$$

Now we can give the definition of $\mathrm{VMO}_\partial(\mathbf{D})$.

DEFINITION. A function $f \in L^1(\mathbf{D}, dA)$ is in $\mathrm{VMO}_\partial(\mathbf{D})$ if

$$\lim_{|z|\to 1^-} \frac{1}{|S_z|} \int_{S_z} \left| f(w) - \frac{1}{|S_z|} \int_{S_z} f(u)dA(u) \right| A(w) = 0,$$

where $|S_z| = (1+|z|)(1-|z|)^2$ is the measure of S_z and VMO_∂ stands for "vanishing mean oscillation near the boundary".

The main theorem of this section is the equality

$$\tilde{Q} = L^\infty(\mathbf{D}) \cap \mathrm{VMO}_\partial(\mathbf{D}).$$

LEMMA 1. *If* $\delta \in (0,1)$ *is close enough to* 1, *then* $|1 - z| \leq |1 - \delta e^{i(1-\delta)}|$ *for all* z *in* S_δ.

PROOF. Given $z \in S_\delta$, write $z = re^{i\theta}$. Then

$$\begin{aligned}|1 - \delta e^{i(1-\delta)}|^2 - |1 - z|^2 &= \delta^2 - 2\delta\cos(1-\delta) - r^2 + 2r\cos\theta\\ &\geq \delta^2 - 2\delta\cos(1-\delta) - r^2 + 2r\cos(1-\delta)\\ &= (\delta - r)(\delta + r) - 2(\delta - r)\cos(1-\delta)\\ &= (r - \delta)(2\cos(1-\delta) - \delta - r)\\ &\geq (r - \delta)(2\cos(1-\delta) - 1 - \delta).\end{aligned}$$

For δ close enough to 1, we have

$$\cos(1-\delta) \geq 1-(1-\delta)^2/2,$$

thus

$$2\cos(1-\delta)-1-\delta \geq 2-(1-\delta)^2-1-\delta = 2\delta-\delta^2-\delta = \delta-\delta^2 > 0.$$

This completes the proof of the lemma.

LEMMA 2. *If $\delta \in (0,1)$ is very close to 1, then $|1-\delta e^{i(1-\delta)}| \leq 2(1-\delta)$.*

PROOF. The equality

$$|1-\delta e^{i(1-\delta)}|^2 = 1+\delta^2-2\delta\cos(1-\delta)$$

and L'Hôpital's rule give us the limit

$$\lim_{\delta\to 1^-} \frac{|1-\delta e^{i(1-\delta)}|^2}{(1-\delta)^2} = 2.$$

So for δ close enough to 1, we must have

$$|1-\delta e^{i(1-\delta)}|^2 \leq 4(1-\delta)^2.$$

LEMMA 3. *For any $\varepsilon > 0$, there are σ and δ_0 in $(0,1)$ such that*

$$\int_{\mathbf{D}-S_{\delta e^{i\theta}}} \frac{(1-|z|^2)^2}{|1-\bar{z}w|^4}\,dA(w) < \varepsilon$$

whenever $z = |z|e^{i\theta} \in \mathbf{D}$, $0 < 1-|z| < \delta_0$, and $1-|z| = \sigma(1-\delta)$.

PROOF. Let $r = |z|$. A change of variable gives

$$\int_{\mathbf{D}-S_{\delta e^{i\theta}}} \frac{(1-|z|^2)^2}{|1-\bar{z}w|^4}\,dA(w) = \int_{\mathbf{D}-S_\delta} \frac{(1-r^2)^2}{|1-rw|^4}\,dA(w) = \frac{1}{\pi}\operatorname{Area} F(\mathbf{D}-S_\delta),$$

where $F\colon \mathbf{D}\to\mathbf{D}$ is the map defined by $F(w) = (r-w)/(1-rw)$.

Notice that we have used the fact that $(1-|z|^2)^2/|1-\bar{z}w|^4$ is the Jacobian of the map $w \mapsto (z-w)/(1-\bar{z}w)$.

Now suppose that σ is any number in $(0,1)$ and $1-r = \sigma(1-\delta)$. We want to estimate $|1-F(w)|$ for all w in $\mathbf{D}-S_\delta$. If $w \in \mathbf{D}-S_\delta$, then either $|w| < \delta$ or $|w| \geq \delta$ but $|\arg w| > 1-\delta$.

Case 1. $|w| < \delta$. In this case, we have

$$\begin{aligned} |1-F(w)| &= \left|1-\frac{r-w}{1-rw}\right| = \frac{(1-r)|1+w|}{|1-rw|} \\ &\leq \frac{2(1-r)}{1-r\delta} = \frac{2\sigma(1-\delta)}{1-\delta(1-\sigma(1-\delta))} \\ &= \frac{2\sigma(1-\delta)}{(1-\delta)(1+\sigma\delta)} = \frac{2\sigma}{1+\sigma\delta} \leq 2\sigma. \end{aligned}$$

Case 2. $|w| \geq \delta$, $|\arg w| > 1-\delta$. In this case, we have

$$\begin{aligned} |1-rw|^2 &\geq 1+r^2|w|^2-2r|w|\cos(1-\delta) \\ &\geq 1+r^2|w|^2-2r|w|(1-(1-\delta)^2/2+(1-\delta)^4/24) \\ &= (1-r|w|)^2+r|w|(1-\delta)^2-r|w|(1-\delta)^4/12 \\ &\geq (1-r)^2+\delta r(1-\delta)^2-\tfrac{1}{12}(1-\delta)^4 \\ &= (1-\delta)^2(\sigma^2+\delta r-\tfrac{1}{12}(1-\delta)^2), \end{aligned}$$

thus

$$\begin{aligned}|1-F(w)| &= \frac{(1-r)|1+w|}{|1-rw|} \le \frac{2(1-\delta)\sigma}{|1-rw|}\\ &\le \frac{2\sigma(1-\delta)}{(1-\delta)\sqrt{\sigma^2+\delta r-\frac{1}{12}(1-\delta)^2}} = \frac{2\sigma}{\sqrt{\sigma^2+\delta r-\frac{1}{12}(1-\delta)^2}}.\end{aligned}$$

Since we are only concerned with δ close to 1, we may assume $\delta > \frac{1}{2}$ ($\Rightarrow r > \delta > \frac{1}{2}$). Thus

$$\sigma^2+\delta r-\tfrac{1}{12}(1-\delta)^2 > \tfrac{1}{2}\cdot\tfrac{1}{2}-\tfrac{1}{12}(1-\tfrac{1}{2})^2 > \tfrac{1}{9},$$

which gives $|1-F(w)| \le 6\sigma$.

Combining Cases 1 and 2, we have proved that $|1-F(w)| \le 6\sigma$ whenever $w \in \mathbf{D}-S_\delta$, $1-r=\sigma(1-\delta)$ ($\delta > \frac{1}{2}$). Therefore, if σ is small enough, $F(\mathbf{D}-S_\delta)$ is concentrated around 1, so Area $F(\mathbf{D}-S_\delta)$ is small. This completes the proof of Lemma 3. [Note: $\delta > \frac{1}{2} \Rightarrow 1-r=\sigma(1-\delta) < \frac{1}{2}\sigma$, so δ_0 can be chosen to be $\frac{1}{2}\sigma$.]

LEMMA 4. *For $f \in L^\infty(\mathbf{D})$, we have*

(1)
$$\begin{aligned}&\frac{1}{|S_z|}\int_{S_z}\left|f(w)-\frac{1}{|S_z|}\int_{S_z}f(u)dA(u)\right|dA(w)\\ &\qquad\le \sqrt{\frac{1}{|S_z|^2}\int_{S_z}\int_{S_z}|f(w)-f(u)|^2dA(w)dA(u)};\end{aligned}$$

(2)
$$\begin{aligned}&\frac{1}{|S_z|^2}\int_{S_z}\int_{S_z}|f(w)-f(u)|^2dA(w)dA(u)\\ &\qquad\le 4\|f\|_\infty\frac{1}{|S_z|}\int_{S_z}\left|f(w)-\frac{1}{|S_z|}\int_{S_z}f(u)dA(u)\right|dA(w).\end{aligned}$$

PROOF. (1) follows from the Cauchy-Schwarz inequality, while (2) follows from the following identity:

$$\begin{aligned}&\frac{1}{|S_z|^2}\int_{S_z}\int_{S_z}|f(w)-f(u)|^2\,dA(w)\,dA(u)\\ &\quad= \frac{2}{|S_z|}\int_{S_z}\bar{f}(w)\left(f(w)-\frac{1}{|S_z|}\int_{S_z}f(u)dA(u)\right)dA(w)\\ &\quad+\frac{2}{|S_z|}\int_{S_z}\bar{f}(w)\,dA(w)\cdot\frac{1}{|S_z|}\int_{S_z}\left(f(w)-\frac{1}{|S_z|}\int_{S_z}f(u)dA(u)\right)dA(w).\end{aligned}$$

COROLLARY. *For $f \in L^\infty(\mathbf{D})$, $f \in \mathrm{VMO}_\partial(\mathbf{D})$ if and only if*

$$\lim_{|z|\to 1^-}\frac{1}{|S_z|^2}\int_{S_z}\int_{S_z}|f(w)-f(u)|^2\,dA(w)\,dA(u)=0.$$

THEOREM 1. $\tilde{Q} = \mathrm{VMO}_\partial(\mathbf{D}) \cap L^\infty(\mathbf{D})$.

PROOF. First we prove the inclusion $\tilde{Q} \subset \mathrm{VMO}_\partial(\mathbf{D})$. Given $z=|z|e^{i\theta} \in \mathbf{D}$ and $f \in \tilde{Q}$,

$$\begin{aligned}\widetilde{|f|^2}(z)-|\tilde{f}(z)|^2 &= \frac{1}{2}(1-|z|^2)^4\int_{\mathbf{D}}\int_{\mathbf{D}}\frac{|f(w)-f(u)|^2}{|1-\bar{z}w|^4|1-\bar{z}u|^4}dA(w)dA(u)\\ &\ge \frac{1}{2}(1-|z|^2)^4\int_{S_z}\int_{S_z}\frac{|f(w)-f(u)|^2}{|1-\bar{z}w|^4|1-\bar{z}u|^4}\,dA(w)\,dA(u).\end{aligned}$$

For $w, u \in S_z$, we have $\bar{z}w$, $\bar{z}u \in S_{|z|^2}$. Thus if $|z|$ is close enough to 1,

$$|1-\bar{z}w| \le |1-|z|^2 e^{i(1-|z|^2)}| \le 2(1-|z|^2)$$

by Lemmas 1 and 2. Similarly, $|1-\bar{z}u| \le 2(1-|z|^2)$. So

$$\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2 \ge \frac{1}{2^9} \cdot \frac{1}{(1-|z|^2)^4} \int_{S_z}\int_{S_z} |f(w)-f(u)|^2\, dA(w)\, dA(u).$$

Notice that

$$|S_z|^2 = (1+|z|)^2(1-|z|)^4 \sim (1-|z|^2)^4.$$

So we have

$$\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2 \to 0 \Rightarrow \lim_{|z|\to 1} \frac{1}{|S_z|^2} \int_{S_z}\int_{S_z} |f(w)-f(u)|^2\, dA(w)\, dA(u) = 0,$$

which means $f \in \tilde{Q} \Rightarrow f \in \mathrm{VMO}_\partial(\mathbf{D})$ by Lemma 4.

Next we prove the other inclusion:

$$\mathrm{VMO}_\partial(\mathbf{D}) \cap L^\infty(\mathbf{D}) \subset \tilde{Q}.$$

Given $z = |z|e^{i\theta} \in \mathbf{D}$, $\delta \in (0,1)$, and $f \in \mathrm{VMO}_\partial(\mathbf{D}) \cap L^\infty(\mathbf{D})$,

$$\begin{aligned}
\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2 &= \frac{1}{2}\int_{\mathbf{D}}\int_{\mathbf{D}} \frac{(1-|z|^2)^4|f(w)-f(u)|^2}{|1-z\bar{w}|^4|1-z\bar{u}|^4}\, dA(w)\, dA(u)\\
&= \frac{1}{2}(1-|z|^2)^4 \int_{S_{\delta e^{i\theta}}}\int_{S_{\delta e^{i\theta}}} \frac{|f(w)-f(u)|^2}{|1-\bar{z}w|^4|1-\bar{z}u|^4}\, dA(w)\, dA(u)\\
&\quad + \frac{1}{2}(1-|z|^2)^4 \left[\int_{\mathbf{D}-S_{\delta e^{i\theta}}}\int_{\mathbf{D}} + \int_{\mathbf{D}}\int_{\mathbf{D}-S_{\delta e^{i\theta}}} - \int_{\mathbf{D}-S_{\delta e^{i\theta}}}\int_{\mathbf{D}-S_{\delta e^{i\theta}}}\right]\\
&\qquad\qquad \cdot \frac{|f(w)-f(u)|^2}{|1-\bar{z}w|^4|1-\bar{z}u|^4}\, dA(w)\, dA(u)\\
&\le \frac{1}{2}(1-|z|^2)^4 \int_{S_{\delta e^{i\theta}}}\int_{S_{\delta e^{i\theta}}} \frac{|f(w)-f(u)|^2}{(1-|z|)^4(1-|z|)^4}\, dA(w)\, dA(u)\\
&\quad + \frac{1}{2}(1-|z|^2)^4 \left[\int_{\mathbf{D}}\int_{\mathbf{D}-S_{\delta e^{i\theta}}} + \int_{\mathbf{D}-S_{\delta e^{i\theta}}}\int_{\mathbf{D}}\right]\\
&\qquad\qquad \cdot \frac{4\|f\|_\infty^2}{|1-\bar{z}w|^4|1-\bar{z}u|^4}\, dA(w)\, dA(u)\\
&= \frac{1}{2}\left(\frac{1+|z|}{1-|z|}\right)^4 \int_{S_{\delta e^{i\theta}}}\int_{S_{\delta e^{i\theta}}} |f(w)-f(u)|^2\, dA(u)\, dA(w)\\
&\quad + 4\|f\|_\infty^2 (1-|z|^2)^2 \int_{\mathbf{D}-S_{\delta e^{i\theta}}} \frac{dA(w)}{|1-\bar{z}w|^4}\\
&= \frac{(1+|z|)^4(1+\delta)^2}{2}\left(\frac{1-\delta}{1-|z|}\right)^4 \frac{1}{|S_{\delta e^{i\theta}}|^2}\\
&\quad \cdot \int_{S_{\delta e^{i\theta}}}\int_{S_{\delta e^{i\theta}}} |f(w)-f(u)|^2\, dA(w)\, dA(u)\\
&\quad + 4\|f\|_\infty \int_{\mathbf{D}-S_{\delta e^{i\theta}}} \frac{(1-|z|^2)^2}{|1-\bar{z}w|^4}\, dA(w).
\end{aligned}$$

Now given any $\varepsilon > 0$, by Lemma 3, there exist $\sigma \in (0,1)$ and $\delta_0 \in (0,1)$ such that

$$\int_{\mathbf{D}-S_{\delta e^{i\theta}}} \frac{(1-|z|^2)^2}{|1-\bar{z}w|^4} dA(w) < \varepsilon$$

whenever $0 < 1-|z| < \delta_0,\ 1-|z| = (1-\delta)\sigma$. Thus

$$\begin{aligned}&\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2 \\ &\quad \le 4\|f\|_\infty^2 \varepsilon + \frac{(1+|z|)^4(1+\delta)^2}{2\sigma^4} \frac{1}{|S_{\delta e^{i\theta}}|^2} \int_{S_{\delta e^{i\theta}}} \int_{S_{\delta e^{i\theta}}} |f(w)-f(u)|^2 \, dA(w)\, dA(u)\end{aligned}$$

whenever $0 < 1-|z| < \delta_0$ and $1-|z| = (1-\delta)\sigma$.

Now using Lemma 4 we get

$$\overline{\lim_{|z|\to 1^-}} (\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2) \le 4\|f\|_\infty^2 \varepsilon.$$

(Note: $|z| \to 1 \Rightarrow \delta \to 1$, σ is fixed.) Since ε is arbitrary, we have

$$\lim_{|z|\to 1^-} (\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2) = 0,$$

and so $f \in \tilde{Q}$. This completes the proof of Theorem 1.

THEOREM 2. *For $f \in L^\infty(\mathbf{D}, dA)$, $f \in \tilde{Q} \cap \tilde{B}$ iff*

$$\lim_{|z|\to 1^-} \frac{1}{|S_z|} \int_{S_z} |f(w)| \, dA(w) = 0. \tag{1}$$

PROOF. Suppose that (1) holds. Consider

$$\widetilde{|f|}(z) = (1-|z|^2)^2 \int_{\mathbf{D}} \frac{|f(w)|}{|1-z\bar{w}|^4} \, dA(w), \qquad z = |z|e^{i\theta}.$$

Given $\varepsilon > 0$, by Lemma 3 there are $\sigma \in (0,1)$ and $\delta_0 \in (0,1)$ such that

$$\int_{\mathbf{D}-S_{\delta e^{i\theta}}} \frac{(1-|z|^2)^2}{|1-\bar{z}w|^4} \, dA(w) < \varepsilon$$

whenever $0 < 1-|z| < \delta_0$ and $1-|z| = (1-\delta)\sigma$, so

$$\begin{aligned}\widetilde{|f|}(z) &\le \|f\|_\infty \varepsilon + (1-|z|^2)^2 \int_{S_{\delta e^{i\theta}}} \frac{|f(w)|}{|1-\bar{z}w|^4} \, dA(w) \\ &\le \|f\|_\infty \varepsilon + \frac{(1-|z|^2)^2}{(1-|z|)^4} \int_{S_{\delta e^{i\theta}}} |f(w)| \, dA(w)\end{aligned}$$

whenever $0 < 1-|z| < \delta_0$ and $1-|z| = (1-\delta)\sigma$. Since

$$\frac{(1-|z|^2)^2}{(1-|z|)^4} = \frac{(1+|z|)^2}{(1-|z|)^2} = \frac{(1+|z|)^2}{\sigma^2(1-\delta)^2} \le \frac{4}{\sigma^2(1-\delta)^2}$$

and $|S_{\delta e^{i\theta}}| = (1+\delta)(1-\delta)^2$, thus

$$\widetilde{|f|}(z) \le \|f\|_\infty \varepsilon + \frac{4(1+\delta)}{\sigma^2 |S_{\delta e^{i\theta}}|} \int_{S_{\delta e^{i\theta}}} |f(w)| \, dA(w) \tag{2}$$

whenever $0 < 1-|z| < \delta_0$ and $1-|z| = \sigma(1-\delta)$. Let $|z| \to 1^-$ in (2). Then

$$\varlimsup_{|z|\to 1^-} \widetilde{|f|}(z) \le \|f\|_\infty \varepsilon.$$

Since ε is arbitrary, we have $|f| \in \tilde{B}$, thus $f \in \tilde{Q} \cap \tilde{B}$.

Conversely, if $f \in \tilde{Q} \cap \tilde{B}$, then $|f| \in \tilde{B}$. But

$$\begin{aligned}\widetilde{|f|}(z) &= (1-|z|^2)^2 \int_{\mathbf{D}} \frac{|f(w)|}{|1-\bar{z}w|^4}\, dA(w) \\ &\ge (1-|z|^2)^2 \int_{S_z} \frac{|f(w)|}{|1-\bar{z}w|^4}\, dA(w),\end{aligned}$$

and $|1-\bar{z}w| \le 2(1-|z|^2)$ for $|z|$ close enough to 1 and $w \in S_z$ by Lemmas 1 and 2. So there is $\delta \in (0,1)$ such that

$$\widetilde{|f|}(z) \ge \frac{1}{2^4(1-|z|^2)^2} \int_{S_z} |f(w)|\, dA(w) \tag{3}$$

for all $\delta < |z| < 1$. Notice that $(1-|z|^2)^2 = (1+|z|)|S_z|$. So (3) says that $\tilde{|f|}(z) \to 0$ $(|z| \to 1^-)$ implies

$$\frac{1}{|S_z|} \int_{S_z} |f(w)|\, dA(w) \to 0 \qquad (|z| \to 1^-).$$

This completes the proof of Theorem 2.

4. ESV($\mathbf{D}$). Let f be in $L^\infty(\mathbf{D}, dA)$. We say f is in ESV($\mathbf{D}$) if for any $\varepsilon > 0$ and $\sigma \in (0,1)$, there is $\delta_0 > 0$ such that $|f(z) - f(w)| < \varepsilon$ whenever $|z|, |w| \in [1-\delta, 1-\sigma\delta]$, $\delta < \delta_0$ and $|\arg z - \arg w| \le \max(1-|z|, 1-|w|)$.

Recall that for $z \in \mathbf{D}$,

$$S_z = \{w \in \mathbf{D} \mid |w| \ge |z|,\ |\arg z - \arg w| \le 1-|z|\}.$$

Then it is clear that $f \in \mathrm{ESV}(\mathbf{D})$ if and only if for any $\varepsilon > 0$, $\sigma \in (0,1)$, there exists $\delta_0 > 0$ such that $|f(z)-f(w)| < \varepsilon$ whenever $w \in S_z$ and $|z|$, $|w| \in [1-\delta, 1-\sigma\delta]$, $\delta < \delta_0$.

It is also easy to see that $f \in \mathrm{ESV}(\mathbf{D})$ if and only if for any $\varepsilon > 0$, $\sigma \in (0,1)$ and for any positive number k, there exists a positive number δ_0 such that $|f(z) - f(w)| < \varepsilon$ whenever $|z|$, $|w| \in [1-\delta, 1-\sigma\delta]$, $\delta < \delta_0$, and $|\arg z - \arg w| \le k \max(1-|z|, 1-|w|)$. In particular, if

$$S'_z = \{w \in \mathbf{D} \mid |w| \ge |z|, |\arg w - \arg z| \le (1-|z|)/2\},$$

then $f \in \mathrm{ESV}(\mathbf{D})$ if and only if for any $\varepsilon > 0$, and $\sigma \in (0,1)$, there exists $\delta_0 > 0$ such that $|f(z) - f(w)| < \varepsilon$ whenever $w \in S'_z$ and $|z|, |w| \in [1-\delta, 1-\sigma\delta]$, $\delta < \delta_0$.

The notation ESV is borrowed from [**6**] and [**7**], where it stands for "eventually slowly varying". In [**22**] and [**23**], Sarason also introduced the concept of ESV in a special case, but used a different notation, namely, SO, standing for "slowly oscillating". ESV is indeed an oscillation condition. It is stronger than the mean-oscillation condition as shown in Theorem 5.

ESV($\mathbf{D}$) is a relatively large class of functions in $L^\infty(\mathbf{D}, dA)$. It is easy to see that $C(\overline{\mathbf{D}})$ is strictly contained in it.

Let $f \in L^\infty(\mathbf{D})$ and $z \in \mathbf{D}$, and define

$$\hat{f}(z) = \frac{1}{|S_z'|}\int_{S_z'} f(w)\,dA(w).$$

REMARK. By the corollary to Lemma 4, it is easy to see that $f \in \tilde{Q}$ if and only if $|\hat{f}|^2(z) - |\hat{f}(z)|^2 \to 0$ as $|z| \to 1^-$.

THEOREM 3. *If* $f \in \tilde{Q} = \mathrm{VMO}_\partial(\mathbf{D}) \cap L^\infty(\mathbf{D})$, *then* $\hat{f} \in \mathrm{ESV}(\mathbf{D})$.

PROOF. Given $\varepsilon > 0$, $\sigma \in (0,1)$, choose $\delta_0 > 0$ such that

$$\frac{1}{|S_z|^2}\int_{S_z}\int_{S_z} |f(w) - f(u)|\,dA(w)\,dA(u) < \frac{\varepsilon\sigma^2}{8}$$

whenever $0 < 1 - |z| < \delta_0$.

Now if $1 - \delta \le |z_1| \le |z_2| \le 1 - \sigma\delta$, $\delta < \delta_0$, and

$$|\arg z_1 - \arg z_2| \le \tfrac{1}{2}\max(1 - |z_1|, 1 - |z_2|) = \tfrac{1}{2}(1 - |z_1|),$$

then $S_{z_2}' \subset S_{z_1}$, so we have

$$\begin{aligned}
|\hat{f}(z_1) - \hat{f}(z_2)| &= \left|\frac{1}{|S_{z_1}'||S_{z_2}'|}\int_{S_{z_1}'}\int_{S_{z_2}'} (f(u) - f(w))\,dA(u)\,dA(w)\right| \\
&\le \frac{1}{|S_{z_1}'||S_{z_2}'|}\int_{S_{z_1}'}\int_{S_{z_2}'} |f(u) - f(w)|\,dA(u)\,dA(w) \\
&\le \frac{1}{|S_{z_1}'||S_{z_2}'|}\int_{S_{z_1}}\int_{S_{z_1}} |f(u) - f(w)|\,dA(u)\,dA(w) \\
&= \frac{4}{|S_{z_1}||S_{z_2}'|}\int_{S_{z_1}}\int_{S_{z_1}} |f(u) - f(w)|\,dA(u)\,dA(w).
\end{aligned}$$

But

$$\frac{|S_{z_1}|}{|S_{z_2}|} = \frac{(1 + |z_1|)(1 - |z_1|)^2}{(1 + |z_2|)(1 - |z_2|)^2} \le 2\left(\frac{1 - |z_1|}{1 - |z_2|}\right)^2 \le \frac{2}{\sigma^2}.$$

(Note: $1 - \delta \le |z_1| \le |z_2| \le 1 - \sigma\delta \Rightarrow (1 - |z_1|)/(1 - |z_2|) \le 1/\sigma$.) So we have

$$|\hat{f}(z_1) - \hat{f}(z_2)| \le \frac{8}{\sigma^2}\cdot\frac{1}{|S_{z_1}|^2}\int_{S_{z_1}}\int_{S_{z_1}} |f(u) - f(w)|\,dA(u)\,dA(w) \le \frac{8}{\sigma^2}\cdot\frac{\sigma^2\varepsilon}{8} = \varepsilon.$$

Thus $\hat{f}$ is in $\mathrm{ESV}(\mathbf{D})$.

LEMMA 5. *If* $f \in \mathrm{ESV}(\mathbf{D})$, *then*

$$\lim_{|z| \to 1^-} \frac{1}{|S_z|}\int_{S_z} |f(z) - f(u)|\,dA(u) = 0.$$

PROOF. For $0 < |z| < \delta < 1$, let

$$\begin{aligned}
A_1 &= \{w \in S_z \mid |w| \le \delta\}, \\
A_2 &= \{w \in S_z \mid |w| \ge \delta\}.
\end{aligned}$$

Then $S_z = A_1 \cup A_2$ and

$$\begin{aligned}\frac{1}{|S_z|}\int_{S_z} &|f(z)-f(u)|\,dA(u)\\ &\le \frac{1}{|A_1|}\int_{A_1}|f(z)-f(u)|\,dA(u)+\frac{1}{|S_z|}\int_{A_2}|f(z)-f(u)|\,dA(u).\end{aligned}$$

Given any $\varepsilon \in (0,1)$, choose $\delta_0 \in (0,1)$ such that

$$(*) \qquad |f(r_1e^{i\theta_1}) - f(r_2e^{i\theta_2})| < \varepsilon$$

whenever $r_1,\ r_2 \in [1-r, 1-\varepsilon r]$, $r \le \delta_0$, and $|\theta_1-\theta_2| \le \max(1-r_1, 1-r_2)$.

Now let $|z| > 1-\delta_0$ and $\delta = 1-\varepsilon(1-|z|)$. Then $|z|<\delta<1$. For any $u \in A_1$, $|f(z)-f(u)|<\varepsilon$ by $(*)$. So for each $|z|>1-\delta_0$, we have

$$\begin{aligned}\frac{1}{|S_z|}\int_{S_z}|f(z)-f(u)|\,dA(u) &\le \varepsilon + \frac{1}{|S_z|}\int_{A_2}|f(z)-f(u)|\,dA(u)\\ &\le \varepsilon + 2|A_2|/|S_z|\|f\|_\infty.\end{aligned}$$

Since

$$\begin{aligned}&|S_z| = (1+|z|)(1-|z|)^2,\\ &|A_2| = (1-|z|)(1-\delta^2) = \varepsilon(1-|z|)^2(2-\varepsilon(1-|z|)),\\ &\frac{|A_2|}{|S_z|} \le \frac{2\varepsilon(1-|z|)^2}{(1+|z|)(1-|z|)^2} \le 2\varepsilon,\end{aligned}$$

$|z|>1-\delta_0$ implies

$$\frac{1}{|S_z|}\int_{S_z}|f(z)-f(u)|dA(u) \le (1+4\|f\|_\infty)\varepsilon.$$

This completes the proof of Lemma 5.

THEOREM 4. *If* $f \in \tilde{Q} = \mathrm{VMO}_\partial(\mathbf{D}) \cap L^\infty(\mathbf{D})$, *then* $f-\hat{f} \in \tilde{Q}\cap\tilde{B}$.

PROOF. Let $g = f-\hat{f}$. Then

$$\begin{aligned}\frac{1}{|S_z|}\int_{S_z}|g(w)|\,dA(w) &\le \frac{1}{|S_z|}\int_{S_z}|f(w)-\hat{f}(z)|\,dA(w)\\ &\quad+\frac{1}{|S_z|}\int_{S_z}|\hat{f}(z)-\hat{f}(w)|\,dA(w).\end{aligned}$$

The second term goes to 0 as $|z|\to 1^-$ by Lemma 5 and Theorem 3. Next we estimate the first term.

$$\begin{aligned}\frac{1}{|S_z|}&\int_{S_z}|f(w)-\hat{f}(z)|\,dA(w)\\ &= \frac{1}{|S_z|}\int_{S_z}\left|\frac{1}{|S_z'|}\int_{S_z'}(f(w)-f(u))\,dA(u)\right|dA(w)\\ &\le \frac{1}{|S_z||S_z'|}\int_{S_z}\int_{S_z'}|f(w)-f(u)|dA(u)\\ &\le \frac{2}{|S_z|^2}\int_{S_z}\int_{S_z}|f(w)-f(u)|dA(w)dA(u),\end{aligned}$$

this goes to 0 as $|z|\to 1$ since f is in $\mathrm{VMO}_\partial(\mathbf{D})$.

LEMMA 6. $\mathrm{ESV}(\mathbf{D}) \subset \tilde{Q} = \mathrm{VMO}_\partial(\mathbf{D}) \cap L^\infty(\mathbf{D})$.

PROOF. Let $f \in \mathrm{ESV}(\mathbf{D})$. Then $|f|^2 \in \mathrm{ESV}(\mathbf{D})$ since $\mathrm{ESV}(\mathbf{D})$ is a C^*-subalgebra of $L^\infty(\mathbf{D})$. By the corollary to Lemma 5, we have $f(z) - \hat{f}(z) \to 0$ as $|z| \to 1^-$, $\widehat{|f|^2}(z) - |f|^2(z) \to 0$ as $|z| \to 1^-$. Moreover,

$$\begin{aligned} \left||\hat{f}(z)|^2 - |f(z)|^2\right| &= (|\hat{f}(z)| + |f(z)|)\left||\hat{f}(z)| - |f(z)|\right| \\ &\le 2\|f\|_\infty |\hat{f}(z) - f(z)| \to 0 \quad \text{as } |z| \to 1^-. \end{aligned}$$

So

$$\widehat{|f|^2}(z) - |\hat{f}(z)|^2 = (\widehat{|f|^2}(z) - |f(z)|^2) + (|f(z)|^2 - |\hat{f}(z)|^2) \to 0 \quad \text{as } |z| \to 1^-,$$

and we have $f \in \mathrm{VMO}_\partial(\mathbf{D})$. Since $\mathrm{ESV}(\mathbf{D}) \subset L^\infty(\mathbf{D})$, $f \in \mathrm{VMO}_\partial(\mathbf{D}) \cap L^\infty(\mathbf{D}) = \tilde{Q}$.

Summarizing the above results, we have proved the following theorem.

THEOREM 5. $\tilde{Q} = \mathrm{ESV}(\mathbf{D}) + \tilde{Q} \cap \tilde{B}$. *A decomposition is given by* $f = \hat{f} + (f - \hat{f})$.

COROLLARY 1.

$$\mathrm{ESV}(\mathbf{D}) \cap \tilde{B} = \{f \in L^\infty(\mathbf{D}, dA) \mid f(z) \to 0 \text{ as } |z| \to 1^-\}.$$

PROOF. If $f(z) \to 0$ as $|z| \to 1^-$, then obviously $f \in \mathrm{ESV}(\mathbf{D})$ (just by definition). On the other hand,

$$\tilde{f}(z) = \int_{\mathbf{D}} f\left(\frac{z - w}{1 - \bar{z}w}\right) dA(w) \to 0 \qquad (|z| \to 1^-)$$

by the dominated convergence theorem. So $f \in \tilde{B}$, and hence $f \in \mathrm{ESV}(\mathbf{D}) \cap \tilde{B}$.

Conversely, if $f \in \mathrm{ESV}(\mathbf{D}) \cap \tilde{B}$, then $f(z) - \hat{f}(z) \to 0$ as $|z| \to 1^-$ and

$$\begin{aligned} |\hat{f}(z)| &\le \frac{1}{|S_z'|} \int_{S_z'} |f(w)|\, dA(w) \\ &\le \frac{2}{|S_z|} \int_{S_z} |f(w)|\, dA(w) \to 0 \quad \text{as } |z| \to 1^- \end{aligned}$$

since $f \in \tilde{Q} \cap \tilde{B}$. Therefore, $f(z) = (f(z) - \hat{f}(z)) + \hat{f}(z) \to 0$ as $|z| \to 1^-$.

COROLLARY 2. *For* $f \in \tilde{Q} = \mathrm{VMO}_\partial(\mathbf{D}) \cap L^\infty(\mathbf{D})$, $f \in \mathrm{ESV}(\mathbf{D})$ *iff* $f(z) - \hat{f}(z) \to 0$ $(|z| \to 1^-)$.

PROOF. The "only if" part follows from the corollary to Lemma 5. If $f(z) - \hat{f}(z) \to 0$ as $|z| \to 1^-$, then $f - \hat{f} \in \mathrm{ESV}(\mathbf{D})$. So $f = (f - \hat{f}) + \hat{f} \in \mathrm{ESV}(\mathbf{D})$.

THEOREM 6. *If* $f \in \mathrm{ESV}(\mathbf{D})$, *then* $f(z) - \tilde{f}(z) \to 0$ *as* $|z| \to 1^-$.

PROOF.

$$\begin{aligned} f(z) - \tilde{f}(z) &= (1 - |z|^2)^2 \int_{\mathbf{D}} \frac{f(z) - f(w)}{|1 - \bar{z}w|^4}\, dA(w), \\ |f(z) - \tilde{f}(z)| &\le (1 - |z|^2)^2 \int_{\mathbf{D}} \frac{|f(z) - f(w)|}{|1 - \bar{z}w|^4}\, dA(w). \end{aligned}$$

Given $\varepsilon > 0$, by Lemma 3, there are σ and δ_0 in (0,1) such that

$$\int_{\mathbf{D}-S_{\delta e^{i\theta}}} \frac{(1-|z|^2)^2}{|1-\bar{z}w|^4} dA(w) < \varepsilon \qquad (z = |z|e^{i\theta})$$

whenever $0 < 1-|z| < \delta_0$ and $1-|z| = (1-\delta)\sigma$. But $f \in \mathrm{ESV}(\mathbf{D})$, so there exists $\delta_1 \in (0,1)$ such that

$$|f(r_1 e^{i\theta_1}) - f(r_2 e^{i\theta_2})| < \varepsilon\sigma^2 \tag{4}$$

whenever $r_1, r_2 \in [1-\lambda, 1-\sigma\lambda]$, $\lambda < \delta_1$, and $|\theta_1 - \theta_2| \le \max(1-r_1, 1-r_2)$.

Let $\delta_2 = \min(\delta_0, \delta_1)$. Then for $0 < 1-|z| < \delta_2$ and $1-|z| = \sigma(1-\delta)$, we have

$$\begin{aligned}
|f(z) - \tilde{f}(z)| &\le 2\|f\|_\infty \int_{\mathbf{D}-S_{\delta e^{i\theta}}} \frac{(1-|z|^2)^2}{|1-\bar{z}w|^4} dA(w) \\
&\quad + (1-|z|^2)^2 \int_{S_{\delta e^{i\theta}}} \frac{|f(z)-f(w)|}{|1-\bar{z}w|^4} dA(w) \\
&\le 2\|f\|_\infty \varepsilon + \frac{(1-|z|^2)^2}{(1-|z|)^4} \int_{S_{\delta e^{i\theta}}} |f(z)-f(w)| dA(w) \\
&\le 2\|f\|_\infty \varepsilon + \frac{4}{(1-|z|)^2} \int_{S_{\delta e^{i\theta}}} |f(w) - f(\delta e^{i\theta})| dA(w) \\
&\quad + \frac{4}{(1-|z|)^2} \int_{S_{\delta e^{i\theta}}} |f(z) - f(\delta e^{i\theta})| dA(w).
\end{aligned}$$

Since $|f(z) - f(\delta e^{i\theta})| < \varepsilon\sigma^2$ by (4), and

$$\frac{|S_{\delta e^{i\theta}}|}{(1-|z|)^2} = \frac{(1+\delta)(1-\delta)^2}{(1-|z|)^2} = \frac{1+\delta}{\sigma^2} \le \frac{2}{\sigma^2},$$

we must have

$$|f(z) - \tilde{f}(z)| \le 2\|f\|_\infty \varepsilon + \frac{\delta}{\sigma^2} \cdot \frac{1}{|S_{\delta e^{i\theta}}|} \int_{S_{\delta e^{i\theta}}} |f(w) - f(\delta e^{i\theta})| d(w) + \delta\varepsilon \qquad (0 < 1-|z| < \delta_2).$$

Because

$$\lim_{|z|\to 1^-} \frac{1}{|S_{\delta e^{i\theta}}|} \int_{S_{\delta e^{i\theta}}} |f(w) - f(\delta e^{i\theta})| dA(w) = 0$$

by Lemma 5, we have

$$\overline{\lim_{|z|\to 1^-}} |f(z) - \tilde{f}(z)| \le 2\|f\|_\infty \varepsilon + \delta\varepsilon.$$

This completes the proof of $\lim_{|z|\to 1^-}(f(z) - \tilde{f}(z)) = 0$ for any $f \in \mathrm{ESV}(\mathbf{D})$.

THEOREM 7. *For $f \in \tilde{Q} = \mathrm{VMO}_\partial(\mathbf{D}) \cap L^\infty(\mathbf{D}, dA)$, we have*
(1) $\tilde{f} \in \mathrm{ESV}(\mathbf{D})$;
(2) $f - \tilde{f} \in \tilde{Q} \cap \tilde{B}$.

PROOF. (1). By Theorem 5, $f = f_1 + f_2$, where $f_1 \in \mathrm{ESV}(\mathbf{D})$ and $f_2 \in \tilde{Q} \cap \tilde{B}$. Taking the Berezin symbol of f, we get $\tilde{f} = \tilde{f}_1 + \tilde{f}_2$.

Now $f_2 \in \tilde{Q} \cap \tilde{B}$ implies $\tilde{f}_2(z) \to 0$ as $|z| \to 1^-$, so $\tilde{f}_2 \in \mathrm{ESV}(\mathbf{D})$. $f_1 \in \mathrm{ESV}(\mathbf{D})$ implies $f_1(z) - \tilde{f}_1(z) \to 0$ as $|z| \to 1^-$, so $f_1 - \tilde{f}_1 \in \mathrm{ESV}$. Thus

$$\tilde{f}_1 = (\tilde{f}_1 - f_1) + f_1 \in \mathrm{ESV}(\mathbf{D}).$$

Hence

$$\tilde{f} = \tilde{f}_1 + \tilde{f}_2 \in \mathrm{ESV}(\mathbf{D}).$$

(2). $f - \tilde{f} = f_1 + f_2 - \tilde{f}_1 - \tilde{f}_2 = (f_1 - \tilde{f}_1) + f_2 - \tilde{f}_2$. $f_1(z) - \tilde{f}_1(z) \to 0$ (as $|z| \to 1^-$) implies $f_1 - \tilde{f}_1 \in \tilde{Q} \cap \tilde{B}$. $\tilde{f}_2(z) \to 0$ (as $|z| \to 1^-$) implies $\tilde{f}_2 \in \tilde{Q} \cap \tilde{B}$. So $f - \tilde{f} \in \tilde{Q} \cap \tilde{B}$.

COROLLARY 1. *If $f \in \tilde{Q}$, then $\tilde{f}(z) - \hat{f}(z) \to 0$ as $|z| \to 1^-$.*

PROOF. If $f \in \tilde{Q}$, then $\tilde{f}$ and $\hat{f}$ are $\mathrm{ESV}(\mathbf{D})$ by Theorems 3 and 7, so $\tilde{f} - \hat{f}$ are $\mathrm{ESV}(\mathbf{D})$. On the other hand,

$$\tilde{f} - \hat{f} = (\tilde{f} - f) + (f - \hat{f}) \in \tilde{Q} \cap \tilde{B}$$

by Theorems 4 and 7. Therefore,

$$\tilde{f} - \hat{f} \in \mathrm{ESV}(\mathbf{D}) \cap \tilde{Q} \cap \tilde{B} = \mathrm{ESV}(\mathbf{D}) \cap \tilde{B}.$$

Applying Corollary 1 to Theorem 5, we get $\tilde{f}(z) - \hat{f}(z) \to 0$ as $|z| \to 1^-$.

COROLLARY 2. *For $f \in \tilde{Q}$, we have $f \in \mathrm{ESV}(\mathbf{D})$ iff $f(z) - \tilde{f}(z) \to 0$ as $|z| \to 1$.*

PROOF. If $f \in \mathrm{ESV}(\mathbf{D})$, then $f(z) - \tilde{f}(z) \to 0$ ($|z| \to 1$) by Theorem 6. If $f(z) - \tilde{f}(z) \to 0$, then $f - \tilde{f} \in \mathrm{ESV}(\mathbf{D})$, but $\tilde{f} \in \mathrm{ESV}(\mathbf{D})$ by Theorem 7, so $f = (f - \tilde{f}) + \tilde{f} \in \mathrm{ESV}(\mathbf{D})$.

REMARK. For the identity $\tilde{Q} = \mathrm{ESV}(\mathbf{D}) + \tilde{Q} \cap \tilde{B}$, we have found two canonical decompositions:

$$f = \tilde{f} + (f - \tilde{f}) \quad \text{and} \quad f = \hat{f} + (f - \hat{f}).$$

THEOREM 8. *For $f \in L^\infty(\mathbf{D})$, we have $f \in \mathrm{ESV}(\mathbf{D})$ iff $\|f(z) - f \circ b_z\|_{L^2} \to 0$ as $|z| \to 1^-$, where $b_z(w) = (z - w)/(1 - \bar{z}w)$, and the norm is just the usual L^2-norm.*

PROOF. For $f \in L^\infty(\mathbf{D})$, it is easy to check the following identity:

$$\|f(z) - f \circ b_z\|_{L^2}^2 = \widetilde{|f|^2}(z) - |\tilde{f}(z)|^2 + |\tilde{f}(z) - f(z)|^2. \tag{5}$$

If the left-hand side of (5) goes to 0 as $|z| \to 1^-$, then $\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2 \to 0$ ($|z| \to 1$) and $|\tilde{f}(z) - f(z)| \to 0$. The first limit says that f is in $\tilde{Q}$, the second limit and Corollary 2 to Theorem 7 imply that f in in $\mathrm{ESV}(\mathbf{D})$.

Conversely, if $f \in \mathrm{ESV}(\mathbf{D}) \subset \tilde{Q}$, the $\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2 \to 0$ and $|f(z) - \tilde{f}(z)| \to 0$ as $|z| \to 1^-$, so the left-hand side of (5) goes to 0 as $|z| \to 1^-$.

LEMMA 7. *There is an absolute constant C such that*

$$\int_{\mathbf{D}} |f(z) - f(0)|^2\, dA(z) \le C \int_{\mathbf{D}} (1 - |z|^2)^2 |f'(z)|^2\, dA(z)$$

for all $f \in H^\infty(\mathbf{D})$.

PROOF. Using Green's formula, we can easily prove (see p. 236 of [13])

$$\int_{|z|<r} |f'(z)|^2 \log \frac{r}{|z|}\, dA(z) = \frac{1}{4\pi} \int_0^{2\pi} |f(re^{i\theta}) - f(0)|^2\, d\theta.$$

It is also known (p. 237 of [13]) that

$$\int_{|z|<r} |f'(z)|^2 \log \frac{r}{|z|}\, dA(z) \le C \int_{|z|<r} |f'(z)|^2 \left(1 - \left|\frac{z}{r}\right|^2\right) dA(z)$$

for all $f \in H^\infty(\mathbf{D})$, where C is an absolute constant, i.e. C does not depend on f. Now integrating the above inequality with respect to $r\,dr$, we get

$$\frac{1}{4\pi} \int_0^1 r\,dr \int_0^{2\pi} |f(re^{i\theta}) - f(0)|^2\, d\theta \le C \int_0^1 r\,dr \int_{|z|<r} |f'(z)|^2 \left(1 - \left|\frac{z}{r}\right|^2\right) dA(z),$$

or

$$\frac{1}{4} \int_{\mathbf{D}} |f(z) - f(0)|^2 dA(z) \le \frac{C}{2} \int_{\mathbf{D}} [1 - |z|^2 + |z|^2 \log |z|^2] |f'(z)|^2 dA(z).$$

Power series expansion shows that

$$1 - |z|^2 + |z|^2 \log |z|^2 \le (1 - |z|^2)^2,$$

so we have

$$\int_{\mathbf{D}} |f(z) - f(0)|^2 dA(z) \le 2C \int_{\mathbf{D}} |f'(z)|^2 (1 - |z|^2)^2 dA(z).$$

THEOREM 9. *For $f \in H^\infty(\mathbf{D})$, the following are all equivalent:*
(1) $f \in \mathrm{ESV}(\mathbf{D})$;
(2) $f \in \mathrm{VMO}_\partial(\mathbf{D})$;
(3) $f \in \tilde{Q}$;
(4) $f \in \mathcal{B}_0$, *where $\mathcal{B}_0$ is the "little Bloch" space consisting of all the analytic functions g on $\mathbf{D}$ such that $|g'(z)|(1 - |z|^2) \to 0$ as $|z| \to 1^-$.*

PROOF. (2) and (3) are equivalent by Theorem 1. That (1) implies (3) follows from the fact that $\mathrm{ESV}(\mathbf{D}) \subset \tilde{Q}$. If $f \in \tilde{Q}$, then

$$\|f(z) - f \circ b_z\|_{L^2}^2 = \widetilde{|f|^2}(z) - |\tilde{f}(z)|^2 + |\tilde{f}(z) - f(z)|^2 \to 0$$

as $|z| \to 1^-$ since $\tilde{f} = f$ for $f \in H^\infty(\mathbf{D})$, so $f \in \mathrm{ESV}(\mathbf{D})$ by Theorem 8. Thus we have proved that (3) implies (1).

Next we prove the equivalence of (3) and (4).

If we replace f by $f \circ b_{z_0}$ in Lemma 7, then the inequality becomes

$$\widetilde{|f|^2}(z_0) - |\tilde{f}(z_0)|^2 \le C \int_{\mathbf{D}} \left(1 - \left|\frac{z_0 - z}{1 - \bar{z}_0 z}\right|^2\right)^2 \left|f'\left(\frac{z_0 - z}{1 - \bar{z}_0 z}\right)\right|^2 dA(z).$$

Now if $f \in \mathcal{B}_0$, then

$$\left(1 - \left|\frac{z_0 - z}{1 - \bar{z}_0 z}\right|^2\right)^2 \left|f'\left(\frac{z_0 - z}{1 - \bar{z}_0 z}\right)\right|^2 \to 0$$

as $|z_0| \to 1^-$ for any fixed $z \in \mathbf{D}$. Thus by the dominated convergence theorem, we have $\widetilde{|f|^2}(z_0) - |\tilde{f}(z_0)|^2 \to 0$ as $|z_0| \to 1^-$. This shows that (4) implies (3).

To prove (3) implies (4), we use the Bergman formula

$$f(z) - f(0) = \int_{\mathbf{D}} \frac{f(w) - f(0)}{(1 - z\bar{w})^2} \, dA(w), \qquad f \in H^\infty(\mathbf{D}).$$

Taking derivative on both sides, we get

$$f'(z) = \int_{\mathbf{D}} \frac{2\bar{w}(f(w) - f(0))}{(1 - z\bar{w})^3} \, dA(w).$$

Let $z = 0$, then

$$|f'(0)|^2 \le 4 \int_{\mathbf{D}} |f(w) - f(0)|^2 \, dA(w), \qquad f \in H^\infty(\mathbf{D}).$$

Replacing f by $f \circ b_z$, we get

$$|f'(z)|^2(1 - |z|^2)^2 \le 4(\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2), \qquad z \in \mathbf{D}.$$

This completes the proof of Theorem 9.

5. Symbol calculus of Toeplitz operators. In this section, we are going to determine the largest C^*-subalgebra Q of $L^\infty(\mathbf{D}, dA)$ such that the map $\xi\colon Q \to \mathcal{B}(L^2_a(\mathbf{D}))/\mathcal{K}$, defined by $\xi(f) = T_f + \mathcal{K}$, is a C^*-algebra homomorphism, where $\mathcal{K}$ is the compact ideal of the full algebra $\mathcal{B}(L^2_a(\mathbf{D}))$ of bounded linear operators on $L^2_a(\mathbf{D})$. First we establish the existence of such an algebra.

Let

$$\begin{aligned} \Gamma &= \{f \in L^\infty(\mathbf{D}) | T_g T_f - T_{gf} \in \mathcal{K} \text{ for all } g \in L^\infty(\mathbf{D})\}, \\ Q &= \Gamma \cap \overline{\Gamma}, \\ B &= \{f \in L^\infty(\mathbf{D}) \mid T_f \in \mathcal{K}\}. \end{aligned}$$

PROPOSITION 5. *For $f \in L^\infty(\mathbf{D})$, the following are all equivalent:*
(1) $f \in \Gamma$;
(2) H_f *is compact;*
(3) $T_{|f|^2} - T_{\bar{f}} T_f$ *is compact.*

PROOF. The proof is the same as in [**6**].

PROPOSITION 6. *For $f \in L^\infty(\mathbf{D})$, the following are all equivalent:*
(1) $f \in Q$;
(2) H_f *and* $H_{\bar{f}}$ *are compact;*
(3) $T_{|f|^2} - T_f T_{\bar{f}}$ *and* $T_{|f|^2} - T_{\bar{f}} T_f$ *are compact.*

PROOF. The proof follows from Proposition 5.

PROPOSITION 7. *Q is a C^*-subalgebra of $L^\infty(\mathbf{D})$; $Q \cap B$ is a closed selfadjoint ideal of Q.*

PROOF. The proof is the same as in [**6**] and [**7**].

REMARK. Propositions 6 and 7 imply that Q is the largest C^*-subalgebra of $L^\infty(\mathbf{D})$ such that the map $\xi\colon Q \to \mathcal{B}(L^2_a(\mathbf{D}))/\mathcal{K}$ is a C^*-algebra homomorphism.

The kernel of this homomorphism is $Q \cap B$. Thus if we let $\tau(Q)$ denote the C^*-subalgebra of $\mathcal{B}(L_a^2(\mathbf{D}))$ generated by all the operators T_f with $f \in Q$, then

$$Q/Q \cap B \cong \tau(Q)/\mathcal{K} \tag{6}$$

as C^*-algebras. (6) is traditionally called the symbol calculus of Toeplitz operators. So far Q has only been defined abstractly. Next we want to determine Q. The main theorem is that for a function $f \in L^\infty(\mathbf{D})$. $f \in Q$ if and only if $f \in \mathrm{VMO}_\partial(\mathbf{D})$; $f \in Q \cap B$ if and only if $f \in \tilde{Q} \cap \tilde{B}$.

PROPOSITION 8 (C. A. BERGER). *The operator $P\colon L^\infty(\mathbf{D}) \to L_a^2(\mathbf{D})$ is compact.*

PROOF. Given a bounded sequence $\{f_n\}$ in $L^\infty(\mathbf{D})$, say, $\|f_n\|_\infty \le M$ ($n = 1, 2, \ldots$). We want to find a subsequence $\{f_{n_k}\}$ such that $\{Pf_{n_k}\}$ converges in $L_a^2(\mathbf{D})$.

Recall that

$$Pf_n(z) = \int_{\mathbf{D}} \frac{f_n(w)}{(1 - z\bar{w})^2}\, dA(w), \qquad z \in \mathbf{D}.$$

Now if $|z| \le \delta < 1$, then

$$|Pf_n(z)| \le \int_{\mathbf{D}} \frac{M\,dA(w)}{|1 - z\bar{w}|^2} \le \frac{M}{(1-\delta)^2}, \qquad n = 1, 2, \ldots.$$

So $\{Pf_n\}$ is uniformly bounded on every compact subset of $\mathbf{D}$. Since $\{Pf_n\}$ is a sequence of analytic functions on $\mathbf{D}$, by Arzela's theorem, there is a subsequence $\{Pf_{n_k}\}$ which converges to $h \in L_a^2(\mathbf{D})$ uniformly on every compact subset of $\mathbf{D}$. (Note: $h \in L_a^2(\mathbf{D})$ by Fatou's lemma.) It remains to prove that

$$\|Pf_{n_k} - h\|_{L^2} \to 0 \qquad (k \to +\infty). \tag{7}$$

For any $z \in \mathbf{D}$, we have

$$|Pf_n(z)| \le M \int_{\mathbf{D}} \frac{dA(w)}{|1 - z\bar{w}|^2} = -\frac{M}{2|z|^2} \ln(1 - |z|^2).$$

Since $\int_{\mathbf{D}} (\frac{1}{2}|z|^{-2} \ln(1 - |z|^2))^2\, dA(z) < +\infty$, (7) follows from the dominated convergence theorem.

LEMMA 8. *If $\{f_n\}$ is a sequence of real-valued functions in $L^2(\mathbf{D})$ such that $\|f_n - h\|_{L^2} \to 0$ ($n \to +\infty$) for some $h \in L_a^2(\mathbf{D})$, then h is a constant.*

PROOF. Write $h = u + iv$. Then

$$|f_n(z) - h(z)|^2 = (f_n(z) - u(z))^2 + (v(z))^2,$$

so

$$\|f_n - h\|_{L^2}^2 = \|f_n - u\|_{L^2}^2 + \|v\|_{L^2}^2 \ge \|v\|_{L^2}^2.$$

Let $n \to +\infty$, we have $v = 0$. Thus h is real-valued. Since h is analytic, h must be a constant.

THEOREM 10. $Q \subset \tilde{Q} = \mathrm{VMO}_\partial(\mathbf{D}) \cap L^\infty(\mathbf{D})$.

PROOF. Given $f \in Q$, we want to prove $\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2 \to 0$ ($|z| \to 1^-$). Since Q and $\tilde{Q}$ are selfadjoint, we might as well assume that f is real-valued.

It is easy to check that

$$\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2 = \|\tilde{f}(z) - f \circ b_z\|_{L^2}^2 \geq 0,$$

where $b_z(w) = (z-w)/(1-\bar{z}w)$. We prove the theorem by contradiction.

Suppose

$$\overline{\lim_{|z|\to 1^-}} \|\tilde{f}(z) - f \circ b_z\|^2 > 0.$$

Then there exists $\rho > 0$ and $|z_n| \to 1^-$ such that

$$\|\tilde{f}(z_n) - f \circ b_{z_n}\|^2 > \rho, \qquad n = 1, 2, \ldots. \tag{8}$$

Because $f \in Q$, $H_f = (I-P)M_f P$ is compact, so

$$\|(I-P)fk_z\|_{L^2} \to 0 \qquad (|z| \to 1^-) \tag{9}$$

since $k_z \to 0$ weakly as $|z| \to 1^-$.

For each $z \in \mathbf{D}$, define a unitary operator U_z on $L^2(\mathbf{D})$ as follows:

$$U_z f(w) = \frac{1-|z|^2}{(1-\bar{z}w)^2} f\left(\frac{z-w}{1-\bar{z}w}\right), \qquad w \in \mathbf{D},\ f \in L^2(\mathbf{D}).$$

It is easy to check that $U_z^* = U_z$ and $L_a^2(\mathbf{D})$ is a reducing subspace of U_z, so $U_z P = P U_z$.

Now using the equality $fk_z = U_z(f \circ b_z)$ and (9), we get

$$\|(I-P)U_z(f \circ b_z)\|_{L^2} \to 0 \quad (\text{as } |z| \to 1^-).$$

But $(I-P)U_z = U_z(I-P)$ and U_z is unitary, so we must have

$$\|(I-P)f \circ b_z\|_{L^2} \to 0 \quad (\text{as } |z| \to 1^-). \tag{10}$$

Notice that $\|f \circ b_z\|_\infty = \|f\|_\infty$ for all $z \in \mathbf{D}$, so by Proposition 8, there is a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ and $h \in L_a^2(\mathbf{D})$ such that

$$\|P(f \circ b_{z_{n_k}}) - h\|_{L^2} \to 0 \quad (k \to +\infty). \tag{11}$$

Now (10) + (11) implies that

$$\|f \circ b_{z_{n_k}} - h\|_{L^2} \to 0 \quad (k \to +\infty). \tag{12}$$

By Lemma 8, h is a constant. Therefore,

$$\tilde{f}(z_{n_k}) = \langle f \circ b_{z_{n_k}}, 1\rangle \to \langle h, 1\rangle = h$$

as $k \to +\infty$. Thus

$$\begin{aligned}\|f \circ b_{z_{n_k}} - \tilde{f}(z_{n_k})\|_{L^2} &\leq \|f \circ b_{z_{n_k}} - h\|_{L^2} + \|h - \tilde{f}(z_{n_k})\|_{L^2} \\ &= \|f \circ b_{z_{n_k}} - h\|_{L^2} + |\tilde{f}(z_{n_k}) - h| \to 0 \quad \text{as } k \to +\infty,\end{aligned}$$

a contradiction to (8).

REMARK. The proof of Theorem 10 is a modification of the corresponding result in an early version of [**7**].

PROPOSITION 9. *Let*

$$M_p = \sup_{z \in \mathbf{D}} \int_{\mathbf{D}} \frac{dA(w)}{|1 - \bar{z}w|^p (1 - |w|^2)^{p/2}}.$$

Then $M_p < +\infty$ *for* $p < \frac{4}{3}$.

PROOF. See [**1**], or 1.4.10 of [**21**].

LEMMA 9. *Let* $\mathbf{D}_\delta = \{z \in \mathbf{D} \mid |z| < \delta\}$, $\delta \in (0,1)$. *Then* $M_{\chi_{\mathbf{D}_\delta}} H_f$ *is Hilbert-Schmidt as an operator from* $L_a^2(\mathbf{D})$ *to* $L^2(\mathbf{D})$ *for all* f *in* $L^\infty(\mathbf{D})$.

PROOF. For $|z| \geq \delta$, $M_{\chi_{\mathbf{D}_\delta}} H_f g(z) = 0$. For $|z| < \delta$,

$$M_{\chi_{\mathbf{D}_\delta}} H_f g(z) = H_f g(z) = \int_{\mathbf{D}} \frac{f(z) - f(w)}{(1 - z\bar{w})^2} g(w)\, dA(w).$$

Thus for all $z \in \mathbf{D}$ and $g \in L_a^2(\mathbf{D})$,

$$|M_{\chi_{\mathbf{D}_\delta}} H_f g(z)| \leq \int_{\mathbf{D}} \frac{|f(z) - f(w)|}{(1 - \delta)^2} |g(w)| dA(w). \tag{13}$$

The operator A on $L^2(\mathbf{D})$ defined by

$$Ag(z) = \int_{\mathbf{D}} |f(z) - f(w)| g(w) dA(w)$$

is Hilbert-Schmidt since the integral kernel is in $L^2(\mathbf{D} \times \mathbf{D})$, so $M_{\chi_{\mathbf{D}_\delta}} H_f$ is Hilbert-Schmidt by (13).

THEOREM 11. $\mathrm{ESV}(\mathbf{D}) \subset Q$.

PROOF. Let $f \in \mathrm{ESV}(\mathbf{D})$. Then by Theorem 8, $\|f(z) - f \circ b_z\|_{L^2} \to 0$ as $|z| \to 1^-$. Given $\varepsilon > 0$, choose $\delta \in (0,1)$ such that $\|f(z) - f \circ b_z\|_{L^2} < \varepsilon^3$ for all $\delta \leq |z| < 1$. Then for all $\delta \leq |z| < 1$, we have

$$\begin{aligned}
\int_{\mathbf{D}} \frac{|f(z) - f(w)|}{|1 - z\bar{w}|^2 \sqrt{1 - |w|^2}}\, dA(w) &= \frac{1}{\sqrt{1 - |z|^2}} \int_{\mathbf{D}} \frac{|f(z) - f \circ b_z(w)|}{|1 - \bar{z}w| \sqrt{1 - |w|^2}}\, dA(w) \\
&\leq \frac{M}{\sqrt{1 - |z|^2}} \left(\int_{\mathbf{D}} |f(z) - f \circ b_z(w)|^6 dA(w) \right)^{1/6} \\
&\leq \frac{M(2\|f\|_\infty)^{2/3}}{\sqrt{1 - |z|^2}} \left(\int_{\mathbf{D}} |f(z) - f \circ b_z(w)|^2 dA(w) \right)^{1/6} \\
&= \frac{M(2\|f\|_\infty)^{2/3}}{\sqrt{1 - |z|^2}} \|f(z) - f \circ b_z\|_{L^2}^{1/3} < \frac{M(2\|f\|_\infty)^{2/3} \varepsilon}{\sqrt{1 - |z|^2}},
\end{aligned}$$

where $M = M_{6/5}$ in Proposition 9.

It is easy to check that

$$H_f g(z) = \int_{\mathbf{D}} \frac{f(z) - f(w)}{(1 - z\bar{w})^2} g(w) dA(w), \qquad g \in L_a^2.$$

So

$$|H_f g(z)| \leq \int_{\mathbf{D}} \frac{|f(z) - f(w)|}{|1 - z\bar{w}|^2} |g(w)| dA(w).$$

The Cauchy-Schwarz inequality shows that

$$|H_f g(z)|^2 \le \int_{\mathbf{D}} \frac{|f(z)-f(w)|}{|1-z\bar{w}|^2\sqrt{1-|w|^2}}\,dA(w) \cdot \int_{\mathbf{D}} \frac{|f(z)-f(w)|}{|1-z\bar{w}|^2}\sqrt{1-|w|^2}|g(w)|^2\,dA(w).$$

Thus for all $\delta \le |z| < 1$,

$$\begin{aligned}|H_f g(z)|^2 &\le \frac{M(2\|f\|_\infty)^{2/3}}{\sqrt{1-|z|^2}}\varepsilon \int_{\mathbf{D}} \frac{|f(z)-f(w)|}{|1-z\bar{w}|^2}\sqrt{1-|w|^2}|g(w)|^2\,dA \\ &\le \frac{M(2\|f\|_\infty)^{5/3}}{\sqrt{1-|z|^2}}\varepsilon \int_{\mathbf{D}} \frac{\sqrt{1-|w|^2}|g(w)|^2}{|1-z\bar{w}|^2}\,dA(w).\end{aligned}$$

Write $\overline{M} = M(2\|f\|_\infty)^{5/3}$. Then

$$\begin{aligned}&\int_{|z|\ge\delta} |H_f g(z)|^2\,dA(z) \\ &\le \overline{M}\varepsilon \int_{\mathbf{D}} \sqrt{1-|w|^2}|g(w)|^2\,dA(w) \cdot \int_{\mathbf{D}} \frac{dA(z)}{\sqrt{1-|z|^2}|1-\bar{w}z|^2} \\ &= \overline{M}\varepsilon \int_{\mathbf{D}} |g(w)|^2 dA(w) \cdot \int_{\mathbf{D}} \frac{dA(z)}{|1-\bar{w}z|\sqrt{1-|z|^2}} \\ &\le \overline{M}M_1\varepsilon \int_{\mathbf{D}} |g(w)|^2 dA(w) = \overline{M}M_1\varepsilon\|g\|^2.\end{aligned}$$

This implies that $\|M_{\chi_{\mathbf{D}-\mathbf{D}_\delta}} H_f\| \le \overline{M}M_1\varepsilon$, namely, $\|H_f - M_{\chi_{\mathbf{D}_\delta}} H_f\| \le \overline{M}M_1\varepsilon$. Since $M_{\chi_{\mathbf{D}_\delta}} H_f$ is compact and ε is arbitrary, H_f is compact, and so $f \in \Gamma$. Because f is arbitrary and $\mathrm{ESV}(\mathbf{D})$ is selfadjoint, we have proved $\mathrm{ESV}(\mathbf{D}) \subset \Gamma \cap \overline{\Gamma} = Q$.

THEOREM 12. $\tilde{Q} \cap \tilde{B} \subset Q \cap B$.

PROOF. Given $f \in \tilde{Q} \cap \tilde{B}$, we have $|f| \in \tilde{B}$. To prove $f \in Q \cap B$, it suffices to prove $|f| \in B$, that is, $T_{|f|}$ is compact.

Recall that

$$T_{|f|} g(z) = \int_{\mathbf{D}} \frac{|f(w)|}{(1-z\bar{w})^2} g(w)dA(w)$$

for $g \in L_a^2(\mathbf{D})$ and $z \in \mathbf{D}$. So the Cauchy-Schwarz inequality gives

$$|T_{|f|} g(z)|^2 \le \int_{\mathbf{D}} \frac{|f(w)|^2 dA(w)}{|1-z\bar{w}|^2\sqrt{1-|w|^2}} \int_{\mathbf{D}} \frac{\sqrt{1-|w|^2}|g(w)|^2}{|1-z\bar{w}|^2}\,dA(w).$$

But

$$\begin{aligned}\int_{\mathbf{D}} \frac{|f(w)|^2 dA(w)}{|1-z\bar{w}|^2\sqrt{1-|w|^2}} &= \frac{1}{\sqrt{1-|z|^2}} \int_{\mathbf{D}} \frac{|f\circ b_z(w)|^2}{|1-z\bar{w}|\sqrt{1-|w|^2}}\,dA(w) \\ &\le \frac{M_{6/5}}{\sqrt{1-|z|^2}} \left(\int_{\mathbf{D}} |f\circ b_z(w)|^{12} dA(w)\right)^{1/6} \\ &\le \frac{M_{6/5}\|f\|_\infty^{11/6}}{\sqrt{1-|z|^2}} \left(\int_{\mathbf{D}} |f\circ b_z(w)| dA(w)\right)^{1/6},\end{aligned}$$

and we have

$$|T_{|f|}g(z)|^2 \leq \frac{\overline{M}}{\sqrt{1-|z|^2}}(\widetilde{|f|}(z))^{1/6}\int_{\mathbf{D}}\frac{\sqrt{1-|w|^2}|g(w)|^2}{|1-z\bar{w}|^2}\,dA(w),$$

where $\overline{M} = M_{6/5}\|f\|_\infty^{11/6}$.

Given $\varepsilon > 0$, choose $\delta \in (0,1)$ such that $\widetilde{|f|}(z) < \varepsilon^6$ whenever $\delta < |z| < 1$. Then

$$\begin{aligned}\int_{|z|\geq\delta}|T_{|f|}g(z)|^2dA(z) &\leq \overline{M}\varepsilon\int_{\mathbf{D}}|g(w)|^2\sqrt{1-|w|^2}dA(w)\\ &\quad\cdot\int_{\mathbf{D}}\frac{dA(z)}{|1-z\bar{w}|^2\sqrt{1-|z|^2}}\\ &= \overline{M}\varepsilon\int_{\mathbf{D}}|g(w)|^2dA(w)\int_{\mathbf{D}}\frac{dA(z)}{|1-z\bar{w}|\sqrt{1-|z|^2}}\\ &\leq \overline{M}M_1\varepsilon\int_{\mathbf{D}}|g(w)|^2dA(w) = \overline{M}M_1\varepsilon\|g\|_{L^2}^2.\end{aligned}$$

So $\|M_{\chi_{\mathbf{D}-\mathbf{D}_\delta}}T_{|f|}\|^2 \leq \overline{M}M_1\varepsilon$, that is,

$$\|T_{|f|} - M_{\chi_{\mathbf{D}_\delta}}T_{|f|}\|^2 \leq \overline{M}M_1\varepsilon.$$

Since $M_{\chi_{\mathbf{D}_\delta}}T_{|f|}$ is compact as an operator from $L^2_a(\mathbf{D})$ to $L^2(\mathbf{D})$ and ε is arbitrary, $T_{|f|}$ must be compact.

REMARK. Since $k_z \to 0$ weakly as $|z| \to 1^-$, we have $Q \cap B \subset \tilde{Q} \cap \tilde{B}$ trivially.

Theorems 11 and 12 and the decomposition $\tilde{Q} = \mathrm{ESV}(\mathbf{D}) + \tilde{Q} \cap \tilde{B}$ show that $\tilde{Q} \subset Q$. In summary, we have proved the following main theorem.

THEOREM 13. (1) $Q = \tilde{Q} = \mathrm{VMO}_\partial(\mathbf{D}) \cap L^\infty(\mathbf{D})$.
(2) $Q \cap B = \tilde{Q} \cap \tilde{B}$.

COROLLARY 1 (S. AXLER [1]). *Let $f \in H^\infty(\mathbf{D})$. Then $H_{\bar{f}}$ is compact if and only if f is in the "little Bloch" space $\mathcal{B}_0$.*

PROOF. It follows from Theorems 9 and 13 and the fact that $H_f = 0$.

COROLLARY 2. *Q and $Q \cap B$ are invariant under Möbius transformations.*

PROOF. This follows from the facts that $Q = \tilde{Q}$ and $Q \cap B = \tilde{Q} \cap \tilde{B}$ and $\tilde{f}(b_\lambda(z)) = \widetilde{f \circ b_\lambda}(z)$ (simply a change of variable formula), where the b_λ's are Möbius transformations.

6. Fredholm theory of Toeplitz operators with symbols in Q. The isomorphism $Q/Q \cap B \cong \tau(Q)/\mathcal{K}$ and the decomposition $Q = \mathrm{ESV} + Q \cap B$ will serve as basic tools for our study of Fredholm theory of Toeplitz operators with symbols in Q. Let $\mathrm{BC}(\mathbf{D})$ be the C^*-algebra of all bounded continuous functions on $\mathbf{D}$, and $C_0(\mathbf{D})$ be the space of continuous functions f on $\mathbf{D}$ with the property that $f(z) \to 0$ as $|z| \to 1^-$. Consider the algebra BCESV defined as $\mathrm{BC}(\mathbf{D}) \cap \mathrm{ESV}$. Since $\tilde{f} \in \mathrm{BC}(\mathbf{D})$ for any $f \in L^\infty(\mathbf{D})$, the equality $f = \tilde{f} + (f - \tilde{f})$ gives a decomposition

$$Q = \mathrm{BCESV} + Q \cap B.$$

Notice that BCESV $\cap (Q \cap B) = C_0(\mathbf{D})$, so we have

$$Q/Q \cap B = (\text{BCESV} + Q \cap B)/Q \cap B \cong \text{BCESV}/C_0(\mathbf{D}).$$

Also we should mention that

$$Q/Q \cap B \cong \text{ESV}/V_0(\mathbf{D}),$$

where $V_0(\mathbf{D})$ consists of all functions f in $L^\infty(\mathbf{D})$ with $f(z) \to 0$ as $|z| \to 1^-$.

Let [23 $\beta\beta\mathbf{D}$ be the Stone-Cěch compactification of $\mathbf{D}$. Any bounded continuous functions f on $\mathbf{D}$ has a unique continuous extension to $\beta\mathbf{D}$: we also denote this extension of f to $\beta\mathbf{D}$ by f, so there should be no confusion about this.

THEOREM 14. *If $f \in Q$, then $\sigma_e(T_f) = \tilde{f}(\beta\mathbf{D} - \mathbf{D})$, where $\sigma_e(T_f)$ is the essential spectrum of T_f.*

PROOF. Since $f \in Q$, we know $T_{f-\tilde{f}}$ is compact. Thus $\sigma_e(T_f) = \sigma_e(T_{\tilde{f}})$. $\tilde{f}$ is in BCESV. Mimicking [7], we can prove that for any $g \in$ BCESV, $g + C_0(\mathbf{D})$ is invertible in BCESV$/C_0(\mathbf{D})$ if and only if there are δ, ε in (0,1) such that $|g(z)| \geq \varepsilon$ for all $\delta \leq |z| < 1$. By the symbol calculus

$$\text{BCESV}/C_0(\mathbf{D}) \cong \tau(\text{BCESV})/\mathcal{K},$$

$T_g + \mathcal{K}$ is invertible in $\tau(\text{BCESV})/\mathcal{K}$ if and only if there are δ, ε in (0,1) such that $|g(z)| \geq \varepsilon$ for all $\delta \leq |z| < 1$. Therefore

$$\sigma_e(T_{\tilde{f}}) = \bigcap_{\delta \in (0,1)} \overline{\tilde{f}(\mathbf{D} - \mathbf{D}_\delta)},$$

where $\mathbf{D}_\delta = \{z \in \mathbf{D} \mid |z| < \delta\}$. The compactness of $\beta\mathbf{D}$ and the continuity of $\tilde{f}$ yield $\tilde{f}\,\overline{(\mathbf{D} - \mathbf{D}_\delta)} = \tilde{f}\,\overline{(\mathbf{D} - \mathbf{D}_\delta)} = \tilde{f}(\beta\mathbf{D} - \mathbf{D}_\delta)$. So we get

$$\sigma_e(T_{\tilde{f}}) = \bigcap_{\delta \in (0,1)} \tilde{f}(\beta\mathbf{D} - \mathbf{D}_\delta).$$

On the other hand, if $\lambda \in \bigcap_{\delta \in (0,1)} \tilde{f}(\beta\mathbf{D} - \mathbf{D}_\delta)$, then $\lambda = \tilde{f}(z_\delta)$, $z_\delta \in \beta\mathbf{D} - \mathbf{D}_\delta$, $\delta \in (0,1)$. Consider the sequence $\{z_{1-1/n}\}$. The compactness of $\beta\mathbf{D}$ implies that there exists a subsequence $\{z_{1-1/n_k}\}$ and $z \in \beta\mathbf{D}$ such that $z_{1-1/n_k} \to z$ as $k \to +\infty$. It is clear that $z \in \beta\mathbf{D} - \mathbf{D}$ since $\mathbf{D}$ is open in $\beta\mathbf{D}$. The continuity of $\tilde{f}$ and the equality $\lambda = f(z_\delta)$ give $\lambda = \tilde{f}(z) \in \tilde{f}(\beta\mathbf{D} - \mathbf{D})$. Hence $\tilde{f}(\beta\mathbf{D} - \mathbf{D}) = \bigcap_{\delta \in (0,1)} \tilde{f}(\beta\mathbf{D} - \mathbf{D}_\delta)$, and the proof is complete.

COROLLARY 1. *For $f \in Q$, T_f is Fredholm if and only if $\tilde{f}$ is nonvanishing on $\{z \mid z \in \mathbf{D},\ |z| \geq \delta\}$ for some $\delta \in (0,1)$.*

COROLLARY 2. *If $f \in$ BCESV, then $\sigma_e(T_f) = f(\beta\mathbf{D} - \mathbf{D})$, hence T_f is Fredholm if and only if f is nonvanishing on $\{z \mid z \in \mathbf{D},\ |z| \geq \delta\}$ for some $\delta \in (0,1)$.*

PROOF. For $f \in$ BCESV, $f - \tilde{f}$ is in $C_0(\mathbf{D})$, so $f(\beta\mathbf{D} - \mathbf{D}) = \tilde{f}(\beta\mathbf{D} - \mathbf{D})$.

COROLLARY 3. *If $f \in Q$, then $\sigma_e(T_f)$ is connected.*

PROOF. $\sigma_e(T_f) = \bigcap_{\delta \in (0,1)} \overline{\tilde{f}(\mathbf{D} - \mathbf{D}_\delta)}$ is the intersection of a nested family of compact connected sets, so it is connected. See [7].

REMARK. As C^*-algebras, $C(\beta\mathbf{D})$ is isomorphic to $\mathrm{BC}(\mathbf{D})$. Under the isomorphism, $C_0(\mathbf{D})$ is the closed ideal of $C(\beta\mathbf{D})$ consisting of functions f on $\beta\mathbf{D}$ such that f is identically zero on $\beta\mathbf{D} - \mathbf{D}$.

If f is in BCESV and T_f is Fredholm, then we know that there are $\delta, \varepsilon \in (0,1)$ such that $|f(z)| \geq \varepsilon$ for all $\delta \leq |z| < 1$. For any $r \in (\delta, 1)$, we have a continuous map $f_r : \partial\mathbf{D} \to C - \{0\}$ defined by $f_r(e^{i\theta}) = f(re^{i\theta})$. Given any two $r_1, r_2 \in [\delta, 1)$, f_{r_1} and f_{r_2} are homotopic in the obvious way. So the winding numbers of f_{r_1} and f_{r_2} are equal and independent of the choice of δ. Denote the common winding number by $\mathcal{N}_f$. Then by monodromy as used in [**7**], we can prove

THEOREM 15. *If $f \in Q$ and T_f is Fredholm, then* $\mathrm{Ind}(T_f) = -\mathcal{N}_{\bar{f}}$, *where* $\mathrm{Ind}(T_f)$ *is the Fredholm index of T_f, i.e.* $\mathrm{Ind}(T_f) =$ *dimension of kernel T_f- dimension of kernel $T_{\bar{f}}$.*

REMARK. BCESV has played a significant role in our analysis. It seems interesting to know the structure of BCESV as a C^*-algebra. BCESV contains $C(\overline{\mathbf{D}})$ as a proper C^*-subalgebra. Let $\mathcal{M}$ be the maximal ideal space of BCESV. $\mathcal{M}$ is connected since for any $f \in$ BCESV, $\sigma(f) = \overline{f(\mathbf{D})}$ is connected (so there is no idempotent in BCESV with spectrum $\{0,1\}$). For any $\lambda \in \mathbf{D}$, the evaluation functional on BCESV at λ is in $\mathcal{M}$, denoted by F_λ. The map $\lambda \mapsto F_\lambda$ is a one-to-one map of $\mathbf{D}$ into $\mathcal{M}$. Let $\mathcal{D}$ be the image of this map. We put the induced topology on $\mathcal{D}$. Let $f_0 \in$ BCESV be the function $f_0(z) = z$ for all $z \in \mathbf{D}$. Then we have the following

THEOREM 16. *Let $F \in \mathcal{M}$ be a multiplicative linear functional on* BCESV. *Then $F \in \mathcal{D}$ if and only if $|F(f_0)| < 1$.*

PROOF. The "only if" part is obvious. We prove the "if" part.

Suppose $|F(f_0)| < 1$. Let $z_0 = F(f_0) \in \mathbf{D}$. We want to prove $F(f) = f(z_0)$ for all f in BCESV. By the Stone-Weierstrass approximation theorem, it is easy to show that $F(f) = f(z_0)$ for all f in $C(\overline{\mathbf{D}})$. Choose a function $\varphi \in (\overline{\mathbf{D}})$ so that $\varphi \equiv 1$ on a neighborhood $U \subset \mathbf{D}$ of z_0 and $\varphi \equiv 0$ on a neighborhood V of $\partial\mathbf{D}$. Now for any $f \in$ BCESV, $f\varphi \in C(\overline{\mathbf{D}})$. Thus $F(f\varphi) = (f\varphi)(z_0) = f(z_0)\varphi(z_0)$. On the other hand, the multiplicativity of F given $F(f\varphi) = F(f)F(\varphi) = F(f)\varphi(z_0) = F(f)$. Hence $F(f) = f(z_0)$ for all $f \in$ BCESV.

COROLLARY. *$\mathcal{D}$ is open in $\mathcal{M}$, and hence* $\mathrm{BCESV}/C_0(\mathbf{D}) \cong C(\mathcal{M} - \mathcal{D})$.

PROOF. The map $F \mapsto F(f_0)$ from $\mathcal{M}$ to C is continuous. By the above theorem, $\mathcal{D}$ is the inverse image of $\mathbf{D}$ under this map, so $\mathcal{D}$ is open in $\mathcal{M}$.

REMARK. This corollary says that $\mathcal{M} - \mathcal{D}$ is homeomorphic to the maximal ideal space of $\mathrm{BCESV}/C_0(\mathbf{D}) \cong \tau(Q)/\mathcal{K}$.

REMARK. $\mathcal{M} - \mathcal{D}$ is connected since $\mathrm{BCESV}/C_0(\mathbf{D})$ has no idempotent element with spectrum $\{0,1\}$ by Corollary 3 to Theorem 14.

7. A conformal invariant description of VMO_∂. In this section, we are going to give another characterization of VMO_∂. Also we will describe the relationship between $\mathrm{VMO}_\partial(\mathbf{D})$ and the usual $\mathrm{VMO}(\mathbf{D})$.

For $z_0 \in \mathbf{D}$ and $r \in (0,1)$, let

$$D(z_0, r) = \{z \in \mathbf{D} \colon \ |(z_0 - z)/(1 - \bar{z}_0 z)| < r\}.$$

$D(z_0, r)$ is called the pseudohyperbolic disc centered at z_0 with radius r. It is actually a Euclidean disc (see [**13**]) contained in $\mathbf{D}$ with center

$$c = \frac{1-r^2}{1-r^2|z_0|^2} z_0$$

and radius

$$R = r\frac{1-|z_0|^2}{1-r^2|z_0|^2}.$$

Thus the normalized Lebesgue measure of $D(z_0, r)$ is

$$|D(z_0, r)| = r^2(1-|z_0|^2)^2/(1-r^2|z_0|^2)^2.$$

THEOREM 17. *For $f \in L^\infty(\mathbf{D}, dA)$, we have $f \in \mathrm{VMO}_\partial$ if and only if*

$$\lim_{|z|\to 1} \frac{1}{|D(z,r)|}\int_{D(z,r)} \left| f(w) - \frac{1}{|D(z,r)|}\int_{D(z,r)} f(u)\,dA(u)\right| dA(w) = 0$$

for each $r \in (0,1)$.

PROOF. Let

$$I(z,r) = \frac{1}{|D(z,r)|^2}\int_{D(z,r)}\int_{D(z,r)} |f(u) - f(w)|^2 \, dA(w)\, dA(u).$$

Since f is bounded, it suffices to show that $f \in \mathrm{VMO}_\partial \Leftrightarrow I(z,r) \to 0$ as $|z| \to 1$ for each $r \in (0,1)$.

A change of variable shows that

$$\begin{aligned} I(z,r) &= \frac{1}{|D(z,r)|^2}\int_{|w|\le r}\int_{|u|\le r}\left| f\left(\frac{z-w}{1-\bar z w}\right) - f\left(\frac{z-u}{1-\bar z u}\right)\right|^2 \\ &\qquad\qquad \cdot \frac{(1-|z|^2)^4 dA(w) dA(u)}{|1-\bar z w|^4 |1-\bar z u|^4} \\ &\le \frac{1}{|D(z,r)|^2}\frac{(1-|z|^2)^4}{(1-r)^8}\int_{\mathbf{D}}\int_{\mathbf{D}}\left| f\left(\frac{z-w}{1-\bar z w}\right) - f\left(\frac{z-u}{1-\bar z u}\right)\right|^2 dA(w)\, dA(u) \\ &= \frac{2(1-r^2|z|^2)^4}{r^4(1-r)^8}\left(\widetilde{|f|^2}(z) - |\tilde f(z)|^2\right) \\ &\le \frac{2}{r^4(1-r)^8}\left(\widetilde{|f|^2}(z) - |\tilde f(z)|^2\right). \end{aligned}$$

Thus $f \in \mathrm{VMO}_\partial \Rightarrow f \in \tilde Q \Rightarrow I(z,r) \to 0$ as $|z| \to 1$ for each $r \in (0,1)$.

On the other hand,

$$\begin{aligned} I(z,r) &\ge \frac{(1-|z|^2)^4}{2^8|D(z,r)|^2}\int_{|w|\le r}\int_{|u|\le r}\left| f\left(\frac{z-w}{1-\bar z w}\right) - f\left(\frac{z-u}{1-\bar z u}\right)\right|^2 dA(w) dA(u) \\ &= \frac{(1-r^2|z|^2)^4}{2^8 r^4}\left[\int_{\mathbf{D}}\int_{\mathbf{D}} - \int_{\mathbf{D}}\int_{\mathbf{D}-\mathbf{D}_r} - \int_{\mathbf{D}-\mathbf{D}_r}\int_{\mathbf{D}_r}\right] \\ &\ge \frac{(1-r^2|z|^2)^4}{2^8 r^4}\left[\int_{\mathbf{D}}\int_{\mathbf{D}} - 8\|f\|_\infty^2 |\mathbf{D} - \mathbf{D}_r|\right], \end{aligned}$$

that is,

$$2(\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2) \leq \frac{2^8 r^4}{(1-r^2|z|^2)^4} I(z,r) + 8\|f\|_\infty^2 |\mathbf{D} - \mathbf{D}_r|.$$

Now if $I(z,r) \to 0$ as $|z| \to 1$ for each $r \in (0,1)$, then

$$2 \varlimsup_{|z|\to 1} (\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2) \leq 8\|f\|_\infty^2 |\mathbf{D} - \mathbf{D}_r|$$

for each $r \in (0,1)$. Letting $r \to 1$ yields

$$\lim_{|z|\to 1} \left(\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2 \right) = 0,$$

namely, $f \in \tilde{Q} = \mathrm{VMO}_\partial \cap L^\infty$.

COROLLARY 1. *For $f \in L^\infty(\mathbf{D}, dA)$, we have $f \in Q \cap B$ if and only if*

$$\lim_{|z|\to 1} \frac{1}{|D(z,r)|} \int_{D(z,r)} |f(w)| dA(w) = 0$$

for each $r \in (0,1)$.

The proof of Corollary 1 is very similar to that of the theorem, so we omit it.

For any $f \in L^\infty(\mathbf{D}, dA)$, define a continuous function $\hat{f}_r(z)$ on $\mathbf{D}$ as follows:

$$\hat{f}_r(z) = \frac{1}{|D(z,r)|} \int_{D(z,r)} f(u) dA(u).$$

Then we have

COROLLARY 2. *For $f \in L^\infty(\mathbf{D}, dA)$, we have*

$$f \in Q \Leftrightarrow \widehat{|f|_r^2}(z) - |\hat{f}_r(z)|^2 \to 0 \ (|z| \to 1) \quad \text{for any} \ \ r \in (0,1),$$
$$f \in Q \cap B \Leftrightarrow \widehat{|f|}_r(z) \to 0 \ (|z| \to 1) \quad \text{for any} \ \ r \in (0,1).$$

PROOF. The second equivalence is just the above Corollary 1. The first equivalence follows from the identity

$$I(z,r) = 2(\widehat{|f|_r^2}(z) - |\hat{f}_r(z)|^2).$$

THEOREM 18. *For $f \in L^\infty(\mathbf{D}, dA)$, we have*

$$f \in Q \Leftrightarrow \lim_{|z|\to 1} \frac{1}{|D(z,r)|} \int_{D(z,r)} |f(w) - \tilde{f}(z)|^2 \, dA(w) = 0,$$
$$f \in \mathrm{ESV} \Leftrightarrow \lim_{|z|\to 1} \frac{1}{|D(z,r)|} \int_{D(z,r)} |f(w) - f(z)|^2 \, dA(w) = 0.$$

PROOF. Recall that

$$f \in Q \Leftrightarrow \|f \circ b_z - \tilde{f}(z)\|_{L^2} \to 0 \quad \text{as } |z| \to 1,$$
$$f \in \mathrm{ESV} \Leftrightarrow \|f \circ b_z - f(z)\|_{L^2} \to 0 \quad \text{as } |z| \to 1.$$

Now the theorem can be proved by using the same techniques as in the proof of Theorem 17.

COROLLARY 1. *For $f \in Q$, we have*

$$f \in \mathrm{ESV} \Leftrightarrow \hat{f}_r(z) - f(z) \to 0 \quad \text{as } |z| \to 1 \text{ for each } r \in (0,1).$$

PROOF. The proof follows from Corollary 2 to Theorem 17 and the identity

$$\frac{1}{|D(z,r)|}\int_{D(z,r)} |f(w) - f(z)|^2 dA(w) = \widehat{|f|^2_r}(z) - |\hat{f}_r(z)|^2 + |\hat{f}_r(z) - f(z)|^2.$$

COROLLARY 2. *For $f \in Q$, we have*

$$\hat{f}_r(z) - \tilde{f}(z) \to 0 \quad \text{as } |z| \to 1 \text{ for each } r \in (0,1).$$

PROOF.

$$|\hat{f}_r(z) - \tilde{f}(z)| \le \frac{1}{|D(z,r)|}\int_{D(z,r)} |f(w) - \tilde{f}(z)|\, dA(w).$$

Now the assertion follows from Theorem 18 and the Schwarz inequality.

COROLLARY 3. *For $f \in Q$, we have $\hat{f}_r \in \mathrm{ESV}$ and $f - \hat{f}_r \in Q \cap B$.*

PROOF. It follows from Corollary 2 and the fact that $\tilde{f} \in \mathrm{ESV}$ and $f - \tilde{f} \in Q \cap B$.

COROLLARY 4. *Given $f \in L^\infty(\mathbf{D})$, we have*

$$f \in \mathrm{ESV} \Leftrightarrow \lim_{|z|\to 1^-} \sup_{w \in D(z,r)} |f(z) - f(w)| = 0$$

for all $r \in (0,1)$.

PROOF. "$\Leftarrow$" follows from the second statement of the theorem.

To prove "$\Rightarrow$", given any $r \in (0, \frac{1}{2})$ and consider $\hat{f}_r(z)$ on $\mathbf{D}$. Suppose $w \in D(z,r)$. Then

$$\begin{aligned} |\hat{f}_r(z) - \hat{f}_r(w)| &\le \frac{1}{|D(z,r)||D(w,r)|}\int_{D(z,r)}\int_{D(w,r)} |f(u) - f(v)| dA(u) dA(v) \\ &\le \frac{|D(z,2r)|^2}{|D(z,r)||D(w,r)|}\frac{1}{|D(z,2r)|^2}\cdot\int_{D(z,2r)}\int_{D(z,2r)} |f(u) - f(v)| dA(u) dA(v). \end{aligned}$$

Since $f \in \mathrm{ESV} \Rightarrow f \in \mathrm{VMO}_\partial(\mathbf{D})$, we have

$$\lim_{|z|\to 1^-} \sup_{w \in D(z,r)} |\hat{f}_r(z) - \hat{f}_r(w)| = 0 \qquad (r \in (0, \tfrac{1}{2})),$$

but $f(z) - \hat{f}_r(z) \to 0$ as $|z| \to 1^-$, hence

$$\lim_{|z|\to 1^-} \sup_{w \in D(z,r)} |f(z) - f(w)| = 0 \qquad (r \in (0, \tfrac{1}{2})).$$

By a finite covering argument, we get

$$\lim_{|z|\to 1^-} \sup_{w \in D(z,r)} |f(z) - f(w)| = 0$$

for all $r \in (0,1)$.

Finally, we discuss the relationship between $\mathrm{VMO}_\partial(\mathbf{D})$ and the usual area $\mathrm{VMO}(\mathbf{D})$. Recall that $f \in \mathrm{VMO}(\mathbf{D})$ if and only if given $\varepsilon > 0$, there is $\delta \in (0,1)$ such that

$$\frac{1}{|D|}\int_D \left| f(w) - \frac{1}{|D|}\int_D f(u) dA(u) \right| dA(w) < \varepsilon$$

whenever D is a disc contained in $\mathbf{D}$ with radius $\leq \delta$. For any $r \in (0,1)$, the pseudohyperbolic disc $D(z,r)$ centered at z is a Euclidean disc contained in $\mathbf{D}$ with radius

$$R = r\frac{1-|z|^2}{1-r^2|z|^2}$$

which goes to 0 as $|z| \to 1$. Thus if $f \in \mathrm{VMO}(\mathbf{D})$, then for any $r \in (0,1)$ we have

$$\lim_{|z|\to 1} \frac{1}{|D(z,r)|}\int_{D(z,r)} \left| f(w) - \frac{1}{|D(z,r)|}\int_{D(z,r)} f(u)\, dA(u)\right| dA(w) = 0.$$

THEOREM 19. *If $f \in L^\infty(\mathbf{D})$, then $f \in \mathrm{VMO}(\mathbf{D}) \Rightarrow f \in \mathrm{VMO}_\partial(\mathbf{D})$.*

REMARK. The converse of Theorem 19 is obviously false. For example, if f is the characteristic function of any closed square contained in $\mathbf{D}$, then $f \in Q \cap B \subset \mathrm{VMO}_\partial(\mathbf{D})$ while $f \notin \mathrm{VMO}(\mathbf{D})$. Even for bounded continuous functions f on $\mathbf{D}$, the converse of Theorem 19 does not hold. However, if $f \in H^\infty(\mathbf{D})$, then $f \in \mathrm{VMO}_\partial(\mathbf{D}) \Leftrightarrow f \in \mathrm{VMO}(\mathbf{D})$.

8. Open questions and possible generalizations. All the results in this paper are concerned with essentially bounded functions on $\mathbf{D}$. It is clear that many concepts and techniques apply to unbounded functions. First we make some definitions.

DEFINITION 1. A function $f \in L^1(\mathbf{D})$ is said to be in $\mathrm{BMO}_\partial(\mathbf{D})$ if

$$\sup_{z\in\mathbf{D}} \frac{1}{|S_z|}\int_{S_z} \left| f(w) - \frac{1}{|S_z|}\int_{S_z} f(u)dA(u)\right| dA(w) < +\infty.$$

It is obvious that $\mathrm{VMO}_\partial(\mathbf{D}) \subset \mathrm{BMO}_\partial(\mathbf{D})$.

For a function $f \in L^2(\mathbf{D}, dA)$, the Toeplitz operator T_f is an unbounded operator in general. However, we always have $k_z \in \mathcal{D}(T_f)$. Thus, the Berezin symbol $\tilde{f}$ is well defined in this case. Also $\hat{f}$ is well defined. Our first problem is to generalize Theorem 1:

Problem 1. For $f \in L^2(\mathbf{D}, dA)$, prove that the following are all equivalent:

(a) H_f and $H_{\bar{f}}$ are compact;

(b) $T_{|f|^2} - T_f T_{\bar{f}}$ and $T_{|f|^2} - T_{\bar{f}} T_f$ are compact;

(c) $f \in \mathrm{VMO}_\partial(\mathbf{D})$;

(d) $\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2 \to 0$ as $|z| \to 1$;

(e) $\widehat{|f|^2}(z) - |\hat{f}(z)|^2 \to 0$ as $|z| \to 1$.

An analogous problem is

Problem 2. For $f \in L^2(\mathbf{D}, dA)$, prove that the following are all equivalent:

(a) H_f and $H_{\bar{f}}$ are bounded;

(b) $T_{|f|^2} - T_f T_{\bar{f}}$ and $T_{|f|^2} - T_{\bar{f}} T_f$ are bounded;

(c) $f \in \mathrm{BMO}_\partial(\mathbf{D})$;

(d) $\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2$ is bounded on $\mathbf{D}$;

(e) $\widehat{|f|^2}(z) - |\hat{f}(z)|^2$ is bounded on $\mathbf{D}$.

For $f \in L^2(\mathbf{D}, dA)$, let

$$\|f\|_1 = \sup_{z\in\mathbf{D}} \frac{1}{|S_z|} \int_{S_z} \left| f(w) - \frac{1}{|S_z|} \int_{S_z} f(u)dA(u) \right| dA(w),$$

$$\|f\|_2 = \sup_{z\in\mathbf{D}} \frac{1}{|S_z|} \sqrt{\int_{S_z} \int_{S_z} |f(u) - f(w)|^2 dA(u)dA(w)},$$

$$\|f\|_3 = \sup_{z\in\mathbf{D}} \sqrt{\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2}.$$

Problem 3. Show that $\| \, \|_i$ $(i = 1, 2, 3)$ are complete norms on $\mathrm{BMO}_\partial(\mathbf{D})$ modula the constant functions and show that they are equivalent.

In the theory of BMO and VMO [**13**], Fefferman's duality theorem is one of the most important and deepest results, so it is very natural to propose:

Problem 4. Formulate and prove a duality theorem about $\mathrm{BMO}_\partial(\mathbf{D})$.

New characterizations of $\mathrm{BMO}_\partial(\mathbf{D})$ and $\mathrm{VMO}_\partial(\mathbf{D})$ are also worth further investigation.

Finally, I am very curious about the possible generalizations of the above concepts and results to general strongly pseudo-convex domains Ω in $\mathbf{C}^n$. The definitions of Berezin symbol, Q, and $\tilde{Q}$ can be carried over word by word. It seems to me that a reasonable definition of $\mathrm{BMO}_\partial(\Omega)$ and $\mathrm{VMO}_\partial(\Omega)$ as well as $\mathrm{ESV}(\Omega)$ should involve the geometry of Ω and $\partial\Omega$. A connection between geometry and operator theory is expected in the further study of this direction.

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