TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 302, Number 2, August 1987

# THE CONNECTEDNESS OF THE GROUP OF AUTOMORPHISMS OF $L^1(0,1)$

F. GHAHRAMANI

ABSTRACT. For each of the radical Banach algebras $L^1(0,1)$ and $L^1(w)$ an integral representation for the automorphisms is given. This is used to show that the groups of the automorphisms of $L^1(0,1)$ and $L^1(w)$ endowed with bounded strong operator topology (BSO) are arcwise connected. Also it is shown that if $|||\cdot|||_p$ denotes the norm of $B(L^p(0,1),\ L^1(0,1))$, $1 < p \leq \infty$, then the group of automorphisms of $L^1(0,1)$ topologized by $|||\cdot|||_p$ is arcwise connected. It is shown that every automorphism $\theta$ of $L^1(0,1)$ is of the form $\theta = e^{\lambda d} \lim e^{q_n}$ (BSO), where each $q_n$ is a quasinilpotent derivation. It is shown that the group of principal automorphisms of $l^1(w)$ under operator norm topology is arcwise connected, and every automorphism has the form $e^{i\alpha d}(e^{\lambda d}e^D e^{-\lambda d})^-$, where $\alpha \in \mathbf{R}$, $\lambda > 0$, and $D$ is a derivation, and where $(e^{\lambda d}e^D e^{-\lambda d})^-$ denotes the extension by continuity of $e^{\lambda d}e^D e^{-\lambda d}$ from a dense subalgebra of $l^1(w)$ to $l^1(w)$.

**0. Introduction.** Suppose in the Banach space $L^1(0,1)$ we define the product $*$ by

$$(f * g)(x) = \int_0^x f(x-y)g(y)\,dy \qquad (f, g \in L^1(0,1),\ \text{a.e. } x \in (0,1)).$$

With this "convolution" product $L^1(0,1)$ becomes a radical Banach algebra [**12**]. In [**12**] Kamowitz and Scheinberg studied the derivations and automorphisms of $L^1(0,1)$, and asked whether the group of automorphisms is connected. We prove that this group is connected in the bounded strong operator topology (BSO) and for topologies induced by the norm of $B(L^p(0,1),\ L^1(0,1))$, where $1 < p \leq \infty$.

The class of weighted convolution algebras has been studied by several authors from different viewpoints [**1**, **3**, **4** and **5**]. Suppose $w$ is a continuous and positive function on $\mathbf{R}^+$ with $w(0) = 1$ and $w(s+t) \leq w(s)w(t)$, and let $L^1(w)$ be the Banach space of all equivalence classes of Lebesgue measurable functions $f$ with

$$\|f\| = \int_0^\infty |f(x)|\, w(x)\,dx < \infty,$$

with convolution product $*$ defined by

$$(f * g)(x) = \int_0^x f(x-y)g(y)\,dy;$$

$L^1(w)$ is a Banach algebra. We prove that the group of automorphisms of $L^1(w)$ endowed with the topology (BSO) is connected.

Received by the editors March 3, 1986 and, in revised form, June 20, 1986.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46H99; Secondary 43A20, 43A22.

*Key words and phrases.* Automorphism, derivation, connectedness, quasinilpotent derivation.

As S. Grabiner has discussed in [**5**], analogies exist between weighted discrete and nondiscrete convolution algebras. Let $w$ be a weight function (a positive function on $\mathbf{Z}^+$, satisfying $w(0) = 1$ and $w(m+n) \leq w(m)w(n)$), and let $l^1(w)$ be the class of all sequences $x = (x(n))$ with $\|x\| = \sum_{n=0}^{\infty} |x(n)|w(n) < \infty$, with product as convolution of sequences. We show here that the group of principal automorphisms of $l^1(w)$ under the operator norm topology is connected, and every automorphism is of the form $e^{i\alpha d}(e^{\lambda d}e^{D}e^{-\lambda d})$. S. Grabiner has made an extensive study of the automorphisms and derivations of the formal power series [**6**]. For some classes of weight functions $w$, the connectedness of the group of principal automorphisms of $l^1(w)$ could already be observed from Grabiner's work on formal power series.

We will use the fact that the derivations and automorphisms on the algebras $L^1(0,1)$ and $L^1(w)$ are continuous [**10**, Remark 3a].

**1. The automorphisms of** $L^1(0,1)$. We employ the techniques of [**3** and **4**]; every automorphism $\theta$ on $L^1(0,1)$ has an extension to an automorphism $\bar{\theta}$ of the multiplier algebra of $L^1(0,1)$. The multiplier algebra can be identified with the algebra $M[0,1)$ of all complex regular Borel measures on $[0,1)$ [**12**, Remark 10]. To define the product of two measures in $M[0,1)$, let $C_0[0,1)$ be the Banach space of all continuous functions $f$ on $[0,1)$ with $\lim_{x\to 1^-} f(x) = 0$, and with $\|f\| = \sup|f(x)|$. Then we have $(C_0[0,1))^* = M[0,1)$, and the product $*$ can be defined by

$$\int_{[0,1)} \psi(x)d(u*\nu)(x) = \int_{[0,1)}\int_{[0,1-y)} \psi(x+y)d\mu(x)d\nu(y) \qquad (\psi \in C_0[0,1)).$$

The mapping $f \mapsto f dx$ is an isometric embedding of $L^1(0,1)$ into $M[0,1)$ and maps $L^1(0,1)$ onto an ideal of $M[0,1)$.

In $M[0,1)$ we consider three topologies other than the norm topology: (I) The topology $\sigma = \sigma(M[0,1), C_0[0,1))$. (II) The topology (so); a net $(\mu_\alpha)$ tends to a measure $\mu$ in (so) if $\mu_\alpha * f \xrightarrow{\|\cdot\|} \mu * f$ for every $f \in L^1(0,1)$. (III) The topology (bso); a net $(\mu_\alpha)$ tends to $\mu$ in (bso) if $\mu_\alpha \xrightarrow{(\text{so})} \mu$ and is bounded.

The proofs of Lemmas 1.1 and 1.2 below follow, respectively, the same lines as the proofs of Lemma 1.1.1 and Lemma 1.1.3 in [**7**], and are therefore omitted.

LEMMA 1.1. *The topology* (bso) *is stronger than the topology* $\sigma$.

LEMMA 1.2. *The closed unit ball of* $M[0,1)$ *is equal to the closed convex hull of the set* $\{\lambda\delta_x : |\lambda| = 1,\ x \in [0,1)\}$, *with the closure taken in the* (so) *topology.*

PROPOSITION 1.1. *Suppose* $\theta$ *is an automorphism of* $L^1(0,1)$. *Then the formula* $\bar{\theta}(\mu)(f) = \theta(\mu * \theta^{-1}(f))$ $(\mu \in M[0,1),\ f \in L^1(0,1))$ *defines a continuous automorphism* $\bar{\theta}\colon M[0,1) \to M[0,1)$ *which extends* $\theta$.

PROOF. The automorphisms of $L^1(0,1)$ extend directly by conjugation to automorphisms of the multiplier algebra of $L^1(0,1)$ [**11**, §8]. The identification of $M[0,1)$ with the multiplier algebra of $L^1(0,1)$ gives the result in a standard manner.

In the following lemma, for $\mu \neq 0, \alpha(\mu)$ denotes the infimum of the support of $\mu$. It follows from Titchmarsh's convolution theorem that if $\mu \neq 0$, $\nu \neq 0$, and $\alpha(\mu) + \alpha(\nu) < 1$, then $\alpha(\mu * \nu) = \alpha(\mu) + \alpha(\nu)$.

LEMMA 1.3. *Suppose $\theta$ is an automorphism of $L^1(0,1)$ and $\bar{\theta}$ is its extension as described in Proposition* 1.1. *Then for every $\mu \neq 0$,*

$$\alpha(\bar{\theta}(\mu)) = \alpha(\mu).$$

PROOF. Let the function $\beta\colon [0,1) \to [0,1)$ be defined by $\beta(x) = \alpha(\bar{\theta}(\delta_x))$. Then for every $x, y \in [0,1)$, if $x + y < 1$, then

$$\begin{aligned} \beta(x+y) &= \alpha(\bar{\theta}(\delta_{x+y})) = \alpha(\bar{\theta}(\delta_x) * \bar{\theta}(\delta_y)) \\ &= \alpha(\bar{\theta}(\delta_x)) + \alpha(\bar{\theta}(\delta_y)) = \beta(x) + \beta(y). \end{aligned} \tag{1}$$

Next we prove that $\beta$ is continuous from the right at every $x \in [0,1)$. It suffices to do this for $x = 0$. Let $x_n > 0$ and $x_n \to 0$. Then $\delta_{x_n} \xrightarrow{\text{(bso)}} \delta_0$. Since $\bar{\theta}$ is an automorphism we have

$$\bar{\theta}(\delta_{x_n}) \xrightarrow{\text{(bso)}} \bar{\theta}(\delta_0) = \delta_0, \tag{2}$$

whence

$$\bar{\theta}(\delta_{x_n}) \xrightarrow{\sigma} \delta_0, \tag{3}$$

by Lemma 1.1. This implies $\beta(x_n) = \alpha(\bar{\theta}(\delta_{x_n})) \to 0$, for otherwise there exists a positive number $b < 1$ such that for infinitely many values of $n$, $\alpha(\bar{\theta}(\delta_{x_n})) > b$. Then if $f$ is a continuous function whose support is contained in $[0,b)$ with $f(0) = 1$ we get $\int_{[0,1)} f(t)\,d\bar{\theta}(\delta_{x_n})(t) = 0$ for infinitely many values of $n$, while $\int_{[0,1)} f(t)\,d\delta_0(t) = f(0) = 1$, and this contradicts (3). Hence $\beta$ is continuous from the right, whence there exists $A_\theta \geq 0$ such that $\alpha(\bar{\theta}(\delta_x)) = \beta(x) = A_\theta x$ for every $x \in [0,1)$.

Next we prove that $A_\theta > 0$. If $A_\theta = 0$, then $\alpha(\bar{\theta}(\delta_x)) = 0$ for every $x \in [0,1)$. We prove that this implies $\alpha(\theta(f)) = 0$ for every $f \in L^1(0,1)$. Let $f \in L^1(0,1)$ with $\alpha(f) = 0$. Then $((L^1(0,1))*f)^-$ is a closed ideal, hence $(L^1(0,1)*\theta(f))^- = L^1(0,1)$ [**2**, Example 3], whence $\alpha(\theta(f)) = 0$. On the other hand, if $\alpha(f) = a > 0$, then there exists $g \in L^1(0,1)$, with $\alpha(g) = 0$ and $f = \delta_a * g$. Thus $\alpha(\theta(g)) = 0$, whence

$$\alpha(\theta(f)) = \alpha(\theta(g)) + \alpha(\bar{\theta}(\delta_a)) = 0. \tag{4}$$

Then, for every $f \in L^1(0,1)$, $\alpha(\theta(f)) = 0$, a contradiction. Thus $A_\theta > 0$.

Now suppose $\alpha(f) = a > 0$. Then, as above, we consider $g$ such that $f = g * \delta_a$, with $\alpha(g) = 0$. We have $\theta(f) = \theta(g * \delta_a) = \theta(g) * \bar{\theta}(\delta_a)$. Thus,

$$\alpha(\theta(f)) = \alpha(\theta(g)) + \alpha(\bar{\theta}(\delta_a)) = 0 + A_\theta a = A_\theta \alpha(f). \tag{5}$$

For a measure $\mu \in (M[0,1)\backslash\{0\})$, let $f \in (L^1(0,1)\backslash\{0\})$ be such that $\mu * f \neq 0$. Then

$$\alpha(\bar{\theta}(\mu)) + \alpha(\theta(f)) = \alpha(\theta(\mu * f)) = A_\theta \alpha(u * f) = A_\theta \alpha(\mu) + A_\theta \alpha(f). \tag{6}$$

Canceling $\alpha(\theta(f)) = A_\theta \alpha(f)$ from two sides of (6) we obtain $\alpha(\bar{\theta}(\mu)) = A_\theta \alpha(\mu)$.

Since $\beta$ is a function from $[0,1)$ into $[0,1)$ we have $A_\theta \leq 1$. To show that $A_\theta = 1$, let $A_{\theta^{-1}}$ be such that $\alpha(\bar{\theta}^{-1}(\delta_x)) = A_{\theta^{-1}} x$ $(x \in [0,1))$. Then

$$\alpha(\bar{\theta}\bar{\theta}^{-1}(\delta_x)) = \alpha(\delta_x) = x,$$

which, when compared to $\alpha(\bar{\theta}(\bar{\theta})^{-1}(\delta_x)) = A_\theta \alpha((\bar{\theta})^{-1}(\delta_x)) = A_\theta A_{\theta^{-1}} x$, implies $A_{\theta^{-1}} = 1/A_\theta$. Then, from $A_{\theta^{-1}} \leq 1$, it follows $A_\theta \geq 1$, and the proof is complete.

LEMMA 1.4. *Suppose $\mu, \nu \in (M[0,1)\backslash\{0\})$ with $\alpha(\mu) + \alpha(\nu) < 1$. Then $\mu * \nu$ has a nonzero mass at $\alpha(\mu * \nu)$ if and only if $\mu$ has a nonzero mass at $\alpha(\mu)$ and $\nu$ has a nonzero mass at $\alpha(\nu)$.*

PROOF. We have $(\mu * \nu)(\{\alpha(\mu * \nu)\}) = \mu(\{\alpha(\mu)\})\nu(\{\alpha(\nu)\})$, and the lemma is proved.

LEMMA 1.5. *Suppose $\theta$ is an automorphism of $L^1(0,1)$ and $\bar{\theta}$ is its extension as described in Proposition* 1.1. *Then for every $x \in [0,1)$, $\bar{\theta}(\delta_x)$ has a nonzero mass at $\alpha(\bar{\theta}(\delta_x))$.*

PROOF. Let $x \in [0,1)$ and suppose that $\bar{\theta}(\delta_x)$ has zero mass at $\alpha(\bar{\theta}(\delta_x))$. Then, we first prove that $\bar{\theta}(\delta_y)$ has zero mass at $\alpha(\bar{\theta}(\delta_y))$ $(0 \leq y < 1)$. If $y > x$, then $\bar{\theta}(\delta_y) = \bar{\theta}(\delta_y) * \bar{\theta}(\delta_{y-x})$, whence by Lemma 1.4, $\bar{\theta}(\delta_y)$ has zero mass at $y$. On the other hand, if $0 < y < x$, let $n$ be a positive integer such that $x/n < y$. Then, by Lemma 1.4, $\bar{\theta}(\delta_{x/n})$ has zero mass at $x/n$, and since $\bar{\theta}(\delta_y) = \bar{\theta}(\delta_{x/n}) * \bar{\theta}(\delta_{y-x/n})$ another application of Lemma 1.4 implies that $\bar{\theta}(\delta_y)$ has zero mass at $y$. Next we prove that this implies that if $\mu \in M[0,1)$ and $\alpha(\mu) = a > 0$, then $\bar{\theta}(\mu)$ has zero mass at $\alpha(\bar{\theta}(\mu)) = a$. There exists $\nu \in M[0,1)$ such that $\mu = \delta_a * \nu$, whence $\bar{\theta}(\mu) = \bar{\theta}(\delta_a) * \bar{\theta}(\nu)$, and the discussion in the above paragraph together with Lemma 1.4 implies that $\bar{\theta}(\mu)$ has zero mass at $\alpha(\bar{\theta}(\mu))$. Now $\alpha((\bar{\theta})^{-1}(\delta_{1/2})) = 1/2$. Hence $\delta_{1/2} = \bar{\theta}((\bar{\theta})^{-1}(\delta_{1/2}))$ has zero mass at $\alpha(\bar{\theta}((\bar{\theta})^{-1}(\delta_{1/2}))) = 1/2$. From this contradiction we conclude that $\bar{\theta}(\delta_x)$ has a nonzero mass at $x$ for every $x \in [0,1)$.

REMARK 1.1. Suppose $\theta$ is an automorphism of $L^1(0,1)$ and $\bar{\theta}$ is its extension. Then for every $x \in [0,1)$, by Lemma 1.5, we have $\bar{\theta}(\delta_x) = k(x)\delta_x + \mu_x$, where $k(x) \neq 0$ is the mass of $\bar{\theta}(\delta_x)$ at $x$, $\alpha(\mu_x) \geq x$ and $\mu_x$ has zero mass at $x$.

LEMMA 1.6. *Suppose $k(x)$ is as described in Remark* 1.1. *Then there exists a complex number $z$ such that $k(x) = e^{zx}$ $(x \in [0,1))$.*

PROOF. If $x$ and $y$ are in $[0,1)$ with $x+y < 1$, then from $\bar{\theta}(\delta_{x+y}) = \bar{\theta}(\delta_x) * \bar{\theta}(\delta_y)$ it follows that

$$(1) \quad k(x+y)\delta_{x+y} + \mu_{x+y} = k(x)k(y)\delta_{x+y} + k(x)\delta_x * \mu_y + k(y)\delta_y * \mu_x + \mu_x * \mu_y.$$

By Remark 1.1 and Lemma 1.4 the measures $k(x)\delta_x * \mu_y + k(y)\delta_y * \mu_x + \mu_x * \mu_y$ and $\mu_{x+y}$ have zero mass at $x+y$. Thus, on equating the coefficients of $\delta_{x+y}$ on both sides of (1) we obtain

$$(2) \qquad k(x+y) = k(x)k(y) \qquad (0 \leq x+y < 1).$$

Thus if we prove that $x \mapsto k(x)$ is Lebesgue measurable on $[0,1)$, the lemma will be proved.

Since $\bar{\theta}$ is an automorphism it is continuous from $(M[0,1),(\mathrm{bso}))$ into $(M[0,1),(\mathrm{bso}))$; for, if $\mu_\alpha \xrightarrow{(\mathrm{bso})} \mu$, then $\mu_\alpha * \theta^{-1}(f) \xrightarrow{\|\cdot\|} \mu * \theta^{-1}(f)$ for every $f \in L^1(0,1)$. Hence

$$(3) \qquad \bar{\theta}(\mu_\alpha) * f = \theta(\mu_\alpha * \theta^{-1}(f)) \xrightarrow{\|\cdot\|} \theta(\mu * \theta^{-1}(f)) = \theta(\mu) * f.$$

Thus, by Lemma 1.1, $\bar{\theta}$ is continuous from $(M[0,1),(\mathrm{bso}))$ into $(M[0,1),\sigma)$. From this and the fact that $x \mapsto \delta_x$ is continuous from $[0,1)$ into $(M[0,1),(\mathrm{bso}))$ it follows that $x \mapsto \bar{\theta}(\delta_x)$ is continous from $[0,1)$ into $M([0,1),\sigma)$, by Lemma 1.1. Hence for every $\psi \in C_0[0,1)$ the function

$$x \mapsto \langle k(x)\delta_x + \mu_x, \psi\rangle$$

is continuous on $[0,1)$.

Let $f_n$ be a continous function with support in $(-1/n, 1/n)$, $0 \le f_n \le 1$, and $f_n(0) = 1$. Let $f_{n,x}$ be $f_n$ shifted $x$ to the right so that $f_{n,x}(x) = 1$ for all $x \in (0,1)$. Regard $f_{n,x}$ as a function on $[0,1)$ by defining it to be 0 wherever it is not defined. From the decomposition $\bar{\theta}(\delta_x) = k(x)\delta_x + \mu_x$, it follows that $\lim_{n\to\infty}\langle\bar{\theta}(\delta_x), f_{n,x}\rangle = k(x)$ for $x \in (0,1)$, so to prove that $k(x)$ is measurable on $(0,1)$ it is sufficient to show that $x \mapsto \langle\bar{\theta}(\delta_x), f_{n,x}\rangle$ is continuous on $(1/n, 1-1/n)$, for $n$ large enough. Now $x \mapsto \bar{\theta}(\delta_x)$ is $\sigma$-continous and bounded, and $x \mapsto f_{n,x}$ into $C_0(0,1)$ is continuous, so $x \mapsto \langle\bar{\theta}(\delta_x), f_{n,x}\rangle$ is continuous.

THEOREM 1.1. *Suppose $\theta$ is an automorphism of $L^1(0,1)$, and suppose its extension $\bar{\theta}$ to $M[0,1)$ satisfies*

$$\bar{\theta}(\delta_x) = e^{zx}\delta_x + \mu_x \qquad (x \in [0,1)),$$

*with $\mu_x(\{x\}) = 0$ and $\alpha(\mu_x) \ge x$. Then, for every $f \in L^1(0,1)$, we have*

$$\theta(f) = e^{zd}f + \int_0^1 f(x)\mu_x\,dx,$$

*where $e^{zd}f(x) = e^{zx}f(x)$ ($f \in L^1(0,1)$, a.e. $x \in (0,1)$), and the integral is an* (so) *Bochner integral.*

PROOF. The maps $x \mapsto \delta_x$ and $x \mapsto \bar{\theta}(\delta_x)$ are continuous from $[0,1)$ into $(M[0,1),(\text{bso}))$. Hence $x \mapsto \mu_x$ is continuous from $[0,1)$ into $(M[0,1),(\text{bso}))$. If $f \in L^1(0,1)$, then

$$f = \int_0^1 f(x)\delta_x\,dx, \tag{1}$$

where the integral is an (so) Bochner integral whose existence follows from [**9**, Theorem 3.8.2], and the equality in (1) can be proved by noting that (1) is equivalent to

$$f * g = \int_0^1 f(x)\delta_x * g\,dx \qquad (g \in L^1(0,1)), \tag{2}$$

and then (2) can be shown to hold by applying both sides of it to an arbitrary $\psi \in C_0[0,1)$.

Since $e^{zd}$ is an automorphism, $A = e^{-zd}\theta$ is an automorphism too and we have

$$\overline{A}(\delta_x) = \delta_x + e^{-zd}\mu_x \qquad (x \in [0,1)). \tag{3}$$

Applying $\overline{A}$ to both sides of (1) and using (3) we obtain

$$\begin{aligned} Af &= \int_0^1 f(x)\overline{A}\delta_x\,dx = \int_0^1 f(x)(\delta_x + e^{-zd}\mu_x)\,dx \\ &= \int_0^1 f(x)\delta_x\,dx + \int_0^1 f(x)e^{-zd}\mu_x\,dx = f + \int_0^1 f(x)(e^{-zd}\mu_x)\,dx. \end{aligned} \tag{4}$$

Then, if we apply $e^{zd}$ to the two sides of (4) we obtain,

$$\theta(f) = e^{zd} f + \int_0^1 f(x)\mu_x \, dx \qquad (f \in L^1(0,1)),$$

and the theorem is proved.

In the following theorem we assume that (SO) is the strong operator topology on the group of automorphisms of $L^1(0,1); \theta_\alpha \xrightarrow{\text{(SO)}} \theta$ if and only if $\theta_\alpha(f) \to \theta(f)$ in the norm topology of $B(L^1(0,1))$ for every $f \in L^1(0,1)$; and (BSO) is the bounded strong operator topology; a net $(\theta_\alpha)$ of automorphisms tends to an automorphism $\theta$(BSO) if and only if $(\theta_\alpha)$ tends to $\theta$(SO) and is bounded.

THEOREM 1.2. *The group of automorphisms of $L^1(0,1)$ endowed with the topology* (BSO) *is connected.*

PROOF. We show that every automorphism $\theta$ can be connected to $I$ by an arc. Suppose $\bar{\theta}$ is the extension of $\theta$ to an automorphism on $M[0,1)$. Then by Lemma 1.6 there exists a complex number $z$ such that

$$\bar{\theta}(\delta_x) = e^{zx}\delta_x + \mu_x \qquad (x \in [0,1)).$$

The map $t \mapsto e^{-tzd}\theta$ $(0 \le t \le 1)$ is continuous from $[0,1]$ into the group of automorphisms endowed with (BSO), and connects $\theta$ to $e^{-zd}\theta$. Now $e^{-zd}\bar{\theta}(\delta_x) = \delta_x + e^{-zx}\mu_x$. Thus we lose no generality if we assume that

$$\bar{\theta}(\delta_x) = \delta_x + \mu_x.$$

We now consider the map

$$\Phi(t) = \begin{cases} e^{-td/(1-t)}\theta e^{td/(1-t)}, & 0 \le t < 1, \\ I, & t = 1, \end{cases}$$

and we note that $\Phi(0) = \theta$, so it suffices to show that $\Phi$ is (BSO) continuous.

Suppose that $0 \le t_0 < 1$. If $f \in L^1(0,1)$, then

$$\left\| (e^{-td/(1-t)} - e^{-t_0 d/(1-t_0)})(f) \right\| = \int_0^1 |e^{-tx/(1-t)} - e^{-t_0 x/(1-t_0)}| \, |f(x)| \, dx \to 0,$$

as $t \to t_0$, by Lebesgue's dominated convergence theorem. Similarly

$$\|(e^{td/(1-t)} - e^{t_0 d/(1-t_0)})(f)\| \to 0 \quad \text{as } t \to t_0.$$

Thus $\Phi$ is (BSO) continuous on $[0,1)$.

To show that $\Phi(t) \xrightarrow{\text{(BSO)}} I$, as $t \to 1^-$, first we note that

$$\begin{aligned} \overline{\Phi(t)}(\delta_x) &= e^{-td/(1-t)}\bar{\theta}(e^{tx/(1-t)}\delta_x) = e^{tx/(1-t)}e^{-td/(1-t)}(\delta_x + \mu_x) \\ &= \delta_x + e^{tx/(1-t)}e^{-td/(1-t)}\mu_x. \end{aligned} \tag{1}$$

Also

$$\left\| e^{tx/(1-t)}e^{-td/(1-t)}\mu_x \right\| = \int_{(x,1)} e^{-t(y-x)/(1-t)} d\,|\mu_x|\,(y) \to 0, \tag{2}$$

as $t \to 1^-$, by Lebesgue's dominated convergence theorem. Then from (1), by Theorem 1.1, we have

$$\Phi(t)(f) = f + \int_0^1 f(x)e^{tx/(1-t)}e^{-td/(1-t)}\mu_x \, dx \qquad (f \in L^1(0,1)). \tag{3}$$

Hence

$$(4) \qquad \|(\Phi(t)-I)(f)\| = \left\| \int_0^1 f(x) e^{tx/(1-t)} e^{-td/(1-t)} \mu_x \, dx \right\| \leq \int_0^1 |f(x)| \left\| e^{tx/(1-t)} e^{-td/(1-t)} \mu_x \right\| dx.$$

Now from equation (2) it can be seen that

$$\left\| e^{tx/(1-t)} e^{-td/(1-t)} \mu_x \right\| \leq \int_{(x,1)} d|\mu_x| = \|\mu_x\| \leq \|\bar{\theta}(\delta_x)\| \leq \|\bar{\theta}\|.$$

Hence an application of Lebesgue's dominated convergence theorem and (4) imply

$$\|(\Phi(t)-I)f\| \to 0, \qquad \text{as } t \to 1^-.$$

To complete the proof we show that $\Phi(t)$ remains bounded as $t \to 1^-$. By Lemma 1.8 and equation (1) for $0 \leq t \leq 1$ we have

$$\begin{aligned} \|\Phi(t)\| &\leq \left\| \overline{\Phi(t)} \right\| = \sup\left\{ \left\| \overline{\Phi(t)}(\delta_x) \right\| : x \in [0,1) \right\} \\ &= \sup\left\{ 1 + \int_{(x,1)} e^{-(y-x)t/(1-t)} d|\mu_x|(t) \colon x \in [0,1) \right\} \\ &\leq \sup\{1 + \|\mu_x\| \colon x \in [0,1)\} = \sup\left\{ \|\bar{\theta}(\delta_x)\| \colon x \in [0,1) \right\} = \|\bar{\theta}\|, \end{aligned}$$

and the proof is complete.

Next we prove the connectedness of the group of automorphisms, when it is topologized by the following metrics: for each $1 < p \leq \infty$, let $|||\cdot|||_p$ be defined on $B(L^1(0,1))$ (i.e., the space of bounded linear operators on $L^1(0,1)$) by

$$|||T|||_p = \sup\{\|Tf\|_1/\|f\|_p \colon f \in L^p(0,1), f \neq 0\}.$$

Since $L^p(0,1)$ is dense in $L^1(0,1)$ it can be verified that $|||\cdot|||_p$ is in fact a norm.

THEOREM 1.3. *The group of automorphisms of $L^1(0,1)$, with the topology induced by $|||\cdot|||_p$ $(1 < p \leq \infty)$ is arcwise connected.*

PROOF. Suppose $\theta$ is an automorphism of $L^1(0,1)$, and let $1 < p \leq \infty$ be fixed. As in the proof of Theorem 1.2 we can assume that

$$\bar{\theta}(\delta_x) = \delta_x + \mu_x, \qquad x \in [0,1),$$

and we define

$$\varphi(t) = \begin{cases} e^{-td/(1-t)} \theta e^{td/(1-t)}, & 0 \leq t < 1, \\ I, & t = 1. \end{cases}$$

As in Theorem 1.2 it can be shown that $\varphi(t)$ is continuous for $0 \leq t < 1$. Thus, it suffices to prove that $\lim_{t \to 1^-} \varphi(t) = I$. By equation (3) of Theorem 1.2 we have

(1)

$$\left( e^{-td/(1-t)} \theta e^{td/(1-t)} \right)(f) = f + \int_0^1 f(x) e^{tx/(1-t)} e^{-td/(1-t)} \mu_x dx \quad (f \in L^1(0,1)).$$

Now suppose that $f \in L^p(0,1)$. Then from equation (1) it follows that

(2)
$$\left\|\left(e^{-td/(1-t)}\theta e^{td/(1-t)} - I\right) f\right\|_1 \leq \int_0^1 |f(x)| \|e^{tx/(1-t)} e^{-td/(1-t)} \mu_x\| \, dx$$
$$\leq \|f\|_p \left(\int_0^1 \|e^{tx/(1-t)} e^{-td/(1-t)} \mu_x\|^q \, dx\right)^{1/q},$$

where $q$ is the conjugate exponent of $p$. Thus

$$(3) \qquad |||e^{-td/(1-t)}\theta e^{td/(1-t)} - I|||_p \leq \left(\int_0^1 \|e^{tx/(1-t)} e^{-td/(1-t)} \mu_x\|^q \, dx\right)^{1/q}.$$

Since $\|e^{tx/(1-t)} e^{-td/(1-t)} \mu_x\| \leq \|(e^{-td/(1-t)}\theta e^{td/(1-t)})(\delta_x)\| \leq \|\bar{\theta}\|$, an application of Lebesgue's dominated convergence theorem together with equation (3) yield

$$\lim_{t \to 1^-} |||e^{-td/(1-t)}\theta e^{td/(1-t)} - I|||_p = 0,$$

and continuity of $\varphi$ follows.

THEOREM 1.4. *Every automorphism $\theta$ of $L^1(0,1)$ is of the form*

$$e^{\lambda d} \lim_n e^{q_n} (\text{BSO}),$$

*where $\lambda$ is a constant, $d$ is the derivation $df(x) = xf(x)$ and each $q_n$ is a quasinilpotent derivation.*

PROOF. By Lemma 1.6, there exists a complex number $\lambda$ such that $\bar{\theta}(\delta_x) = e^{\lambda x}\delta_x + \mu_x$ $(x \in [0,1))$. Let $\theta_1 = e^{-\lambda d}\theta$. Then $\theta_1(\delta_x) = \delta_x + \nu_x$, where $\alpha(\nu_x) \geq x$, and $\nu_x$ has zero mass at $x$.

Since the connected component of the identity is a normal subgroup of the group of automorphisms [**8**, Theorem 7.1], from [**12**, Theorem 9] it follows that for each positive integer $n$, there exists a complex number $\alpha_n$ and a quasinilpotent derivation $q_n$ such that

$$(1) \qquad \theta_1^{-1} e^{nd} \theta_1 = e^{\alpha_n d} e^{q_n}.$$

First we show that $\alpha_n = n$. Equation (1) can be written as

$$(2) \qquad e^{nd}\theta_1 = \theta_1 e^{\alpha_n d} e^{q_n}.$$

Now suppose that the two sides of (2) are extended to automorphisms of $M[0,1)$. Then, if we apply the two sides of (2) to $\delta_x$ we obtain

$$e^{nd}(\delta_x + \nu_x) = \bar{\theta}_1 e^{\alpha_n d}\left(\delta_x + q_n(\delta_x) + \frac{q_n^2(\delta_x)}{2!} + \cdots\right),$$

or, equivalently,

$$(3) \qquad \begin{aligned} e^{nx}\delta_x + e^{nd}\nu_x &= \bar{\theta}_1 \left[e^{\alpha_n x}\delta_x + e^{\alpha_n d}\left(q_n\delta_x + \frac{q_n^2\delta_x}{2!} + \cdots\right)\right] \\ &= e^{\alpha_n x}\delta_x + e^{\alpha_n x}\nu_x + \bar{\theta}_1 e^{\alpha_n d}\left(q_n\delta_x + \frac{q_n^2}{2!}\delta_x + \cdots\right). \end{aligned}$$

Since $q_n$ is a quasinilpotent derivation the measure $\mu_n$, for which $q_n(f) = df * \mu_n$, has zero mass at $x = 0$, therefore, the measure

$$e^{\alpha_n d}\left(q_n\delta_x + \frac{q_n^2\delta_x}{2!} + \cdots\right)$$

has zero mass at $x$, which implies

$$\theta_1 e^{\alpha_n d}\left(q_n\delta_x + \frac{q_n^2\delta_x}{2!} + \cdots\right)$$

has zero mass at $x$. Now, if we compare the coefficients of $\delta_x$ on both sides of (3) we get $\alpha_n = n$. Then (1) can be written as

$$(e^{-nd}\theta_1^{-1}e^{nd})\theta_1 = e^{q_n}. \tag{4}$$

Let $f \in L^1(0,1)$. Then from (4) it follows that

$$(e^{-nd}\theta_1^{-1}e^{nd})(\theta_1 f) = e^{q_n} f. \tag{5}$$

Now, if in (5) we let $n \to \infty$ and use the argument in the proof of Theorem 1.2 for $\theta_1^{-1}$, we obtain

$$\theta_1 f = \lim_{n\to\infty}(e^{-nd}\theta_1^{-1}e^{nd})(\theta_1 f) = \lim_{n\to\infty} e^{q_n} f.$$

From equation (4) we obtain

$$\|e^{q_n}\| \leq \|\theta_1\|\|e^{-nd}\theta_1^{-1}e^{nd}\| \leq \|\theta_1\|\|\theta_1^{-1}\|,$$

and the proof is complete.

REMARK. From equation (4) of the proof of Theorem 1.4 it also follows $\|e^{-q_n}\| \leq \|\theta_1\|\|\theta_1^{-1}\|$. We will need this fact in the proof of next theorem.

THEOREM 1.5. *Suppose $\theta = e^{\lambda d}\lim e^{q_n}$(BSO), where $(q_n)$ is as obtained in Theorem 1.4. If $(\|q_n\|)$ has a bounded subsequence, then $\theta = e^{\lambda d}e^q$ for some quasinilpotent derivation $q$, and thus $\theta$ is in the componet of identity.*

PROOF. A continuous derivation $p$ on $L^1(0,1)$ may be extended to a continuous derivation $p$ on $M[0,1)$; for example, by defining $p(\mu) * f = p(u * f) - \mu * p(f)$ for each $\mu \in M[0,1)$ and $f \in L^1(0,1)$. Denoting the extended derivation on $M[0,1)$ by the same symbol, it follows that $q_n$ is a bounded sequence of derivations on $M[0,1)$. Because $M[0,1)$ is the dual of $C_0[0,1)$, it follows that the close unit ball of $B(M[0,1))$ is compact in the topology induced by the set of seminorms $\{|\langle T_\mu, f\rangle| \colon \mu \in M[0,1),\ f \in C_0[0,1)\}$. Thus there is a subnet of $(q_n)$ converging in this topology to an operator $q$ on $M[0,1)$. A direct calculation using the seminorms $|\langle T\mu, f\rangle|$ and the separate $\sigma$-continuity of $*$ shows that $q$ is a derivation on $M[0,1)$. If $f \in L^1(0,1)$, then there are $g, h \in L^1(0,1)$ such that $f = g * h$ (by Cohen's factorization theorem) so $q(f) = q(g) * h + g * q(h) \in L^1(0,1)$. Hence $q$ restricts to a derivation on $L^1(0,1)$.

It is sufficient to show that on $L^1(0,1)$ the derivation $q$ is the (BSO) limit of a subnet of $q_n$, because this and $\theta = e^{\lambda d}\lim e^{q_n}$(BSO) implies that $\theta = e^{\lambda d}e^q$. Let $(q_\alpha)$ be a subnet of $(q_n)$ such that $\langle q_\alpha(g), f\rangle \to \langle q(g), f\rangle$ for all $g \in L^1(0,1)$ and $f \in C_0(0,1)$. Let $C_0(0,1)$ be the set of continuous functions on $(0,1)$ vanishing at 0 and 1. Convolution products of the form $g * h$ with $g, h \in C_0(0,1)$ are dense in $L^1(0,1)$ in the $L^1$-norm. Thus it is sufficient to show that $\|q_\alpha(g*h) - q(g*h)\|_1 \to 0$

for all $g, h \in C_0(0,1)$. The derivation property $q(g*h) = q(g)*h + q(h)*g$ and $\|\cdot\|_1 \leq \|\cdot\|_\infty$ show that it is sufficient to prove that $\|(q_\alpha - q)(h) * g\|_\infty \to 0$ for all $g, h \in C_0(0,1)$. Let $\varepsilon > 0$ and assume that $\|h\|_1 \leq 1, \|q\| \leq 1$, and $\|q_\alpha\| \leq 1$ for all $\alpha$. The uniform continuity of $g$ on $(0,1)$ implies that there is a $\delta > 0$ such that $|g(x) - g(y)| < \varepsilon$ for $x, y \in (0,1)$ with $|x-y| < \delta$. Then $|(q_\alpha - q)(h)*g(x) - (q_\alpha - q)(h)*g(y)| \leq \varepsilon \cdot \|(q_\alpha - q)(h)\|_1 \leq 2\varepsilon$ provided $|x-y| < \delta$.

Choose $x_0 = 0 < x_1 < \cdots < x_{m+1} = 1$ such that $x_{j+1} - x_j < \delta$, and regard $t \mapsto g(x_j - t)$ as an element of $C_0(0,1)$ by taking it to be zero for $x_j < t \leq 1$. Choose $\alpha$ so that

$$\left| \int_0^1 (q_\alpha - q)h(t)g(x_j - t)\,dt \right| < \varepsilon$$

for $j = 1, \ldots, m$. If $x \in (x_j, x_{j+1})$, then

$$\begin{aligned}|(q_\alpha - q)(h) * g(x)| &\leq 2\varepsilon + |(q_\alpha - q)(h) * g(x_j)| \\ &= 2\varepsilon + \left| \int_0^{x_j} (q_\alpha - q)(h)(t) g(x_j - t)\,dt \right| < 3\varepsilon.\end{aligned}$$

Thus $\|(q_\alpha - q)(h) * g\|_\infty \leq 3\varepsilon$, and the proof is complete.

**2. The group of automorphisms of $L^1(w)$.** In [**3**] we gave a description of the automorphisms of a semisimple $L^1(w)$, which shows that the group of automorphism is connected. In the next theorem we assume that $L^1(w)$ is a radical Banach algebra, or equivalently, $\lim_{t\to\infty} w(t)^{1/t} = 0$.

THEOREM 2.1. *The group of automorphisms of $L^1(w)$ endowed with the topology* (BSO) *is arcwise connected.*

PROOF. Suppose that $\theta$ is an automorphims of $L^1(w)$ and let $\bar\theta$ be its extension to $M(w)$ [**4**, Proposition 1]. Then we have

$$\bar\theta(\delta_x) = e^{zx}\delta_x + \mu_x \qquad (x \in \mathbf{R}^+),$$

where $\alpha(\mu_x) \geq x$, and $\mu_x(\{x\}) = 0$ [**4**, Lemma 3 and Corollary 3] and Lemma 1.2. Since for large values of $x$

$$\|\bar\theta\| \geq \frac{\|\bar\theta(\delta_x)\|}{\|\delta_x\|} \geq \frac{\|e^{zx}\delta_x\|}{\|\delta_x\|} = |e^{zx}|,$$

we conclude that $z = i\alpha$, for some real number $\alpha$. Then we lose no generality if we assume that

$$\bar\theta(\delta_x) = \delta_x + \mu_x \qquad (x \in \mathbf{R}^+).$$

An argument similar to that of Theorem 1.1 will then show that

$$\theta(f) = f + \int_0^\infty f(t)\mu_t\,dt, \tag{1}$$

where the integral converges in the (bso) topology of $M(w)$. Now for every $0 \leq t < 1$, let $\mathcal{D}_t = \{f \in L^1(w) \colon e^{td/(1-t)}f \in L^1(w)\}$. Then $\mathcal{D}_t$ is a dense subalgebra of $L^1(w)$, since it contains all functions with compact support. We now define $\varphi_t$ from $\mathcal{D}_t$ into $L^1(w)$ by $\varphi_t = e^{-td/(1-t)}\theta e^{td/(1-t)}$. It is easily verified that each $\varphi_t$

is a homomorphism. We prove that each $\varphi_t$ is a continuous operator. By (1) for $f \in \mathcal{D}_t$ we have

$$\varphi_t(f) = f + \int_0^\infty f(x)e^{tx/(1-t)}e^{-td/(1-t)}\mu_x\,dx, \tag{2}$$

from which, as in the proof of Theorem 1.1, it follows that

$$\|\varphi_t(f)\| \le \|f\|\|\bar{\theta}\| \qquad (f \in \mathcal{D}_t),$$

so $\varphi_t$ is continuous on $\mathcal{D}_t$. Let $\bar{\varphi}_t$ be the extension of $\varphi_t$ to $L^1(w)$ by continuity. Then $\bar{\varphi}_t$ is obviously a homomorphism. We prove that $\bar{\varphi}_t$ is an automorphism. Consider $\theta^{-1}$ and let $\psi_t = e^{-td/(1-t)}\theta^{-1}e^{td/(1-t)}$ be defined on $\mathcal{D}_t$, and denote by $\bar{\psi}_t$ the extension by continuity of $\psi_t$ to $L^1(w)$. We now show that $\bar{\psi}_t\bar{\varphi}_t = \bar{\varphi}_t\bar{\psi}_t = I$. In fact, if $f \in L^1(w)$ has compact support, then

$$\bar{\psi}_t\bar{\varphi}_t(f) = \bar{\psi}_t(e^{-td/(1-t)}\theta e^{td/(1-t)})(f) \tag{1}$$

and since $e^{td/(1-t)}[e^{-td/(1-t)}\theta e^{td/(1-t)}](f) \in L^1(w)$, from the definition of $\psi_t$ it follows that

(2)

$$\bar{\psi}_t(e^{-td/(1-t)}\theta e^{td/(1-t)})(f) = e^{-td/(1-t)}\theta^{-1}e^{td/(1-t)}(e^{-td/(1-t)}\theta e^{td/(1-t)})(f) = f.$$

Thus $\bar{\psi}_t\bar{\varphi}_t = I$. It can similarly be shown that $\bar{\varphi}_t\bar{\psi}_t = I$.

It can be easily verified that $t \mapsto \bar{\varphi}_t$ for $0 \le t < 1$ is (BSO) continuous. We show that $\lim_{t\to 1^-} \bar{\varphi}_t = I$(BSO); if we use the representation

$$\bar{\varphi}_t(f) = f + \int_0^\infty f(x)e^{tx/(1-t)}e^{-td/(1-t)}\mu_x\,dx \qquad (f \in L^1(w)), \tag{3}$$

then an argument similar to that of Theorem 1.2 will show that $\lim_{t\to 1^-} \bar{\varphi}_t(f) = f$ $(f \in L^1(w))$. Also, this representation can be used to show that $\|\bar{\varphi}_t\|$ remains bounded as $t \to 1^-$, and thus the arc

$$\gamma(t) = \begin{cases} \bar{\varphi}_t, & 0 \le t < 1, \\ I, & t = 1, \end{cases}$$

is (BSO) continuous and connects $\theta$ to $I$.

REMARK. As in the case of $L^1(0,1)$, we can define for each $1 < p \le \infty$ a norm $|||\cdot|||_p$ on $B(L^1(w))$ by

$$|||T|||_p = \sup\left\{\frac{\|Tf\|_1}{\|f\|_p} : f \in L^1(w) \cap L^p(w),\ f \ne 0\right\}$$

and show that the group of automorphisms of $L^1(w)$ with the topology induced by $|||\cdot|||_p$ is connected.

**3. The group of automorphisms of $l^1(w)$.** Suppose $w$ is a radical weight on $\mathbf{Z}^+$, i.e., $\inf_{n\ge 0} w(n)^{1/n} = 0$. Then $l^1(w)$ is a local algebra. We denote $x = (x_n) \in l^1(w)$ by $x = \sum_{n=0}^\infty x(n)X^n$. Following S. Grabiner [**6**], we call $\theta$ a scalar automorphism if, for some real number $\alpha$, $\theta(X^n) = e^{i\alpha n}X^n$, $n = 1, 2, \ldots$, and we say $\theta$ is a principal automorphism if $\theta(X) = X + \sum_{n=2}^\infty x(n)X^n$. An argument similar to that of Theorem 2.1 may be employed to show that every automorphism on $l^1(w)$ factors as a product of a scalar automorphism and a principal automorphism. The group of scalar automorphisms is discrete in the operator norm topology [**6**, Theorem 7.2].

THEOREM 3.1. *For every radical weight $w$ the group of principal automorphisms of $l^1(w)$ endowed with operator norm topology is arcwise connected.*

PROOF. Suppose $\theta$ is a principal automorphism of $l^1(w)$. Then

$$\theta(X^n) = X^n + \mu_n, \qquad n = 0, 1, 2, \ldots,$$

where $\mu_n = \sum_{k=n+1}^{\infty} a_{k,n} X^k$. From this it follows that if $x = \sum_{n=0}^{\infty} x(n) X^n$ is in $l^1(w)$, then $\theta(x) = x + \sum_{n=1}^{\infty} x_n \mu_n$ (note that $\mu_0 = 0$), where the series converges in the norm topology of $l^1(w)$. Now as in Theorem 2.1, for $t \in [0,1)$ we first define a homomorphism $\varphi_t = e^{-td/(1-t)} \theta e^{td/(1-t)}$ on a dense subalgebra $\mathcal{D}_t$ of $l^1(w)$, and then extend it by continuity to an automorphism $\bar{\varphi}_t$ of $l^1(w)$. Now we show that the arc

$$\gamma(t) = \begin{cases} \bar{\varphi}_t, & 0 \le t < 1, \\ I, & t = 1, \end{cases}$$

is continuous. It is easy to show that $t \mapsto \bar{\varphi}_t$ $(0 \le t < 1)$ is continuous. We show that $\lim_{t \to 1^-} \bar{\varphi}_t = I$. We have

$$\begin{aligned}
\|\bar{\varphi}_t - I\| &= \sup_n \left\| (\bar{\varphi}_t - I) \left( \frac{1}{w(n)} X^n \right) \right\| \\
&= \sup_{n \ge 1} \left\| \frac{e^{tn/(1-t)}}{w(n)} \sum_{k=n+1} a_{k,n} e^{-tk/(1-t)} X^k \right\| \\
&\le \sup_{n \ge 1} \frac{e^{tn/(1-t)}}{w(n)} \sum_{k=n+1} e^{-tk/(1-t)} |a_{k,n}| w(k) \\
&\le e^{-t/(1-t)} \sup_{n \ge 1} \frac{1}{w(n)} \sum_{k=n+1} |a_{k,n}| w(k) \\
&= e^{-t/(1-t)} \sup_{n \ge 1} \left\| \frac{\mu_n}{w(n)} \right\| \le e^{-t/(1-t)} \sup_{n \ge 1} \left\| \theta \left( \frac{X^n}{w(n)} \right) \right\| \\
&\le e^{-t/(1-t)} \|\theta\| \to 0, \qquad \text{as } t \to 1^-,
\end{aligned}$$

and the proof is complete.

THEOREM 3.2. *Suppose $\theta$ is an automorphism of $l^1(w)$. Then there exist real numbers $\alpha, \lambda > 0$ and a derivation $D$ on $l^1(w)$ such that $\theta = e^{i\alpha d}(e^{\lambda d} e^D e^{-\lambda d})^-$, where $(e^{\lambda d} e^D e^{-\lambda d})^-$ is the extension of $e^{\lambda d} e^D e^{-\lambda d}$ from a dense subalgebra of $l^1(w)$ to $l^1(w)$.*

PROOF. As in the proof of Lemma 1.6 and Remark 1.1 we have

$$\theta(X^n) = e^{i\alpha n} X^n + \mu_n, \qquad n = 0, 1, 2, \ldots,$$

for some real number $\alpha$. Let $\theta_1 = e^{-i\alpha d}\theta$. Then

$$\theta_1(X^n) = X^n + \nu_n, \qquad n = 0, 1, 2, \ldots,$$

where $\nu_n$ is of the form $\sum_{k=n+1} a_{k,n} X^k$. For each $\lambda > 0$, now consider $e^{-\lambda d}\theta_1 e^{\lambda d}$ defined on a dense subalgebra $\mathcal{D}_\lambda$ of $l^1(w)$, and let $(e^{-\lambda d}\theta_1 e^{\lambda d})^-$ be the extension by continuity of $e^{-\lambda d}\theta_1 e^{\lambda d}$ to an automorphism of $l^1(w)$. The proof of Theorem 3.1 then shows that if $\lambda$ is large enough $\|(e^{-\lambda d}\theta_1 e^{\lambda d})^- - I\| < 1$, whence, for some derivation $D$ on $l^1(w)$, $(e^{-\lambda d}\theta_1 e^{\lambda d})^- = e^D$. On $\mathcal{D}_\lambda$, then, we have $e^{-\lambda d}\theta_1 e^{\lambda d} = e^D$, from which follows $\theta_1 = (e^{\lambda d} e^D e^{-\lambda d})^-$, and the proof is complete.

Now we obtain an alternative proof for a result due to Scheinberg [**13**].

COROLLARY 3.1 (SCHEINBERG). *The algebra $l^1(w)$ has a nonscalar automorphism if and only if there exists a positive integer $k$ such that* $\sup_n nw(n+k)/w(n) < \infty$.

PROOF. If this conditions holds, then $l^1(w)$ has a nonzero derivation $D$ defined by $D(x) = (dx)X^k$ $(x \in l^1(w))$, and then $e^D$ is a nonscalar automorphism of $l^1(w)$. If, on the other hand, $l^1(w)$ has a nonscalar automorphism $\theta$, then the automorphism $\theta_1$ of the proof of Theorem 3.2 gives rise to a nonzero derivation $D$ and this in turn implies that $\sup_n nw(n+k)/w(n) < \infty$, for some positive integer $k$ [**6**, Theorem 4.9] and [**12**], and the proof is complete.

ACKNOWLEDGMENTS. The author is grateful to the referee for many interesting suggestions; the whole proof of Theorem 1.5, which is much shorter than the author's proof, is due to the referee. The author would like to thank Professor Sandy Grabiner for pointing out an error in the earlier version of the paper. Thanks are also due to the Department of Mathematics at the University of Alberta, especially Professor Anthony T. Lau, for their hospitality.

## REFERENCES

1. H. G. Dales, *Convolution algebras on the real line*, Radical Banach algebras and automatic continuity, Proc. Long Beach 1981 (J. M. Bachar et al., eds.), Springer-Verlag, Berlin and New York, 1983.
2. W. F. Donoghue, Jr., *The lattice of invariant subspaces of a completely continuous quasi-nilpotent transformation*, Pacific J. Math. **7** (1957), 1031–1035.
3. F. Ghahramani, *Homomorphisms and derivations on weighted convolution algebras*, J. London Math. Soc. (2) **21** (1980), 149–161.
4. ——, *Isomorphisms between radical weighted convolution algebras*, Proc. Edinburgh Math. Soc. **26** (1983), 343–351.
5. S. Grabiner, *Weighted convolution algebras as analogues of Banach algebras of power series*, Radical Banach Algebras and Automatic Continuity, Proc. Long Beach 1981 (J. M. Bachar et al. eds.), Springer-Verlag, Berlin and New York, 1983.
6. ——, *Derivations and automorphisms of Banach algebras of power series*, Mem. Amer. Math. Soc. No. 146, 1974.
7. F. P. Greenleaf, *Normdecreasing homomorphisms of group algebras*, Pacific J. Math. **15** (1965), 1187–1219.
8. E. Hewitt and K. A. Ross, *Abstract harmonic analysis.* I, Springer-Verlag, Berlin, Heidelberg, New York, 1963.
9. E. Hille and R. S. Philips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ., vol. 31, rev. ed., Providence, R.I., 1957.
10. N. P. Jewell and A. M. Sinclair, *Epimorphisms and derivations on $L^1(0,1)$ are continuous*, Bull. London Math. Soc. **8** (1976), 135–139.
11. B. E. Johnson, *An introduction to the theory of centralizers*, Proc. London Math. Soc. **14** (1964), 299–320.
12. H. Kamomitz and S. Scheinberg, *Derivations and automorphisms of $L^1(0,1)$*, Trans. Amer. Math. Soc. **135** (1969), 415–427.
13. S. Scheinberg, *Automorphims of commutative Banach algebras*, Problems in Analysis, A Symposium in Honor of Salomon Bochner (R. C. Gunning, ed.), Princeton Univ. Press, Princeton, N. J., 1970.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA, CANADA T6G 2G1