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CONVEX SUBCONES OF THE CONTINGENT CONE IN NONSMOOTH CALCULUS AND OPTIMIZATION

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ABSTRACT. The tangential approximants most useful in nonsmooth analysis and optimization are those which lie "between" the Clarke tangent cone and the Bouligand contigent cone. A study of this class of tangent cones is undertaken here. It is shown that although no convex subcone of the contingent cone has the isotonicity property of the contingent cone, there are such convex subcones which are more "accurate" approximants than the Clarke tangent cone and possess an associated subdifferential calculus that is equally strong. In addition, a large class of convex subcones of the contingent cone can replace the Clarke tangent cone in necessary optimality conditions for a nonsmooth mathematical program. However, the Clarke tangent cone plays an essential role in the hypotheses under which these calculus rules and optimality conditions are proven. Overall, the results obtained here suggest that the most complete theory of nonsmooth analysis combines a number of different tangent cones.

1. Introduction. Research in convex and nonsmooth analysis has, over the past quarter century, considerably broadened the scope of optimization theory. Indeed, optimization theory has grown during this period to encompass, successively, problems involving

(i) convex functions [**24**].

(ii) locally Lipschitzian functions [**5**, **11**].

(iii) certain classes of locally lower semicontinuous functions [**25**, **5**, **15**, **1**, **28**, **33**].

The analysis developed for stages (ii) and (iii) centers around local approximations to sets called tangent cones. A plethora of these tangential approximants have been defined (see for instance [**6**, **18**, **19**, **30**, **22**, **32**]), of which a few have proven to be particularly useful. We review below the definitions of three of them. Here and throughout the paper, E will denote a Banach space.

DEFINITION 1.1. Let $C \subset E$ and x_0 an element of the closure of C (hereafter denoted $\operatorname{cl} C$).

(a) The *contingent cone* to C at x_0 is the set

$$K_C(x_0) := \{y \in E \mid \exists t_k \downarrow 0, \exists y_k \to y, x_0 + t_k y_k \in C\}.$$

(b) The *Ursescu tangent cone* to C at x_0 is the set

$$k_C(x_0) := \{y \in E \mid \forall t_k \downarrow 0, \exists y_k \to y, x_0 + t_k y_k \in C\}.$$

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(c) The *Clarke tangent cone* to C at x_0 is the set

$$T_C(x_0) := \{y \in E \mid \forall x_k \to x_0 \text{ with } x_k \in C, \forall t_k \downarrow 0, \exists y_k \to y, x_k + t_k y_k \in C\}.$$

It follows from Definition 1.1 that each of these cones is always a closed set, and that the inclusions

$$T_C(x_0) \subset k_C(x_0) \subset K_C(x_0) \tag{1.1}$$

are true in general. The Ursescu tangent cone, perhaps the least well known of the three, has received increasing attention in recent years [**30**, **6**, **19**, **7**, **8**, **33**, **22**]. Each of these cones has an alternate definition valid in any locally convex topological vector space [**25**, **30**], but since the main results to be presented here are Banach space results, we will conduct our entire discussion in a Banach space setting.

Each of these tangent cones has strengths and weaknesses. For example, the contingent cone is *isotone* with respect to inclusion; i.e.,

$$K_C(x_0) \subset K_D(x_0) \quad \text{whenever } C \subset D.$$

A rudimentary theory of necessary optimality conditions can be built upon this property [**31**, **34**]. The Ursescu tangent cone is also isotone, but the Clarke tangent cone is not ([**31**]; see also Theorem 1.2 below).

On the other hand, the Clarke tangent cone is always a convex cone [**19**, **26**, **5**, **6**] and is thus a powerful analytical tool. The contingent and Ursescu tangent cones are not always convex, however, a fact that somewhat restricts their usefulness.

One can construct a closed, convex, isotone tangent cone by taking the closed convex hull of the contingent cone. The resulting object, called the *pseudotangent cone*, is useful in differentiable programming [**10**]; however, it is too "large" to play a corresponding role in nonsmooth optimization where convex subcones of the contingent cone become important.

In this paper, we investigate the convex cones A which satisfy the inclusions

$$T_C(x_0) \subset A_C(x_0) \subset K_C(x_0). \tag{1.2}$$

The preceding paragraphs suggest that we begin with the following question:

(Q1) Is there some "sensible" tangent cone satisfying (1.2) which is both convex and isotone?

This question has a definite negative answer, as we now demonstrate. In the statement of this theorem, we denote by $\mathcal{P}(\mathbb{R}^n)$ the *power set* of $\mathbb{R}^n$.

THEOREM 1.2. *There is no mapping* $A\colon \mathcal{P}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ *which has all of the following properties*:

(a) *A is isotone.*

(b) *A is convex.*

(c) $A(C, x_0) \subset K_C(x_0)$ *for all* $C \subset \mathbb{R}^n$ *and* $x_0 \in \operatorname{cl} C$.

(d) $A(C, x_0) \supset C$ *whenever C is a one-dimensional subspace of* $\mathbb{R}^n$ *and* $x_0 \in C$.

PROOF. Consider the subsets of $\mathbb{R}^n$ defined by $C_1\colon \mathbb{R} \times \{0\}$ and $C_2\colon = \{0\} \times \mathbb{R}$. If A has property (c), $A(C_1 \cup C_2, (0,0)) \subset C_1 \cup C_2$. On the other hand, if A has properties (d) and (a), $C_1 \cup C_2 \subset A(C_1, 0) \cup A(C_2, 0) = A(C_1 \cup C_2, (0,0))$. Thus $A(C_1 \cup C_2, (0,0)) = C_1 \cup C_2$ if A has properties (a), (c) and (d). Such an A cannot have property (b). □

REMARK 1.3. (a) One can easily find mappings A which possess three of the properties listed in Theorem 1.2. All tangent cones satisfying the first inclusion of (1.2) also satisfy (d) of Theorem 1.2.

(b) A number of other "tangent cone impossibility theorems" are collected in [**32**].

Given a negative answer to (Q1), we shift our attention to a broader question:

(Q2) Are there convex cones satisfying (1.2) which are more accurate approximants than T and possess the analytical strengths of T?

We will give a qualified affirmative answer to (Q2). Specifically, we show that two particular convex tangent cones satisfying (1.2) have an associated subdifferential calculus as extensive as that for the Clarke tangent cone. In addition, we demonstrate that large classes of convex subcones of K and k can replace T in necessary conditions for optimality in nonsmooth mathematical programming. We hasten to add, however, that these results seem to require assumptions involving the Clarke tangent cone. Our theorems and examples indicate that the Clarke tangent cone plays a special role in nonsmooth analysis.

Here is an outline of the remainder of the paper: In §2, we define and examine some basic properties of three convex tangent cones satisfying (1.2). In §3, we review a Liusternik-type theorem which enables us to prove key tangent cone inclusions. We present in §4 a sort of "algorithm" for generating subdifferential calculus formulae. This procedure, which reduces the proofs of calculus rules to the verification that a tangent cone has three specific properties, was used quite successfully in [**33**]. We apply this algorithm in §5 to establish a calculus for the directional derivatives and subgradients associated with the tangent cones discussed in §2. In §6 we apply our directional derivative calculus formulae to derive necesary optimality conditions for a nonsmooth mathematical program. These conditons sharpen, in a Banach space setting, optimality conditions given in [**25**, **33**, and **21**].

At this juncture we compile a list of notations used in this paper. For an extended-real-valued function $f\colon E \to \overline{\mathbf{1R}}$, we denote by epi f the *epigraph* of f. By the *domain* of f, we mean the set

$$\operatorname{dom} f := \{x \in E \mid f(x) < +\infty\}.$$

We say that f is *proper* if $\operatorname{dom} f$ is nonempty and f never takes on the value $-\infty$. If f is convex, $\partial f(x_0)$ will denote the subgradient of f at x_0 [**24**].

We say that a mapping $A\colon \mathcal{P}(E) \times E \to \mathcal{P}(E)$ is a *tangent cone* if $A(C, x_0)$ (which we will usually write $A_C(x_0)$) is a (possibly empty) cone for all nonempty $C \subset E$ and $x_0 \in \operatorname{cl} C$. As in Theorem 1.2, we will say that A has a certain property if $A_C(x_0)$ has that property for all nonempty $C \subset E$ and $x_0 \in \operatorname{cl} C$. For two tangent cones A and A', we say $A' \subset A$ if the inclusion $A'_C(x_0) \subset A_C(x_0)$ is true for all nonempty $C \subset E$ and $x_0 \in \operatorname{cl} C$. We will denote by $A_f(x_0)$ the set $A_{\operatorname{epi} f}(x_0, f(x_0))$.

We denote the dual space of a Banach space E by E'. For $\delta > 0$, we define

$$N_\delta(x) := \{y \in E \mid \|x - y\| < \delta\}.$$

For a nonempty set $C \subset E$, we denote the *interior* of C by $\operatorname{int} C$. By the *recession cone* of C, we mean the set

$$0^+C := \{y \in E \mid C + y \subset C\};$$

by the *polar* of C, the set

$$C^0 = \{x \in E \mid \langle x, y\rangle \le 0 \text{ for all } y \in C\};$$

and by the *indicator function* of C, we mean the function $i_C\colon E \to \overline{\mathbf{R}}$ defined by

$$i_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{else.}\end{cases}$$

We denote the *nonnegative orthant* in $\mathbf{R}^n$ by $\mathbf{R}^n_+$.

2. Some convex tangent cones. In this section we discuss three convex tangent cones which satisfy (1.2) and have recently been studied by Penot [**22**], Frankowska [**7**, **8**], and others [**9**, **20**, **29**].

To begin, we observe that one way to produce a closed convex cone is to take the recession cone of a closed cone. Let us then define, for $C \subset E$ and $x_0 \in \operatorname{cl} C$, the closed convex tangent cones

$$K_C^\infty(x_0) := \{y \in E \mid K_C(x_0) + y \subset K_C(x_0)\}, \tag{2.1}$$

$$k_C^\infty(x_0) := \{y \in E \mid k_C(x_0) + y \subset k_C(x_0)\}. \tag{2.2}$$

It follows readily from (2.1) and (2.2) that $T \subset K^\infty \subset K$ and $T \subset k^\infty \subset k$ [**19**, Theorems 1, 2]. An example of C and x_0 for which $T_C(x_0), K_C^\infty(x_0)$, and $k_C^\infty(x_0)$ are distinct is given in [**19**]. Interestingly, k^∞ is not always contained in K^∞, even though $k \subset K$. For example, define $f\colon \mathbf{R} \to \mathbf{R}$ by

$$f(x) := \begin{cases} 0 & \text{if } x = 0, \\ -2^{-(n+1)} & \text{if } 2^{-(n+1)} \le |x| < 2^{-n}, \qquad n = 0, \pm 1, \pm 2, \ldots,\end{cases}$$

and let $C := \operatorname{epi} f$ and $x_0 := (0,0)$. Here

$$K_C(x_0) = \{(x,y) \mid y \ge -|x|\},$$
$$k_C(x_0) = \{(x,y) \mid y \ge -|x|/2\},$$

so that

$$K_C^\infty(x_0) = \{(x,y) \mid y \ge |x|\},$$
$$k_C^\infty(x_0) = \{(x,y) \mid y \ge |x|/2\}.$$

As we will see presently, k^∞ is somewhat easier to work with than K^∞ is, and it has received more attention in the literature [**7**, **8**, **22**]. In particular, Frankowska has applied k^∞ in the study of a general Bolza problem in the calculus of variations [**8**].

In [**22**], Penot gives an interesting alternate definition of k^∞:

$$k_C^\infty(x_0) = \{y \mid \forall (x_n, t_n) \to (x_0, 0^+), \forall z \in k_C(x_0) \text{ with } x_n \in C, t_n^{-1}(x_n - x_0) \to z, \exists y_n \to y, x_n + t_n y_n \in C\}. \tag{2.3}$$

Equation (2.3) makes clear the fact that $T \subset k^\infty$. It also suggests the experiment of replacing "$z \in k_C(x_0)$" in (2.3) with "$z \in S$" for others sets S, and studying the resulting objects. Of course the larger the set S, the smaller the object obtained. For example, the choice $S := T_C(x_0)$ gives a cone which always contains k^∞. This cone is not necessarily convex, however. On the other hand, the choice $S := K_C(x_0)$ gives a convex tangent cone which is always contained in k^∞. When $\{x_n\} \subset C$, $\{t_n^{-1}(x_n - x_0)\}$ converges if and only if it converges to an element of $K_C(x_0)$, so

$S := E$ gives the same cone as $S := K_C(x_0)$. The resulting tangent cone is Penot's *prototangent cone*

$$(2.4)\quad P_C(x_0) := \{y \mid \forall (x_n, t_n) \to (x_0, 0^+) \text{ with } x_n \in C \text{ and } t_n^{-1}(x_n - x_0) \text{ convergent}, \exists y_n \to y,\ x_n + t_n y_n \in C\}.$$

If follows from (2.4) that $T \subset P \subset k^\infty \subset k \subset K$, with $P_C(x_0) = k_C^\infty(x_0) = K_C^\infty(x_0)$ whenever $k_C(x_0) = K_C(x_0)$.

REMARK 2.1. (a) In a forthcoming paper [**29**], Treiman defines a tangent cone in Banach space which reduces to P in the finite-dimensional case and whose polar cone, like that of T, has a useful characterization in terms of "proximal normals".

(b) In (2.3) and (2.4), one may replace "$x_n + t_n y_n \in C$" with "$x_{\sigma(n)} + t_{\sigma(n)} y_{\sigma(n)} \in C$ for some subsequence $\{x_{\sigma(n)} + t_{\sigma(n)} y_{\sigma(n)}\}$".

An analogous statement is true for T [**12**] and k.

In §§4 and 5, we will be particularly concerned with two questions for each of the tangent cones A that we have defined:

(a) Does the inclusion

$$(2.5)\qquad A_C(x_0) \times A_D(y_0) \subset A_{C\times D}(x_0, y_0)$$

hold in general?

(b) For what linear mappings $M\colon E \to E_1$ and $(C, z_0) \subset \mathcal{P}(E) \times E$ does the inclusion

$$(2.6)\qquad M(A_C(z_0)) \subset A_{M(C)}(Mz_0)$$

hold?

We give below the answer to question (a). The proofs, which are completely straightforward, are left to the reader.

PROPOSITION 2.2. *Let C and D be nonempty subsets of E and E_1, respectively, and let $x_0 \in \operatorname{cl} C$ and $y_0 \in \operatorname{cl} C$. Then*

$$(2.7)\qquad k_C(x_0) \times k_D(y_0) = k_{C\times D}(x_0, y_0).$$

$$(2.8)\qquad T_C(x_0) \times T_D(y_0) = T_{C\times D}(x_0, y_0).$$

$$(2.9)\qquad k_C^\infty(x_0) \times k_D^\infty(y_0) = k_{C\times D}^\infty(x_0, y_0).$$

$$(2.10)\qquad P_C(x_0) \times P_D(y_0) \subset P_{C\times D}(x_0, y_0),$$

$$(2.11)\qquad K_C(x_0) \times k_D(y_0) \subset K_{C\times D}(x_0, y_0).$$

It is not possible to combine K^∞ with any of the other tangent cones above to produce an analogue of (2.11), a defect of K^∞ which will limit its usefulness in the sequel. For example, define

$$C := \{x \in \mathbf{1R} \mid x = 2^{-2n}, n = 1, 2, 3, \dots\} \cup \{0\},$$
$$D := \{x \in \mathbf{1R} \mid x = 2^{-2n+1}, n = 1, 2, 3, \dots\} \cup \{0\},$$

and let $(x_0, y_0) = (0, 0)$. Then $K_C(0) = K_D(0) = \mathbf{1R}_+$, while $k_C(0) = k_D(0) = \{0\}$ and $K_{C\times D}^\infty(x_0, y) = \{(0,0)\}$ (see [**2**]). As a result, the inclusion $K_C^\infty(0) \times A_D(0) \subset K_{C\times D}^\infty(0,0)$ is not true for $A = K,\ K^\infty, k, k^\infty, P$, or T.

The cones K and k satisfy (2.6) for any nonempty C, $x_0 \in \operatorname{cl} C$, and continuous linear M (see for example [**33**]). Conditions under which T satisfies (2.6) are given in [**2** and **33**]. We now present conditions sufficient to give (2.6) for k^∞, K^∞, and P.

LEMMA 2.3. *Let $C \subset E$ be nonempty with $z_0 \in \operatorname{cl} C$, and let $M: E \to E_1$ be linear and continuous. The following implications hold:*

(1) *If $k_{M(C)}(Mz_0) \subset M(k_C(z_0))$, then $M(k_C^\infty(z_0)) \subset k_{M(C)}^\infty(Mz_0)$.*

(2) *If $K_{M(C)}(Mz_0) \subset M(K_C(z_0))$, then $M(K_C^\infty(z_0)) \subset K_{M(C)}^\infty(Mz_0)$.*

PROOF. Let $y \in k_C^\infty(z_0)$, and call $z = M(y)$. Let $w \in k_{M(C)}(Mz_0)$. By hypothesis, $w = M(v)$ for some $v \in k_C(z_0)$. Hence $z + w = M(y + v)$, and since $y + v \in k_C(z_0)$, we have $z + w \in M(k_C(z_0)) \subset k_{M(C)}(Mz_0)$. Therefore $z \in k_{M(C)}^\infty(Mz_0)$, and implication (1)is established. The proof of (2) is completely analogous to that of (1). □

PROPOSITION 2.4. *Let C be a nonempty subset of E, $z_0 \in \operatorname{cl} C$, and $M: E \to E_1$ a continuous linear mapping satisfying the following condition:*

(2.12) *Whenever $(w_n, t_n) \to (Mz_0, 0^+)$ such that $w_n \in M(C)$ and $t_n^{-1}(w_n - Mz_0)$ converges, there exists $z_n \in C$ with $w_n = M(z_n)$ and $t_n^{-1}(z_n - z_0)$ convergent.*

Then for $A := P$, K^∞, and k^∞,

$$M(A_C(z_0)) \subset A_{M(C)}(Mz_0). \tag{2.13}$$

PROOF. Let $v \in M(P_C(z_0))$. Then $v = M(y)$ with $y \in P_C(z_0)$. Suppose $(w_n, t_n) \to (Mz_0, 0^+)$ such that $w_n \in M(C)$ and $t_n^{-1}(w_n - Mz_0)$ converges. By (2.12), there exists $z_n \in C$ with $w_n = M(z_n)$ and $t_n^{-1}(z_n - z_0)$ convergent. There then exists $y_n \to y$ such that $z_n + t_n y_n \in C$. Hence $My_n \to My$ and $w_n + t_n M(y_n) = M(z_n + t_n y_n) \in M(C)$. Therefore $v \in P_{M(C)}(Mz_0)$ and (2.13) is established for $A := P$. To prove (2.13) for $A := k^\infty$, it suffices by Lemma 2.3 to show that $k_{M(C)}(Mz_0) \subset M(k_C(z_0))$. Let $v \in k_{M(C)}(Mz_0)$ and $t_n \downarrow 0$. There exists $v_n \to v$ such that $w_n := Mz_0 + t_n v_n \in M(C)$. Since $v_n = t_n^{-1}(w_n - Mz_0)$ converges, there exists by (2.12) a sequence $\{z_n\} \subset C$ such that $w_n = M(z_n)$ and $y_n := t_n^{-1}(z_n - z_0)$ converges to some $y \in E$. Now $v_n = M(y_n)$, so $v = M(y)$, and since $x_0 + t_n y_n = z_n \in C$, $y \in k_C(z_0)$. Thus $v \in M(k_C(z_0))$. The proof for the $A := K^\infty$ case is completely analogous to that for k^∞. □

We will make use of Proposition 2.4 in proving calculus formulae in §5.

3. The inversion theorem and tangent cone inclusions. A key ingredient in the proofs of the subdifferential calculus formulae we will present in §4 is a tangent cone inclusion derived by means of an "inversion theorem", a special case of [**4**, Theorem 4.1] (see also [**13**, **2**, **1**]).

We begin with some preliminary definitons.

DEFINITION 3.1. Let E and E_1 be Banach spaces. A function $G: E \to E_1$ is said to be *strictly differentiable* at $x_0 \in E$ if there exists a continuous linear mapping $\nabla G(x_0): E \to E_1$ such that

$$\lim_{(x,y',t)\to(x_0,y,0^+)} t^{-1}[G(x + ty') - G(x)] = \nabla G(x_0)y$$

for all $y \in E$. It is *Hadamard differentiable* at x_0 if for all $y \in E$,

$$\lim_{(y',t)\to(y,0^+)} t^{-1}[G(x_0 + ty') - G(x_0)] = \nabla G(x_0)y.$$

DEFINITION 3.2 (CF. [**27**]). A nonempty $C \subset E$ is closed near $x_0 \in \operatorname{cl} C$ if $N_\varepsilon(x_0) \cap C$ is closed for some $\varepsilon > 0$. A function $f\colon E \to \overline{\mathbf{IR}}$ which is finite at x_0 is said to be *strictly l.s.c.* at x_0 if epi f is closed near $(x_0, f(x_0))$.

DEFINITION 3.3 [**3**]. Let C be a nonempty subset of E that is closed near x.

(a) The set C is said to be *epi-Lipschitz-like* at x if there exist $\delta > 0$, a convex set Ω with Ω^0 weak-star locally compact, and $\lambda > 0$ such that for all $t \in (0, \lambda)$, $C \cap N_\delta(x) + t\Omega \subset C$.

(b) Let $f\colon E \to \overline{\mathbf{IR}}$ be strictly l.s.c. at x. The function f is said to be *Lipschitz-like* at x if epi f is *epi-Lipschitz-like* at $(x, f(x))$.

Observe that if E is finite-dimensional, any locally closed set is epi-Lipschitz-like, since Ω may be chosen to be $\{0\}$. An epi-Lipschitzian set in a normed space E is epi-Lipschitz-like, since Ω may be chosen to be a neighborhood of some point. Thus strictly l.s.c. functions with finite-dimensional domains and epi-Lipschitzian functions with normed space domains are Lipschitz-like. Also, we note that products of epi-Lipschitz-like sets are epi-Lipschitz-like. This fact will be important in §4.

The inversion theorem of [**4**, Theorem 4.1] unifies the finite-dimensional and Banach space cases treated separately in [**2**]. We will use the following special case of this theorem.

THEOREM 3.4 [**4**]. *Let E and E_1 be Banach spaces, and let $G\colon E \to E_1$ be strictly differentiable on $N_\lambda(x_0)$ for some $\lambda > 0$, where $x_0 \in G^{-1}(0) \cap D$ and D is epi-Lipschitz-like at x_0. Suppose*

$$\nabla G(x_0) T_D(x_0) = E_1. \tag{3.1}$$

Then there exist $K > 0$ and $\delta > 0$ such that for each $x \in D \cap N_\delta(x_0)$, there exists $d \in D \cap G^{-1}(0)$ satisfying $\|x - d\| \le K\|G(x)\|$.

Theorem 3.4 may be proved by means of Ekeland's variational principle [**1**]. Because of the presence of $T_D(x_0)$ in assumption (3.1), the hypotheses of all our main results will involve the Clarke tangent cone. We will later give an example (Remark 4.5(d)) which shows that T cannot be replaced by K, k, K^∞, k^∞, or P in (3.1). (See [**2**] for a similar example.) This seems to indicate that the Clarke tangent cone occupies a special position in the theory developed here.

We will now utilize Theorem 3.4 to prove a number of tangent cone inclusions.

THEOREM 3.5. *Under the hypotheses of Theorem* 3.4,

$$A_D(x_0) \cap \nabla G(x_0)^{-1}(0) \subset A_{D \cap G^{-1}(0)}(x_0) \tag{3.2}$$

for $A := T$, P, k^∞, K^∞, k, and K.

PROOF. The cases $A = T$, K are proven in [**2**, Theorem 4.1]. We include here the proof of the $A := k^\infty$ case. Let $y \in k_D^\infty(x_0) \cap \nabla G(x_0)^{-1}(0)$, and suppose $z \in k_{D\cap G^{-1}(0)}(x_0)$. It suffices to show that $y + z \in k_{D\cap G^{-1}(0)}$. Since k is isotone, $z \in k_D(x_0)$ and $z \in k_{G^{-1}(0)}(x_0) \subset \nabla G(x_0)^{-1}(0)$. So if $t_n \to 0^+$, there exists $w_n \to y + z$ such that $x_0 + t_n w_n \in D$. Now since G is strictly differentiable at x_0,

$$t_n^{-1}(G(x_0 + t_n w_n) - G(x_0)) \to \nabla G(x_0)(y + z) = 0.$$

It follows, then, from Theorem 3.4 that there exists $d_n \in D \cap G^{-1}(0)$ such that $t_n^{-1}(d_n - x_0 - t_n w_n) \to 0$ also. Let $v_n := t_n^{-1}(d_n - x_0)$. Then $v_n \to y + z$ and

$x_0 + t_n v_n = d_n \in D \cap G^{-1}(0)$. Therefore $y + z \in k_{D\cap G^{-1}(0)}(x_0)$, and the proof is complete. The proofs for the cases $A := k,\ K^\infty$, and P are quite similar to this one. $\square$

4. Calculus for directional derivatives and subgradients. We begin this section by reviewing the now familiar idea of associating directional derivatives and subgradients with tangent cones [**12**, **26**].

DEFINITION 4.1. Let $f\colon E \to \overline{1\mathbf{R}}$ be finite at $x_0 \in E$. For a tangent cone A the A *directional derivative* of f at x_0 in the direction y is defined by

$$f^A(x_0; y) := \inf\{r | (y, r) \in A_f(x_0)\}. \tag{4.1}$$

The A subgradient of f at x_0 is the set

$$\partial^A f(x_0) := \{x' \in E' \mid \langle x', y\rangle \leq f^A(x_0; y) \text{ for all } y \in E\}. \tag{4.2}$$

Definition 4.1 is designed precisely so that

$$\operatorname{epi} f^A(x_0; \cdot) = A_f(x_0) \tag{4.3}$$

if A is a closed tangent cone (in particular, for $A := K,\ k,\ K^\infty,\ k^\infty,\ P$ and T).

It is well known that if $G\colon E \to \overline{1\mathbf{R}}$ is strictly differentiable at x_0,

$$\partial^A G(x_0) = \{\nabla G(x_0)\} \tag{4.4}$$

for any A satisfying (1.2). If G is merely Hadamard differentiable, (4.4) remains true for A such that $P \subset A \subset K$ [**22**]. This is one advantage of $P,\ k^\infty,\ K^\infty,\ k$, and K over T.

Equation (4.3) and the tangent cone properties discussed in §§2 and 3 can be combined to prove calculus formulae for f^A, which will in turn produce corresponding formulae for $\partial^A f$ if A is convex. This was demonstrated in detail for $A := T,\ k$, and K in [**33**]. Roughly speaking, if a closed tangent cone A satisfies (2.5), (2.6) for the appropriate M, C, and z_0 and (3.2) (under assumption (3.1)), then f^A will have an extensive calculus including rules for sums and pointwise maxima of functions, products of positive-valued functions, and compositions $f = g \circ F$ where either $g\colon 1\mathbf{R}^m \to \overline{1\mathbf{R}}$, $F\colon E \to 1\mathbf{R}^m$, and g is nondecreasing or $g\colon E_1 \to \overline{1\mathbf{R}}$, $F\colon E \to E_1$, and F is strictly differentiable. In other words, an "algorithm" for generating a calculus for f^A consists simply of checking (2.5), (2.6), and Theorem 3.4 for A. We establish the details of this procedure in this section and apply it to k^∞ and P in §5. In this section, we will assume that A and A' are closed tangent cones which satisfy $T \subset A' \subset A \subset K$ and

$$A_C(x_0) \times A'_D(y_0) \subset A_{C\times D}(x_0, y_0) \tag{4.5}$$

in general. We assume in addition that (3.2) is true for A under condition (3.1). For example, $A = A' = T,\ k,\ k^\infty,\ P$, as well as $A = K,\ A' = k$ fit this description.

We now consider the first of two chain rule formulations. Here, as in [**25**], we adopt the convention that $\infty - \infty = \infty$.

THEOREM 4.2. *Let E and E_1 be Banach spaces and $F\colon E \to E_1$ strictly differentiable on $N_\delta(x_0)$ for some $\delta > 0$. Let $f_1\colon E \to \overline{1\mathbf{R}}$ be finite and Lipschitz-like at x_0 and $f_2\colon E_1 \to \overline{1\mathbf{R}}$ finite and Lipschitz-like at $F(x_0)$. Suppose A and A' are* *closed tangent cones satisfying* $T \subset A' \subset A \subset K$ *and* (4.5), *with* (3.2) *valid for* A *under conditon* (3.1). *Assume*

$$\nabla F(x_0) \operatorname{dom} f_1^T(x_0, \cdot) - \operatorname{dom} f_2^T(F(x_0); \cdot) = E_1. \tag{4.6}$$

Define $M\colon E \times 1\mathbf{R} \times E_1 \times 1\mathbf{R} \to E \times 1\mathbf{R}$ *by* $M(x,y,z,r) = (x, y+r)$ *and* $G\colon E \times 1\mathbf{R} \times E_1 \times 1\mathbf{R} \to E$, *by* $G(x,y,z,r) = F(x) - z$. *Assume that* (2.6) *holds for* M *as above,* $z_0 := (x_0, f_1(x_0), F(x_0), f_2(F(x_0)))$, *and* $C := (\operatorname{epi} f_1 \times \operatorname{epi} f_2) \cap G^{-1}(0)$. *Then for all* $y \in E$,

$$(f_1 + f_2 \circ F)^A(x_0; y) \leq f_1^A(x_0; y) + f_2^{A'}(F(x_0); \nabla F(x_0) y). \tag{4.7}$$

Moreover, if A *and* A' *are convex, then*

$$\partial^A (f_1 + f_2 \circ F)(x_0) \subset \partial^A f_1(x_0) + \nabla F(x_0)^* \partial^{A'} f_2(F(x_0)). \tag{4.8}$$

Equality holds in (4.8) *if* $f_1^A(x_0; \cdot) = f_1^K(x_0; \cdot)$ *and* $f_2^{A'}(F(x_0); \nabla F(x_0)(\cdot)) = f_2^K(F(x_0); \nabla F(x_0)(\cdot))$. *Equality holds in* (4.7) *if in addition* $f_1^K(x_0; \cdot)$ *and* $f_2^K(F(x_0); \cdot)$ *are proper.*

PROOF. Call $f := f_1 + f_2 \circ F$. Then

$$\operatorname{epi} f = \{(x_1, r_1 + r_2) \mid f_1(x_1) \leq r_1,\ f_2(x_2) \leq r_2,\ F(x_1) - x_2 = 0 \text{ for some } x_2 \in E_1\}.$$

Define $D := \operatorname{epi} f_1 \times \operatorname{epi} f_2$. Note that D is epi-Lipschitz-like at z_0. By our definitions, $M(D \cap G^{-1}(0)) = \operatorname{epi} f$, and so

$$\operatorname{epi} f^A(x_0; \cdot) = A_{M(D \cap G^{-1}(0))}(x_0, f(x_0)) \supset M(A_{D \cap G^{-1}(0)}(z_0))$$

by hypotheses. Next observe that (4.6) and (2.8) ensure that $\nabla G(z_0) T_D(z_0) = E_1$. Since A satisfies (3.2) under this condition,

$$A_{D \cap G^{-1}(0)}(z_0) \supset A_D(z_0) \cap \nabla G(z_0)^{-1}(0).$$

Thus

$$\begin{aligned} M(A_{D \cap G^{-1}(0)}(z_0)) &\supset M(A_D(z_0) \cap \nabla G(z_0)^{-1}(0)) \\ &\supset M((\operatorname{epi} f_1^A(z_0 \colon \cdot) \times \operatorname{epi} f_2^{A'}(F(x_0); \cdot)) \cap \nabla G(z_0)^{-1}(0)) \end{aligned}$$

by (4.5)

$$\begin{aligned} &= M(\{(h_1, r_1, h_2, r_2) \mid f_1^A(x_0; h_1) \leq r_1, f_2^{A'}(F(x_0); h_2) \leq r_2, \nabla F(x_0) h_1 = h_2\}) \\ &= \{(h, r_1 + r_2) \mid f_1^A(x_0; h) \leq r_1, f_2^{A'}(F(x_0); \nabla F(x_0) h) \leq r_2\} \\ &= \operatorname{epi}[f_1^A(x_0; \cdot) + f_2^{A'}(F(x_0); \nabla F(x_0)(\cdot))]. \end{aligned}$$

Therefore (4.7) holds. If A and A' are convex, set $p_1(\cdot) := f_1^A(x_0; \cdot)$ and $p_2(\cdot) := f_2^{A'}(F(x_0); \cdot)$. If either $p_1(0)$ or $p_2(0)$ is $-\infty$, (4.7) shows that both sides of (4.8) wil then be empty. We may assume, then, that $p_1(0) = p_2(0) = 0$. Then

$$\begin{aligned} \partial^A f(x_0) &= \{z \in E' \mid (f_1 + f_2 \circ F)^A(x_0; y) \geq \langle z, y \rangle \text{ for all } y \in E\} \\ &= \{z | p_1(y) + p_2(\nabla F(x_0) y) \geq \langle z, y \rangle \text{ for all } y \in E\} \\ &= \partial(p_1 + p_2 \circ \nabla F(x_0))(0). \end{aligned}$$

Since $T \subset A' \subset A$ and (4.6) holds, it follows that $\nabla F(x_0) \operatorname{dom} p_1 - \operatorname{dom} p_2 = E_1$. Now p_1 and p_2 are proper and sublinear, so we have by the subdifferential calculus of sublinear functions (see [**17**, 1.2.5], or in finite dimensions [**24**, Theorems 23.8, 23.9]) that

$$\begin{aligned}\partial(p_1 + p_2 \circ \nabla F(x_0))(0) &= \partial p_1(0) + \nabla F(x_0)^* \partial p_2(0)\\ &= \partial^A f_1(x_0) + \nabla F(x_0)^* \partial^{A'} f_2(F(x_0)),\end{aligned}$$

and so (4.8) holds. Finally, if $f_1^A(x_0;\cdot) = f_1^K(x_0;\cdot)$ and $f_2^{A'}(F(x_0);\nabla F(x_0)(\cdot)) = f_2^K(F(x_0);\nabla F(x_0)(\cdot))$,

$$\begin{aligned}(f_1 + f_2 \circ F)^A(x_0;\cdot) &\geq (f_1 + f_2 \circ F)^K(x_0;\cdot)\\ &\geq f_1^K(x_0;\cdot) + f_2^K(F(x_0);\nabla F(x_0)(\cdot))\end{aligned}$$

(if $f_1^K(x_0;\cdot)$ and $f_2^K(F(x_0);\cdot)$ are proper)

$$= f_1^A(x_0;\cdot) + f_2^{A'}(F(x_0);\nabla F(x_0)(\cdot)).$$

Hence equality holds in (4.7) and (4.8) under the stated assumptions. □

REMARK 4.3. (a) Condition (4.6) is satisfied by quite general classes of functions. For example, this conditon holds in any of the following cases:

(i) $\nabla F(x_0)$ is surjective and f_1 is locally Lipschitzian near x_0.

(ii) f_2 is locally Lipschitzian near $F(x_0)$.

(iii) f_2 is directionally Lipschitzian at $F(x_0)$ and

$$\nabla F(x_0)\,\mathrm{dom}\, f_1^T(x_0;\cdot) \cap \mathrm{int}\,\mathrm{dom}\, f_2^T(F(x_0);\cdot) \neq \emptyset.$$

For further discussion, see [**33**].

(b) If $A \subset k$, the conditions for equality in (4.7) and (4.8) can be sharpened [**33**]. In this case, either $f_1^A(x_0;\cdot) = f_1^k(x_0;\cdot)$ and

$$f_2^{A'}(F(x_0);\nabla F(x_0)(\cdot)) = f_2^K(F(x_0);\nabla F(x_0)(\cdot))$$

or

$$f_1^A(x_0;\cdot) = f_1^K(x_0;\cdot) \quad \text{and} \quad f_2^{A'}(F(x_0);\nabla F(x_0)(\cdot)) = f_2^k(F(x_0);\nabla F(x_0)(\cdot))$$

will guarantee equality in (4.8).

(c) The roles of A and A' may be reversed in the right-hand side of (4.7) and (4.8).

The special cases of Theorem 4.2 where $A := K$, $A' := k$, $A = A' := k$, and $A = A' := T$ are discussed in detail in [**33**]. A number of corollaries of Theorem 4.2, analogous to those listed for the $A = A' := T$ case in [**33**, §3] can be proven. We will concentrate our attention here on just one of them, after making some preliminary definitions.

DEFINITON 4.4. Let C be a nonempty subset of E. Define

$$\Delta^n C := \{(x_1, \ldots, x_n) \in C \mid x_1 = x_2 = \cdots = x_n\}.$$

DEFINITION 4.5. Let $C_i \subset E$, $i = 1, \ldots, n$, be nonempty convex sets. These sets are said to be in *strong general position* [**36**] if

$$0 \in \mathrm{int}\left[\Delta^{n-1} C_1 - \prod_{j=2}^{n} C_j\right]. \tag{4.9}$$

It is shown in [**36**] that (4.9) is equivalent to

$$0 \in \operatorname{int}\left[\Delta^n E - \prod_{j=1}^{n} C_j\right]. \tag{4.10}$$

If the sets C_i, $i = 1, \ldots, n$, are cones, then (4.9) can be written

$$\Delta^{n-1} C_1 - \prod_{j=2}^{n} C_j = E^{n-1}. \tag{4.11}$$

PROPOSITION 4.6 (CF. [**33**, PROPOSITION 3.10, COROLLARY 6.15]). *Let A and A' be tangent cones as in Theorem* 4.2, *and let $D_i \subset E$, $i = 1, \ldots, n$, be epi-Lipschitz-like at $y_0 \in \bigcap_{i=1}^n D_i$. Assume $T_{D_i}(y_0)$, $i = 1, \ldots, n$, are in strong general position. Then*

$$A_{D_1 \cap \cdots \cap D_n}(y_0) \supset A_{D_1}(y_0) \cap \left(\bigcap_{i=2}^{n} A'_{D_i}(y_0)\right). \tag{4.12}$$

Moreover, if A and A' are convex, then

$$(A_{D_1 \cap \cdots \cap D_n}(y_0))^0 \subset (A_{D_1}(y_0))^0 + \sum_{i=2}^{n} (A'_{D_i}(y_0))^0. \tag{4.13}$$

Equality holds in (4.12) *and* (4.13) *if*

$$A_{D_1}(y_0) = K_{D_1}(y_0) \quad \text{and} \quad A'_{D_i}(y_0) = K_{D_i}(y_0), \qquad i = 2, \ldots, n.$$

PROOF. Define $f_1 := i_{D_1 \times \cdots \times D_n}$ and $f_2 := i_{\{0\}}$, where $\{0\}$ denotes the origin in E^{n-1}. Define $F\colon E^n \to E^{n-1}$ by $F(x_1, \ldots, x_n) := (x_1 - x_2, \ldots, x_1 - x_n)$. Observe that the relationships $i_C^A(x'; \cdot) = i_{A_C(x)}(\cdot)$ and $\partial^A i_C(x) = (A_C(x))^0$ hold for any nonempty $C \subset E^n$ and $x \in C$, and for A' as well as A. Apply Theorem 4.2 with $x_0 := (y_0, \ldots, y_0)$. Since T satisfies (2.5), the strong general position assumption guarantees that (4.6) holds. Then (4.12) and (4.13) follow from (4.7) and (4.8), respectively, since A and A' satisfy (4.5). Since $i_C^K(x_0; \cdot)$ is proper for any nonempty C and $x \in C$, the stated conditons for equality follow from those in Theorem 4.2. □

REMARK 4.7. (a) An application of Proposition 4.6 with $D_i := \operatorname{epi} f_i$ will give a calculus rule for f^A and $\partial^A f$ where $f(x) := \max_{1 \le i \le n} f_i(x)$ (see [**33**, Proposition 3.14]).

(b) If $A \subset k$, then the conditions for equality in Proposition 4.6 can be sharpened to $A_{D_1}(y_0) = k_{D_1}(y_0)$ and $A'_{D_i}(y_0) = k_{D_i}(y_0)$, $i = 2, \ldots, n$.

Under these conditions,

$$A_{D_i \cap \cdots \cap D_n}(y_0) \subset k_{D_1 \cap \cdots \cap D_n}(y_0) \subset \bigcap_{i=1}^{n} k_{D_i}(y_0)$$

since k is isotone.

(c) Inclusion (4.12) for $A = A' = T$ was established by Watkins in [**35**] to show that the Clarke tangent cone satisfies the intersection principle of Martin, Gardner, and Watkins [**18**]. As a result, a Dubovitskii-Milyutin approach may be used to prove quite general Fritz John type Lagrange multiplier rules involving $\partial^T f$ [**35**, **33**].

(d) We will see in §5 that $A = A' := P$ and $A = A' := k^\infty$ can be used in Theorem 4.2 and Proposition 4.6. However, in the hypotheses of these results, T cannot be replaced by k^∞ or P, as we now demonstrate. Define

$$D_1 := \{(x,y) \in \mathbf{1R}^2_+ \mid x + y = 1/n,\ n \text{ odd}\} \cup \{0,0\}$$

and

$$D_2 := \{(x,y) \in \mathbf{1R}^2_+ \mid x + y = 1/n,\ n \text{ even}\} \cup \{(0,0)\}.$$

Then $k_{D_i}(0,0) = \mathbf{1R}^2_+$, $i = 1,2$, so that $k^\infty_{D_i}(0,0) = P_{D_i}(0,0) = \mathbf{1R}^2_+$. In this example, $A_{D_i}(0,0), i = 1,2$, are in strong general position (i.e., $A_{D_1}(0,0) - A_{D_2}(0,0) = \mathbf{1R}^2$) for $A := k^\infty, P$. Inclusion (4.12) with $n = 2$ does not hold for $A = A' := P$ or $A = A' := k^\infty$, though, since $D_1 \cap D_2 = \{(0,0)\}$. This example also demonstrates that T cannot be replaced by k^∞ or P in (3.1), and that P and k^∞ do not satisfy the intersection principle mentioned in (c).

DEFINITION 4.8. Let $x = (x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be elements of $\mathbf{1R}^n$. We say $x \leq y$ if $x_i \leq y_i$ for each i. The function $F\colon \mathbf{1R}^n \to \overline{\mathbf{1R}}$ is *isotone* on $B \subset \mathbf{1R}^n$ if $F(x) \leq F(y)$ whenever $x, y \in B$ and $x \leq y$.

We now establish a chain rule for compositions of the form $F \circ f$, where $f := (f_1, \ldots, f_n)$ each $f_i\colon E \to \overline{\mathbf{1R}}$ is finite and Lipschitz-like at x_0, and $F\colon \mathbf{1R}^n \to \overline{\mathbf{1R}}$ is finite at $f(x_0)$, l.s.c., and isotone on $N_\delta(f(x_0)) \cup B$ for some $\delta > 0$, with

$$B\colon = \{y \in \mathbf{1R}^n \mid f(x) \leq y \text{ for some } x \in E\}.$$

In such a composition, we define

$$(F \circ f)(x) = \inf\{F(y) \mid f(x) \leq y, y \in \mathbf{1R}^n\},$$

and set $F(f(x)) = +\infty$ if some $f_i(x) = +\infty$. The proof of this chain rule will depend on another special case of (2.6).

THEOREM 4.9. *Let F and f be defined as in the preceding paragraph. Suppose A and A' are closed tangent cones satisfying $T \subset A' \subset A \subset K$ and* (4.5), *with* (3.2) *valid for A under condition* (3.1). *Assume that $F^{A'}(f(x_0);\cdot)$ is isotone on $\mathbf{1R}^n$, that $f_1^A(x_0;\cdot)$ and $f_i^{A'}(x_0;\cdot)$, $i = 2, \ldots, n$, are proper, and that*

$$(\Delta^n E \times \operatorname{dom} F^T(f(x_0);\cdot)) - S = E^n \times \mathbf{1R}^n, \tag{4.14}$$

where

$$S := \{(y_1, \ldots, y_n, r_1, \ldots, r_n) \mid (y_i, r_i) \in \operatorname{epi} f_i^T(x_0;\cdot), i = 1, \ldots, n\}.$$

Define $M\colon (E \times \mathbf{1R})^n \times \mathbf{1R}^{n+1} \to E \times \mathbf{1R}$ by

$$M(x_1, y_1, \ldots, x_n, y_n, z_1, \ldots, z_n, r) = (x_1, r)$$

and $G\colon (E \times \mathbf{1R})^n \times \mathbf{1R}^{n+1} \to E^{n-1} \times \mathbf{1R}^n$ by

$$G(x_1, y_1, \ldots, x_n, y_n, z_1, \ldots, z_n, r) = (x_1 - x_2, \ldots, x_1 - x_n, y_1 - z_1, \ldots, y_n - z_n).$$

Assume that (2.6) *holds for this M and $C\colon = (\prod_{i=1}^n \operatorname{epi} f_i \times \operatorname{epi} F) \cap G^{-1}(0)$, $z_0\colon = (x_0, f_1(x_0), \ldots, x_0, f_n(x_0), f_1(x_0), \ldots, f_n(x_0), F(f(x_0)))$. Then for all $y \in E$,*

$$(F \circ f)^A(x_0; y) \leq F^{A'}(f(x_0); f_1^A(x_0, y), f_2^{A'}(x_0; y), \ldots, f_n^{A'}(x_0; y)). \tag{4.15}$$

Moreover, if A and A' are convex, then

$$\text{(4.16)}\quad \partial^A(F\circ f)\subset\{\lambda\cdot(\partial^A f_1(x_0),\partial^{A'}f_2(x_0),\dots,\partial^{A'}f_n(x_0))|\lambda\geq 0^+,\ \lambda\in\partial k^{A'}F(f(x_0))\}.$$

Equality holds in (4.16) *if* $F^{A'}(f(x_0);\cdot)=F^K(f(x_0);\cdot), f_1^A(x_0;\cdot)=f_1^K(x_0;\cdot)$, *and* $f_i^{A'}(x_0;\cdot)=f_i^K(x_0;\cdot), i=2,\dots,n$. *Equality holds in* (4.15) *if in addition* $F^K(f(x_0);\cdot)$ *is proper.*

PROOF. Call $h:=F\circ f$. Since F is isotone on B,

$$\operatorname{epi} h=\{(x,r)\mid\exists(y_1,\dots,y_n)\in 1\mathbf{R}^n\text{ with }F(y_1,\dots,y_n)\leq r,\ f_i(x)\leq y_i, i=1,\dots,n\}.$$

Define $D:=\operatorname{epi} f_1\times\cdots\times\operatorname{epi} f_n\times\operatorname{epi} F$, so that $C=D\cap G^{-1}(0)$. Then $\operatorname{epi} h=M(C)$, and we have

$$\begin{aligned}\operatorname{epi} h^A(x_0;\cdot)&=A_{M(C)}(x_0,h(x_0))\\ &\supset M(A_C(z_0))\quad\text{by hypothesis}\\ &\supset M(A_D(z_0)\cap\nabla G(z_0)^{-1}(0)),\end{aligned}$$

since (4.14) ensures that (3.1) is satisfied

$$\begin{aligned}&\supset\{(x,r)\mid\exists y\in 1\mathbf{R}^n\text{ with }f_1^A(x_0;x)\leq y_1,\\ &\qquad f_1^{A'}(x_0;x)\leq y_i, i=2,\dots,n,\ F^{A'}(f(x_0);y)\leq r\}\quad\text{by (4.5)}\\ &=\operatorname{epi} F^{A'}(f(x_0);f_1^A(x_0;\cdot),f_2^{A'}(x_0;\cdot),\dots,f_n^{A'}(x_0;\cdot))\\ &\qquad\qquad\text{since }F^{A'}(f(x_0);\cdot)\text{ is isotone.}\end{aligned}$$

Thus (4.15) holds. Now suppose A and A' are convex. If $F^{A'}(f(x_0);0)=-\infty$, (4.15) shows that both sides of (4.16) will be empty. We may assume, then, that $F^{A'}(f(x_0);\cdot)$ is proper. Condition (4.14) implies $\operatorname{dom} f_1^A(x_0;\cdot)$ and $\operatorname{dom} f_i^{A'}(x_0;\cdot)$, $i=2,\dots,n$, are in strong general position, and that

$$\operatorname{int}\operatorname{dom} F^{A'}(f(x_0);\cdot)\cap\{(r_1,\dots,r_n)\mid f_1^A(x_0;y)\leq r_1, f_i^{A'}(x_0;y)\leq r_i,\ i=2,\dots,n,\text{ for some }y\in E\}\neq\varnothing$$

(see [**33**, Remark 2.11]). By the corresponding result from the subdifferential calculus for sublinear functions (the approriate analogue of [**33**, Theorem 2.10]),

$$\begin{aligned}\partial(F\circ f)(x_0)&=\{z\mid(F\circ f)^A(x_0;y)\geq\langle z,y\rangle\forall y\in E\}\\ &\subset\{z\mid F^{A'}(f(x_0);f_1^A(x_0;y),f_2^{A'}(x_0;y),\dots,f_n^{A'}(x_0,y))\\ &\qquad\geq\langle z,y\rangle\forall y\in E\}\\ &=\partial((F^{A'}(f(x_0);\cdot))\circ(f_1^A(x_0;\cdot),f_2^{A'}(x_0;\cdot),\dots,f_n^{A'}(x_0;\cdot)))(0)\\ &=\{\lambda\cdot(\partial f_1^A(x_0;\cdot)(0),\partial f_2^{A'}(x_0;\cdot)(0),\dots,\partial f_n^{A'}(x_0;\cdot)(0))|\\ &\qquad\lambda\geq 0^+,\lambda\in\partial F^{A'}(f(x_0);\cdot)(0)\}\\ &=\{\lambda\cdot(\partial^A f_1(x_0),\partial^{A'}f_2(x_0),\dots,\partial^{A'}f_n(x_0))|\\ &\qquad\lambda\geq 0^+,\lambda\in\partial^{A'}F(f(x_0))\}.\end{aligned}$$

Finally, assume that the stated conditions for equality hold. Then for all $y \in E$,

$$\begin{aligned}(F\circ f)^A(x_0;y) &\geq (F\circ f)^K(x_0;y)\\ &\geq F^K(f(x_0);f_1^K(x_0;y),\dots,f_n^K(x_0;y))\\ &\qquad\qquad\qquad\qquad\text{by [33, Proposition 6.1]}\\ &= F^{A'}(f(x_0);f_1^A(x_0;y),f_2^{A'}(x_0;y),\dots,f_n^{A'}(x_0;y)).\end{aligned}$$

Therefore equality holds in (4.15) and (4.16).

The special cases of Theorem 4.9 where $A = A' = T$, $A = A' = k$, and $A = K$, $A' = k$ are covered in [**33**]. Corollaries of Theorem 4.9 include formulae for directional derivatives and subgradients of sums and pointwise maxima of functions, as well as product and quotient rules for positive-valued functions. Proposition 4.6 can be rederived via Theorem 4.9.

5. Calculus for P, k^∞, and K^∞. In this section, we establish the special cases of Theorems 4.2 and 4.9 involving $A = A' := P$ and $A = A' := k^\infty$. We already have much relevant information about these tangent cones from Proposition 2.2 and Theorem 3.5; all that remains to be checked is inclusion (2.6) for the appropriate choices of M, C, and z_0. The case $A := K^\infty$ is less satisfactory, since as we saw in §2 there is no A' to pair with $A := K^\infty$ which will satisfy (4.5). Nevertheless, some results can be derived from known formulae for the $A := K, A' := k$ case.

We now present the $A := P$, k^∞ cases of Theorem 4.2. In the proofs, we use the fact that

$$f^K(x_0;y) = \liminf_{(y',t)\to(y,0^+)} t^{-1}[f(x_0+ty') - f(x_0)]. \tag{5.1}$$

THEOREM 5.1. *Let $F\colon E \to E_1$ be strictly differentiable on some $N_\delta(x_0)$, $f_1\colon E \to \overline{\mathrm{IR}}$ finite and Lipschitz-like at x_0, and $f_2\colon E_1 \to \overline{\mathrm{IR}}$ finite and Lipschitz-like at $F(x_0)$. Assume that* (4.6) *holds, and that $f_1^K(x_0;\cdot)$ and $(f_2\circ F)^K(x_0;\cdot)$ are proper. Then for all $y \in E$,*

$$(f_1 + f_2\circ F)^{k\infty}(x_0;y) \leq f_1^{k\infty}(x_0;y) + f_2^{k\infty}(F(x_0);\nabla F(x_0)y. \tag{5.2}$$

$$(f_1 + f_2\circ F)^{P}(x_0;y) \leq f_1^{P}(x_0;y) + f_2^{P}(F(x_0);\nabla F(x_0)y). \tag{5.3}$$

Moreover,

$$\partial^{k\infty}(f_1 + f_2\circ F)(x_0) \subset \partial^{k\infty} f_1(x_0) + \nabla F(x_0)^*\partial^{k\infty} f_2(F(x_0)). \tag{5.4}$$

$$\partial^{P}(f_1 + f_2\circ F)(x_0) \subset \partial^{P} f_1(x_0) + \nabla F(x_0)^*\partial^{P} f_2(F(x_0)). \tag{5.5}$$

Equality holds in (5.3) *and* (5.5) *if $f_1^P(x_0;\cdot) = f_1^k(x_0;\cdot)$ and*

$$f_2^P(F(x_0);\nabla F(x_0)(\cdot)) = f^K(F(x_0);\nabla F(x_0)(\cdot)),$$

or if $f_1^P(x_0;\cdot) = f_1^K(x_0;\cdot)$ and $f_2^P(F(x_0);\nabla F(x_0)(\cdot)) = f_2^k(F(x_0);\nabla F(x_0)(\cdot))$. Replacing "$P$" with "$k^\infty$" in these conditions gives conditions for equality in (5.2) *and* (5.4).

PROOF. Define M, G, C, and z_0 as in Theorem 4.2. It suffices to prove (2.12). To that end, suppose $w_n := (v_n,d_n) \in M(C)$ converges to $Mz_0 = (x_0, f_1(x_0) + f_2(F(x_0)))$ and $t_n \downarrow 0$ such that $t_n^{-1}(w_n - Mz_0)$ converges. Then $(v_n,d_n) = M(x_n,y_n,a_n,r_n)$ with $f_1(x_n) \leq y_n$, $f_2(a_n) \leq r_n$, $a_n = F(x_n)$, and

$v_n = x_n$, $d_n = y_n + r_n$. Since $t_n^{-1}(w_n - Mz_0)$ converges, it follows that $t_n^{-1}(x_1 - x_0)$ and $t_n^{-1}(d_n - f_1(x_0) - f_2(F(x_0)))$ converge. It remains to show that $t_n^{-1}(y_n - f_1(x_0))$, $t_n^{-1}(r_n - f_2(F(x_0)))$, and $t_n^{-1}(a_n - F(x_0))$ converge. Since F is strictly differentiable, $t_n^{-1}(a_n - F(x_0)) = t_n^{-1}(F(x_n) - F(x_0))$ converges. Since $f_1^K(x_0;\cdot)$ and $(f_2 \circ F)^K(x_0;\cdot)$ are proper, the sequences $t_n^{-1}(y_n - f_1(x_0))$ and $t_n^{-1}(r_n - f_2(F(x_0)))$ are bounded below by (5.1). If $t_n^{-1}(y_n - f_1(x_0))$ were not bounded above, then

$$t_n^{-1}(d_n - f_1(x_0) - f_2(F(x_0))) = t_n^{-1}(y_n - f_1(x_0)) + t_n^{-1}(r_n - f_2(F(x_0)))$$

would also not be bounded above, a contradiction. Thus $t_n^{-1}(y_n - f_1(x_0))$ is bounded, and taking a subsequence if necessary, we may assume (because of Remark 2.1(b)) that it converges. Then $t_n^{-1}(r_n - f_2(F(x_0)))$ must converge also. We have established (2.12). By Theorem 4.2 with $A = A' := k^\infty$ and $A = A' = P$, (5.2) through (5.5) hold. The conditions for equality follow from [**33**, Propositions 6.1, 6.2]. □

REMARK 5.2. In the case in which $E = E_1$ and F is the identity mapping on E, (4.6) reduces to

$$\operatorname{dom} f_1^T(x_0;\cdot) - \operatorname{dom} f_2^T(x_0;\cdot) = E, \tag{5.6}$$

and (5.3) and (5.5) become

$$(f_1 + f_2)^P(x_0; y) \le f_1^P(x_0; y) + f_2^P(x_0; y) \tag{5.7}$$

and

$$\partial^P(f_1 + f_2)(x_0) \subset \partial^P f_1(x_0) + \partial f_2(x_0), \tag{5.8}$$

respectively. Penot [**22**, Proposition 5.4] has proven (5.7) and (5.8) under different hypotheses. In [**22**], (5.6) is replaced by

$$\operatorname{dom} f_1^P(x_0;\cdot) \cap \operatorname{dom} f_2^{IP}(x_0;\cdot) \ne \varnothing \tag{5.9}$$

where IP is the *interiorly prototangent cone*

$$IP_C(x_0) := \{y \mid \forall (x_n, t_n) \to (x_0, 0^+) \text{ such that } x_n \in C \text{ and } t_n^{-1}(x_n - x_0) \\ \text{converges}, \forall y_n \to y, x_n + t_n y_n \in C \text{ for } n \text{ sufficiently large}\}.$$

Condition (5.9) is sometimes more restictive, sometimes less restrictive, than (5.6). For example, suppose $f_1\colon \mathbf{1R}^2 \to \mathbf{1R}$ and $f_2\colon \mathbf{1R}^2 \to \mathbf{1R}$ are defined by $f_1(x,y) = |x|^{1/2}$ and $f_2(x,y) = |y|^{1/2}$. Then at $x_0 = (0,0)$, $\operatorname{dom} f_1^T(x_0;\cdot) = 0 \times \mathbf{1R}$, $\operatorname{dom} f_2^T(x_0;\cdot) = \mathbf{1R} \times 0$, and $\operatorname{dom} f^{IP}(x_0;\cdot) = \varnothing$, so that (5.6) holds while (5.9) does not. On the other hand, suppose $f_i\colon \mathbf{1R} \to \mathbf{1R}$, $i = 1, 2$, are both defined by

$$f_i(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1/n & \text{if } 1/(n+1) < |x| \le 1/n, \qquad n = 1,2,3,4,\ldots, \\ |x| & \text{if } |x| > 1. \end{cases}$$

Then $\operatorname{dom} f_i^T(0;\cdot) = \{0\}$ and $\operatorname{dom} f_i^P(0;\cdot) = \operatorname{dom} f_i^{IP}(0;\cdot) = \mathbf{1R}$, so that (5.9) holds at $x_0 = 0$ while (5.6) does not.

One advantage of [**22**, Proposition 5.4] is that it is applicable to functions with general domain spaces. Assumption (5.9) is analogous to

$$\operatorname{dom} f_1^T(x_0;\cdot) \cap \operatorname{int} \operatorname{dom} f_2^T(x_0;\cdot) \ne \varnothing, \tag{5.10}$$

the assumption under which the sum formula for f^T can be proven in general spaces [**25**, Theorem 2]. Interestingly, while (5.10) can be weakened to (5.6) in finite dimensions [**33**], we have already seen in Remark 4.7(d) that (5.9) cannot be correspondingly weakened to

$$\operatorname{dom} f_1^P(x_0;\cdot) - \operatorname{dom} f_2^P(x_0;\cdot) = E.$$

We next establish the $A = A' := P, k^\infty$ cases of Theorem 4.9, beginning with a technical lemma.

LEMMA 5.3. *Let $F\colon 1\mathbf{R}^n \to \overline{1\mathbf{R}}$ be finite at x_0 and isotone on $N_\delta(x_0)$ for some $\delta > 0$. Then $F^{k\infty}(x_0;\cdot)$ and $F^P(x_0;\cdot)$ are isotone on $1\mathbf{R}^n$.*

PROOF. Let $y_1, y_2 \in 1\mathbf{R}^n$ with $y_1 \leq y_2$, and suppose that $F^{k\infty}(x_0; y_2) \leq d$. It suffices to show that $F^{k\infty}(x_0; y_1) \leq d$. To this end, let $(z, r) \in k_F(x_0)$. Then $(z + y_2, d + r) \in k_F(x_0)$. Let $t_n \downarrow 0$. There exist $(w_n, a_n) \to (0, 0)$ such that $(x_0, F(x_0)) + t_n(z + y_2 + w_n, d + r_n + a_n) \in \operatorname{epi} F$; i.e.,

$$t_n^{-1}[F(x_0 + t_n(z + y_2 + w_n)) - F(x_0)] \leq d + r + a_n.$$

For n large enough, both $x_0 + t_n(z + y_i + w_n)$, $i = 1, 2$, lie in $N_\delta(x_0)$. By the isotonicity of F, then, $t_n^{-1}[F(x_0 + t_n(z + y_1 + w_n)) - F(x_0)] \leq d + r + a_n$. Thus $(z + y_1, d + r) \in k_F(x_0)$, and it follows that $F^{k\infty}(x_0; y_1) \leq d$. The $A := P$ case can be proved in a similar fashion. □

THEOREM 5.4. *Let $f_i\colon E \to \overline{1\mathbf{R}}$, $i = 1, \ldots, n$, be finite and Lipschitz-like at x_0, and define $f := (f_1, \ldots, f_n)$. Let $F\colon 1\mathbf{R}^n \to \overline{1\mathbf{R}}$ be finite at $f(x_0)$, isotone on $N_\delta(x_0) \cup B$ for some $\delta > 0$, and l.s.c. Assume that* (4.14) *holds, that each $f_i^K(x_0;\cdot)$ is proper, and that* $\limsup_{k\to\infty}[(F \circ f)(x_0 + t_k y_k) - (F \circ f)(x_0)]t_k^{-1} = +\infty$ *whenever $t_k \downarrow 0$, $y_k \to y$ and* $\limsup_{k\to\infty} t_k^{-1}[f_i(x_0 + t_k y_k) - f_i(x_0)] = +\infty$ *for some i. Then for all $y \in E$,*

$$(F \circ f)^{k\infty}(x_0; y) \leq F^{k\infty}(f(x_0); f_1^{k\infty} f(x_0; y), \ldots, f_n^{k\infty}(x_0; y)), \tag{5.11}$$

$$(F \circ f)^P(x_0; y) \leq F^P(f(x_0); f_1^P(x_0; y), \ldots, f_n^P(x_0; y)). \tag{5.12}$$

Moreover,

$$\partial^{k\infty}(F \circ f)(x_0) \subset \{\lambda \cdot (\partial^{k\infty} f_1(x_0), \ldots, \partial^{k\infty} f_n(x_0)) \mid \lambda \in \partial^{k\infty} F(f(x_0)), \lambda \geq 0^+\}, \tag{5.13}$$

$$\partial^P(F \circ f)(x_0) \subset \{\lambda \cdot (\partial^P f_1(x_0), \ldots, \partial^P f_n(x_0)) \mid \lambda \in \partial^P F(f(x_0)), \lambda \geq 0^+\}. \tag{5.14}$$

Equality holds in (5.14) *if $F^P(f(x_0);\cdot) = F^K(f(x_0);\cdot), f_1^P(x_0;\cdot) = f_1^k(x_0;\cdot)$, and $f_i^P(x_0;\cdot) = f_i^K(x_0;\cdot)$, $i = 2, \ldots, n$. Equality holds in* (5.12) *if in addition $F^K(f(x_0);\cdot)$ is proper. The replacement of P by k^∞ in these conditions gives condition for equality in* (5.13) *and* (5.11).

PROOF. Define M, G, C, and z_0 as in Theorem 4.9. It suffices to prove (2.12). Suppose $w_k := (v_k, d_k) \in M(C)$ converges to Mz_0 and $t_k \downarrow 0$ such that $t_k^{-1}(w_k - Mz_0)$ is convergent. Then

$$(v_k, d_k) = M(v_k, f_1(v_k), \ldots, v_k, f_n(v_k), f_1(v_k), \ldots, f_n(v_k), d_k)$$

with $d_k \geq (F \circ f)(v_k)$ and $y_k := t_k^{-1}(v_k - x_0)$ and $t_k^{-1}(d_k - (F \circ f)(x_0))$ convergent. It remains to show that $t_k^{-1}(f_i(x_0 + t_k y_k) - f_i(x_0))$ converges for each i. Each of these sequences is bounded below since $f_i^K(x_0; \cdot)$ is proper. Suppose one of them is not bounded above. Then $t_k^{-1}[(F \circ f)(x_0 + t_k y_k) - (F \circ f)(x_0)]$ is also not bounded above, a contradiction of the fact that $t_k^{-1}[d_k - (F \circ f)(x_0)]$ converges. Taking a subsequence if necessary, we may assume that $t_k^{-1}[f_i(x_0 + t_k y_k) - f_i(x_0)]$ converges. Therefore (2.12) holds, and (5.11) through (5.14) follow from Theorem 4.9 and Proposition 2.4. The conditions for equality are a consequence of [**33**, Proposition 6.1]. □

Important special cases of Theorem 5.4 include $F(z_1, \ldots, z_n) = \sum_{i=1}^n z_i$ and $F(z_1, \ldots, z_n) = \max\{z_1, \ldots, z_n\}$. Another corollary of Theorem 5.4 is a product rule for positive-valued functions.

COROLLARY 5.5. *Let $f_i \colon E \to \overline{\mathbb{R}}$, $i = 1, \ldots, n$, be nonnegative on E and Lipschitz-like and positive at $x_0 \in \operatorname{dom} f_i$. Assume that each $f_i^K(x_0; \cdot)$ is proper and that* $\operatorname{dom} f_i^T(x_0; \cdot)$, $i = 1, \ldots, n$, *are in strong general position. Then for all $y \in E$,*

$$\left(\prod_{i=1}^n f_i\right)^{k\infty}(x_0; y) \leq \sum_{i=1}^n \left(\prod_{j \neq i} f_j(x_0)\right) f_i^{k\infty}(x_0; y), \tag{5.15}$$

$$\left(\prod_{i=1}^n f_i\right)^{P}(x_0; y) \leq \sum_{i=1}^n \left(\prod_{j \neq i} f_j(x_0)\right) f_i^{P}(x_0; y). \tag{5.16}$$

Moreover,

$$\partial^{k\infty}\left(\prod_{i=1}^n f_i\right)(x_0) \subset \sum_{i=1}^n \left(\prod_{j \neq 1} f_j(x_0)\right) \partial^{k\infty} f_i(x_0). \tag{5.17}$$

$$\partial^{P}\left(\prod_{i=1}^n f_i\right)(x_0) \subset \sum_{i=1}^n \left(\prod_{j \neq i} f_j(x_0)\right) \partial^{P} f_i(x_0). \tag{5.18}$$

Equality holds in (5.16) *and* (5.18) *if $f_i^P(x_0; \cdot) = f_i^k(x_0; \cdot)$ for some i and $f_j^P(x_0; \cdot) = f_j^K(x_0; \cdot)$ for each $j \neq i$. The replacement of P by k^∞ in these conditions gives conditions for equality in* (5.15) *and* (5.17).

PROOF. In Theorem 5.4, define $F \colon \mathbb{R}^n \to \mathbb{R}$ by $f(z_1, \ldots, z_n) = \prod_{i=1}^n z_i$. Condition (4.14) in this case reduces to $\operatorname{dom} f_i^T(x_0; \cdot)$, $i = 1, \ldots, n$, being in strong general position. To verify the remaning hypothesis of Theorem 5.4, suppose $t_k \downarrow 0$ and $y_k \to y$. Since each f_i is strictly l.s.c. at x_0, for each $\delta > 0$ there exists m such that for all $k \geq m$, $f_i(x_0 + t_k y_k) \geq f_i(x_0) - \delta$. Let $\varepsilon > 0$ be given. Then there exists n_0 such that for all $k \geq n_0$,

$$\begin{aligned} &t_k^{-1}[(F \circ f)(x_0 + t_k y_k) - (F \circ f)(x_0)] \\ &\quad \geq \sum_{i=1}^n \left(\prod_{j \neq i} f_j(x_0)\right) t_k^{-1}[f_i(x_0 + t_k y_k) - f_i(x_0)] - \varepsilon. \end{aligned}$$

If for some i, $\limsup_{k\to\infty} t_k^{-1}[f_i(x_0+t_ky_k)-f_i(x_0)] = +\infty$, it follows from the above inequality that

$$\limsup_{k\to\infty} t_k^{-1}[(F\circ f)(x_0+t_ky_k)-(F\circ f)(x_0)] = +\infty.$$

Thus the hypotheses of Theorem 5.4 are satisfied and (5.15) through (5.18) follow from (5.14). □

Some calculus formulae for $f^{K\infty}$ can be proven by means of our results for the $A := K$, $A' := k$ case.

PROPOSITION 5.6. *Let $F\colon E\to E_1$ be strictly differentiable on some $N_\delta(x_0)$ and $f\colon E_1\to \overline{\mathbf{IR}}$ Lipschitz-like and finite at $F(x_0)$. Assume*

$$\nabla F(x_0)E - \operatorname{dom} f^T(x_0;\cdot) = E_1. \tag{5.19}$$

Then for all $y\in E$,

$$(f\circ F)^{K\infty}(x_0;y) = f^{K\infty}(F(x_0);\nabla F(x_0)y). \tag{5.20}$$

Moreover,

$$\partial^{K\infty}(f\circ F)(x_0) = \nabla F(x_0)^*\partial^{K\infty}f(F(x_0)). \tag{5.21}$$

PROOF. The inequality $(f\circ F)^K(x_0;y)\ge f^K(F(x_0);\nabla F(x_0)y)$ is true in general [**33**, Proposition 6.2], and $(f\circ F)^K(x_0;y)\le f^K(F(x_0);\nabla F(x_0)y)$ holds under assumption (5.19) by Theorem 4.2 (and Remark 4.3(c)) with $A := K$, $A' := k$, and $f_1\equiv 0$. Thus (5.20) holds. Since $T\subset K^\infty$, (5.19) implies that $\nabla F(x_0)E - \operatorname{dom} f^{K\infty}(x_0;\cdot) = E_1$. Equation (5.21) then follows from (5.20) and the corresponding convex analysis result. □

THEOREM 5.7. *Let $f_1\colon E\to\overline{\mathbf{IR}}$ and $f_2\colon E\to\overline{\mathbf{IR}}$ be finite and Lipschitz-like at x_0, and suppose (5.6) holds. Assume in addition that $f_2^K(x_0;\cdot) = f_2^k(x_0;\cdot)$ and that $f_1^K(x_0;\cdot)$ and $f_2^K(x_0;\cdot)$ are proper. Then for all $y\in E$,*

$$(f_1+f_2)^{K\infty}(x_0;y)\le f_1^{K\infty}(x_0;y)+f_2^{K\infty}(x_0;y). \tag{5.22}$$

Moreover,

$$\partial^{K\infty}(f_1+f_2)(x_0)\subset \partial^{K\infty}f_1(x_0)+\partial^{K\infty}f_2(x_0). \tag{5.23}$$

Equality holds in (5.22) *and* (5.23) *if in addition $f_i^{K\infty}(x_0;\cdot) = f_i^K(x_0;\cdot), i=1,2$.*

PROOF. Let $(y,r_i)\in K_{f_i}^\infty(x_0)$, $i=1,2$. To prove (5.22), it suffices to show that $(y,r_1+r_2)\in K_{f_1+f_2}^\infty(x_0)$. Suppose $(z,s)\in K_{f_1+f_2}(x_0)$. Since $f_1^K(x_0;\cdot)$ and $f_2^K(x_0;\cdot)$ are proper, there exist $s_1,s_2\in\mathbf{1R}$ such that $s=s_1+s_2$ and $(z,s_i)\in K_{f_i}(x_0)$. Then $(y+z,r_i+s_i)\in K_{f_i}(x_0)$. By hypothesis, $K_{f_2}(x_0)=k_{f_2}(x_0)$. Apply Theorem 4.2 with $E=E_1$, $A := K$, $A' := k$, and F the identity mapping on E to deduce that $(y+z,r_1+r_2+s)\in K_{f_1+f_2}(x_0)$. Therefore $(y,r_1+r_2)\in K_{f_1+f_2}^\infty(x_0)$, and the proof of (5.22) is complete. Inclusion (5.23) follows from (5.22) and [**17**, 1.2.5] □

An analogue of Theorem 5.4 for K^∞ can be derived by the above method under the assumption that $f_i^K(x_0;\cdot) = f_i^k(x_0;\cdot)$, $i=2,\dots,n$. Details are left to the reader.

6. Necessary conditions for optimality. The case $A := K$, $A' := k$ in Theorem 4.2 can be applied to prove quite general necessary conditions for local optimality in the abstract mathematical program

$$\text{(P)} \qquad \min\{f(x) | x \in C\}.$$

Our results will rely on the fact that if x_0 is an unconstrained local minimizer for $f\colon E \to \overline{\mathbf{1R}}$, then $f^K(x_0; y) \geq 0$ for all $y \in E$ (see for example [**26**]). We begin with a refinement of optimality conditions given in [**21**, Proposition 4.1; **13**, Theorem 5; **34**, Corollary 3.3].

THEOREM 6.1. *Let $C \subset E$ be epi-Lipschitz-like at $x_0 \in C$, and suppose $f\colon E \to \overline{\mathbf{1R}}$ is finite and Lipschitz-like at x_0, a local minimizer for* (P). *Assume*

$$\text{(6.1)} \qquad \operatorname{dom} f^T(x_0; \cdot) - T_C(x_0) = E.$$

Then

$$\text{(6.2)} \qquad f^K(x_0; y) \geq 0 \quad \textit{for all } y \in k_C(x_0),$$

$$\text{(6.3)} \qquad f^k(x_0; y) \geq 0 \quad \textit{for all } y \in K_C(x_0).$$

PROOF. The point x_0 is an unconstrained local minimizer for the function $f + i_C$. Hence for all $y \in E$, $0 \leq (f + i_C)^K(x_0; y)$. Assumption (6.1) allows us to apply Theorem 4.2 with $A := K$, $A' := k$, $E = E_1$ and F the identity mapping E to obtain

$$0 \leq f^K(x_0; y) + i_C^k(x_0; y),$$
$$0 \leq f^k(x_0; y) + i_C^K(x_0; y)$$

for all $y \in E$. Now if $y \in k_C(x_0)$, $i_C^k(x_0; y) = 0$ and (6.2) follows from the first of these inequalities. Condition (6.3) follows in like manner from the second inequality. □

REMARK 6.2. Theorem 6.1 generates a whole family of necessary conditions. If $A \subset K$ and $A' \subset k$ are tangent cones,

$$\text{(6.4)} \qquad f^A(x_0; y) \geq 0 \quad \text{for all } y \in A'_C(x_0),$$

$$\text{(6.5)} \qquad f^{A'}(x_0; y) \geq 0 \quad \text{for all } y \in A_C(x_0)$$

are necessary conditions for local optimality in (P) under assumption (6.1). In other words, Theorem 6.1 expands the class of "upper convex approximants" [**23**, **14**, **34**] or "approximate quasidifferentials" [**16**] for which optimality conditions can be stated. The cases $A = A' := T$, P, or k^∞ can of course be alternately derived from sum formulae for f^T, f^P, and $f^{k\infty}$.

One important special case of problem (P) is that in which $C := \{x \mid g_i(x) \leq 0, i = 1, \ldots, n\}$, the set of points satisfying a finite number of inequality constraints. Tangent cones of such sets have been calculated (with the help of special cases of Proposition 4.6) in [**33** and **34**]. We list the basic result below.

PROPOSITION 6.3 [**33**, **34**]. *Let $g\colon E \to \overline{\mathbf{1R}}$ be Lipschitz-like at $x_0 \in g^{-1}(0)$. Suppose there exists $y \in E$ with $g^T(x_0; y) < 0$* (*or equivalently, that $0 \notin \partial^T g(x_0)$*). *Define $C := \{x \in E \mid g(x) \leq 0\}$. Then*

$$\text{(6.6)} \qquad K_C(x_0) = \{y \in E \mid g^K(x_0; y) \leq 0\},$$

$$\text{(6.7)} \qquad k_C(x_0) = \{y \in E \mid g^k(x_0; y) \leq 0\},$$

$$\text{(6.8)} \qquad T_C(x_0) \supset \{y \in E \mid g^T(x_0; y) \leq 0\}.$$

PROPOSITION 6.4. *Let $f\colon E \to \overline{\mathbb{R}}$ and $g_i\colon E \to \overline{\mathbb{R}}$, $i = 1, \ldots, n$, be Lipschitz-like at x_0, a local minimizer for*

$$\min\{f(x) \mid g_i(x) \le 0, i = 1, \ldots, n\}. \tag{6.9}$$

Define $I(x_0) := \{j \mid g_j(x_0) = 0\}$. Assume tht g_j is continuous at x_0 for each $j \notin I(x_0)$, that $0 \notin \partial^T g_j(x_0)$ for each $j \in I(x_0)$, and that $\operatorname{dom} f^T(x_0;\cdot)$ *and $\{y \mid g_j^T(x_0; y) \le 0\}$, $j \in I(x_0)$, are in strong general position. Then*

$$f^K(x_0; y) \ge 0 \quad \textit{whenever } g_j^k(x_0; y) \le 0 \textit{ for all } j \in I(x_0), \tag{6.10}$$

$$f^k(x_0; y) \ge 0 \quad \textit{whenever } g_i^K(x_0; y) \le 0 \textit{ for some } i \in I(x_0) \textit{ and } g_j^k(x_0; y) \le 0 \textit{ for all } j \in I(x_0) \setminus \{i\}. \tag{6.11}$$

PROOF. Let $C := \{x \mid g_j(x_0) \le 0, j = 1, \ldots, n\}$ in problem (P), and call $D_j := \{x \mid g_j(x_0) \le 0\}$. For $j \notin I(x_0)$, $T_{D_j}(x_0) = E$ since g_j is continuous at x_0. By (6.8), $T_{D_j}(x_0)$, $j = 1, \ldots, n$, are in strong general position, so we may apply Proposition 4.6 with $A = A' := T$ to deduce that

$$T_C(x_0) \supset \{y \mid g_j^T(x_0; y) \le 0, j \in I(x_0)\}.$$

This inclusion and our strong general position assumption imply that (6.1) holds. Then (6.10) and (6.11) follow from (6.2), (6.3) and Proposition 4.6. □

One can also derive optimality conditions in the form of subgradient inclusions for problem (6.9).

THEOREM 6.5. *Let x_0 be a local minimizer for* (6.9), *and suppose A is a convex tangent cone satisfying* (1.2) *and A_j, $j \in I(x_0)$, are convex tangent cones such that $T \subset A_j \subset k$. In addition to the hypotheses of Proposition* 6.4, *assume that $\partial^{Aj} g_j(x_0) \notin \emptyset$ for each $j \in I(x_0)$. Then there exist $\lambda_j \ge 0$, $j \in I(x_0)$, such that*

$$0 \in \partial^A f(x_0) + \sum_{I(x_0)} (\lambda_j \partial^{Aj} g_j(x_0) \cup 0^+ \partial^{Aj} g_j(x_0)). \tag{6.12}$$

PROOF. Define C and $D_j, j = 1, \ldots, n$, as in Proposition 6.4. As in the proof of Theorem 6.1, for all $y \in E$

$$0 \le f^K(x_0; y) + i_C^k(x_0; y) = f^K(x_0; y) + \sum_{I(x_0)} i_{D_j}^k(x_0; y)$$

since $k_C(x_0) = \bigcap_{I(x_0)} k_{D_j}(x_0)$ by Proposition 4.6. Now call $S_j = \{x \mid g_j^{Aj}(x_0; y) \le 0\}$ for $j \in I(x_0)$. By (6.7), $S_j \subset k_{D_j}(x_0)$. Thus $0 \le f^A(x_0; y) + \sum_{I(x_0)} i_{S_j}(y)$ for all $y \in E$. It follows that

$$0 \in \partial\left(f^A(x_0; \cdot) + \sum_{I(x_0)} i_{S_j}(\cdot)\right)(0) = \partial^A f(x_0) + \sum_{I(x_0)} S_j^0$$

since our strong general position assumption allows us to apply [**17**, Theorem 1.2.5]. By [**24**, Theorems 23.7, 9.6], (6.12) holds. □

The subdifferential calculus developed here and in [**33**] also enables one to handle objective functions f of various forms—for example, $f := g \circ G$, where G is strictly differentiable ([**14**] shows the importance of this form); $f := g/h$, where g and h are positive-valued and h is continuous [**33**]; and $f := \max_{1 \le i \le n} h_i$.

7. Conclusions. We have demonstrated that two tangent cones, k^∞ and P, which can give a sharper local approximation to a set than the Clarke tangent cone T, have as strong a subdifferential calculus as T. We have also shown that any pair of convex tangent cones A and A' with $T \subset A \subset K$, $T \subset A' \subset k$ can be used in place of T in necessary optimality conditions. Nevertheless, the hypotheses in our results seem to necessarily involve T (as do those in [**15**]), which suggests that the Clarke tangent cone occupies a unique and essential position in the theory of nonsmooth analysis. In summary, no one tangent cone possesses all desirable properties; the most complete theory of nonsmooth analysis and optimization combines several tangent cones.

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