

SCALAR CURVATURE AND WARPED PRODUCTS OF RIEMANN MANIFOLDS

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ABSTRACT. We establish the relationship between the scalar curvature of a warped product $M \times_f N$ of Riemann manifolds and those ones of M and N . Then we search for weights f to obtain constant scalar curvature on $M \times_f N$ when M is compact.

1. Introduction. Let $M = (M_m, g)$ and $N = (N_n, h)$ be two Riemann manifolds. For $f \in C^\infty(M)$, $f > 0$ on M , we consider the *warped product* $M \times_f N = ((M \times N)_{m+n}, g + f^2h)$ and show the relationship between the scalar curvatures R on M , H on N and \tilde{R} on $M \times_f N$. This relationship is a nonlinear partial differential equation satisfied by a power of the *weight* f . In the case M is compact and connected, we obtain a geometric interpretation of the principal eigenfunction and eigenvalue of the canonical elliptic operator $-\Delta + R/2$, where Δ denotes the laplacian on M . Finally we consider the question of finding a weight f such that $M \times_f N$ has constant scalar curvature. This question is equivalent to find a positive solution to a nonlinear eigenvalue problem.

The notion of warped product $M \times_f N$ generalizes that of a surface of revolution. It was introduced in [B-O] for studying manifolds of negative curvature (cf. [Z] for other applications). The Riemann metric $\tilde{g} = g + f^2h$ on $M \times_f N$ is defined for pairs of vector fields \tilde{X}, \tilde{Y} on $M \times N$ by

$$\tilde{g}(\tilde{X}, \tilde{Y}) = g(\pi_*\tilde{X}, \pi_*\tilde{Y}) + f^2(\pi(\cdot))h(\omega_*\tilde{X}, \omega_*\tilde{Y})$$

where π and ω are the canonical projections over M and N respectively.

We denote by Δ_g or Δ the laplacian (or Laplace-Beltrami) operator on (M_m, g) with local expression $\Delta_g u = \nabla^i \nabla_i u = |g|^{-1/2} \partial_i (g^{ij} |g|^{1/2} \partial_j u)$, for $u \in C^2(M)$ (cf. [Au1, B-G-M]). Thus $\Delta u = u''$ for a real valued function u on $M = \mathbb{R}$.

2. The equation. Given a metric $g' = kg$ with $k \in C^\infty(M)$, $k > 0$ on M , g' is said to be *conformal* to g . It is known that the scalar curvature R' on (M_m, g') is related to R , the one on (M_m, g) , by the Yamabe equation

$$(Ya) \quad -\frac{4(m-1)}{m-2} \Delta_g u + Ru = R' u^{(m+2)/(m-2)}$$

where $k = u^{4/(m-2)}$, whenever $m \geq 3$ [4, Au1, p. 126].

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If we consider now the case of a warped product we have

THEOREM 2.1. *Let R, H and \tilde{R} denote the scalar curvature on M, N and $M \times_f N$ respectively. Then the following equality holds:*

$$(2.1) \quad -\frac{4n}{n+1} \Delta_g u + Ru + Hu^{(n-3)/(n+1)} = \tilde{R}u$$

where $u = f^{(n+1)/2}$.

PROOF. We write $\tilde{g} \equiv g + f^2 h = f^2(f^{-2}g + h)$, so \tilde{g} is conformal to $\tilde{g} = f^{-2}g + h$ on $M \times N$ and $f^{-2}g$ is conformal to g on M .

Supposing $m \geq 3$, we apply (Ya) in M to obtain that f satisfies

$$(2.2) \quad -\frac{4(m-1)}{m-2} \Delta_g \eta + R\eta = \eta^{(m+2)/(m-2)} \hat{R}$$

with $\eta^{4/(m-2)} = f^{-2}$ and where \hat{R} denotes the scalar curvature on $(M_m, f^{-2}g)$. As $m+n \geq 3$, we use (Ya) in $M \times N$. Hence f also satisfies

$$(2.3) \quad -\frac{4(m+n-1)}{m+n-2} \Delta_{\tilde{g}} \psi + \tilde{R}\psi = \tilde{R}\psi^{(m+n+2)/(m+n-2)}$$

with $\psi^{4/(m+n-2)} = f^2$, where \tilde{R} denotes the scalar curvature on $((M \times N)_{m+n}, \tilde{g})$ and $\Delta_{\tilde{g}}$ the corresponding laplacian.

From $\psi \in C^\infty(M)$ we deduce that $\Delta_{\tilde{g}} \psi \equiv \Delta_{f^{-2}g+h} \psi = \Delta_{f^{-2}g} \psi$. Working in local coordinates

$$\Delta_{f^{-2}g} \psi = |f^{-2}g|^{-1/2} \partial_i [(f^{-2}g)^{ij} |f^{-2}g|^{1/2} \partial_j \psi]$$

with $|f^{-2}g| = \det(f^{-2}g_{ij}) = f^{-2m}|g|$ and $(f^{-2}g)^{ij} = f^2 g^{ij}$. Hence

$$(2.4) \quad \begin{aligned} \Delta_{f^{-2}g} \psi &= \eta^{-2m/(m-2)} |g|^{-1/2} \partial_i [\eta^2 g^{ij} |g|^{1/2} \partial_j \psi] \\ &= [\eta \Delta_g \psi + 2g^{ij} \partial_i \eta \partial_j \psi] \eta^{-(m+2)/(m-2)}. \end{aligned}$$

On the other hand

$$\Delta_g(\eta\psi) = \eta \Delta_g \psi + \psi \Delta_g \eta + 2g^{ij} \partial_i \eta \partial_j \psi$$

and from (2.4) we get

$$(2.5) \quad \eta^{(m+2)/(m-2)} \Delta_{\tilde{g}} \psi = \Delta_g(\eta\psi) - \psi \Delta_g \eta.$$

We have that $\tilde{R} = \hat{R} + H$, because we consider the usual product of $(M_m, f^{-2}g)$ by (N_n, h) . Using this in (2.3), multiplying by $\eta^{(m+2)/(m-2)}$, we obtain from (2.5)

$$\begin{aligned} &-\frac{4(m+n-1)}{m+n-2} (\Delta_g(\eta\psi) - \psi \Delta_g \eta) + \eta^{(m+2)/(m-2)} (\hat{R} + H)\psi \\ &= \tilde{R}\psi^{(m+n+2)/(m+n-2)} \eta^{(m+2)/(m-2)}. \end{aligned}$$

From (2.2) we arrive at

$$\begin{aligned} &-\frac{4n}{(m+n-2)(m-2)} \psi \Delta_g \eta - \frac{4(m+n-1)}{(m+n-2)} \Delta_g(\eta\psi) + R\eta\psi + H\eta^{(m+2)/(m-2)} \psi \\ &= \tilde{R}\psi\eta\psi^{4/(m+n-2)} \eta^{4/(m-2)}. \end{aligned}$$

Recalling that $\psi^{4/(m+n-2)}\eta^{4/(m-2)} = 1$, denoting $u = f^{(n+1)/2}$, replacing in terms of u in this last equality, then multiplying by $u^{1-n/(n+1)}$, we obtain

$$(2.6) \quad \begin{cases} -\frac{4n}{(m+n-2)(m-2)}u^{(m+n-1)/(n+1)}\Delta_g u^{-(m-2)/(n+1)} \\ -\frac{4(m+n-1)}{m+n-2}u^{1/(n+1)}\Delta_g u^{n/(n+1)} + Ru + Hu^{(n-3)/(n+1)} = \tilde{R}u. \end{cases}$$

For any $\alpha \neq 0$, $v \in C^\infty(M)$, $v > 0$ on M satisfies

$$(2.7) \quad \Delta_g v^\alpha = \alpha(\alpha - 1)v^{\alpha-2}\nabla^i v \nabla_i v + \alpha v^{\alpha-1}\Delta_g v.$$

Choosing first $\alpha = n/(n+1)$, then $\alpha = -(m-2)/(n+1)$ in (2.7) (for $v = u$) we obtain from (2.6) the desired result (2.1).

REMARK. Formula (2.1) holds even for $m = 1, 2$. For $m = 1$, it is easily deduced from a similar formula in [E-1]. For $m = 2$, the proof is similar but Yamabe equation is for the conformal change $g' = e^u g$:

$$(Ya)_2 \quad -\Delta_g u + R = R'e^u$$

(cf. [M, Au1, p. 119]). (2.1) could also be deduced from a formula in [B-O, p. 26] but this formula is a consequence of an unwritten 15-term calculation and besides our method of proof is different.

3. Constant scalar curvature. Let the scalar curvatures R of M and H of N be fixed, we look for a weight $f \in C^\infty(M)$, $f > 0$, on M such that the warped product $M \times_f N$ has constant scalar curvature \tilde{R} and then which constants are attained. Taking account of $R \in C^\infty(M)$, $u = f^{(n+1)/2} \in C^\infty(M)$ and $H \in C^\infty(N)$, it follows easily from Theorem 2.1 that for \tilde{R} to be a constant λ , it is necessary that N have constant scalar curvature, still denoted H .

The simplest case is $H = 0$.

THEOREM 3.1. *Let M be compact and connected. Suppose N of zero scalar curvature. Then there exists a weight f such that the scalar curvature \tilde{R} on $M \times_f N$ is a constant λ_1 . f is unique up to a positive multiplicative constant, λ_1 is unique and is given by*

$$\lambda_1 = \inf \left\{ \int_M \left(\frac{4n}{n+1} |\nabla v|^2 + Rv^2 \right) dV; v \in H^1(M), \int_M v^2 dV = 1 \right\}$$

where $H^1(M) = \{v \in L^2(M); |\nabla v|^2 \equiv \nabla^i v \nabla_i v \in L^1(M)\}$ is the Sobolev space of order 1.

PROOF. From (2.1) we deduce that we search $\lambda \in \mathbf{R}$ and $u \in C^\infty(M)$, $u > 0$ on M such that

$$(3.1) \quad Lu = \lambda u \quad \text{on } M$$

where $Lu = -4n\Delta_g u/(n+1) + Ru$.

It is well known that this linear eigenvalue problem on M compact and connected has only one nonnegative solution u_1 with $\max_M u_1 = 1$; cf. [Au1, p. 137] (in fact $u_1 > 0$ on M) u_1 is the so-called *principal eigenfunction* of the elliptic operator L . The corresponding eigenvalue λ_1 is simple and is called the *principal eigenvalue*.

Hence $f = u_1^{2/(n+1)} > 0$ on M is the weight we are searching. Any other solution is of the form rf , $r \in \mathbf{R}_0^+$, because λ_1 is simple. The formula for λ_1 is classical.

The case of warping M with a circle, i.e. $M \times_f S^1$ gives a geometric interpretation of the principal eigenfunction u_1 of $-\Delta + R/2$ and its corresponding eigenvalue λ_1 , which in the special case of M 2-dimensional can be expressed in terms of the classical Gaussian curvature on M .

COROLLARY. *Given a compact, connected $M = (M_2, g)$ with Gaussian curvature K and Laplace-Beltrami operator Δ , then the principal eigenfunction u_1 of the canonical elliptic operator $-\Delta + K$ holds the property that the warped product $M \times_{u_1} S^1$ has a scalar curvature constantly equal to the principal eigenvalue λ_1 of $-\Delta + K$.*

PROOF. Take $n = 1$ in (2.1). Recall that $H = 0$ in S^1 and $R/2 = K$ in a 2-dimensional manifold.

The case $H \in \mathbf{R}$, $H < 0$ is similar to $H = 0$ because the half-line $\{(\lambda_1, ru_1), r > 0\}$ is deformed into a curve $\{(\lambda, f(\lambda)); \lambda < \lambda_1\}$:

THEOREM 3.2. *Let $M = (M_m, g)$ be compact and connected. Suppose $N = (N_n, h)$ of constant negative scalar curvature H and assume $n \geq 3$. Let λ_1 denote the principal eigenvalue of $-4n\Delta_g/(n + 1) + R$. Then for each $\lambda < \lambda_1$ there exists a unique weight $f = f(\lambda)$ such that $M \times_f N$ has constant scalar curvature λ . No constant $\geq \lambda_1$ may be curvature of $M \times_f N$ for any weight f .*

PROOF. We look for positive solutions u in $C^\infty(M)$ of

$$(3.2) \quad Lu + Hu^\alpha = \lambda u \quad \text{on } M$$

where $Lu = -4n\Delta_g u/(n + 1) + Ru$, $0 \leq \alpha = (n - 3)/(n + 1) < 1$ (cf. (2.1)).

Let us still denote u_1 the positive eigenfunction of $Lu = \lambda u$ but with L^2 -norm 1, i.e. $\int_M u_1^2 dV = 1$. If u is a positive solution of (3.2), then multiplying (3.2) by u_1 and integrating by parts (L is selfadjoint), we obtain

$$\lambda_1 \int_M uu_1 dV + H \int_M u^\alpha u_1 dV = \lambda \int_M uu_1 dV.$$

Hence

$$(3.3) \quad (\lambda - \lambda_1) \int_M uu_1 dV = H \int_M u^\alpha u_1 dV.$$

Then $H < 0$ necessarily gives $\lambda < \lambda_1$.

Let us fix $\lambda < \lambda_1$. As $0 \leq \alpha < 1$, we have that

$$(L - \lambda I)\underline{t}u_1 \leq |H|\underline{t}^\alpha u_1^\alpha$$

for $\underline{t} \in \mathbf{R}^+$ small enough, so $\underline{t}u_1$ is a subsolution of (3.2). Also

$$(L - \lambda I)\bar{t}u_1 \geq |H|\bar{t}^\alpha u_1^\alpha$$

for $\bar{t} \in \mathbf{R}^+$ big enough, so we have a supersolution $\bar{t}u_1 \geq \underline{t}u_1$.

The operator $(L - \lambda I): C^{2,\beta}(M) \rightarrow C^\beta(M)$ is an isomorphism and its inverse $(L - \lambda I)^{-1}$ is continuous for the C^0 -norm on $C^\beta(M)$ and the $C^{1,\beta}$ -norm on $C^{2,\beta}(M)$. Besides $(L - \lambda I)^{-1}$ is strongly positive, i.e. $w \in C^\beta(M)$, $w \geq 0$, $w \not\equiv 0$, implies $(L - \lambda I)^{-1}w > 0$ on M . So it extends uniquely to a compact map, still denoted

$(L - \lambda I)^{-1}$ from $C(M)$ into $C(M)$, which is still strongly positive (cf. [Am]). For each $w \in C(M)$, $(L - \lambda I)^{-1}w$ is a weak solution of $Lu - \lambda u = w$.

The nonlinear compact and order preserving operator $v \rightarrow (L - \lambda I)^{-1}(|H||v|^\alpha): C(M)^+ \rightarrow C(M)^+$ leaves invariant the order interval $[\underline{t}u_1, \bar{t}u_1] \subset C(M)$, so it has a fixed point $u \in C(M)$, with $0 < \underline{t}u_1 \leq u \leq \bar{t}u_1$, i.e. $u = (L - \lambda I)^{-1}(|H||u|^\alpha)$, hence by a classical bootstrap argument $u \in C^\infty(M)$; in particular u is a classical positive solution of (3.2) (cf. [Am] in the Neumann case for details). Finally the nonlinearity $|H|t^\alpha$ in (3.2) is such that t^α/t is strictly decreasing in $t > 0$; hence u is unique [L, Bere]. As $f = u^{2/(n+1)}$ we obtain the uniqueness.

The situation seems more complicated when $H > 0$.

THEOREM 3.3. *Let $M = (M_m, g)$ be compact and connected. Suppose $N = (N_n, h)$ of constant positive curvature H and assume $n \geq 3$. Let λ_1 denote the principal eigenvalue of $-4n\Delta_g/(n + 1) + R$. Then for each λ in some interval $(\lambda_1, \lambda_1 + \delta)$ there exists a weight $f = f(\lambda)$ such that $M \times_f N$ has λ as scalar curvature. No constant $\leq \lambda_1$ may be curvature of $M \times_f N$ for any f .*

PROOF. If $u > 0$ on M is a solution of (3.2) with $H > 0$, then (3.3) gives that $\lambda > \lambda_1$ is a necessary condition.

Denote $\|u\| = \max_M |u|$, $v = u/\|u\|^2$ for $u \neq 0$, i.e. $u = v/\|v\|^2$. We will obtain solutions as a bifurcation from infinity near $\lambda = \lambda_1$ (cf. [R-2]). Multiplying our equation (3.2) by $1/\|u\|^2$, replacing by v , we are reduced to finding positive solutions of

$$(3.4) \quad Lv + av = \lambda v - H\|v\|^{2(1-\alpha)}v^\alpha + av$$

where $\lambda \in \mathbf{R}$ is a parameter and $a \in \mathbf{R}^+$ is a fixed number chosen big enough so that the operator $(L + aI): C^{2,\beta}(M) \rightarrow C^\beta(M)$ is an isomorphism with positive inverse. Hence, as in the preceding proof, $(L + aI)^{-1}: C(M) \rightarrow C(M)$ is a linear compact strongly positive operator such that $(L + aI)^{-1}v > 0$ on M if $v \geq 0$, $v \not\equiv 0$ on M .

We now search for pairs $(\lambda, v) \in \mathbf{R} \times C(M)$ with $v > 0$ on M , solutions of

$$(3.5) \quad v = \mu(L + aI)^{-1}v - H(L + aI)^{-1}f(v)$$

where $\mu = \lambda + a$, $H > 0$, and $f(v) = \|v\|^{2(1-\alpha)}|v|^\alpha$. We easily see that $H(L + aI)^{-1}f(v) = o(\|v\|)$ in $C(M)$ for $\|v\|$ near 0. Moreover from $Lu_1 = \lambda_1 u_1$, it follows that $\mu_1 = \lambda_1 + a$ is a simple characteristic value of the compact map $(L + aI)^{-1}$. Then by the Rabinowitz bifurcation theorem [R-1], there exists a maximal connected closed subset C_{μ_1} of

$$S = \text{adh}_{\mathbf{R} \times C(M)}\{(\lambda, v) \text{ solutions of (3.5) with } v \neq 0\}$$

such that $(\mu_1, 0) \in C_{\mu_1}$ and either

- (i) C_{μ_1} is bounded in $\mathbf{R} \times C(M)$ or
- (ii) C_{μ_1} meets $(\hat{\mu}, 0)$ with $\mu_1 \neq \hat{\mu} \in \{\lambda + a; \lambda \text{ eigenvalue of } L\}$.

We write $v = \gamma u_1 + w$, with $\gamma = v(x_o)$, where $u_1(x_o) = \|u_1\|$ and $w(x_o) = 0$. For $\xi > 0$, $0 < \eta < 1$, the open sets in $\mathbf{R} \times C(M)$: $K_{\xi,\eta}^+, K_{\xi,\eta}^-$ defined by

$$(3.6) \quad K_{\xi,\eta}^\pm = \{(\mu, v) \in \mathbf{R} \times C(M); |\mu - \mu_1| < \xi, \pm\gamma < \eta\|v\|\}$$

satisfy both $K_{\xi,\eta}^\pm \cap C_{\mu_1} \neq \emptyset$ for ξ small enough as in Theorem 1.25 in [R-1]. Consequently the maximal connected closed subset (or continuum) of C_{μ_1} contained

in $\{(\mu, \gamma u_1 + w); \gamma > 0\} \cup \{(\mu_1, 0)\}$ is nontrivial; we denote it as $C_{\mu_1}^+$. Besides from $H(L + aI)^{-1}f(v) = o(\|v\|)$ we deduce that $\|w\| = o(\gamma)$ near $\gamma = 0$, so $v = \gamma u_1 + w \in C_{\mu_1}^+$ is (strictly) positive on M for $\gamma > 0$ small enough.

We have then obtained a “branch” of $C(M)$ -solutions (μ, v) , $v > 0$ on M of (3.5). Going back to $u = v/\|v\|^2$, $\lambda = \mu - a$, we have weak $C(M)$ -solutions of (3.2): (λ, u) for λ near λ_1 , $u > 0$ on M , $\{\|u\|\}$ unbounded. By a regularity argument, each u is a C^∞ classical solution and by the necessary condition we have $\lambda > \lambda_1$.

REMARK 1. The nonlinearity $v \rightarrow (L + aI)^{-1}f(v) = \|v\|^{2(1-\alpha)}(L + aI)^{-1}(|v|^\alpha)$ is not always differentiable for $\|v\|$ near 0, so we cannot directly apply the results in [R-1, 2].

REMARK 2. The operator $F(\mu, v) = (L + aI)^{-1}(\mu v - Hf(v))$, $H > 0$, generally transforms a positive v into a nonpositive $F(\mu, v)$, so some known results on maps of cones into cones [L, T] do not apply. More precisely, when N is 3-dimensional, (3.2) can be written as

$$(3.7) \quad (-3\Delta_g + RI - \lambda I)u = -H$$

and Theorem 3.3 is a consequence of the antimaximum principle (cf. [C-P]) which says that $\lambda > \lambda_1$ and near λ_1 implies that negative data $-H$ on M gives a positive solution u , on the contrary of the maximum principle.

Let us denote

$$C_\infty = \{(\lambda, u); u \geq 0, u \neq 0, (\lambda + a, u/\|u\|^2) \in C_{\mu_1}^+\} \subset \mathbf{R} \times C(M)^+,$$

the nonnegative weak $C(M)$ -solutions bifurcating from infinity. We know now that $H > 0$ fixed implies that $u > 0$ for $\lambda > \lambda_1$, λ near λ_1 . We can say more on $\|u\|$ in C_∞ .

THEOREM 3.4. *Let M, N and λ_1 be as in Theorem 3.3. For any $0 < \varepsilon < A$, the set $\{(\lambda, u) \in C_\infty; \lambda_1 + \varepsilon \leq \lambda \leq \lambda_1 + A\}$ is bounded in $\mathbf{R} \times C(M)$.*

PROOF. Suppose not. Then there exists a sequence $(\lambda_n, u_n) \in C_\infty$ with $\lambda_n \in [\lambda_1 + \varepsilon, \lambda_1 + A]$, $u_n \geq 0$ on M , $u_n \neq 0$, $\lim_n \lambda_n = \lambda$ and $\lim_n \|u_n\| = +\infty$, which satisfies

$$u_n/\|u_n\| = (\lambda_n + a)(L + aI)^{-1}(u_n/\|u_n\|) - H(L + aI)^{-1}(u_n^\alpha/\|u_n\|).$$

$(L + aI)^{-1}: C(M) \rightarrow C(M)$ being compact we may suppose, up to a subsequence, that $\lim_n (L + aI)^{-1}(u_n/\|u_n\|) = \tilde{u}$ in $C(M)$. From $\lim_n (u_n^\alpha/\|u_n\|) = 0$ in $C(M)$, we obtain that $u_n/\|u_n\|$ tends to some u in $C(M)$, $u \geq 0$, $\|u\| = 1$, so we have

$$u = (\lambda + a)(L + aI)^{-1}u$$

i.e., by regularity properties, $-4n\Delta u/(n + 1) + Ru = \lambda u$, $u \geq 0$, $u \neq 0$, hence $u = u_1$ and $\lambda = \lambda_1$ by uniqueness. This contradicts $\lambda \geq \lambda_1 + \varepsilon$.

REMARK. If the set C_∞ meets $(\hat{\mu}, 0) \in \mathbf{R} \times C(M)$ so does $C_{\mu_1}^+$. Then $\hat{\mu} \neq \mu_1$ and $\hat{\mu}$ is necessarily a characteristic value of $(L + aI)^{-1}$ (cf. [R-1]). Hence $\hat{\mu} = \hat{\lambda} + a$, $\hat{\lambda} > \lambda_1$, $\hat{\lambda}$ an eigenvalue of $-4n\Delta/(n + 1) + R$. We would then have a sequence $(\mu_k, v_k) \in C_{\mu_1}^+$, such that $\lim_k \mu_k = \hat{\mu}$, $\lim_k v_k = \lim_k u_k/\|u_k\|^2 = 0$ in $C(M)$, $v_k \neq 0$, i.e. $(\lambda_k, u_k) \in C_\infty$, with $\lambda_k = \mu_k - a \geq \lambda_1 + \varepsilon$, $\{\lambda_k\}$ bounded and $\lim_k \|u_k\| = +\infty$, contradicting Theorem 3.4. Then C_∞ never meets some $(\lambda, 0)$.

For $n = 3$, $H > 0$ we are able to characterize the case that for any $\lambda > \lambda_1$ there is a $u(\lambda) > 0$ on M with $(\lambda, u(\lambda)) \in C_\infty$.

THEOREM 3.5. *Let M, N and λ_1 be as in Theorem 3.3 with $\dim N = 3$. Then any constant $\lambda \in (\lambda_1, \infty)$ is scalar curvature of $M \times_f N$ for some weight f if and only if the scalar curvature R on M is constant.*

PROOF. If R is constant and $\lambda > \lambda_1$, $u(\lambda) = H/(\lambda - \lambda_1)$ is a solution of (3.7), hence $f = u^{1/2}$ is a constant weight which gives λ as scalar curvature on $M \times_f N$.

Conversely, suppose that for each $\lambda > \lambda_1$ there corresponds a weight f i.e. a solution u of (3.7). This means for $\lambda = \lambda_k$ ($k > 1$) an eigenvalue of the selfadjoint operator $-3\Delta_g + R$, that the second member in equality (3.7) is orthogonal to the corresponding eigenspace. Hence H is orthogonal to all eigenfunctions except the principal one u_1 , so $H = tu_1$ with $t > 0$ by the completeness of an orthonormal system of eigenfunctions, i.e. u_1 is constant and (3.7) with $u = u_1$ gives R constant.

Ejiri proves in [E-2] that there exist countable immersions of $S^1 \times S^n$ into S^{n+2} such that $S^1 \times S^n$ is a warped product of constant scalar curvature $n(n + 1)$ with respect to the induced metric.

On one hand, if we consider equation (2.1) in $S^1 \times S^n$, we have $R = 0$ on S^1 and $H = n(n - 1)$ on S^n with the usual metric. So [E-2] gives a countable number of positive solutions of

$$(3.8) \quad -\frac{4n}{n + 1}u'' + Hu^{(n-3)/(n+1)} = n(n + 1)u.$$

In particular for $n = 3$, this equation becomes

$$(3.9) \quad -3u'' - 12u = -H$$

with $H = 6$.

On the other hand, equation (3.7) reduces on $S^1 \times S^3$ to

$$(3.10) \quad -3u'' - \lambda u = -H.$$

This last equation has a unique solution $u = u(\lambda)$ for $\lambda > \lambda_1$, λ not an eigenvalue. But $\lambda = 12$ is an eigenvalue of (-3) times the laplacian on S^1 , the eigenvalues being $\{\lambda_k = 3(k - 1)^2, k = 1, 2, \dots\}$, so the countable solutions of (3.9) given in [E-2] appear for $\lambda = \lambda_3$. But then we have uncountable positive solutions of (3.10) for $\lambda = \lambda_k, k > 1$, they are

$$(3.11) \quad v = H/\lambda_k + tu_k$$

where $-3u''_k = \lambda_k u_k$ and $|t|$ small enough so that v is positive. We have proved

THEOREM 3.6. *Let M, N and λ_1 be as in Theorem 3.3 with $\dim M = 1$ and $\dim N = 3$. Given an eigenvalue $\lambda_k, k > 1$, of $-3\Delta_g$ there exist uncountable weights f such that $M \times_f N$ has λ_k as scalar curvature.*

REMARK. The $u = f^2$ from this theorem are secondary bifurcations of the branch $\{H/\lambda\}$ of solutions considered up to now, at points $\lambda = \lambda_k$. In the case $M = S^1$ and $N = S^n$ with $n > 3$, the curvature on $S^1 \times_f S^n$ given in [E-2] is $n(n + 1)$ with $u = f^{(n+1)/2}$ a solution of (3.8). This constant $n(n + 1)$ is far from the scalar curvatures obtained from Theorem 3.3 which are near zero, i.e. our solutions of (2.1) and those in [E-2] on $S^1 \times S^n$ seem to be of different type.

Theorem 3.2 still holds for N of dimension 2 with a different proof inspired from [C-R-T].

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